

On the convexity of the value function for a class of nonconvex variational problems: existence and optimality conditions ^{*}

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Abstract

In this paper we study a class of perturbed constrained nonconvex variational problems. Its (optimal) value function is proved to be convex and then several related properties are obtained. Existence, strong duality results and necessary/sufficient optimality conditions are established. Moreover, it is shown that local minima are global. Such results are given in terms of the Hamiltonian function. Finally various examples are exhibited showing the wide applicability of our main results.

Key words. Nonconvex variational problems, Liapunov theorem, existence of minima, Hamiltonian, strong duality, local minima.

1 Introduction and formulation of the problem

We consider, given $a \in \mathbb{R}^m$, the following class of minimization problems ($P(a)$):

$$\inf \left\{ \int_0^1 f_0(t, z(t)) dt : z \in K(a) \right\}, \quad (1.1)$$

where

$$K(a) \doteq \left\{ z \in L^1([0, 1], \mathbb{R}^n) : z \in \mathcal{Z}, \int_0^1 g_0(t, z(t)) dt \in -W + a \right\}. \quad (1.2)$$

^{*}This research, for the first author, was supported in part by CONICYT-Chile through FONDECYT 112-0980 and BASAL projects, CMM, Universidad de Chile. The first two authors thank the ECOS project for their support.

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Here, $W \subseteq \mathbb{R}^m$ is nonempty, closed and convex; \mathcal{Z} is the set of functions $z \in L^1([0, 1], \mathbb{R}^n)$ satisfying $z(t) \in Z(t)$ for a. e. $t \in [0, 1]$, with $Z : [0, 1] \rightrightarrows \mathbb{R}^n$ a measurable set-valued mapping with nonempty and closed values; $f_0 : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, $g_0 : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $f_0(t, \cdot)$ is lower semicontinuous and $g_0(t, \cdot)$ is continuous for a.e. $t \in [0, 1]$; f_0 is a Borel function and $g_0(\cdot, z)$ is measurable (with respect to Lebesgue measure) for all $z \in \mathbb{R}^n$ such that $g_0(\cdot, z(\cdot)) \in L^1([0, 1], \mathbb{R}^m)$ for all $z \in \mathcal{Z}$. We consider the functions $f : L^1([0, 1], \mathbb{R}^n) \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : L^1([0, 1], \mathbb{R}^n) \rightarrow \mathbb{R}^m$ defined by

$$f(z) \doteq \int_0^1 f_0(t, z(t))dt, \quad g(z) \doteq \int_0^1 g_0(t, z(t))dt.$$

Furthermore, we impose the following assumptions on f_0 :

- there exists $z_0 \in \mathcal{Z}$ such that

$$\int_0^1 f_0(t, z_0(t))dt < +\infty; \quad (1.3)$$

- there exist $\alpha \in \mathbb{R}^n$ and $\beta \in L^1([0, 1], \mathbb{R})$ satisfying

$$f_0(t, z) \geq \langle \alpha, z \rangle + \beta(t), \quad \text{for a.e. } t \in [0, 1], \text{ all } z \in \mathbb{R}^n. \quad (1.4)$$

Under the latter assumption, $f(z) > -\infty$ for all $z \in L^1([0, 1], \mathbb{R}^n)$.

Problems like (1.1) subsume an important class of variational problems, namely

$$\inf \left\{ \int_0^1 f_0(t, \dot{x}(t))dt : x \in W^{1,1}([0, 1], \mathbb{R}^n) : x(0) = a, x(1) = b \right\}, \quad (1.5)$$

where $W^{1,1}([0, 1], \mathbb{R}^n)$ denotes the space of absolutely continuous functions from $[0, 1]$ to \mathbb{R}^n , and a, b are given vectors in \mathbb{R}^n .

Several models in Mathematical economics can be written in the form of (1.1), see [1] for instance. The classical existence result due to Tonelli requires the convexity and superlinear growth assumptions on $f_0(t, \cdot)$, which imply the weak lower semicontinuity of the integral functional and the weak compactness of its sublevel sets. Our goal is to avoid such assumptions by analyzing carefully the value function associated with (1.1) depending on a , which allows us to consider integrands with slow growth.

The particular case $g_0(t, z) = z$ and $W = \{0\}$ was considered in [5], and under convexity of $f_0(t, \cdot)$ in [7].

One of the main goals of the present paper is, after a carefully analysis of the value function, to provide a necessary and sufficient optimality condition of zero-order for a feasible solution to problem $(P(0))$ to be optimal (Corollary 3.7), along with sufficient and/or necessary conditions for the same problem via the Hamiltonian.

The structure of the paper is as follows. Next section deals with some basic notations, definitions and preliminaries; in particular, a variant of the Liapunov convexity

theorem (Theorem 2.1), suitable for our purpose, is recalled. Section 3 describes the most important properties of the value function (including convexity) to be used in subsequent sections. In Section 4, we establish necessary optimality conditions via the subdifferential of the optimal value function. Some of the results of the preceding section are applied in Section 5 to prove that local minimality implies global for the problem (1.1). In connection to assumption (5.1), Section 6 provides several equivalent conditions implying the previous assumption. Finally Section 7 establishes some formulas for computing the value function via the Hamiltonian, and some existence results as well.

2 Basic notations and preliminaries

Given two vectors a, b in \mathbb{R}^n , $\langle a, b \rangle$ denotes for its inner or scalar product. A set $P \subseteq \mathbb{R}^n$ is said to be a cone if $tP \subseteq P$, for all $t \geq 0$. For a given $A \subseteq \mathbb{R}^n$: \bar{A} , $\text{co}(A)$, $\text{int } A$, $\text{bd } A$, stand for the closure of A , the convex hull of A (the smallest convex set containing A , topological interior of A , boundary of A , respectively. Furthermore $\text{cone}(A)$ denotes the smallest cone containing A , that is,

$$\text{cone}(A) = \bigcup_{t \geq 0} tA,$$

whereas $\overline{\text{cone}}(A)$ denotes the smallest closed cone containing A : obviously $\overline{\text{cone}}(A) = \overline{\text{cone}(A)}$.

Moreover, $x \in A$ is said to be a relative interior point of A if $\overline{\text{cone}}(A - x)$ is vector space (see for instance [2]). The set of relative interior points of A is denoted by $\text{ri } A$. It is well-known that, in case A is convex, $x \in \text{ri } A$ if and only if x is an interior point with respect to the affine hull of A , or equivalently if $N_A(x)$ is a vector space, where $N_A(x) \doteq \{\xi \in \mathbb{R}^n : \langle \xi, a - x \rangle \leq 0, \forall a \in A\}$, is the (outward) normal cone to A at $x \in A$.

The positive polar of the convex cone $P \subseteq \mathbb{R}^n$ is defined by:

$$P^* \doteq \{y^* \in \mathbb{R}^n : \langle y^*, x \rangle \geq 0, \forall x \in P\}.$$

We now recall an extension of the Lyapunov theorem proved in [11]. This plays an important role in the existence theory for optimal control problems without convexity assumptions, see for instance [5] and references therein.

Given a set $K \subseteq L^1([0, 1], \mathbb{R}^k)$, define the set

$$I(K) \doteq \left\{ \int_0^1 \phi(t) dt : \phi \in K \right\}.$$

K is said to be decomposable if, for every measurable set $B \subseteq [0, 1]$ and all $u, v \in K$:

$$u \cdot \chi_B + v \cdot \chi_{[0,1] \setminus B} \in K,$$

where χ_B is the characteristic function of the set B , i.e. $\chi_B(x) = 1$ if $x \in B$, and $\chi_B(x) = 0$ elsewhere.

The next theorem is taken from [11] and provides a simple existence result for problem (1.1) as a consequence of Corollary 3.3, as Remark 3.4 shows.

Theorem 2.1. ([11, Theorem 4]) *If $K \subseteq L^1([0, 1], \mathbb{R}^k)$ is decomposable, then $I(K)$ is convex and $I(K) = I(\text{co } K)$. If, in addition, K is (strong) closed and the closure of $I(K)$ contains neither a line nor an extremal halfline, then $I(K)$ is closed.*

In what follows, we recall some main facts about envelopes of functions. Given $h : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$, \bar{h} , $\overline{\text{co}} h$ stand for the greatest lower semicontinuous function not larger than h and for the greatest convex and lower semicontinuous function not larger than h , respectively. To be coherent with our previous notation we need the following definition of epigraph of a function

$$\text{epi } h \doteq \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : h(x) \leq t\}.$$

Denotes also

$$\text{epi}^0 h \doteq \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : h(x) < t\}.$$

In case h is convex, we have ([14, Lemma 7.3])

$$\text{ri}(\text{epi } h) = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : x \in \text{ri}(\text{dom } h), h(x) < t\}. \quad (2.1)$$

It is known that

$$\text{epi } \bar{h} = \overline{\text{epi } h}; \quad \overline{\text{co}}(\text{epi } h) = \text{epi } \overline{\text{co}} h.$$

Moreover, if $\overline{\text{co}} h(x) > -\infty$ for all $x \in \mathbb{R}^n$ then $\overline{\text{co}} h(x) = h^{**}(x)$ for all $x \in \mathbb{R}^n$, where $h^{**} \doteq (h^*)^*$ is the bipolar or biconjugate of h , that is, the conjugate of h^* . There are examples showing the assumption $\overline{\text{co}} h(x) > -\infty$ for all $x \in \mathbb{R}^n$ is necessary to get the previous equality. In general we have $h^{**} \leq \overline{\text{co}} h \leq h$.

