A residual-based a posteriori error estimator for the plane linear elasticity problem with pure traction boundary conditions *

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Abstract

In this work we consider the two-dimensional linear elasticity problem with pure non-homogeneous Neumann boundary conditions, and derive a reliable and efficient residual-based a posteriori error estimator for the corresponding stress-displacement-rotation dual-mixed variational formulation. The proof of reliability makes use of a suitable auxiliary problem, the continuous inf-sup conditions satisfied by the bilinear forms involved, and the local approximation properties of the Clément and Raviart-Thomas interpolation operators. In turn, inverse and discrete trace inequalities, and the localization technique based on triangle-bubble and edge-bubble functions, are the main tools yielding the efficiency of the estimator. Several numerical results confirming the properties of the estimator and illustrating the performance of the associated adaptive algorithm are also reported.

Key words: elasticity equation, pure Neumann conditions, mixed finite element method, a posteriori error estimator, PEERS

Mathematics subject classifications (1991): 65N15, 65N30, 65N50, 74B20

1 Introduction

The possibility of introducing further unknowns of physical interest, such as stresses and rotations, and the need of locking-free numerical schemes when the corresponding Poisson ratio approaches 1/2, constitute the main reasons for the utilization of dual-mixed variational formulations and the associated mixed finite element methods to solve elasticity problems. Consequently, the derivation of appropriate finite element subspaces yielding well posed Galerkin schemes has been extensively studied and several choices, including the classical PEERS element and recent approaches, are already available in the literature (see, e.g. [4], [5], [6], [7], [8], [15], [32], [39], and [42]). It is also well known that, within the framework of dual-mixed formulations, and on the contrary to the usual primal ones, the Dirichlet and Neumann data exchange their roles and become now natural and essential boundary conditions, respectively. In particular, non-homogeneous Neumann data usually lead to non-conforming Galerkin schemes and respective consistency terms, which need to be suitably estimated to be able to prove stability and convergence of the discrete methods. These facts explain why most of the works dealing with dual-mixed finite element methods in continuum mechanics consider either

*This research was partially supported by BASAL project CMM, Universidad de Chile, by Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, and by CONICYT project Anillo ACT1118 (ANANUM).

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pure Dirichlet or mixed boundary conditions with homogeneous Neumann datum, thus avoiding the additional difficulties arising from the presence of non-homogeneous essential boundary conditions. Nevertheless, one way of successfully handling these conditions consists of the introduction of appropriate Lagrange multipliers enforcing them weakly, as done originally in [9] for the primal finite element method with non-homogeneous Dirichlet boundary conditions. The extension of the method from [9] to a large class of dual-mixed variational formulations was studied in [10], where a second order elliptic equation in divergence form with mixed boundary conditions and non-homogeneous Neumann datum was considered as a model problem.

In turn, the extension of the results from [10] to the dual-mixed variational formulation of the linear elasticity problem in the plane was performed in [28]. More precisely, the stress-displacement-rotation formulation for the case of non-homogeneous pure traction boundary conditions was considered in [28], and a new dual-mixed finite element method for approximating its solution was developed there. The main novelty of the approach in [28] lies on the weak enforcement of the non-homogeneous Neumann boundary condition, similarly as done in [10], through the introduction of the boundary trace of the displacement as a Lagrange multiplier. In addition, since the rigid body motions solve the associated homogeneous boundary value problem, the displacements lie in the respective orthogonal complement and are computed through the introduction of an artificial unknown as an additional Lagrange multiplier. A suitable combination of PEERS and continuous piecewise linear functions on the boundary are employed to define the dual-mixed finite element scheme, and the classical Babuška-Brezzi theory is applied to show the well-posedness of the continuous and discrete formulations. A priori rates of convergence of the method, including an estimate for the global error when the stresses are measured with the L^2-norm, are also derived in [28]. It is important to remark that this work is actually the first one dealing with the dual-mixed finite element method for the above mentioned boundary value problem, in which the stress-displacement-rotation formulation and triangular elements are employed. Moreover, the analysis of the corresponding continuous variational formulation, which is also provided there, was not available before. On the contrary, the analysis of the continuous and discrete primal variational formulations for the linear elasticity problem with pure Neumann boundary conditions is nowadays very well established (see, e.g. [14, Chapter 9], [13], [23], and [33] for detailed analyses).

On the other hand, in order to guarantee a good convergence behaviour of the finite element solutions, particularly under the presence of singularities, one usually needs to apply an adaptive strategy based on a posteriori error estimates. These are usually represented by global quantities $\boldsymbol{\theta}$ that are expressed in terms of local estimators θ_T defined on each element T of a given triangulation of the domain. The estimator $\boldsymbol{\theta}$ is said to be reliable (resp. efficient) if there exists $C_{\mathtt{rel}} > 0$ (resp. $C_{\mathtt{eff}} > 0$), independent of the meshsizes, such that

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} + \text{h.o.t.} \;\leq\; \|error\| \;\leq\; C_{\mathtt{rel}}\,\boldsymbol{\theta} + \text{h.o.t.}\,,$$

where h.o.t. is a generic expression denoting one or several terms of higher order. Most of the a posteriori error estimators for the mixed finite element formulation of the linear elasticity problem are derived similarly as those for elliptic partial differential equations of second order in divergence form (see. e.g. [2] where estimators based on residuals and on the solution of local problems, using Raviart-Thomas and Brezzi-Douglas-Marini spaces, are provided). In connection with Raviart-Thomas spaces, one may also refers to [12], [16], and [27], where reliable and efficient residual-based a posteriori error estimators for the Poisson problem are obtained. The main tools of the corresponding analyses include Helmholtz decompositions, the localization technique based on bubble functions, discrete trace and inverse inequalities, and the approximation properties of the Clément interpolant. The extension of the results in [16] to the linear elasticity problem is developed in [18] and [34]. In addition, energy norm a posteriori error estimates based on postprocessing are obtained in [35], and functional-type error estimates are presented in [37].

Motivated by the preceding remarks, the main purpose of the present paper is to consider the plane linear elasticity problem with pure traction boundary conditions and derive a reliable and efficient residual-based a posteriori error estimator for the corresponding dual-mixed finite element method introduced and analyzed in [28]. The rest of this work is organized as follows. In Section 2 we recall from [28] the boundary value problem of interest and its dual-mixed variational formulation. In Section 3 we reconsider the mixed finite element scheme from [28] and introduce some improvements in its definition and solvability analysis that have arisen in recent related works. The core of the present work is Section 4, where we develop the announced a posteriori error analysis. The reliability and efficiency of the proposed estimator are proved in Sections 4.1 and 4.2, respectively. Finally, several numerical examples confirming these properties and showing the good performance of the associated adaptive algorithm, are provided in Section 5.

We end this section with further notations to be used below. In what follows, $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$, tr denotes the matrix trace, $^{\mathtt{t}}$ stands for the transpose of a matrix, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we define the deviator tensor $\boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}$, and the tensor product $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$. Also, we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, $\mathcal{S}$ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}, \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^2.$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote H^2 and $H^{2\times 2}$, respectively. In addition, we use $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ to denote the usual duality pairings between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, and between $\mathbf{H}^{-1/2}(\mathcal{S})$ and $\mathbf{H}^{1/2}(\mathcal{S})$. Furthermore, the Hilbert space

$$\mathbf{H}(\operatorname{div};\mathcal{O}) := \left\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \operatorname{div}\mathbf{w} \in L^2(\mathcal{O}) \right\},$$

is standard in the realm of mixed problems (see [15], [31]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\operatorname{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O})$, where $\mathbf{div}$ stands for the usual divergence operator div acting on each row of the tensor, The Hilbert norms of $\mathbf{H}(\operatorname{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\operatorname{div},\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div},\mathcal{O}}$, respectively. Finally, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The boundary value problem

In this section we recall from [28] the boundary value problem of interest, its associated dual-mixed variational formulation, and the corresponding well-posedness result. To this end, we let Ω be a bounded and simply connected polygonal domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary Γ. Our goal is to determine the displacement $\mathbf{u}$ and stress tensor $\boldsymbol{\sigma}$ of a linear elastic material occupying the region Ω and which is subject to a volume force and pure traction boundary conditions. In other words, given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{-1/2}(\Gamma)$, we seek a symmetric tensor field $\boldsymbol{\sigma}$ and a vector field $\mathbf{u}$ such that

$$\boldsymbol{\sigma} = \mathcal{C}\,\boldsymbol{\epsilon}(\mathbf{u}), \quad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega, \quad \text{and} \quad \boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{g} \quad \text{on} \quad \Gamma, \tag{2.1}$$

where $\mathcal{C}$ is the elasticity operator determined by Hooke's law, that is, given Lamé constants λ, $\mu > 0$,

$$\mathcal{C}\,\boldsymbol{\zeta} \;:=\; \lambda\,\mathrm{tr}(\boldsymbol{\zeta})\,\mathbf{I} \;+\; 2\,\mu\,\boldsymbol{\zeta} \qquad \forall\,\boldsymbol{\zeta}\,\in\,\mathbb{L}^2(\Omega)\,, \tag{2.2}$$

$\boldsymbol{\epsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\right)$ is the strain tensor of small deformations, and $\boldsymbol{\nu}$ is the unit outward normal to Γ. Concerning the existence of solution of (2.1), we first recall (see, e.g. [14, Theorem 9.2.30]) that this problem is solvable if and only if

$$\int_\Omega \mathbf{f}\cdot\boldsymbol{\chi} \,+\, \langle\mathbf{g},\boldsymbol{\chi}\rangle_\Gamma \;=\; 0 \qquad \forall\,\boldsymbol{\chi}\,\in\,\mathbb{RM}(\Omega)\,, \tag{2.3}$$

where $\mathbb{RM}(\Omega)$, the space of rigid body motions in Ω, is defined as

$$\mathbb{RM}(\Omega) := \left\{\boldsymbol{\chi}:\Omega\to\mathbb{R}^2 \;:\; \boldsymbol{\chi}(\mathbf{x}) = \begin{pmatrix} a \\ b \end{pmatrix} + c\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \;\forall\,\mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\in\Omega\,,\; a,\,b\,,\,c\in\mathbb{R}\right\}.$$

Hence, throughout the rest of the paper we assume that the compatibility condition (2.3) holds.