For any nonempty set $W \subseteq \mathbb{R}^m$, and $a, b \in \mathbb{R}^m$, we write $a \leq_W b$ (or equivalently, $b \geq_W a$) if $b - a \in W$.

The indicator function ι_S of the set S is defined by $\iota_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise.

The space of absolutely continuous functions from $[0, 1]$ into \mathbb{R}^k is denoted by $W^{1,1}([0, 1], \mathbb{R}^k)$, and it is equipped with the norm

$$\|x\|_{1,1} = \|x(0)\| + \int_0^1 \|\dot{x}(t)\| dt.$$

It is well-known that $W^{1,1}([0, 1], \mathbb{R}^k)$ is a Banach space. Set $W_k^{1,1} \doteq W^{1,1}([0, 1], \mathbb{R}^k)$ and $L^1 \doteq L^1([0, 1], \mathbb{R}^n)$. As usual, the norm on the product space $W_k^{1,1} \times L^1$ is the sum of the norms of $W_k^{1,1}$ and L^1 , and it will be denoted by $\|\cdot\|$.

Set $\mathbb{R}_+ \doteq [0, +\infty[$, $\mathbb{R}_{++} \doteq]0, +\infty[$, $\mathbb{R}_{--} = -\mathbb{R}_{++}$.

3 The convexity of the optimal value function and related properties

We associate with problem (1.1) the optimal value function $\psi : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined as follows

$$\psi(a) = \begin{cases} \inf \left\{ \int_0^1 f_0(t, z(t)) dt : g(z) \in -W + a, z \in \mathcal{Z} \right\} & \text{if } K(a) \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases}$$

Consider the Lagrangian dual of problem $(P(0))$ ($a = 0$ in (1.1)) and defined by

$$v_D \doteq \sup_{\lambda \in W^*} \inf_{z \in \mathcal{Z}} L(\lambda, z). \quad (3.1)$$

Here $L(\lambda, z) = f(z) + \langle \lambda, g(z) \rangle$, $\lambda \in W^*$, $z \in L^1([0, 1], \mathbb{R}^n)$.

We consider a classic result (see e.g., [13, Theorem 7]), although proved under convexity conditions, which relates the optimal value v_D of (3.1) with the biconjugate of the value function ψ .

In what follows we use the convention $+\infty - (+\infty) = +\infty$.

Theorem 3.1. *Assume that f, g and W are as described above, with W being in addition a cone, and $K(0) \neq \emptyset$. Then $v_D = \psi^{**}(0)$.*

Proof. Define $F : \mathcal{Z} \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\pm\infty\}$:

$$F(z, a) = \begin{cases} f(z), & \text{if } g(z) \in -W + a; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the Lagrangian function can be written as

$$L(\lambda, z) = \inf_{a \in \mathbb{R}^m} \{F(z, a) + \langle \lambda, a \rangle\} = \begin{cases} f(z) + \langle \lambda, g(z) \rangle, & \text{if } \lambda \in W^*, \\ -\infty & \text{if } \lambda \notin W^*, f(z) < +\infty; \\ +\infty, & \text{if } \lambda \notin W^*, f(z) = +\infty, \end{cases} \quad (3.2)$$

and the value function as,

$$\psi(a) = \inf_{z \in \mathcal{Z}} F(z, a).$$

Therefore

$$\inf_{z \in \mathcal{Z}} L(\lambda, z) = \inf_{a \in \mathbb{R}^m} \{ \inf_{z \in \mathcal{Z}} F(z, a) + \langle \lambda, a \rangle \} = - \sup_{a \in \mathbb{R}^m} \{ \langle -\lambda, a \rangle - \psi(a) \} = -\psi^*(-\lambda).$$

Then,

$$\sup_{\lambda \in W^*} \inf_{z \in \mathcal{Z}} L(\lambda, z) = \sup_{\lambda \in \mathbb{R}^m} \inf_{z \in \mathcal{Z}} L(\lambda, z) = \sup_{\lambda \in \mathbb{R}^m} [-\psi^*(-\lambda)] = \psi^{**}(0).$$

□

Set $C_0 \doteq \text{dom } f \cap \mathcal{Z} = \{z \in \mathcal{Z} : f(z) < +\infty\}$ and

$$K_0 \doteq \left\{ (u, v) \in L^1([0, 1], \mathbb{R}^{1+m}) : \exists z \in \mathcal{Z}, u(t) \geq f_0(t, z(t)), \right. \\ \left. v(t) \geq_W g_0(t, z(t)), \text{ for a.e. } t \in [0, 1] \right\}. \quad (3.3)$$

We get the following result which is important by itself.

Theorem 3.2. *Let $F(z) \doteq (f(z), g(z))$ with f, g being as above and $W \subseteq \mathbb{R}^m$ being any nonempty closed and convex set. The following assertions hold.*

(a) *The set K_0 is decomposable, $I(K_0)$ is convex and $I(K_0) = F(C_0) + (\mathbb{R}_+ \times W) \subseteq \text{epi } \psi$.*

(b) $(r, a) \in \text{epi } \psi \iff (r + \frac{1}{k}, a) \in F(C_0) + (\mathbb{R}_+ \times W), \quad \forall k \in \mathbb{N}$.

Consequently, the function $\psi : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is convex, and

$$I(K_0) \subseteq \text{epi } \psi \subseteq \overline{I(K_0)}. \quad (3.4)$$

(c) *We have*

$$(r, a) \in \text{ri}(\text{epi } \bar{\psi}) \iff a \in \text{ri}(\text{dom } \bar{\psi}) \text{ and } \exists k_0 \in \mathbb{N}, (r - \frac{1}{k}, a) \in \text{epi } \psi, \quad \forall k \geq k_0.$$

As a consequence, if $A \subseteq \text{ri}(\text{dom } \bar{\psi})$ then,

$$(\text{ri } \text{epi } \bar{\psi}) \cap (\mathbb{R} \times A) = \text{epi}^0 \psi \cap (\mathbb{R} \times A). \quad (3.5)$$

$$\text{epi } \bar{\psi} \cap (\mathbb{R} \times A) = \text{epi } \psi \cap (\mathbb{R} \times A). \quad (3.6)$$

Proof. (a): We observe first that K_0 is a decomposable set. Indeed, let $(u_i, v_i) \in K_0$, $i = 1, 2$ and $B \subseteq [0, 1]$ a measurable set. Then, for some $z_i \in L^1$, $i = 1, 2$, we have

$$u_i(t) \geq f_0(t, z_i(t)), \quad v_i(t) \geq_W g_0(t, z_i(t)), \quad z_i(t) \in Z(t), \text{ for a.e. } t \in [0, 1].$$

Clearly $z_i \in C_0$ for $i = 1, 2$. Setting $\tilde{z} \doteq z_1 \cdot \chi_B + z_2 \cdot \chi_{[0,1] \setminus B} \in L^1$, we have for a.e. $t \in [0, 1]$: $\tilde{z}(t) \in Z(t)$ and

$$u_1(t) \cdot \chi_B(t) + u_2(t) \cdot \chi_{[0,1] \setminus B}(t) \geq f_0(t, \tilde{z}(t)), \quad v_1(t) \cdot \chi_B(t) + v_2(t) \cdot \chi_{[0,1] \setminus B}(t) \geq_W g_0(t, \tilde{z}(t)),$$

i.e. $(u_1, v_1) \cdot \chi_B + (u_2, v_2) \cdot \chi_{[0,1] \setminus B} \in K_0$, proving the decomposability of K_0 . Thus the convexity of $I(K_0)$ follows from Theorem 2.1.

To prove the equality $I(K_0) = F(C_0) + (\mathbb{R}_+ \times W)$, first notice that $I(K_0) \subseteq F(C_0) + (\mathbb{R}_+ \times W)$ is straightforward by the convexity and closedness of W . For the reverse inclusion it is enough to observe that if $(u, v) \in F(C_0) + (\mathbb{R}_+ \times W)$, then, for some $z \in C_0$ and $(h, p) \in (\mathbb{R}_+ \times W)$,

$$(u, v) = \left(\int_0^1 [f_0(t, z(t)) + h] dt, \int_0^1 [g_0(t, z(t)) + p] dt \right) \in I(K_0),$$

proving the equality in (a). This also shows that $F(C_0) + (\mathbb{R}_+ \times W) \subseteq \text{epi } \psi$.