Next, following the usual procedure for the stress-displacement-rotation formulation of the elasticity problem (see, e.g. [4], [15], [39]), that is defining the rotation $\boldsymbol{\gamma} := \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}) \,\in\, \mathbb{L}^2_{\mathtt{skew}}(\Omega)$ as an auxiliary unknown, where

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega) \;:=\; \left\{\boldsymbol{\tau}\,\in\,\mathbb{L}^2(\Omega): \quad \boldsymbol{\tau} + \boldsymbol{\tau}^{\mathtt{t}} \;=\; \mathbf{0}\right\}$$

is the space of skew-symmetric tensors, and introducing the trace $\boldsymbol{\varphi} \;:=\; -\,\mathbf{u} \,\in\, \mathbf{H}^{1/2}(\Gamma)$ as an additional Lagrange multiplier, we obtain, at first instance, the dual-mixed variational formulation: Find $(\boldsymbol{\sigma},(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}))\in\mathbb{H}(\mathbf{div};\,\Omega)\times\mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau},(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \;&=\; 0 \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\,\Omega)\,,\\ \mathbf{b}(\boldsymbol{\sigma},(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) \;&=\; -\int_\Omega \mathbf{f}\cdot\mathbf{v} \,+\, \langle\mathbf{g},\boldsymbol{\psi}\rangle_\Gamma \qquad \forall\,(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})\in\mathbf{Q}\,, \end{aligned} \tag{2.4}$$

where

$$\mathbf{Q} \;:=\; \mathbf{L}^2(\Omega)\times\mathbf{H}^{1/2}(\Gamma)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)\,,$$

and $\mathbf{a}:\mathbb{H}(\mathbf{div};\,\Omega)\times\mathbb{H}(\mathbf{div};\,\Omega)\to\mathbb{R}$ and $\mathbf{b}:\mathbb{H}(\mathbf{div};\,\Omega)\times\mathbf{Q}\to\mathbb{R}$ are the bilinear forms given by

$$\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) \;:=\; \int_\Omega \mathcal{C}^{-1}\boldsymbol{\zeta}:\boldsymbol{\tau} \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau})\in\mathbb{H}(\mathbf{div};\,\Omega)\times\mathbb{H}(\mathbf{div};\,\Omega)\,, \tag{2.5}$$

and

$$\mathbf{b}(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) \;:=\; \int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} \,+\, \langle\boldsymbol{\tau\nu},\boldsymbol{\psi}\rangle_\Gamma \,+\, \int_\Omega \boldsymbol{\tau}:\boldsymbol{\eta} \qquad \forall\,(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta}))\in\mathbb{H}(\mathbf{div};\,\Omega)\times\mathbf{Q}\,. \tag{2.6}$$

However, it is easy to see that, given any $\boldsymbol{\chi}\in\mathbb{RM}(\Omega)$, $(\mathbf{0},(\boldsymbol{\chi},-\boldsymbol{\chi}|_\Gamma,\nabla\boldsymbol{\chi}))$ solves the homogeneous system associated to (2.4), and therefore, in order to avoid these spurious solutions, we now look for displacements $\mathbf{u}$ in the orthogonal complement of the rigid body motions. According to the foregoing analysis, we arrive at the following dual-mixed variational formulation of (2.1): Find $((\boldsymbol{\sigma},\boldsymbol{\rho}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}))$ $\in\mathbf{H}\times\mathbf{Q}$ such that

$$\begin{aligned} \mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}),(\boldsymbol{\tau},\boldsymbol{\chi})) \,+\, \mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \;&=\; 0 \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\chi})\in\mathbf{H}\,,\\ \mathbf{B}((\boldsymbol{\sigma},\boldsymbol{\rho}),(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) \;&=\; -\int_\Omega \mathbf{f}\cdot\mathbf{v} \,+\, \langle\mathbf{g},\boldsymbol{\psi}\rangle_\Gamma \qquad \forall\,(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})\in\mathbf{Q}\,, \end{aligned} \tag{2.7}$$

where

$$\mathbf{H} := \mathbb{H}(\mathbf{div};\Omega) \times \mathbb{RM}(\Omega)\,,$$

and $\mathbf{A} : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ and $\mathbf{B} : \mathbf{H} \times \mathbf{Q} \to \mathbb{R}$ are the bilinear forms given by

$$\mathbf{A}((\boldsymbol{\zeta},\boldsymbol{\varrho}),(\boldsymbol{\tau},\boldsymbol{\chi})) := \mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\varrho}\cdot\boldsymbol{\chi} \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\varrho}),\,(\boldsymbol{\tau},\boldsymbol{\chi}) \in \mathbf{H}\,, \tag{2.8}$$

and

$$\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) := \mathbf{b}(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) + \int_\Omega \boldsymbol{\chi}\cdot\mathbf{v} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\chi}) \in \mathbf{H}\,,\quad \forall\,(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta}) \in \mathbf{Q}\,. \tag{2.9}$$

The following lemmas are needed to establish the well-posedness of (2.7) and also to carry on the announced a posteriori error analysis in Section 4.

Lemma 2.1. *Let* $\mathbf{V} := \{\,(\boldsymbol{\tau},\boldsymbol{\chi}) \in \mathbf{H} : \quad \mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) = 0 \quad \forall\,(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta}) \in \mathbf{Q}\,\}$. *Then there holds*

$$\mathbf{V} = V \times \{\mathbf{0}\}\,, \tag{2.10}$$

with

$$V := \Big\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad in \quad \Omega\,, \quad \boldsymbol{\tau}\boldsymbol{\nu} = \mathbf{0} \quad on \quad \Gamma\,, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \quad in \quad \Omega\Big\}\,, \tag{2.11}$$

and there exists $\alpha > 0$, *independent of* λ, *such that*

$$\mathbf{A}((\boldsymbol{\tau},\boldsymbol{\chi}),(\boldsymbol{\tau},\boldsymbol{\chi})) \;\geq\; \alpha\,\|(\boldsymbol{\tau},\boldsymbol{\chi})\|^2_{\mathbf{H}} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\chi}) \in \mathbf{V}\,.$$

Proof. See [28, Lemma 3.3]. □

Lemma 2.2. *There exists* $\beta > 0$, *independent of* λ, *such that*

$$\sup_{\substack{(\boldsymbol{\tau},\boldsymbol{\chi})\in\mathbf{H}\\(\boldsymbol{\tau},\boldsymbol{\chi})\neq\mathbf{0}}} \frac{|\,\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta}))\,|}{\|(\boldsymbol{\tau},\boldsymbol{\chi})\|_{\mathbf{H}}} \;\geq\; \beta\,\|(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})\|_{\mathbf{Q}} \qquad \forall\,(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta}) \in \mathbf{Q}\,.$$

Proof. See [28, Lemma 3.4]. □

The well-posedness of the variational formulation (2.7) is stated as follows.

Theorem 2.1. *There exists a unique solution* $((\boldsymbol{\sigma},\boldsymbol{\rho}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \in \mathbf{H}\times\mathbf{Q}$ *to* (2.7). *In addition,* $\boldsymbol{\rho} = \mathbf{0}$ *and there exists* $C > 0$, *independent of* λ, *such that*

$$\|(\boldsymbol{\sigma},(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}))\|_{\mathbb{H}(\mathbf{div};\Omega)\times\mathbf{Q}} \;\leq\; C\,\Big\{\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{-1/2,\Gamma}\Big\}\,. \tag{2.12}$$

Proof. See [28, Theorem 3.1]. □

Actually, thanks to Lemmas 2.1 and 2.2, we can establish the following more general result.

Theorem 2.2. *Given* $\bar{F} \in \mathbf{H}'$ *and* $\bar{G} \in \mathbf{Q}'$, *there exists a unique* $((\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}}),(\bar{\mathbf{u}},\bar{\boldsymbol{\varphi}},\bar{\boldsymbol{\gamma}})) \in \mathbf{H}\times\mathbf{Q}$ *such that*

$$\begin{aligned} \mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}}),(\boldsymbol{\tau},\boldsymbol{\chi})) + \mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\bar{\mathbf{u}},\bar{\boldsymbol{\varphi}},\bar{\boldsymbol{\gamma}})) &= \bar{F}\big((\boldsymbol{\tau},\boldsymbol{\chi})\big) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\chi}) \in \mathbf{H}\,,\\ \mathbf{B}((\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}}),(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) &= \bar{G}\big((\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})\big) \qquad \forall\,(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta}) \in \mathbf{Q}\,. \end{aligned} \tag{2.13}$$

In addition, there exists $C > 0$, *depending only on* β, α, $\|\mathbf{a}\|$, *and* $\|\mathbf{b}\|$, *such that*

$$\|(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}} + \|(\bar{\mathbf{u}},\bar{\boldsymbol{\varphi}},\bar{\boldsymbol{\gamma}})\|_{\mathbf{Q}} \;\leq\; C\,\Big\{\|\bar{F}\|_{\mathbf{H}'} + \|\bar{G}\|_{\mathbf{Q}'}\Big\}\,. \tag{2.14}$$

We end this section with the converse of the derivation of (2.7). Indeed, the following theorem establishes that the unique solution of (2.7) solves the original boundary value problem (2.1). This result will be used later on in Section 4.2 to prove the efficiency of the a posteriori error estimator.

Theorem 2.3. *Let* $((\boldsymbol{\sigma}, \boldsymbol{\rho}), (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma})) \in \mathbf{H} \times \mathbf{Q}$ *be the unique solution of* (2.7)*. Then* $\boldsymbol{\rho} = \mathbf{0}$ *in* Ω, $\mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f}$ *in* Ω, $\nabla \mathbf{u} = \mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{\gamma}$ *in* Ω *(which yields* $\mathbf{u} \in \mathbf{H}^1(\Omega)$*),* $\mathbf{u} = -\boldsymbol{\varphi}$ *on* Γ, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathtt{t}}$ *in* Ω, $\gamma = \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}\right)$ *in* Ω *(which yields* $\boldsymbol{\sigma} = \mathcal{C}\,\boldsymbol{\epsilon}(\mathbf{u})$*), and* $\boldsymbol{\sigma}\,\boldsymbol{\nu} = \mathbf{g}$ *on* Γ.

Proof. It suffices to apply integration by parts backwardly in (2.7) and then use suitable test functions. Further details are omitted. □

3 The mixed finite element scheme

We now recall from [28] the mixed finite element scheme for (2.7). As said there, we could define this discrete scheme by utilizing any of the classical finite element subspaces available in the literature (see, e.g. [15] and the references therein), or those that have emerged recently from the finite element exterior calculus (see, e.g. [6], [7]). However, for simplicity of the presentation, we consider in what follows the well known PEERS elements. To this end, we first let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the polygonal region $\bar{\Omega}$ by triangles T of diameter h_T with global mesh size $h := \max\{\, h_T : \quad T \in \mathcal{T}_h \,\}$, such that they are quasi-uniform around Γ. In what follows, given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^2$, $P_\ell(S)$ denotes the space of polynomials defined in S of total degree $\leq \ell$. Recall that, according to the notation convention explained in the introduction, we denote $\mathbf{P}_\ell(S) := [P_\ell(S)]^2$. Furthermore, given $T \in \mathcal{T}_h$ and $\mathbf{x} := (x_1, x_2)^{\mathtt{t}}$ a generic vector of $\mathbb{R}^2$, we let $\mathrm{RT}_0(T) := \mathrm{span}\Big\{(1,0),(0,1),(x_1,x_2)\Big\}$ be the local Raviart-Thomas space of order 0 (cf. [15], [38]), and let $\mathbf{curl}^{\mathtt{t}}\, b_T := \left(\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1}\right)$, where b_T is the usual cubic bubble function on T. Then we define the finite element subspaces $H_h^{\boldsymbol{\sigma}}$, $Q_h^{\mathbf{u}}$, and $Q_h^{\boldsymbol{\gamma}}$, associated with the unknowns $\boldsymbol{\sigma}$, $\mathbf{u}$, and $\boldsymbol{\gamma}$, respectively, as follows:

$$H_h^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \ \mathbf{c}^{\mathtt{t}}\,\boldsymbol{\tau}_h|_T \in \mathrm{RT}_0(T) \oplus P_0(T)\,\mathbf{curl}^{\mathtt{t}}\, b_T \quad \forall\, T \in \mathcal{T}_h\,, \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \right\}, \tag{3.1}$$

$$Q_h^{\mathbf{u}} := \left\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h \right\}, \tag{3.2}$$

and

$$Q_h^{\boldsymbol{\gamma}} := \left\{ \begin{pmatrix} 0 & \eta_h \\ -\eta_h & 0 \end{pmatrix} : \quad \eta_h \in C(\bar{\Omega})\,, \quad \eta_h|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h \right\}. \tag{3.3}$$

Note here that $H_h^{\boldsymbol{\sigma}} \times Q_h^{\mathbf{u}} \times Q_h^{\boldsymbol{\gamma}}$ constitutes the classical PEERS introduced in [4] for a mixed finite element aproximation of the linear elasticity problem with Dirichlet boundary conditions. Next, in order to set the finite dimensional subspace $Q_h^{\boldsymbol{\varphi}}$ associated with the unknown $\boldsymbol{\varphi}$, we let Γ_h be the partition of Γ inherited from the triangulation $\mathcal{T}_h$, and suppose, without loss generality, that the numbers of edges of Γ_h is even. The case of an odd number of edges is easily reduced to the even case (see [30, remark at the end of Section 5.3] for details). Then, we let Γ_{2h} be the partition of Γ arising by joining pairs of adjacent edges of Γ_h. Because of the assumptions on the triangulations, Γ_h is automatically of bounded variation, and, therefore, so is Γ_{2h}. Hence, we now define

$$Q_h^{\boldsymbol{\varphi}} := \Big\{ \boldsymbol{\psi}_h \in \mathbf{C}(\Gamma) : \quad \boldsymbol{\psi}_h|_e \in \mathbf{P}_1(e) \quad \forall e \text{ edge of } \Gamma_{2h} \Big\}. \tag{3.4}$$

It is important to remark at this point that the above choice of $Q_h^{\boldsymbol{\varphi}}$, using the "*double*" partition Γ_{2h} instead of an independent partition $\Gamma_{\hat{h}}$ of Γ as in the original work [28], constitutes a clear simplification of the discrete analysis of our problem. In fact, thanks to the recent results obtained in [30, Section 5.3, particularly Lemma 5.2] (see also [25, Section 4.4]), the restriction on the mesh sizes given by $h \le C_0\,\hat{h}$, with an unknown constant C_0, which is required in [28, Lemmas 4.2 and 4.3] to prove the discrete inf-sup condition for $\mathbf{B}$, is not needed any more. Moreover, the aforementioned requirement of quasi-uniformity of the triangulations around Γ, which is a key ingredient in [30], was removed recently in [36, Sections 4 and 5] for the 2D case. However, we prefer to keep it here since the a posteriori error analysis to be developed below can also be extended to three-dimensional problems, for which that assumption is still necessary.

According to the foregoing analysis, we introduce the product spaces

$$\mathbf{H}_h := H_h^{\boldsymbol{\sigma}} \times \mathbb{RM}(\Omega) \quad \text{and} \quad \mathbf{Q}_h := Q_h^{\mathbf{u}} \times Q_h^{\boldsymbol{\varphi}} \times Q_h^{\boldsymbol{\gamma}},$$

and consider the following Galerkin approximation of (2.7): Find $((\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$\begin{aligned} \mathbf{A}((\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\boldsymbol{\tau}_h, \boldsymbol{\chi}_h)) + \mathbf{B}((\boldsymbol{\tau}_h, \boldsymbol{\chi}_h), (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)) &= 0, \\ \mathbf{B}((\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\mathbf{v}_h, \boldsymbol{\psi}_h, \boldsymbol{\eta}_h)) &= -\int_\Omega \mathbf{f} \cdot \mathbf{v}_h + \langle \mathbf{g}, \boldsymbol{\psi}_h \rangle_\Gamma, \end{aligned} \tag{3.5}$$

for all $((\boldsymbol{\tau}_h, \boldsymbol{\chi}_h), (\mathbf{v}_h, \boldsymbol{\psi}_h, \boldsymbol{\eta}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$. Concerning the analysis of (3.5) we remark that, besides the advances arising from the results in [30, Section 5.3], the asymptotic equivalence of norms given in [28, Lemma 4.4], which is actually taken from [24, Lemma 4.4], has also been improved lately to the case of arbitrary mesh sizes (see [22, Lemma 4.9]). Consequently, instead of the original result provided in [28, Theorem 4.1], the well-posedness of the Galerkin scheme (3.5) is now stated as follows.

Theorem 3.1. *There exists a unique* $((\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *solution of* (3.5). *Moreover, there exist* $C\,,\ \widetilde{C} > 0$, *independent of* h *and* λ, *such that*

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} + \|\boldsymbol{\rho}_h\|_{0,\Omega} + \|\mathbf{u}_h\|_{0,\Omega} + \|\boldsymbol{\varphi}_h\|_{1/2,\Gamma} + \|\boldsymbol{\gamma}_h\|_{0,\Omega} \le C \Big\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{-1/2,\Gamma} \Big\}, \tag{3.6}$$

and

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} + \|\boldsymbol{\rho}_h\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Gamma} + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega} \\ &\le \widetilde{C} \left\{ \operatorname{dist}(\boldsymbol{\sigma}, H_h^{\boldsymbol{\sigma}}) + \operatorname{dist}(\mathbf{u}, Q_h^{\mathbf{u}}) + \operatorname{dist}(\boldsymbol{\varphi}, Q_h^{\boldsymbol{\varphi}}) + \operatorname{dist}(\boldsymbol{\gamma}, Q_h^{\boldsymbol{\gamma}}) \right\}, \end{aligned} \tag{3.7}$$

where $((\boldsymbol{\sigma}, \mathbf{0}), (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma})) \in \mathbf{H} \times \mathbf{Q}$ *is the unique solution of* (2.7).