(b): By taking into account the inclusion in (a), the “only if” part is easily obtained. Let $\psi(a) \leq r < +\infty$. Then $K(a) \neq \emptyset$, and for all $k \in \mathbb{N}$ there exists $z_k \in C_0$ such that $f(z_k) < r + \frac{1}{k}$ and $g(z_k) \leq_W a$. Thus

$$\left(r + \frac{1}{k}, a \right) = (f(z_k), g(z_k)) + \left(r + \frac{1}{k} - f(z_k), a - g(z_k) \right) \in F(C_0) + (\mathbb{R}_+ \times W),$$

which completes the proof of the equivalence.

The convexity of ψ follows from the (a) which asserts the convexity of $I(K_0) = F(C_0) + (\mathbb{R}_+ \times W)$.

Combining (a) and the last equivalence, we get (3.4).

(c): Let $(r, a) \in \text{ri}(\text{epi } \bar{\psi})$. By (2.1), $a \in \text{ri}(\text{dom } \psi)$ and $\bar{\psi}(a) < r$. For $k_0 \in \mathbb{N}$ sufficiently large, we have $\bar{\psi}(a) < r - \frac{1}{k} < r$ for all $k \geq k_0$. Thus, for such $k \in \mathbb{N}$, one obtains

$$\left(r - \frac{1}{k}, a \right) \in \text{ri}(\text{epi } \bar{\psi}) = \text{ri } \overline{I(K_0)} = \text{ri } I(K_0).$$

Moreover, by convexity again, $\text{ri}(\text{epi } \psi) = \text{ri}(\overline{\text{epi } \psi}) = \text{ri}(\text{epi } \bar{\psi})$, proving one implication of the equivalence. The other is trivial.

One inclusion in (3.5) follows from the previous equivalence and the other is straightforward.

For (3.6) we need to check the inclusion “ \subseteq ”. Let take any $(r, a) \in \text{epi } \bar{\psi}$ with $a \in A$. Then, we have two possibilities: $\bar{\psi}(a) < r$ or $\bar{\psi}(a) = r$. In the first case, we get $(r, a) \in \text{ri}(\text{epi } \bar{\psi})$ and so $(r, a) \in \text{epi } \psi$. In the second case, since $\bar{\psi}(a) = r < r + \frac{1}{k}$, we obtain $(r + \frac{1}{k}, a) \in \text{ri}(\text{epi } \bar{\psi})$. By (3.5), $\psi(a) < r + \frac{1}{k}$ for all $k \in \mathbb{N}$, and the conclusion follows. \square

It is not difficult to check that

$$\text{dom } \psi = g(C_0) + W. \tag{3.7}$$

Thus, since $F(C_0) + (\mathbb{R}_+ \times W)$ is convex, we obtain immediately the convexity of $g(C_0) + W$, i. e., $\text{dom } \psi$ is convex, which is also a consequence of the convexity of ψ .

This along with other results, which follow from (3.6), are summarized in the following corollary.

Corollary 3.3. *Under the above assumptions, the following hold:*

$$(a) \text{ ri}(\text{dom } \bar{\psi}) = \text{ri}(\text{dom } \psi), \quad \overline{\text{dom } \bar{\psi}} = \overline{\text{dom } \psi} \quad \text{and} \quad \bar{\psi}(a) = \psi(a) \quad \forall a \in \text{ri}(\text{dom } \bar{\psi}).$$

Consequently,

$$\text{ri}(\text{epi } \bar{\psi}) = \text{ri}(\text{epi } \psi) = \{(r, a) \in \mathbb{R} \times \text{ri}(\text{dom } \psi) : \psi(a) < r\}. \quad (3.8)$$

(b) For $a \in \text{ri}(\text{dom } \psi)$ with $\psi(a) \in \mathbb{R}$, one has

$$\begin{aligned} \psi(a) &= \min\{r \in \mathbb{R} : (r, a) \in \overline{I(K_0)}\} = \inf\{r \in \mathbb{R} : (r, a) \in I(K_0)\}. \\ &= \min\{r \in \mathbb{R} : (r, a) \in \text{epi } \psi\}. \end{aligned}$$

Proof. (a): Let $a \in \text{ri}(\text{dom } \bar{\psi})$ and take any $r \in \mathbb{R}$ satisfying $\bar{\psi}(a) < r < +\infty$. Then $(r, a) \in \text{ri}(\text{epi } \bar{\psi})$, and by (3.5), $\psi(a) < r$, implying $a \in \text{dom } \psi$. This proves that $\text{ri}(\text{dom } \bar{\psi}) \subseteq \text{dom } \psi$, showing the desired result. This also proves that $\bar{\psi}(a) = \psi(a)$ for all $a \in \text{ri}(\text{dom } \bar{\psi})$.

Let us check the second equality. Since $\text{dom } \psi \subseteq \text{dom } \bar{\psi}$, we obtain

$$\overline{\text{dom } \bar{\psi}} \subseteq \overline{\text{dom } \psi} = \overline{\text{ri}(\text{dom } \bar{\psi})} = \overline{\text{ri}(\text{dom } \psi)} = \overline{\text{dom } \psi},$$

and the conclusion follows.

The last part is a consequence of (a) and (3.6).

(b): For $a \in \text{ri}(\text{dom } \psi)$, one obtains,

$$\begin{aligned} \psi(a) &= \bar{\psi}(a) = \min\{r \in \mathbb{R} : (r, a) \in \overline{\text{epi } \bar{\psi}}\} = \min\{r \in \mathbb{R} : (r, a) \in \overline{I(K_0)}\} \\ &\leq \inf\{r \in \mathbb{R} : (r, a) \in \text{epi } \psi\} \leq \inf\{r \in \mathbb{R} : (r, a) \in I(K_0)\} \doteq r_0. \end{aligned}$$

Assume that $\psi(a) < r_0$. There exists $r_k \downarrow \psi(a)$ such that $(r_k, a) \in \text{epi } \psi$. By (a) of the previous proposition, we get $(r_k + \frac{1}{k}, a) \in I(K_0)$ for all $k \in \mathbb{N}$. This means that $r_0 \leq r_k + \frac{1}{k}$, which implies $r_0 \leq \psi(a)$, which is impossible, proving that $\psi(a) = r_0$, and the conclusion follows. \square

Remark 3.4. From (b) of Corollary 3.3, we obtain an existence result to problem $(P(a))$ (see (1.1)), namely: if $a \in \text{ri}(g(C_0) + W)$, $\psi(a) > -\infty$, and $I(K_0)$ is closed, then $(P(a))$ admits at least a solution.

Conditions implying the closedness of $I(K_0)$ are given in Theorem 2.1; whereas the nonemptiness of \mathcal{H} yields $\psi(a) > -\infty$, as Theorem 7.1 shows.

Theorems 3.1 and 3.2 lead to the following characterization of lower semicontinuity of ψ at 0. Certainly, by Corollary 3.3, ψ is lsc in $\text{ri}(\text{dom } \psi)$.

Proposition 3.5. *Assume that $\mu \doteq \psi(0) < +\infty$ and that the assumptions of Theorem 3.2 hold. Then,*

(a) if $\psi(0) > -\infty$,

$$\bar{\psi}(0) = \psi(0) \iff [\overline{I(K_0)} - \mu(1, 0)] \cap (-\mathbb{R}_{++} \times \{0\}) = \emptyset. \quad (3.9)$$

$$\iff [\text{cone}(\overline{I(K_0)} - \mu(1, 0))] \cap (-\mathbb{R}_{++} \times \{0\}) = \emptyset.$$

(b) $\psi(0) = -\infty$ if and only if

$$[I(K_0) - \rho(1, 0)] \cap -(\mathbb{R}_{++} \times \{0\}) \neq \emptyset, \quad \forall \rho \in \mathbb{R}. \quad (3.10)$$

Proof. (a): It follows by noticing that $\overline{I(K_0)} = \text{epi } \bar{\psi}$.

(b): Simply consider the definition of $\psi(0)$. □

We now characterize the zero duality gap for our problem ($P(0)$) in term of the lower semicontinuity of ψ at 0. In particular, if $0 \in \text{ri}(g(C_0) + W)$ then there is no duality gap.

Theorem 3.6. *Assume that W is additionally a cone, and $\psi(0) < +\infty$, then*

(a) $v_D = \bar{\psi}(0)$;

(b) the duality gap between ($P(0)$) and (3.1) is zero, i. e., $\mu = v_D$, if and only if $\psi(0) = \bar{\psi}(0)$.

Proof. (a): In view of Theorem 3.1, we need to check that $\psi^{**}(0) = \bar{\psi}(0)$. If $\bar{\psi}(0) = -\infty$ then $\psi^{**}(0) = -\infty$ since $\psi^{**} \leq \bar{\psi}$. If $\bar{\psi}(0) \in \mathbb{R}$, due to the lower semicontinuity and convexity of $\bar{\psi}$, we conclude that $\bar{\psi} (= \overline{\text{co } \psi})$ never takes the value $-\infty$, and therefore $(\overline{\text{co } \psi}) \bar{\psi} = \psi^{**}$.

(b): It follows from (a). □

We will see next that the condition $0 \in \text{ri}(g(C_0) + W)$ not only implies zero duality gap but also the existence of solution for the dual problem provided $\psi(0)$ is finite. This is due to the important result derived from the convexity of ψ (see Corollary 3.3): the nonemptiness of $\partial\psi(a)$ whenever $a \in \text{ri}(\text{dom } \psi) = \text{ri}(g(C_0) + W)$.

Thus, on combining the previous theorem and Corollary 3.3, we establish the following result on strong duality for ($P(0)$) without any coercivity or convexity assumption.

Corollary 3.7. *Assume that $\mu \in \mathbb{R}$, with W being additionally a cone, $K(0) \neq \emptyset$ and $0 \in \text{ri}(g(C_0) + W)$. Then, there exists $\lambda_0 \in W^*$ such that*

$$\inf_{z \in K(0)} \int_0^1 f_0(t, z(t)) dt = \inf_{z \in \mathcal{Z}} \int_0^1 [f_0(t, z(t)) + \langle \lambda_0, g_0(t, z(t)) \rangle] dt. \quad (3.11)$$

For such λ_0 , we have

$$\inf_{z \in K(0)} \int_0^1 f_0(t, z(t)) dt = \inf_{\substack{\langle \lambda_0, g(z) \rangle \leq 0 \\ z \in \mathcal{Z}}} \int_0^1 f_0(t, z(t)) dt. \quad (3.12)$$

Hence,

$$\bar{z} \text{ solves } (P(0)) \iff \begin{cases} \langle \lambda_0, \int_0^1 g_0(t, \bar{z}(t)) dt \rangle = 0, \\ \int_0^1 f_0(t, \bar{z}(t)) dt = \inf_{z \in \mathcal{Z}} \int_0^1 [f_0(t, z(t)) + \langle \lambda_0, g_0(t, z(t)) \rangle] dt. \end{cases}$$

Proof. By the previous theorem and Corollary 3.3, we get the zero duality gap. Moreover, since $\psi(0)$ is finite and $0 \in \text{ri}(\text{dom } \psi) = \text{ri}(g(C_0) + W)$, a simple application of a convex separation theorem, allows us to conclude that $\partial\psi(0) \neq \emptyset$. Let $\lambda^* \in \partial\psi(0)$. Then, $\psi(a) - \psi(0) \geq \langle \lambda^*, a \rangle$ for all $a \in \mathbb{R}^m$. Since W is a convex cone, we get $K(0) \subseteq K(a)$ for all $w \in W$; this along with the previous inequality imply that $\langle \lambda^*, a \rangle \leq 0$ for all $a \in W$, yielding $-\lambda^* \in W^*$. We need only to check that $-\lambda^*$ is a solution to problem (3.1).

Let us take any $z \in \mathcal{Z}$. For $a = \int_0^1 g_0(t, z(t)) dt \in \mathbb{R}^m$, we obtain

$$\int_0^1 f_0(t, z(t)) dt - \langle \lambda^*, \int_0^1 g_0(t, z(t)) dt \rangle \geq \psi(a) - \langle \lambda^*, a \rangle \geq \psi(0), \quad \forall z \in \mathcal{Z},$$

which proves one inequality in (3.11) for $\lambda_0 = -\lambda^*$, the other is trivial.

Equality (3.12) and the remaining equivalence are not difficult to check. \square

In case $g_0(t, z) = z$ and $\mathcal{Z} = L^1([0, 1], \mathbb{R}^n)$, we obtain $g(L^1([0, 1], \mathbb{R}^m)) = \mathbb{R}^m$. Thus, $0 \in \text{ri}(g(C_0) + W)$ trivially holds whatever W is.

Related strong duality results were established in [3, 12].

Remark 3.8. Example 7.6 shows that the single condition $a \in \text{ri}(\text{dom } \psi)$ does not imply, in general, existence of solutions even when the optimal value is finite.

4 The subdifferential of the value function and necessary optimality conditions

Our aim in this section is to exploit the convexity property of the value function ψ in order to compute its subdifferential, we know that $\partial\psi(a)$ is nonempty, convex and

compact whenever $a \in \text{ri}(\text{dom } \psi)$.

To that purpose, with the same assumptions on f_0, g_0, W and Z , let us consider problem $(P(0))$, i. e., (1.1) with $a = 0$, and its associated Hamiltonian function $H : [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined by

$$H(t, p) \doteq \sup_{\xi \in Z(t)} \{\langle p, g_0(t, \xi) \rangle - f_0(t, \xi)\}. \quad (4.1)$$

Obviously $H(t, \cdot)$ is convex and lsc for all $t \in [0, 1]$, and because of (1.3), we have for all $p \in \mathbb{R}^m$

$$H(t, p) \geq \langle p, g_0(t, z_0(t)) \rangle - f_0(t, z_0(t)), \text{ a. e. } t \in [0, 1]. \quad (4.2)$$

Set

$$\mathcal{U} \doteq \left\{ (x, z) \in W_m^{1,1} \times L^1 : \dot{x}(t) = g_0(t, z(t)), z(t) \in Z(t), \text{ a. e. } t \in [0, 1], x(0) = 0 \right\}.$$

The next theorem does not require that W be a cone as in Corollary 3.7.

Theorem 4.1. *Let $\bar{z} \in K(0)$. Then the following assertions are equivalent:*

- (a) $p \in \partial\psi(0)$ and \bar{z} solves $(P(0))$;
- (b) $p \in N(W, -\int_0^1 g_0(t, \bar{z}(t))dt)$ and

$$H(t, p) = \langle p, g_0(t, \bar{z}(t)) \rangle - f_0(t, \bar{z}(t)), \text{ a. e. } t \in [0, 1]. \quad (4.3)$$

Proof. (a) \Rightarrow (b): Let $p \in \partial\psi(0)$, or equivalently

$$\langle p, a \rangle \leq \psi(a) - \psi(0) \quad \forall a \in \mathbb{R}^m. \quad (4.4)$$

For any fixed $w \in W$, set $a \doteq \int_0^1 g_0(t, \bar{z}(t))dt + w$. Then, we have $\int_0^1 g_0(t, \bar{z}(t))dt \in -W + a$. Replacing a in (4.4) and taking into account that \bar{z} is a solution to $(P(0))$, we get

$$\langle p, w + \int_0^1 g_0(t, \bar{z}(t))dt \rangle \leq 0,$$

proving the first result in (b).

To establish (4.3), pick any $z \in L^1$, with $z(t) \in Z(t)$ a. e. $t \in [0, 1]$. Then

$$\int_0^1 g_0(t, z(t))dt \in -W + a,$$

where $a = \int_0^1 g_0(t, z(t))dt - \int_0^1 g_0(t, \bar{z}(t))dt$. Using (4.4), we obtain

$$\langle p, \int_0^1 g_0(t, z(t))dt - \int_0^1 g_0(t, \bar{z}(t))dt \rangle \leq \int_0^1 f_0(t, z(t))dt - \psi(0).$$

Thus, since \bar{z} is a solution of the problem $(P(0))$ we have $\psi(0) = \int_0^1 f_0(t, \bar{z}(t))dt$ and hence \bar{z} solves the problem

$$\min \left\{ \int_0^1 f_0(t, z(t))dt - \langle p, \int_0^1 g_0(t, z(t))dt \rangle : z \in \mathcal{Z} \right\}, \quad (4.5)$$

or equivalently (\bar{x}, \bar{z}) , where $\bar{x}(t) = \int_0^t g_0(s, \bar{z}(s))ds$, is a solution of the following problem

$$\min_{(x,z) \in \mathcal{U}} \int_0^1 f_0(t, z(t))dt - \langle p, x(1) \rangle.$$

The maximum principle (Theorem 6.1 in [8]) yields $q \in W_m^{1,1}$ such that

$$\dot{q}(t) = 0 \text{ a. e. } t \in [0, 1]; \quad -q(1) \in \partial \langle -p, \cdot \rangle(\bar{x}(1)) = \{-p\}$$

and

$$H(t, p) = \langle p, g_0(t, \bar{z}(t)) \rangle - f_0(t, \bar{z}(t)), \text{ a. e. } t \in [0, 1],$$

proving (4.3).