4 A residual-based a posteriori error estimator

In this section we derive reliable and efficient residual based a posteriori error estimators for (3.5). We begin by introducing several notations. We let $\mathcal{E}_h$ be the set of all edges of the triangulation $\mathcal{T}_h$, and given $T \in \mathcal{T}_h$, we let $\mathcal{E}(T)$ be the set of its edges. Then we write $\mathcal{E}_h = \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$, where $\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h : \quad e \subseteq \Omega\}$ and $\mathcal{E}_h(\Gamma) := \{e \in \mathcal{E}_h : \quad e \subseteq \Gamma\}$. In what follows, h_e stands for the length of a given edge e. Also, for each edge $e \in \mathcal{E}_h$ we fix a unit normal vector $\boldsymbol{\nu}_e := (\nu_1, \nu_2)^{\mathbf{t}}$, and let $\mathbf{s}_e := (-\nu_2, \nu_1)^{\mathbf{t}}$ be the corresponding fixed unit tangential vector along e. However, when no confusion arises, we simple write $\boldsymbol{\nu}$ and $\mathbf{s}$ instead of $\boldsymbol{\nu}_e$ and $\mathbf{s}_e$, respectively. Now, let $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$ such that $\boldsymbol{\tau}|_T \in \mathbb{C}(T)$ on each $T \in \mathcal{T}_h$. Then, given $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)$, we denote by $[\boldsymbol{\tau}\,\mathbf{s}]$ the tangential jump of $\boldsymbol{\tau}$ across e, that is $[\boldsymbol{\tau}\,\mathbf{s}] := (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e\,\mathbf{s}$, where T and T' are the triangles of $\mathcal{T}_h$ having e as a common edge. Similar definitions hold for the tangential jumps of scalar fields

$v \in L^2(\Omega)$ such that $v|_T \in C(T)$ on each $T \in \mathcal{T}_h$. Finally, given scalar, vector and tensor valued fields v, $\boldsymbol{\varphi} := (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ and $\boldsymbol{\tau} := (\tau_{ij})$, respectively we let

$$\mathbf{curl}(v) := \begin{pmatrix} \frac{\partial v}{\partial x_2} \\ -\frac{\partial v}{\partial x_1} \end{pmatrix}, \quad \underline{\mathbf{curl}}(\boldsymbol{\varphi}) := \begin{pmatrix} \frac{\partial \boldsymbol{\varphi}_1}{\partial x_2} & -\frac{\partial \boldsymbol{\varphi}_1}{\partial x_1} \\ \frac{\partial \boldsymbol{\varphi}_2}{\partial x_2} & -\frac{\partial \boldsymbol{\varphi}_2}{\partial x_1} \end{pmatrix}, \quad \text{and} \quad \mathrm{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \\ \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

Next, letting $\big((\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\mathbf{u}_h, \boldsymbol{\varphi}_h, \gamma_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ be the unique solution of (3.5), we define for each $T \in \mathcal{T}_h$ the a posteriori error indicator:

$$\begin{aligned} \theta_T^2 \;:=\; & \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T}^2 + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,T}^2 + \|\boldsymbol{\rho}_h\|_{0,T}^2 + h_T^2 \,\|\mathrm{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \gamma_h\big)\|_{0,T}^2 \\ & + \; h_T^2 \,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \gamma_h\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} h_e \,\|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \gamma_h)\,\mathbf{s}]\|_{0,e}^2 \\ & + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \left\|(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \gamma_h)\,\mathbf{s} + \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}}\right\|_{0,e}^2 \\ & + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \,\|\mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \,\|\boldsymbol{\varphi}_h + \mathbf{u}_h\|_{0,e}^2 \,, \end{aligned} \tag{4.1}$$

and introduce the global a posteriori error estimator

$$\boldsymbol{\theta} \;:=\; \left\{ \sum_{T \in \mathcal{T}_h} \theta_T^2 \right\}^{1/2}.$$

Then, the following theorem constitutes the main result of this paper.

Theorem 4.1. *Let $((\boldsymbol{\sigma}, \boldsymbol{\rho}), (\mathbf{u}, \boldsymbol{\varphi}, \gamma)) \in \mathbf{H} \times \mathbf{Q}$ and $((\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\mathbf{u}_h, \boldsymbol{\varphi}_h, \gamma_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ be the unique solutions of* (2.7) *and* (3.5)*, respectively. Then, there exists constants $C_{\mathtt{rel}} > 0$ and $C_{\mathtt{eff}} > 0$, independent of h, such that*

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} \;\le\; \|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h)\|_{\mathbf{H}} + \|(\mathbf{u}, \boldsymbol{\varphi}, \gamma) - (\mathbf{u}_h, \boldsymbol{\varphi}_h, \gamma_h)\|_{\mathbf{Q}} \;\le\; C_{\mathtt{rel}}\,\boldsymbol{\theta}\,. \tag{4.2}$$

The efficiency of the global a posteriori error estimator (lower bound in (4.2)) is proved below in Subsection 4.2, whereas the corresponding reliability (upper bound in (4.2)) is derived next.

4.1 Reliability of the a posteriori error estimator

We begin with the following preliminary estimate for the partial error $\|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h)\|_{\mathbf{H}}$.

Lemma 4.1. *Let $S_h : \mathbb{H}(\mathbf{div}; \Omega) \to \mathbb{R}$ be the functional defined by*

$$S_h(\boldsymbol{\tau}) \;:=\; \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, (\mathbf{u}_h, \boldsymbol{\varphi}_h, \gamma_h)) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; \Omega)\,,$$

and let $S_h|_V$ be its restriction to V, the first component of the kernel $\mathbf{V}$ of $\mathbf{B}$ (cf. (2.11)*). Then, there exists $C > 0$, independent of h, such that*

$$\begin{aligned} \|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h)\|_{\mathbf{H}} \;\le\; & C \Big\{ \|S_h|_V\|_{V'} + \|\mathbf{f} + \mathbf{div}\boldsymbol{\sigma}_h\|_{0,\Omega} \\ & + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,\Omega} + \|\boldsymbol{\rho}_h\|_{0,\Omega} + \|\mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}\|_{-1/2,\Gamma} \Big\}, \end{aligned} \tag{4.3}$$

and there holds $S_h(\boldsymbol{\tau}_h) = 0$ for each $\boldsymbol{\tau}_h \in H_h^{\boldsymbol{\sigma}}$.

Proof. We make use of a particular problem of the form (2.13). More precisely, let $((\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}}),(\bar{\mathbf{u}},\bar{\boldsymbol{\varphi}},\bar{\boldsymbol{\gamma}})) \in \mathbf{H}\times\mathbf{Q}$ be the unique solution of problem (2.13) with $\bar{F}\in\mathbf{H}'$ and $\bar{G}\in\mathbf{Q}'$ defined by

$$\bar{F}(\boldsymbol{\tau},\boldsymbol{\chi}) := 0 \quad \forall\,(\boldsymbol{\tau},\boldsymbol{\chi})\in\mathbf{H} \quad \text{and} \quad \bar{G}(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta}) := B((\boldsymbol{\sigma},\boldsymbol{\rho})-(\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) \quad \forall\,(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})\in\mathbf{Q}\,.$$

According to the second equation of (2.7) and the definition of $\mathbf{B}$ (cf. (2.9)), we easily find that

$$\bar{G}(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta}) \,=\, -\int_\Omega \mathbf{v}\cdot\big(\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h\big) \,-\, \int_\Omega \boldsymbol{\sigma}_h:\boldsymbol{\eta} \,-\, \int_\Omega \boldsymbol{\rho}_h\cdot\mathbf{v} \,+\, \langle \mathbf{g}-\boldsymbol{\sigma}_h\boldsymbol{\nu},\boldsymbol{\psi}\rangle_\Gamma\,,$$

which, noting that $\displaystyle\int_\Omega \boldsymbol{\sigma}_h:\boldsymbol{\eta} = \frac{1}{2}\int_\Omega(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^{\mathtt{t}}):\boldsymbol{\eta}$, yields

$$\|\bar{G}\|_{\mathbf{Q}'} \,\le\, C\left\{\|\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,\Omega} \,+\, \|\boldsymbol{\rho}_h\|_{0,\Omega} \,+\, \|\mathbf{g}-\boldsymbol{\sigma}_h\boldsymbol{\nu}\|_{-1/2,\Gamma}\right\}.$$

Then, the continuous dependence result (2.14) and the above estimate for $\|\bar{G}\|_{\mathbf{Q}'}$ imply

$$\|(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}} \,\le\, C\left\{\|\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,\Omega} \,+\, \|\boldsymbol{\rho}_h\|_{0,\Omega} \,+\, \|\mathbf{g}-\boldsymbol{\sigma}_h\boldsymbol{\nu}\|_{-1/2,\Gamma}\right\}. \tag{4.4}$$

On the other hand, a straightforward application of the triangle inequality gives

$$\|(\boldsymbol{\sigma},\boldsymbol{\rho}) \,-\, (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}} \,\le\, \|(\boldsymbol{\sigma},\boldsymbol{\rho}) \,-\, (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h) \,-\, (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}} \,+\, \|(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}}\,, \tag{4.5}$$

and hence, thanks to (4.4), it only remains to estimate $\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}}$. To this end, we first observe from the second equation of (2.13) that $(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})$ belongs to $\mathbf{V}$, the kernel of operator $\mathbf{B}$ (cf. (2.10)). Hence, applying the ellipticity of $\mathbf{A}$ on $\mathbf{V}$ (cf. Lemma 2.1), we obtain that

$$\begin{aligned}
\alpha\,\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}}^2 \,&\le\, \mathbf{A}\big((\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}}),(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\big)\\
&\le\, \mathbf{A}\big((\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\big)\\
&\quad+\, \|\mathbf{A}\|\,\|(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}}\,\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}}\,,
\end{aligned}$$

which, dividing by $\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}}$, taking supremum on $\mathbf{V}$, and then recalling from (2.11) (cf. Lemma 2.1) that $\mathbf{V} = V\times\{\mathbf{0}\}$, gives

$$\alpha\,\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}} \,\le\, \sup_{\substack{\boldsymbol{\tau}\in V\\ \boldsymbol{\tau}\neq 0}} \frac{\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\mathbf{0}))}{\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}} \,+\, \|\mathbf{A}\|\,\|(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\rho}})\|_{\mathbf{H}}\,. \tag{4.6}$$

Next, from the first equation of (2.7) we have

$$\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}),(\boldsymbol{\tau},\mathbf{0})) \,=\, -\,\mathbf{B}((\boldsymbol{\tau},\mathbf{0}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \,=\, 0 \qquad \forall\,\boldsymbol{\tau}\in V\,,$$

and then, bearing in mind the definition of $\mathbf{A}$ (cf. (2.8)), we get

$$\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\mathbf{0})) \,=\, -\,\mathbf{A}((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\mathbf{0})) \,=\, -\,\mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in V\,,$$

which, together with the fact that $\mathbf{b}(\boldsymbol{\tau},(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h))$ certainly vanishes for each $\boldsymbol{\tau}\in V$, yields

$$\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\mathbf{0})) \,=\, -\,S_h(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in V\,. \tag{4.7}$$

In this way, (4.3) follows directly from (4.4), (4.5), (4.6), and (4.7). Finally, it is quite clear from the first equation of (3.5) that

$$\begin{aligned}
0 \,&=\, \mathbf{A}((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau}_h,\mathbf{0})) \,+\, \mathbf{B}((\boldsymbol{\tau}_h,\mathbf{0}),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h))\\
&=\, \mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) \,+\, \mathbf{b}(\boldsymbol{\tau}_h,(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \,=\, S_h(\boldsymbol{\tau}_h) \qquad \forall\,\boldsymbol{\tau}_h\in H_h^{\boldsymbol{\sigma}}\,,
\end{aligned}$$

which completes the proof. □

We now aim to estimate

$$\|S_h|_V\|_{V'} \,:=\, \sup_{\substack{\boldsymbol{\tau}\in V\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{S_h(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}}$$

in (4.3), for which, according to the null property of S_h provided by the previous theorem, we will replace $S_h(\boldsymbol{\tau})$ by $S_h(\boldsymbol{\tau}-\boldsymbol{\tau}_h)$ with a suitably chosen $\boldsymbol{\tau}_h \in H_h^{\boldsymbol{\sigma}}$ depending each time on the given $\boldsymbol{\tau} \in V$. To this end, we now let $I_h : H^1(\Omega) \to X_h$ be the Clément interpolation operator (cf. [20]), where

$$X_h \,:=\, \Big\{ v_h \in C(\bar{\Omega}) : \quad v_h|_T \,\in\, P_1(T) \quad \forall\, T \in \mathcal{T}_h \Big\}\,. \tag{4.8}$$

A vectorial version of I_h, say $\mathbf{I}_h : \mathbf{H}^1(\Omega) \to \mathbf{X}_h \,:=\, X_h \times X_h$, which is defined componentwise by I_h, is also required. The following lemma provides the local approximation properties of I_h. Analogue estimates hold for the operator $\mathbf{I}_h$.

Lemma 4.2. *There exist c_1, $c_2 > 0$, independent of h, such that for all $v \in H^1(\Omega)$ there holds*

$$\|v - I_h(v)\|_{0,T} \,\leq\, c_1\, h_T\, \|v\|_{1,\Delta(T)} \qquad \forall\, T \in \mathcal{T}_h$$

and

$$\|v - I_h(v)\|_{0,e} \,\leq\, c_2\, h_e^{1/2}\, \|v\|_{1,\Delta(e)} \qquad \forall\, e \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)\,,$$

where $\Delta(T) \,:=\, \cup\{T' \in \mathcal{T}_h : \quad T' \cap T \neq \emptyset\}$ and $\Delta(e) \,:=\, \cup\{T' \in \mathcal{T}_h : \quad T' \cap e \neq \emptyset\}$.

Proof. See [20]. □

The estimate for $\|S_h|_V\|_{V'}$ is established as follows.

Lemma 4.3. *Let $((\boldsymbol{\sigma},\boldsymbol{\rho}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \,\in\, \mathbf{H}\times\mathbf{Q}$ and $((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \,\in\, \mathbf{H}_h\times\mathbf{Q}_h$ be the unique solutions of* (2.7) *and* (3.5)*, respectively. Then, there exists $C > 0$, independent of h, such that*

$$\|S_h|_V\|_{V'} \,\leq\, C \left\{ \sum_{T\in\mathcal{T}_h} \widetilde{\theta}_T^2 \right\}^{1/2}\,, \tag{4.9}$$

where

$$\begin{aligned}\widetilde{\theta}_T^2 \,:=\, & h_T^2\,\|\mathrm{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h \,+\, \boldsymbol{\gamma}_h\big)\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega)} h_e\,\|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_h \,+\, \boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\,\left\|(\mathcal{C}^{-1}\boldsymbol{\sigma}_h \,+\, \boldsymbol{\gamma}_h)\,\mathbf{s} \,+\, \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}}\right\|_{0,e}^2\,.\end{aligned} \tag{4.10}$$

Proof. Given $\boldsymbol{\tau} \,\in\, V$ (cf. (2.11)) we clearly have $\mathbf{div}\,\boldsymbol{\tau} \,=\, \mathbf{0}$ in Ω, and hence there exists $\boldsymbol{\phi} \,:=\, (\phi_1,\phi_2) \in \mathbf{H}^1(\Omega)$ such that $\int_\Omega \phi_1 \,=\, \int_\Omega \phi_2 \,=\, 0$ and $\boldsymbol{\tau} \,=\, \underline{\mathbf{curl}}\,\boldsymbol{\phi}$. Note that the conditions satisfied by the components of $\boldsymbol{\phi}$ guarantee that $\|\boldsymbol{\phi}\|_{1,\Omega}$ and $|\boldsymbol{\phi}|_{1,\Omega}$ are equivalent. Then, we let $\boldsymbol{\phi}_h \in \mathbf{X}_h$ be the Clément interpolant of $\boldsymbol{\phi}$, that is $\boldsymbol{\phi}_h \,:=\, \mathbf{I}_h(\boldsymbol{\phi})$, and define $\boldsymbol{\tau}_h \,:=\, \underline{\mathbf{curl}}\,\boldsymbol{\phi}_h$ so that $\boldsymbol{\tau} - \boldsymbol{\tau}_h \,=\, \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)$. In turn, it is easy to see that $\boldsymbol{\tau}_h$ belongs to $H_h^{\boldsymbol{\sigma}}$, and therefore the null property satisfied by S_h (cf. Lemma 4.1) implies that

$$S_h(\boldsymbol{\tau}) \,=\, S_h(\boldsymbol{\tau} - \boldsymbol{\tau}_h) \,=\, \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau} - \boldsymbol{\tau}_h) \,+\, \mathbf{b}(\boldsymbol{\tau} - \boldsymbol{\tau}_h, (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h))\,,$$

which, in virtue of the definitions of **a** and **b** (cf. (2.5), (2.6)), gives

$$S_h(\boldsymbol{\tau}) \;=\; \int_\Omega \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h \,+\, \boldsymbol{\gamma}_h\big) : \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \,+\, \big\langle \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,\boldsymbol{\nu}, \boldsymbol{\varphi}_h \big\rangle_\Gamma \,. \tag{4.11}$$

Next, since

$$\underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,\boldsymbol{\nu} \;=\; -\frac{d}{d\mathbf{s}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \quad \text{and} \quad \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}} \in \mathbf{L}^2(\Gamma)\,,$$

we find, integrating by parts on Γ, that

$$\langle \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,\boldsymbol{\nu}, \boldsymbol{\varphi}_h \rangle_\Gamma \;=\; -\,\left\langle \frac{d}{d\mathbf{s}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \boldsymbol{\varphi}_h \right\rangle_\Gamma \;=\; \int_\Gamma \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}} \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,. \tag{4.12}$$