(b) \Rightarrow (a): From (4.3), we obtain for all $z \in \mathcal{Z}$,

$$\langle p, g_0(t, z(t)) \rangle - f_0(t, z(t)) \leq \langle p, g_0(t, \bar{z}(t)) \rangle - f_0(t, \bar{z}(t)), \text{ a. e. } t \in [0, 1],$$

and hence

$$\langle p, \int_0^1 g_0(t, z(t))dt \rangle - \int_0^1 f_0(t, z(t))dt \leq \langle p, \int_0^1 g_0(t, \bar{z}(t))dt \rangle - \int_0^1 f_0(t, \bar{z}(t))dt. \quad (4.6)$$

Now let $z \in K(0)$. Then $\int_0^1 g_0(t, z(t))dt \in -W$, and by the first part in (b),

$$\langle p, -\int_0^1 g_0(t, z(t))dt + \int_0^1 g_0(t, \bar{z}(t))dt \rangle \leq 0.$$

This along with (4.6) imply that for all $z \in K(0)$,

$$\int_0^1 f_0(t, z(t))dt \geq \int_0^1 f_0(t, \bar{z}(t))dt,$$

ensuring that \bar{z} is a solution to $(P(0))$.

We now prove $p \in \partial\psi(0)$. Take any $a \in \mathbb{R}^m$ satisfying $K(a) \neq \emptyset$. Then for all $z \in K(a)$, we have $z \in \mathcal{Z}$ and $\int_0^1 g_0(t, z(t))dt - a \in -W$. On the one hand, the first part of (b) ensures that

$$\langle p, -\int_0^1 g_0(t, z(t))dt + a + \int_0^1 g_0(t, \bar{z}(t))dt \rangle \leq 0.$$

It follows from (4.6)

$$\int_0^1 f_0(t, z(t))dt \geq \int_0^1 f_0(t, \bar{z}(t))dt + \langle p, a \rangle$$

and as $\psi(0) = \int_0^1 f_0(t, \bar{z}(t))dt$ and $z \in K(a)$ is arbitrary, we get

$$\psi(a) - \psi(0) \geq \langle p, a \rangle,$$

or equivalently, $p \in \partial\psi(0)$.

□

We recover Proposition 5.8 in [3].

Corollary 4.2. *Assume that $0 \in \text{ri}(g(C_0) + W)$ and $\bar{z} \in K(0)$. Then the following assertions are equivalent:*

- (a) \bar{z} is a solution to $(P(0))$;
- (b) there exists $p \in N(W, -\int_0^1 g_0(t, \bar{z}(t))dt)$ such that

$$H(t, p) = \langle p, g_0(t, \bar{z}(t)) \rangle - f_0(t, \bar{z}(t)), \quad \text{a. e. } t \in [0, 1]. \quad (4.7)$$

The set of p satisfying (b) is $\partial\psi(0)$.

Proof. It suffices to see that our assumption $0 \in \text{ri}(\text{dom } \psi)$ ensures the existence of $p \in \partial\psi(0)$ and then, we apply Theorem 4.1. □

5 Local minima for the problem (1.1) are global

The aim of the section is to show that under the previous assumptions (except the assumption (1.4)) together with the following constraint qualification

$$0 \in \text{int}[g(C_0) + W] \quad (5.1)$$

ensures that each local minimum for $(P(0))$ is in fact global. Here, we recall that

$$C_0 = \{z \in \mathcal{Z} : f(z) < +\infty\}.$$

Theorem 5.1. *Let f_0 and g_0 satisfying the previous measurability, lower semicontinuity and continuity assumptions and let W be closed and convex. Then, under the qualification condition (5.1), each local minimum for $(P(0))$ is global.*

Before giving the proof of this theorem, we establish a result concerning the necessary optimality conditions of $(P(0))$. These necessary conditions are expressed in terms of the limiting Fréchet (or Mordukhovich [10]) normal cone that we begin by recalling here.

Let C be a closed subset of \mathbb{R}^n containing some point c . The Fréchet normal cone to C at c is the set

$$\hat{N}(C; c) \doteq \left\{ \xi \in \mathbb{R}^n : \liminf_{x \in C \rightarrow c} \frac{\langle -\xi, x - c \rangle}{\|x - c\|} \geq 0 \right\}.$$

The normal cone to C at c is the set

$$N(C; c) \doteq \limsup_{x \in C \rightarrow c} \hat{N}(C, x).$$

Lemma 5.2. *If \bar{z} is a local solution for $(P(0))$, then there exist $p \in \mathbb{R}^m$ and $\lambda \in \{0, 1\}$, with $(p, \lambda) \neq (0, 0)$, such that $p \in N(W, -\int_0^1 g_0(t, \bar{z}(t))dt)$ and*

$$\langle p, g_0(t, \bar{z}(t)) \rangle - \lambda f_0(t, \bar{z}(t)) = \max_{z \in Z(t) \cap \text{dom } f(t, \cdot)} [\langle p, g_0(t, z) \rangle - \lambda f_0(t, z)], \text{ a. e. } t \in [0, 1].$$

Proof. We define the functions $\ell : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $L : [0, 1] \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned} \ell(s_1, s_2, s_3, s_4) &= \iota_{\{0\}}(s_1) + \iota_{\{0\}}(s_2) + \iota_{-W}(s_3) \\ L(t, x, y, u, v) &= \begin{cases} f_0(t, v) & \text{if } v \in Z(t), u = g_0(t, v), \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Let us note that, as L does not depend on (x, y) , then for each element $A \doteq (x, y, u, v, r) \in \text{epi } L(t, \cdot)$ we have

$$(\beta, x^*, y^*, u^*, v^*) \in N(\text{epi } L(t, \cdot), A) \Rightarrow x^* = 0, y^* = 0. \quad (5.2)$$

Put $\bar{x}(t) = \int_0^t g(\tau, \bar{z}(\tau))d\tau$ and $\bar{y}(t) = \int_0^t \bar{z}(\tau)d\tau$. Since \bar{z} is a local solution of the problem $(P(0))$, then (\bar{x}, \bar{y}) is a local solution of the following Bolza problem

$$\left\{ \begin{array}{l} \min_{(x, y) \in W_m^{1,1} \times W_n^{1,1}} \int_0^1 L(t, x(t), y(t), \dot{x}(t), \dot{y}(t))dt + \ell(x(0), y(0), x(1), y(1)) \end{array} \right.$$

It is not difficult to show that all the assumptions of Theorem 4.1.1 in [4] are satisfied and this theorem asserts the existence of an arc $q = (p, p_0) \in W_m^{1,1} \times W_n^{1,1}$ and $\lambda \in \{0, 1\}$ such that

1. $(q, \lambda) \neq 0$;
2. $-p(1) \in N(-W, \bar{x}(1))$, $p_0(1) = 0$;
3. $\dot{q}(t) \in \overline{\text{co}}\{R : (-\lambda, R, q) \in N(\text{epi}L(t, \cdot), (f(t, \bar{z}(t), \bar{x}(t), \bar{y}(t), \dot{\bar{x}}(t), \dot{\bar{y}}(t)))) \text{ a.e. } t \in [0, 1]$ and hence, due to (5.2), $\dot{q} = 0$;
4. for almost every $t \in [0, 1]$ and $(u, v) \in \text{dom}L(t, \bar{x}(t), \bar{y}(t), \cdot, \cdot)$

$$\langle q(t), (u, v) - (\dot{\bar{x}}(t), \dot{\bar{y}}(t)) \rangle \leq \lambda [L(t, \bar{x}(t), \bar{y}(t), (u, v)) - L(t, \bar{x}(t), \bar{y}(t), \dot{\bar{x}}(t), \dot{\bar{y}}(t))].$$

Assertions 2. and 3. assert that $p_0 = 0$ and $p(t) = p(1)$ for all $t \in [0, 1]$ we will put $p(t) = p$ for all $t \in [0, 1]$ while assertion 4. implies the following

$$\langle p, g_0(t, \bar{z}(t)) \rangle - \lambda f_0(t, \bar{z}(t)) = \max_{z \in Z(t) \cap \text{dom } f(t, \cdot)} [\langle p, g_0(t, z) \rangle - \lambda f_0(t, z)], \text{ a. e. } t \in [0, 1],$$

and the proof of the lemma is finished because $-p \in N(-W, \bar{x}(1))$ iff $p \in N(W, -\bar{x}(1))$. \square

Now, we proceed to prove our theorem.