On the other hand, integrating by parts on each $T \in \mathcal{T}_h$, we obtain that

$$\begin{aligned}
\int_\Omega \big\{\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big\} : \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \;&=\; \sum_{T\in\mathcal{T}_h} \int_T \big\{\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big\} : \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \\
&= \sum_{T\in\mathcal{T}_h} \Big\{ - \int_T \operatorname{curl}\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \,+\, \int_{\partial T} \big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s} \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \Big\} \\
&= -\sum_{T\in\mathcal{T}_h} \int_T \operatorname{curl}\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \,+ \sum_{e\in\mathcal{E}_h(\Omega)} \int_e \big[\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\big] \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \\
&\qquad +\; \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s} \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,,
\end{aligned}$$

which, together with (4.12), yields

$$\begin{aligned}
S_h(\boldsymbol{\tau}) \;=\; &-\sum_{T\in\mathcal{T}_h} \int_T \operatorname{curl}\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) + \sum_{e\in\mathcal{E}_h(\Omega)} \int_e \big[\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\big] \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \\
&+\; \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \Big\{ \big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s} \,+\, \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}} \Big\} \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,.
\end{aligned} \tag{4.13}$$

Then, applying Cauchy-Schwarz inequality and the approximation properties of the Clément interpolation operator $\mathbf{I}_h$ (cf. Lemma 4.2), and then using that the number of elements of $\Delta(T)$ is bounded independently of $T \in \mathcal{T}_h$, it follows that

$$\begin{aligned}
\left| \sum_{T\in\mathcal{T}_h} \int_T \operatorname{curl}\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \right| \;&\le\; c_1 \sum_{T\in\mathcal{T}_h} h_T\, \big\|\operatorname{curl}\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\big\|_{0,T}\, \|\boldsymbol{\phi}\|_{1,\Delta(T)} \\
&\le\; C \left\{ \sum_{T\in\mathcal{T}_h} h_T^2\, \big\|\operatorname{curl}\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\big\|_{0,T}^2 \right\}^{1/2} \|\boldsymbol{\phi}\|_{1,\Omega}\,.
\end{aligned} \tag{4.14}$$

Proceeding analogously, and now employing that the number of elements of $\Delta(e)$ is bounded independently of $e \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$, we find that

$$\left| \sum_{e\in\mathcal{E}_h(\Omega)} \int_e \big[\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\big] \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \right| \le C \left\{ \sum_{e\in\mathcal{E}_h(\Omega)} h_e\, \big\|\big[\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\big]\big\|_{0,e}^2 \right\}^{1/2} \|\boldsymbol{\phi}\|_{1,\Omega}\,, \tag{4.15}$$

and

$$\left| \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \left\{ \left(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \gamma_h\right)\mathbf{s} \,+\, \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}} \right\} \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \right| \\ \leq\, C \left\{ \sum_{e\in\mathcal{E}_h(\Gamma)} h_e \left\| \left(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \gamma_h\right)\mathbf{s} \,+\, \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}} \right\|_{0,e}^2 \right\}^{1/2} \|\boldsymbol{\phi}\|_{1,\Omega}\,. \tag{4.16}$$

Finally, (4.13), (4.14), (4.15), and (4.16), together with the fact that

$$\|\boldsymbol{\phi}\|_{1,\Omega} \,\leq\, c\,|\boldsymbol{\phi}|_{1,\Omega} \,=\, \|\underline{\mathbf{curl}}\,\boldsymbol{\phi}\|_{0,\Omega} \,=\, \|\boldsymbol{\tau}\|_{0,\Omega} \,=\, \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,,$$

imply (4.9) and complete the proof. □

Besides Lemmas 4.1 and 4.3, and in order to complete the upper bound for $\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}}$ in terms of local quantities, we need to estimate the boundary term $\|\mathbf{g} - \boldsymbol{\sigma}_h\,\boldsymbol{\nu}\|_{-1/2,\Gamma}$. In fact, we first observe that taking $(\mathbf{v}_h,\boldsymbol{\eta}_h) = (\mathbf{0},\mathbf{0})$ in (3.5), we arrive at

$$\langle \boldsymbol{\sigma}_h\,\boldsymbol{\nu} - \mathbf{g}, \boldsymbol{\psi}_h\rangle_\Gamma \,=\, 0 \qquad \forall \boldsymbol{\psi}_h \in Q_h^{\boldsymbol{\varphi}}\,,$$

which says that each component of $(\boldsymbol{\sigma}_h\,\boldsymbol{\nu} - \mathbf{g})$ is $L^2(\Gamma)$-orthogonal to the continuous piecewise linear functions on the double partition Γ_{2h} of Γ. Consequently, applying [17, Theorem 2] and recalling that Γ_h and Γ_{2h} are of bounded variation, we obtain

$$\|\mathbf{g} - \boldsymbol{\sigma}_h\,\boldsymbol{\nu}\|_{-1/2,\Gamma}^2 \,\leq\, c \sum_{e\in\mathcal{E}_h(\Gamma)} h_e\,\|\mathbf{g} - \boldsymbol{\sigma}_h\,\boldsymbol{\nu}\|_{0,e}^2\,. \tag{4.17}$$

In this way, the a posteriori error estimate for $\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}}$ follows straightforwardly from Lemmas 4.1 and 4.3, and (4.17). More precisely, we have the following result.

Lemma 4.4. *Let* $((\boldsymbol{\sigma},\boldsymbol{\rho}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \in \mathbf{H}\times\mathbf{Q}$ *and* $((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \in \mathbf{H}_h\times\mathbf{Q}_h$ *be the unique solutions of* (2.7) *and* (3.5)*, respectively. Then, there exists a constant* $C > 0$ *independent of* h*, such that*

$$\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}} \,\leq\, C \left\{ \sum_{T\in\mathcal{T}_h} \widehat{\theta}_T^2 \right\}^{1/2}\,, \tag{4.18}$$

where

$$\widehat{\theta}_T^2 \,:=\, \widetilde{\theta}_T^2 \,+\, \|\mathbf{f} + \mathbf{div}\boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,T}^2 \,+\, \|\boldsymbol{\rho}_h\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\,\|\mathbf{g} - \boldsymbol{\sigma}_h\,\boldsymbol{\nu}\|_{0,e}^2 \tag{4.19}$$

for each $T \in \mathcal{T}_h$*, with* $\widetilde{\theta}_T^2$ *defined by* (4.10)*.*

We proceed next to obtain the corresponding upper bound for $\|(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}) - (\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)\|$. For this purpose, we need some additional preliminary results concerning the Helmholtz decomposition of $\mathbb{H}(\mathbf{div};\Omega)$ and the approximation properties of the Raviart-Thomas interpolation operator. We begin with the following lemma.

Lemma 4.5. *For each* $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$ *there exist* $\boldsymbol{\zeta} \in \mathbb{H}^1(\Omega)$ *and* $\boldsymbol{\phi} := (\phi_1,\phi_2)^{\mathtt{t}} \in \mathbf{H}^1(\Omega)$*, with* $\int_\Omega \phi_1 = \int_\Omega \phi_2 = 0$*, such that* $\boldsymbol{\tau} \,=\, \boldsymbol{\zeta} \,+\, \underline{\mathbf{curl}}\,\boldsymbol{\phi}$ *in* Ω *and*

$$\|\boldsymbol{\zeta}\|_{1,\Omega} + \|\boldsymbol{\phi}\|_{1,\Omega} \,\leq\, C\,\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,, \tag{4.20}$$

where C *is a positive constant independent of* $\boldsymbol{\tau}$*.*

Proof. It is an adaptation of the analysis from [26, Section 3.2.2]. See also [21, Lemma 3.4] for full details. □

On the other hand, we also need to introduce the space of pure Raviart-Thomas tensors of order 0, that is

$$\mathrm{RT}_h := \left\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \ \mathbf{c}^{\mathrm{t}}\,\boldsymbol{\tau}_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h\,, \quad \forall\, \mathbf{c} \in \mathbb{R}^2\right\},$$

which is clearly contained in $H_h^{\boldsymbol{\sigma}}$ (cf. (3.1)). Then, we let $\Pi_h : \mathbb{H}^1(\Omega) \to \mathrm{RT}_h$ be the usual Raviart-Thomas interpolation operator, which is characterized by the identity

$$\int_e \Pi_h(\boldsymbol{\zeta})\,\boldsymbol{\nu} = \int_e \boldsymbol{\zeta}\,\boldsymbol{\nu} \qquad \forall\, e \in \mathcal{T}_h\,, \quad \forall\, \boldsymbol{\zeta} \in \mathbb{H}^1(\Omega)\,. \tag{4.21}$$

It is easy to show, using (4.21), that

$$\mathbf{div}(\Pi_h(\boldsymbol{\zeta})) \,=\, \mathcal{P}_h(\mathbf{div}\,\boldsymbol{\zeta}) \qquad \forall\, \boldsymbol{\zeta} \in \mathbb{H}^1(\Omega)\,, \tag{4.22}$$

where $\mathcal{P}_h$ is the $\mathbf{L}^2(\Omega)$-orthogonal projector onto $Q_h^{\mathbf{u}}$ (cf. (3.2)). In addition, it is well known (see, e.g. [15], [25, Lemmas 3.16 and 3.18], and [38]) that Π_h satisfies the following approximation properties

$$\|\boldsymbol{\zeta} - \Pi_h(\boldsymbol{\zeta})\|_{0,T} \,\le\, C\,h_T\,\|\boldsymbol{\zeta}\|_{1,T} \quad \forall\, T \in \mathcal{T}_h\,, \quad \forall\, \boldsymbol{\zeta} \in \mathbb{H}^1(\Omega)\,, \tag{4.23}$$

and

$$\|(\boldsymbol{\zeta} - \Pi_h(\boldsymbol{\zeta}))\,\boldsymbol{\nu}\|_{0,e} \,\le\, C\,h_e^{1/2}\,\|\boldsymbol{\zeta}\|_{1,T_e} \quad \forall\, e \in \mathcal{T}_h, \quad \forall\, \boldsymbol{\zeta} \in \mathbb{H}^1(\Omega)\,, \tag{4.24}$$

where T_e in (4.24) is a triangle of $\mathcal{T}_h$ containing e on its boundary.