Proof. We will show that the multiplier λ in Lemma 5.2 is equal to 1. Indeed, suppose that $\lambda = 0$. Then, in particular

$$\langle p, g_0(t, \bar{z}(t)) \rangle \geq \langle p, g_0(t, z(t)) \rangle, \quad \forall z \in C_0,$$

and hence, by integrating we get

$$\langle p, g(z) - g(\bar{z}) \rangle \leq 0, \quad \forall z \in C_0.$$

On the other hand, $p \in N(W, -\int_0^1 g(t, \bar{z}(t))dt) = N(W, -g(\bar{z}))$, or equivalently

$$\langle p, w + g(\bar{z}) \rangle \leq 0 \quad \forall w \in W.$$

Now adding the two last inequalities, we obtain that

$$\langle p, g(z) + w \rangle \leq 0, \quad \forall z \in C_0, \quad \forall w \in W.$$

Using our constraint qualification (5.1), we conclude that $p = 0$ and this contradicts $(p, \lambda) \neq (0, 0)$. So $\lambda = 1$ and hence the last equality in Lemma 5.2 can be written as

$$\langle p, g_0(t, \bar{z}(t)) \rangle - f_0(t, \bar{z}(t)) = \max_{z \in Z(t)} [\langle p, g_0(t, z) \rangle - f_0(t, z)], \quad \text{a. e. } t \in [0, 1].$$

We now apply Corollary 4.2 to conclude that \bar{z} is a (global) solution to $(P(0))$, and the proof is completed. \square

Now, we can ask when the constraint qualification (5.1) is satisfied. In fact, it is easy to see that the following implication holds true for some $\rho > 0$

$$K(a) \neq \emptyset \quad \forall a \in B(0, \rho) \Rightarrow 0 \in \text{int}[g(C_0) + W]$$

provided $Z \subseteq \text{dom } f$.

Several characterizations concerning the nonemptiness of $K(a)$ around 0 will be presented in the next section.

6 The behaviour of the set-valued mapping K

The main intention of this section is to give sufficient conditions ensuring the nonemptiness and the behaviour of the set-valued mapping K considered in the previous section.

We will consider the set-valued mapping $\tilde{K} : \mathbb{R}^m \rightrightarrows W^{1,1} \times L^1$ defined by

$$\tilde{K}(a) \doteq \left\{ (x, u) \in W^{1,1} \times L^1 : \dot{x}(t) = g_0(t, u(t)), \quad u(t) \in Z(t), \quad \text{a. e. } t \in [0, 1], \right. \\ \left. x(0) = 0, \quad x(1) \in -W + a \right\}.$$

Theorem 6.1. Let $\bar{z} \in K(\bar{a})$ and put $\bar{x}(t) = \int_0^t g_0(\tau, \bar{z}(\tau)) d\tau$, for all $t \in [0, 1]$.

$$\mathcal{U} \doteq \left\{ (x, u) \in W^{1,1} \times L^1 : \dot{x}(t) = g_0(t, u(t)), u(t) \in Z(t), \text{ a. e. } t \in [0, 1], x(0) = 0 \right\}.$$

Suppose that $g_0(t, \cdot)$ is continuous for almost every $t \in [0, 1]$ and W is a closed set in \mathbb{R}^m . Let us consider the following assertions:

i) There is no $p \in \mathbb{R}^m$, with $p \neq 0$, satisfying

$$-p \in N(-W, \bar{x}(1) - \bar{a}), \quad \langle p, g_0(t, \bar{z}(t)) \rangle = \max_{z \in Z(t)} \langle p, g_0(t, z) \rangle, \text{ a. e. } t \in [0, 1].$$

ii) There exist $\alpha > 0$ and $r > 0$ such that

$$d((x, u), \tilde{K}(a)) \leq \alpha d(x(1), -W + a)$$

for all $(x, u) \in B((\bar{x}, \bar{z}), r) \cap \mathcal{U}$ and $a \in B(\bar{a}, r)$.

iii) There exist $\alpha > 0$ and $r > 0$ such that

$$\tilde{K}(a) \cap B((\bar{x}, \bar{z}), r) \subseteq \tilde{K}(a') + \alpha \|a - a'\| B(0, 1).$$

for all $a, a' \in B(\bar{a}, r)$.

iv) There exists $r > 0$ such that

$$\forall a \in B(\bar{a}, r), \quad K(a) \neq \emptyset.$$

Then $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv)$. If moreover W is convex, then $iv) \Rightarrow i)$.

Proof. The implications $ii) \Rightarrow iii) \Rightarrow iv) \Rightarrow i)$ are obvious. We establish only the implication $i) \Rightarrow ii)$: Suppose that $ii)$ does not hold. Then, there are sequences $(y_k, v_k) \in \mathcal{U}$, with $y_k \rightarrow \bar{x}$ and $v_k \rightarrow \bar{z}$, and $a_k \rightarrow \bar{a}$ such that for k large enough

$$d((y_k, v_k), \tilde{K}(a_k)) > kd(y_k(1), -W + a_k). \quad (6.1)$$

It follows that $(y_k, v_k) \notin \tilde{K}(a_k)$. Let us consider the function $f_k : W^{1,1} \times L^1 \rightarrow \mathbb{R}$ defined by

$$f_k(x, u) = d(x(1), -W + a_k).$$

Put $\varepsilon_k = \sqrt{f_k(y_k, v_k)} > 0$ and $\lambda_k = \min\{k\varepsilon_k^2, \varepsilon_k\}$. Then $\varepsilon_k \rightarrow 0$ and $\lambda_k \rightarrow 0$. Obviously,

$$f_k(y_k, v_k) \leq \inf_{(y, u) \in \mathcal{U}} f_k(y, u) + \varepsilon_k^2.$$

Our assumption on g_0 ensures that \mathcal{U} is closed in $W^{1,1} \times L^1$. Now, applying Ekeland variational principle one gets the existence of $(x_k, u_k) \in \mathcal{U}$ such that

$$\|(x_k, u_k) - (y_k, v_k)\| \leq \lambda_k, \quad (6.2)$$

$$f_k(x_k, u_k) \leq f_k(x, u) + s_k \|(x, u) - (x_k, u_k)\|, \quad \forall (x, u) \in \mathcal{U}. \quad (6.3)$$

where $s_k = \frac{\varepsilon_k^2}{\lambda_k}$. Using the inequality (6.3), we obtain that (x_k, u_k) is a solution to the following optimal control problem of Mayer type

$$\begin{cases} \min_{(x,u) \in \mathcal{U}} d(x(1) - a_k, -W) + s_k \|x(0) - x_k(0)\| \\ + s_k \int_0^1 [\|g_0(t, u(t)) - g_0(t, u_k(t))\| + \|u(t) - u_k(t)\|] dt \end{cases}$$

The maximum principle yields an arc $p_k \in W^{1,1}([0, 1], \mathbb{R}^n)$ such that

$$\dot{p}_k(t) = 0, \quad \text{a. e. } t \in [0, 1]; \quad -p_k(1) \in \partial d(\cdot - a_k, -W)(x_k(1))$$

and for a. e. $t \in [0, 1]$

$$\begin{aligned} & \langle p_k(t), g_0(t, u_k(t)) \rangle \\ &= \max_{u \in Z(t)} \langle p_k(t), g_0(t, u) \rangle - s_k [\|u - u_k(t)\| + \|g_0(t, u) - g_0(t, u_k(t))\|], \end{aligned} \quad (6.4)$$

where $\partial d(\cdot - a_k, -W)(x_k(1))$ is the limiting subdifferential ([10]) of the distance function to the set $-W + a_k$. From (6.1) and (6.2) it follows that $x_k(1) \notin -W + a_k$, and this implies that $\|p_k(1)\| = 1$ (see [9]). Since $\lambda_k \rightarrow 0$, (6.2) together with $v_k \rightarrow \bar{z}$ ensure that $u_k \rightarrow \bar{z}$ in $L^1([0, 1], \mathbb{R}^n)$ and hence there exists a subsequence $(u_{\varphi(k)})$ of (u_k) such that

$$u_{\varphi(k)}(t) \rightarrow \bar{z}(t), \quad \text{a. e. } t \in [0, 1]$$

and extracting a subsequence, if necessary, we may also assume that $p_{\varphi(k)}(1) \rightarrow p$, with $p \neq 0$. Because of the closedness of the limiting subdifferential ([10]), $-p \in \partial d(\cdot, -W)(\int_0^1 g_0(t, \bar{z}(t))dt - \bar{a}) \subseteq N(-W, \int_0^1 g_0(t, \bar{z}(t))dt - \bar{a})$. Now, having in mind that $s_{\varphi(k)} = \max(\frac{1}{\varphi(k)}, \varepsilon_{\varphi(k)})$, we get $s_{\varphi(k)} \rightarrow 0$. On the other hand, relation (6.4) and the continuity of $g_0(t, \cdot)$ ensures that

$$\langle p, g_0(t, \bar{z}(t)) \rangle = \max_{u \in Z(t)} \langle p, g_0(t, u) \rangle, \quad \text{a. e. } t \in [0, 1],$$

and this contradicts *i*). □

7 Computing the value function via the Hamiltonian and existence of solutions

This section is devoted to provide conditions under which the value function ψ is minorized by an affine linear function. As a consequence, we find a formula for ψ^* and so of ψ . To that end, let us define the function $G : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\pm\infty\}$ by

$$G(p) = \int_0^1 H(t, p) dt, \quad (7.1)$$

where, H is the Hamiltonian function defined as in (4.1). It follows that G is lsc and convex. Using Theorem 2.2 in [6], we may rewrite the function G as follows :

$$G(p) = \sup_{z \in \mathcal{Z}} \int_0^1 [\langle p, g_0(t, z(t)) \rangle - f_0(t, z(t))] dt. \quad (7.2)$$

The next function will be useful in the sequel

$$G_0(p) = \begin{cases} G(p) & \text{if } p \in -W^*, \\ +\infty & \text{if } p \notin -W^*. \end{cases}$$

Thus, G_0 is lsc and convex. Set

$$\mathcal{H} \doteq \left\{ p \in -W^* : H(\cdot, p) \in L^1([0, 1], \mathbb{R}) \right\}. \quad (7.3)$$

Then, by (4.2), \mathcal{H} is convex and

$$p \in \mathcal{H} \iff p \in -W^* \text{ and } p \in \text{dom } G \iff p \in \text{dom } G_0.$$

The next result generalizes and extends Theorem 3.2 in [5], where the case $W = \{0\}$ and $g_0(t, z) = z$ is considered.

Theorem 7.1. *Assume that W is additionally a cone and $\mathcal{H} \neq \emptyset$. The following statements hold.*

- (a) $\psi(a) \geq \langle p^*, a \rangle - G(p^*) > -\infty, \forall a \in \mathbb{R}^m, \forall p^* \in \mathcal{H}$; consequently $\bar{\psi} = \overline{c\bar{o}} \psi = \psi^{**}$;
- (b) Assume that (1.3) and (1.4) be satisfied. Then $\psi^*(p) = G_0(p), \forall p \in \mathbb{R}^m$, and so $\psi^{**} = \bar{\psi} = G_0^*$, which gives $v_D = G_0^*(0)$.
- (c) $\psi(a) = G_0^*(a)$ for all $a \in \text{ri}(g(C_0) + W)$.

Proof. (a): Let $p^* \in \mathcal{H}$. We have

$$G(p^*) \doteq \int_0^1 H(t, p^*) dt \geq \int_0^1 \langle p^*, g_0(t, z(t)) \rangle dt - \int_0^1 f_0(t, z(t)) dt, \quad \forall z \in \mathcal{Z}.$$

Take any $a \in \mathbb{R}^m$ such that $K(a) \neq \emptyset$. Then for $z \in K(a)$ there exists $p \in W$ such that:

$$\int_0^1 f_0(t, z(t)) dt \geq \int_0^1 \langle p^*, g_0(t, z(t)) \rangle dt - \int_0^1 H(t, p^*) dt \geq \int_0^1 \langle p^*, a - p \rangle dt - G(p^*).$$

Since $p^* \in -W^*$ we have:

$$\psi(a) \geq \int_0^1 \langle p^*, a \rangle dt - G(p^*) = \langle p^*, a \rangle - G(p^*).$$

(b): Let $\lambda \in \mathbb{R}^m$. Then,

$$\begin{aligned} \psi^*(\lambda) &= \sup_{a \in \mathbb{R}^n} \{\langle \lambda, a \rangle - \psi(a)\} = \sup_{\{a: K(a) \neq \emptyset\}} \sup_{z \in K(a)} \{\langle \lambda, a \rangle - f(z)\} \\ &= \sup_{z \in C_0, p \in W} \{\langle \lambda, g(z) \rangle + \langle \lambda, p \rangle - f(z)\} = \sup_{z \in C_0} \{\langle \lambda, g(z) \rangle - f(z)\} + \sup_{p \in W} \langle \lambda, p \rangle \\ &= \begin{cases} \sup_{z \in C_0} \{\langle \lambda, g(z) \rangle - f(z)\} & \text{if } \lambda \in -W^*, \\ +\infty & \text{if } \lambda \notin -W^*. \end{cases} \end{aligned} \quad (7.4)$$

By definition

$$H(t, \lambda) \geq \langle \lambda, g_0(t, z(t)) \rangle - f_0(t, z(t)), \text{ for a.e. } t \in [0, 1], \forall z \in \mathcal{Z},$$

so that for all $\lambda \in \mathbb{R}^m$,

$$G(\lambda) \geq \int_0^1 [\langle \lambda, g_0(t, z(t)) \rangle - f_0(t, z(t))] dt = \langle \lambda, g(z) \rangle - f(z), \forall z \in \mathcal{Z}.$$

Hence, $G(\lambda) \geq \psi^*(\lambda)$ for all $\lambda \in -W^*$ because of (7.4). Suppose that $G(\lambda) > \psi^*(\lambda)$ for some $\lambda \in -W^*$. Then there exists $z \in \mathcal{Z}$ such that

$$\int_0^1 \langle \lambda, g_0(t, z(t)) \rangle dt - \int_0^1 f_0(t, z(t)) dt > \psi^*(\lambda).$$

But relation (1.4) together with the last inequality ensure that $f(z) < \infty$, and hence $z \in C_0$ and this contradiction completes the proof of the equality $G(\lambda) = \psi^*(\lambda)$.

(c): It is a consequence of (b). \square

Next corollary, which is important by itself, provides another formula for the optimal value $\psi(0)$.

Corollary 7.2. *Assume that W is additionally a cone, $\mathcal{H} \neq \emptyset$, $0 \in \text{ri}(g(C_0) + W)$, (1.3) and (1.4) be satisfied. Then there exists $p^* \in -W^*$ such that $\psi(0) = -G(p^*)$.*

Proof. By assumption, it is known that there exists $p^* \in \partial\psi(0)$ and $\psi(0) = \bar{\psi}(0)$ by Corollary 3.3. Recall that by Theorem 3.2, the function ψ is convex and by (a) of Theorem 7.1 it follows that it is proper. Applying Theorem 23.5 in [14], we get

$$p^* \in \partial\psi(0) \Leftrightarrow \psi(0) + \psi^*(p^*) = 0.$$

Then

$$\psi(0) = -\psi^*(p^*). \quad (7.5)$$

From Theorem 7.1 we also have that $\psi^*(p) = G_0(p)$, for all $p \in \mathbb{R}^m$ and $\psi(0) = G_0^*(0)$. Then, taking into account (7.5), we get

$$\psi(0) = G_0^*(0) = -G_0(p^*).$$

Moreover, from (7.5) it also follows that $p^* \in \text{dom } \psi^* = \text{dom } G_0 = \mathcal{H}$; therefore $p^* \in -W^*$ and $G_0(p^*) = G(p^*)$. \square

The following existence theorem subsumes Corollary 3.1 in [5].

Theorem 7.3. *Assume that W is in addition a cone, $\mathcal{H} \neq \emptyset$, and (1.3) along with (1.4) hold. If K_0 is closed and the set $\text{epi } G_0^*$ contains no lines or extremal half-lines, then for every $a \in \mathbb{R}^m$ either $\psi(a) = +\infty$ or $(P(a))$ admits a solution, and the duality gap between $(P(0))$ and (3.1) is zero, i. e., $v_D = \psi(0)$.*

Proof. By Theorem 2.1, $I(K_0)$ is closed since $\overline{I(K_0)} = \text{epi } G_0^*$. Thus, $I(K_0) = F(C_0) + (\mathbb{R}_+ \times W)$ is closed and convex. From (3.4) we obtain

$$\text{epi } \psi = F(C_0) + (\mathbb{R}_+ \times W).$$

which implies that $(P(a))$ admits a solution for every $a \in \mathbb{R}^m$ satisfying $K(a) \neq \emptyset$, and $\psi = \bar{\psi}$. Hence $v_D = \psi^{**}(0) = \bar{\psi}(0)$ by Theorem 3.6(a). \square

Unfortunately we were unable to check that K_0 is closed in this general setting. However if $W = \{0\}$ and $g_0(t, z) = z$, K_0 is closed as one can check it directly. This case was considered in [5].

Next result follows a reasoning similar to that applied in Proposition 3.1 in [5].

Proposition 7.4. *Assume that \mathcal{H} is a nonempty open set, $g(C_0) + W = \mathbb{R}^m$, where W is additionally a cone, K_0 is closed and (1.3) and (1.4) be satisfied. Then $(P(a))$ admits a solution, for every $a \in \mathbb{R}^m$.*

Proof. We will show that the assumptions of Theorem 7.3 are fulfilled. To this aim, it is enough to show that $\text{epi } G_0^*$ contains no lines or extremal half-lines. First of all observe that, since $g(C_0) + W = \mathbb{R}^m$, then $K(a) \neq \emptyset$, for every $a \in \mathbb{R}^m$ and $\text{dom } \psi = g(C_0) + W = \mathbb{R}^m$. It follows that $\text{dom } G_0^* = \text{dom } \bar{\psi} = \mathbb{R}^m$, where the first equality is due to Theorem 7.1(c). This implies that $\text{epi } G_0^*$ cannot have any vertical extremal half-line, i.e., of the form $(t, \bar{\gamma})$, where $\bar{\gamma} \in \mathbb{R}^m$, $t \geq \bar{t} \in \mathbb{R}$.