We are now in a position to establish the remaining a posteriori error estimate.

Lemma 4.6. *Let* $((\boldsymbol{\sigma},\boldsymbol{\rho}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \in \mathbf{H}\times\mathbf{Q}$ *and* $((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \in \mathbf{H}_h\times\mathbf{Q}_h$ *be the unique solutions of* (2.7) *and* (3.5)*, respectively. Then, there exists a constant* $C > 0$*, independent of* h*, such that*

$$\|(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}) - (\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)\|_{\mathbf{Q}} \,\le\, C\left\{\sum_{T\in\mathcal{T}_h} \theta_T^2\right\}^{1/2}, \tag{4.25}$$

where θ_T^2 *is the complete a posteriori error indicator defined by* (4.1)*.*

Proof. We begin by applying the continuous inf-sup condition for $\mathbf{B}$ (cf. Lemma 2.2), which yields

$$\beta\,\|(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}) - (\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)\|_{\mathbf{Q}} \,\le\, \sup_{\substack{(\boldsymbol{\tau},\boldsymbol{\chi})\in\mathbf{H}\\ (\boldsymbol{\tau},\boldsymbol{\chi})\neq 0}} \frac{\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}) - (\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h))}{\|(\boldsymbol{\tau},\boldsymbol{\chi})\|_{\mathbf{H}}}\,. \tag{4.26}$$

Next, using from the first equation of (2.7) that $\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \,=\, -\,\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}),(\boldsymbol{\tau},\boldsymbol{\chi}))$, and then substracting and adding $(\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)$ in the first component, we find that for each $(\boldsymbol{\tau},\boldsymbol{\chi}) \in \mathbf{H}$ there holds

$$\begin{aligned}\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}) - (\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \,&=\, -\,\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\boldsymbol{\chi}))\\ &-\,\mathbf{A}((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\boldsymbol{\chi})) \,-\, \mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h))\,.\end{aligned} \tag{4.27}$$

Then, noting that $\int_\Omega(\mathbf{u}_h + \boldsymbol{\rho}_h)\cdot\boldsymbol{\chi} = 0$, which follows from the first equation of (3.5) when taking $\boldsymbol{\chi}_h = \boldsymbol{\chi}$ and $\boldsymbol{\tau}_h = \mathbf{0}$, and bearing in mind the functional $S_h : \mathbb{H}(\mathbf{div};\Omega) \to \mathbb{R}$ defined in the statement of Lemma 4.1, we obtain that for each $(\boldsymbol{\tau},\boldsymbol{\chi}) \in \mathbf{H}$ there holds

$$-\,\mathbf{A}((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\boldsymbol{\chi})) \,-\, \mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \,=\, -\,\mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) \,-\, \mathbf{b}(\boldsymbol{\tau},(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \,=:\, -\,S_h(\boldsymbol{\tau})\,,$$

whence (4.27) becomes

$$\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u},\boldsymbol{\varphi},\gamma)-(\mathbf{u}_h,\boldsymbol{\varphi}_h,\gamma_h)) \,=\, -\,\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho})-(\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\boldsymbol{\chi})) \,-\, S_h(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\chi})\in\mathbf{H}\,.$$

Thus, replacing the above expression back into (4.26) and using the boundedness of $\mathbf{A}$ (with constant $\|\mathbf{A}\|$), we easily deduce that

$$\beta\,\|(\mathbf{u},\boldsymbol{\varphi},\gamma)-(\mathbf{u}_h,\boldsymbol{\varphi}_h,\gamma_h)\|_{\mathbf{Q}} \,\le\, \|\mathbf{A}\|\,\|(\boldsymbol{\sigma},\boldsymbol{\rho})-(\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}} \,+\, \|S_h\|_{\mathbb{H}(\mathbf{div};\,\Omega)'} \tag{4.28}$$

It remains to bound $\|S_h\|_{\mathbb{H}(\mathbf{div};\,\Omega)'}$ on the right hand side of (4.28), for which we appeal to the Helmholtz decompositions from Lemma 4.5. In other words, given $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\,\Omega)$, we let $\boldsymbol{\zeta}\in\mathbb{H}^1(\Omega)$, $\boldsymbol{\phi}\in\mathbf{H}^1(\Omega)$, and C a positive constant independent of $\boldsymbol{\tau}$, such that $\boldsymbol{\tau}\,=\,\boldsymbol{\zeta}\,+\,\underline{\mathbf{curl}}\,\boldsymbol{\phi}$ in Ω and

$$\|\boldsymbol{\zeta}\|_{1,\Omega}+\|\boldsymbol{\phi}\|_{1,\Omega} \,\le\, C\,\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,. \tag{4.29}$$

Then, we introduce

$$\boldsymbol{\phi}_h \,:=\, \mathbf{I}_h(\boldsymbol{\phi})\in\mathbf{X}_h \quad\text{and}\quad \boldsymbol{\tau}_h \,:=\, \Pi_h(\boldsymbol{\zeta}) \,+\, \underline{\mathbf{curl}}(\boldsymbol{\phi}_h) \,\in\, \mathrm{RT}_h \,\subseteq\, H_h^{\boldsymbol{\sigma}}\,,$$

which yields

$$\boldsymbol{\tau}\,-\,\boldsymbol{\tau}_h \,=\, \boldsymbol{\zeta}\,-\,\Pi_h(\boldsymbol{\zeta}) \,+\, \underline{\mathbf{curl}}\big(\boldsymbol{\phi}-\boldsymbol{\phi}_h\big)\,.$$

It follows using (4.22) that

$$\mathbf{div}\big(\boldsymbol{\tau}\,-\,\boldsymbol{\tau}_h\big) \,=\, \big(\mathbf{I}-\mathcal{P}_h\big)\big(\mathbf{div}\,\boldsymbol{\zeta}\big) \,=\, \big(\mathbf{I}-\mathcal{P}_h\big)\big(\mathbf{div}\,\boldsymbol{\tau}\big)\,,$$

which is $\mathbf{L}^2(\Omega)$-orthogonal to $Q_h^{\mathbf{u}}$, and hence, taking into account from Lemma 4.1 the null property satisfied by S_h, we can write that

$$S_h(\boldsymbol{\tau}) = S_h(\boldsymbol{\tau}-\boldsymbol{\tau}_h) \,=\, S_h\big(\boldsymbol{\zeta}\,-\,\Pi_h(\boldsymbol{\zeta})\big) \,+\, S_h\big(\underline{\mathbf{curl}}\big(\boldsymbol{\phi}-\boldsymbol{\phi}_h\big)\big)\,, \tag{4.30}$$

where, recalling that $S_h(\boldsymbol{\tau}) = \mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau},(\mathbf{u}_h,\boldsymbol{\varphi}_h,\gamma_h))$, we have

$$S_h\big(\boldsymbol{\zeta}\,-\,\Pi_h(\boldsymbol{\zeta})\big) \,=\, \int_\Omega \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h+\gamma_h\big) : \big(\boldsymbol{\zeta}\,-\,\Pi_h(\boldsymbol{\zeta})\big) \,+\, \langle\big(\boldsymbol{\zeta}\,-\,\Pi_h(\boldsymbol{\zeta})\big)\,\boldsymbol{\nu},\boldsymbol{\varphi}_h\rangle_\Gamma\,,$$

and

$$S_h\big(\underline{\mathbf{curl}}\big(\boldsymbol{\phi}-\boldsymbol{\phi}_h\big)\big) \,=\, \int_\Omega \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h+\gamma_h\big) : \underline{\mathbf{curl}}\big(\boldsymbol{\phi}-\boldsymbol{\phi}_h\big) \,+\, \langle\underline{\mathbf{curl}}\big(\boldsymbol{\phi}-\boldsymbol{\phi}_h\big)\,\boldsymbol{\nu},\boldsymbol{\varphi}_h\rangle_\Gamma\,.$$