Suppose that there exists a half-line $s \doteq \{(\alpha + \eta\beta, \xi_0 + \eta z), \eta \geq 0\}$ contained in $\text{bd}(\text{epi } G_0^*)$, $\xi_0, z \in \mathbb{R}^m$, $z \neq 0$, $\alpha, \beta \in \mathbb{R}$.

Let $\Gamma(\xi) \doteq \langle q, \xi \rangle - \gamma_0$, $q \in \mathbb{R}^m$, $\gamma_0 \in \mathbb{R}$ be a supporting hyperplane for $\text{epi } G_0^*$ containing the half-line s . Then $G_0^*(\xi) \geq \Gamma(\xi)$, for all $\xi \in \mathbb{R}^m$, which implies $G_0(q) \leq \Gamma^*(q) = \gamma_0$, yielding $q \in \mathcal{H}$. Moreover,

$$G_0^*(\xi_0 + \eta z) = \Gamma(\xi_0 + \eta z), \quad \forall \eta \geq 0$$

because s is contained in the graph of Γ .

Let $p \in \mathcal{H}$; by the previous relations and recalling that G_0 is a lsc convex function, we get

$$G_0(p) = \sup_{\xi \in \mathbb{R}^m} [\langle p, \xi \rangle - G_0^*(\xi)] \geq \langle p, \xi_0 + \eta z \rangle - G_0^*(\xi_0 + \eta z) = \langle p - q, \xi_0 + \eta z \rangle, \quad \forall \eta \geq 0.$$

Since $G_0(p) < \infty$, it follows that

$$\langle p - q, z \rangle \leq 0, \quad \forall p \in \mathcal{H},$$

which is impossible because $q \in \mathcal{H} = \text{int } \mathcal{H}$ and $z \neq 0$.

Therefore $\text{bd}(\text{epi } G_0^*)$ does not contain any half-line, and since $\text{epi } G_0^*$ is a proper closed convex set, this implies that it cannot contain any line too, which completes the proof. \square

The assumption $g(C_0) + W = \mathbb{R}^m$ is satisfied if $g_0(t, z) = z$, $C_0 = L^1$, in which case $m = n$. Observe also that in such situation K_0 is closed.

Next three examples illustrates the validity of our Theorem 7.3 showing that some of the assumptins are essential.

Example 7.5. Let us consider the problem

$$\psi(a) \doteq \inf \left\{ \int_0^1 -[z(t)]^2 dt \quad s. t. \quad \int_0^1 z(t) dt = a, \quad z \in \mathcal{Z} \right\},$$

where $Z(t) \doteq [\frac{1}{\sqrt{t}}, +\infty)$ for $t \in (0, 1]$. Here $W = \{0\}$. Since $z(t) \geq \frac{1}{\sqrt{t}}$ for a.e. $t \in [0, 1]$, then

$$a = \int_0^1 z(t) dt \geq \int_0^1 \frac{1}{\sqrt{t}} dt = 2, \quad \forall z \in \mathcal{Z}.$$

Obviously $a = 2 \in \text{bd}(\text{dom } \psi)$ and

$$\psi(2) = \int_0^1 -\frac{1}{t} dt = -\infty.$$

We actually have $\psi(a) = -\infty$ for all $a \geq 2$. Indeed, the function $z(t) \doteq \frac{a}{2\sqrt{t}} \in K(a)$, for all $a \geq 2$, and

$$\psi(a) \leq \int_0^1 -\frac{a^2}{4t} dt = -\infty.$$

Note that

$$\bar{\psi}(a) = \psi(a) = \begin{cases} +\infty, & \text{if } a < 2 \\ -\infty, & \text{if } a \geq 2, \end{cases}$$

is convex; whereas $\psi^{**} \equiv -\infty$, $\mathcal{H} = \emptyset$ and $G \equiv +\infty$.

Example 7.6. Consider the problem

$$\psi(a) \doteq \inf \left\{ \int_0^1 e^{-[z_1(t)z_2(t)]^2} dt \quad s. t. \quad \int_0^1 [z_2(t) - z_1(t)]^2 dt \leq a, \quad z \in \mathcal{Z} \right\},$$

where $Z(t) \doteq (\mathbb{R}_+ \times \mathbb{R}_+)$ for $t \in [0, 1]$. Here $W = \mathbb{R}_+$. Obviously $K(a) = \emptyset$ for $a < 0$. Setting $\bar{z}_1(t) = \bar{z}_2(t) = c$ for all $t \in [0, 1]$, then for every $a \geq 0$, and for every $c \geq 0$, $\bar{z} \in K(a)$. Obviously $a = 0 \in \text{bd}(\text{dom } \psi)$ and

$$0 \leq \psi(a) \leq \int_0^1 e^{-c^4} dt = e^{-c^4}, \quad \forall c \geq 0,$$

which implies

$$\psi(a) = 0, \quad \forall a \geq 0.$$

Note that the infimum is never attained and

$$\psi(a) = \begin{cases} +\infty, & \text{if } a < 0, \\ 0, & \text{if } a \geq 0 \end{cases}$$

is convex with $\psi = \bar{\psi}$. Here $\mathcal{H} =]-\infty, 0]$,

$$G(p) = G_0(p) = \begin{cases} +\infty, & \text{if } p > 0, \\ 0, & \text{if } p \leq 0. \end{cases}$$

Thus $\psi = G_0^*$ as expected by Theorem 7.1. Moreover the assumptions of Theorem 2.1 (with K_0 instead of K), or Theorem 7.3 are not fulfilled since an extremal halfline belongs to $\overline{I(K_0)} = \text{epi } \bar{\psi}$.

Example 7.7. Consider the problem

$$\inf \left\{ \int_0^1 e^{-[z(t)]} dt \quad s.t. \quad \int_0^1 \frac{z(t)}{1+z(t)} dt \leq a, \quad z \in \mathcal{Z} \right\},$$

where $Z(t) \doteq [0, +\infty[$ for $t \in [0, 1]$. Here $W = \mathbb{R}_+$. Obviously $K(a) = \emptyset$ for $a < 0$.

Let $a \geq 1$. Setting $\bar{z}(t) = c \geq 0$, $\forall t \in [0, 1]$, then for every $a \geq 1$, and for every $c \geq 0$, $\bar{z} \in K(a)$.

Then,

$$0 \leq \psi(a) \leq \int_0^1 e^{-c} dt = e^{-c}, \quad \forall c \geq 0,$$

which implies $\psi(a) = 0$ for all $a \geq 1$. Note that the infimum is never attained for $a \geq 1$.

Let $0 \leq a < 1$. In such a case we have $\psi(a) > 0$. Indeed $\psi(a) = 0$ if and only if there exists a sequence $z^k \in K(a)$ such that

$$\lim_{k \rightarrow +\infty} z^k(t) = +\infty, \quad \text{for a. e. } t \in [0, 1]$$

but this would imply

$$\lim_{k \rightarrow +\infty} \int_0^1 \frac{z^k(t)}{1+z^k(t)} dt = 1$$

which contradicts that $z^k \in K(a)$ with $a < 1$, for k sufficiently large. Therefore,

$$\psi(a) = \begin{cases} +\infty, & a < 0 \\ \psi(a) > 0, & 0 \leq a < 1, \\ 0, & a \geq 1 \end{cases}$$

Since $\text{ri}(\text{dom } \psi) =]0, +\infty[$, then $\psi(a) = \bar{\psi}(a)$, for every $a \in]0, +\infty[$. Observe that $(a, \psi(a)) = (1, \psi(1))$ is an extreme point of $\text{epi } \bar{\psi}$ with $a = 1$ belonging to $\text{ri}(\text{dom } \psi)$

and such that the infimum value of $(P(1))$ is not attained.

We note that the assumptions of Theorem 2.1, or Theorem 7.3, are not fulfilled since an extremal half-line belongs to $\overline{I(K_0)} = \text{epi } \overline{\psi}$.

Remark 7.8. We notice that the set $I(K_0)$ is closed in the simplest case when K_0 is an affine set, i.e., $\forall x, y \in K_0, \forall \alpha \in \mathbb{R}, \alpha x + (1 - \alpha)y \in K_0$. Then, recalling that $I : K_0 \rightarrow \mathbb{R}^{n+1}$ is linear, $I(K)$ is an affine set in \mathbb{R}^{m+1} and therefore it is closed. Clearly K_0 is affine if $f_0(t, \cdot)$ and $g_0(t, \cdot)$ are linear, for a.e. $t \in [0, 1]$ and C_0 is an affine set in $L^1([0, 1], \mathbb{R}^n)$.

Theorem 7.1 and Theorem 7.3 extend Theorem 3.2 and Corollary 3.1 of [5], respectively. A related existence result may be found in [1].

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