The estimate for the latter term proceeds exactly as in the proof of Lemma 4.3, which gives, using now (4.29), that

$$\big|S_h\big(\underline{\mathbf{curl}}\big(\boldsymbol{\phi}-\boldsymbol{\phi}_h\big)\big)\big| \,\le\, C\,\left\{\sum_{T\in\mathcal{T}_h}\widetilde{\theta}_T^2\right\}^{1/2}\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,, \tag{4.31}$$

with $\widetilde{\theta}_T^2$ defined by (4.10). In turn, for the former term we first notice that the fact that $\boldsymbol{\zeta}$ belongs to $\mathbb{H}^1(\Omega)$ guarantees that $\big(\boldsymbol{\zeta}\,-\,\Pi_h(\boldsymbol{\zeta})\big)\,\boldsymbol{\nu}\in\mathbf{L}^2(\Gamma)$, and then, utilizing additionally the characterization (4.21), we get

$$S_h\big(\boldsymbol{\zeta}\,-\,\Pi_h(\boldsymbol{\zeta})\big) \,=\, \int_\Omega \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h+\gamma_h\big) : \big(\boldsymbol{\zeta}\,-\,\Pi_h(\boldsymbol{\zeta})\big) \,+\, \sum_{e\in\mathcal{E}_h(\Gamma)}\int_e \big(\boldsymbol{\zeta}\,-\,\Pi_h(\boldsymbol{\zeta})\big)\,\boldsymbol{\nu}\cdot\big(\boldsymbol{\varphi}_h+\mathbf{u}_h\big)\,. \tag{4.32}$$

In this way, employing the Cauchy-Schwarz inequality, the approximation properties (4.23) and (4.24), and the estimate (4.29), we deduce from (4.32) that

$$\big|S_h\big(\boldsymbol{\zeta} - \Pi_h(\boldsymbol{\zeta})\big)\big| \le C \left\{ \sum_{T\in\mathcal{T}_h} h_T^2 \, \|\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(\Gamma)} h_e \, \|\boldsymbol{\varphi}_h + \mathbf{u}_h\|_{0,e}^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega} . \quad (4.33)$$

Finally, it follows from (4.30), (4.31), and (4.33) that

$$\|S_h\|_{\mathbb{H}(\mathbf{div};\Omega)'} \le C \left\{ \sum_{T\in\mathcal{T}_h} \left(\widetilde{\theta}_T^2 + h_T^2 \, \|\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\|_{0,T}^2 \right) + \sum_{e\in\mathcal{E}_h(\Gamma)} h_e \, \|\boldsymbol{\varphi}_h + \mathbf{u}_h\|_{0,e}^2 \right\}^{1/2} ,$$

which, together with (4.28) and the estimate for $\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}}$ given by Lemma 4.4, yields (4.25) and completes the proof. □

We end this section by remarking that the reliability of $\boldsymbol{\theta}$, that is the upper bound in (4.2), is a straightforward consequence of Lemmas 4.4 and 4.6.

4.2 Efficiency of the a posteriori error estimators

The goal of this section is to show the efficiency of our a posteriori error estimator $\boldsymbol{\theta}$. In other words, we provide upper bounds depending on the actual errors for the nine terms defining the local indicator θ_T^2 (cf. (4.1)). We begin with the first three ones appearing there. In fact, since $\mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f}$ in Ω, we easily see that

$$\|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T}^2 = \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,T}^2 \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},T}^2 . \quad (4.34)$$

Next, adding and substracting $\boldsymbol{\sigma}$, and using that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathtt{t}}$ in Ω, we obtain

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,T}^2 \le 4\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 . \quad (4.35)$$

Finally, since actually $\boldsymbol{\rho} = \mathbf{0}$ (cf. Theorem 2.1), it is clear that

$$\|\boldsymbol{\rho}_h\|_{0,T}^2 = \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,T}^2 . \quad (4.36)$$

In what follows we give the corresponding upper bounds for the remaining terms in (4.1). Since most of these estimates are already available in the literature or can be easily derived from related ones (see, e.g. [16], [18], [21], [26], and [29]), we either refer to the corresponding proofs or sketch them. The main techniques involved include the localization technique based on triangle-bubble and edge-bubble functions, together with extension operators, discrete trace and inverse inequalities. For a better understanding of them, we now introduce further notations and preliminary results. Given $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T)$, we let ψ_T and ψ_e be the usual triangle-bubble and edge-bubble functions, respectively (see [41, eqs. (1.5) and (1.6)]), which satisfy:

ii) $\psi_T \in P_3(T)$, $\psi_T = 0$ on ∂T, $\mathrm{supp}(\psi_T) \subseteq T$, and $0 \le \psi_T \le 1$ in T.

ii) $\psi_e|_T \in P_2(T)$, $\psi_e = 0$ on $\partial T \setminus e$, $\mathrm{supp}(\psi_e) \subseteq w_e := \cup\{T' \in \mathcal{T}_h : \; e \in \mathcal{E}(T')\}$, and $0 \le \psi_e \le 1$ in w_e.

We also know from [40] that, given $k \in \mathbb{N}\cup\{0\}$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p) \in P_k(T)$ and $L(p)|_e = p$ for all $p \in P_k(e)$. Additional properties of ψ_T, ψ_e and L are collected in the following lemma.

Lemma 4.7. *Given $k \in \mathbb{N} \cup \{0\}$, there exist positive constants c_1, c_2 and c_3, depending only on k and the shape regularity of the triangulations (minimun angle condition), such that for each $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T)$, there hold*

$$\|q\|_{0,T}^2 \leq c_1 \|\psi_T^{1/2} q\|_{0,T}^2 \qquad \forall q \in P_k(T) \tag{4.37}$$

$$\|p\|_{0,e}^2 \leq c_2 \|\psi_e^{1/2} p\|_{0,e}^2 \qquad \forall p \in P_k(e) \tag{4.38}$$

and

$$\|\psi_e^{1/2} L(p)\|_{0,T}^2 \leq c_3 \, h_e \|p\|_{0,e}^2 \qquad \forall p \in P_k(e) \tag{4.39}$$

Proof. See [40, Lemma 1.3]. □

The following inverse and discrete trace inequalities are also employed.

Lemma 4.8. *Let k, l, $m \in \mathbb{N} \cup \{0\}$ such that $l \leq m$. Then there exists $c > 0$, depending only on k, l, m and the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h$ there holds*

$$|q|_{m,T} \; \leq \; c \, h_T^{l-m} \, |q|_{l,T} \qquad \forall \, q \in P_k(T) \, . \tag{4.40}$$

Proof. See [19, Theorem 3.2.6]. □

Lemma 4.9. *There exists $C > 0$, depending only on the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T)$, there holds*

$$\|v\|_{0,e}^2 \; \leq \; C \left\{ h_e^{-1} \, \|v\|_{0,T}^2 \, + \, h_e \, |v|_{1,T}^2 \right\} \quad \forall \, v \in H^1(T) \, . \tag{4.41}$$

Proof. See [1, Theorem 3.10] or [3, eq. (2.4)]. □

The upper bounds for the terms involving only the tensor $\mathcal{C}^{-1} \boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h$, whose proofs make use of the techniques and results described above, are given next.

Lemma 4.10. *There exists $C > 0$, independent of h and λ, such that for each $T \in \mathcal{T}_h$ there holds*

$$h_T^2 \, \|\mathrm{curl} \, (\mathcal{C}^{-1} \boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\|_{0,T}^2 \; \leq \; C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \, + \, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\} .$$

Proof. See [18, Lemma 6.3] or [11, Lemma 4.7]. □

Lemma 4.11. *There exists $C > 0$, independent of h and λ, such that for each $T \in \mathcal{T}_h$ there holds*

$$h_T^2 \, \|\mathcal{C}^{-1} \boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\|_{0,T}^2 \; \leq \; C \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \, + \, h_T^2 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \, + \, h_T^2 \, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\} .$$

Proof. See [18, Lemma 6.6]. □

Lemma 4.12. *There exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Omega)$ there holds*

$$h_e \, \|[(\mathcal{C}^{-1} \boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h) \mathbf{s}]\|_{0,e}^2 \; \leq \; C \sum_{T \subseteq \omega_e} \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \, + \, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\} ,$$

where $\omega_e := \cup \big\{ T' \in \mathcal{T}_h : \quad e \in \mathcal{E}(T') \big\}$.

Proof. See [18, Lemma 6.4]. □

The upper bound for the term involving the tensor $\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h$ and the tangential derivative of $\boldsymbol{\varphi}_h$ is given now.

Lemma 4.13. *There exists $C > 0$, independent of h and λ, such that*

$$\sum_{e\in\mathcal{E}_h(\Gamma)} h_e \left\| (\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\,\mathbf{s} + \frac{d\,\boldsymbol{\varphi}_h}{d\mathbf{s}} \right\|_{0,e}^2 \leq C \left\{ \sum_{e\in\mathcal{E}_h(\Gamma)} \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T_e}^2 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T_e}^2 \right\} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Gamma}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Gamma)$, T_e is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. We refer to [26, Lemma 20] where this result was established and proved. The proof makes use of the vector version of the extension operator $L : C(e) \to C(T)$, the fact that $\nabla\mathbf{u} = \mathcal{C}^{-1}\,\boldsymbol{\sigma} + \boldsymbol{\gamma}$ in Ω, the boundedness of the tangential derivative $\frac{d}{d\mathbf{s}} : \mathbf{H}^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}(\Gamma)$, the inverse and the Cauchy-Schwarz inequalities, and the bound for $h_{T_e}^2\,\|\mathrm{curl}\,(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\|_{0,T_e}^2$ provided by Lemma 4.10. □

While the estimate given by Lemma 4.13 is of non local character, it certainly suffices to conclude the efficiency of $\boldsymbol{\theta}$. However, the following lemma establishes that, under an additional regularity assumption on $\boldsymbol{\varphi}$, a corresponding local estimate can also be obtained.

Lemma 4.14. *Assume that $\boldsymbol{\varphi}|_e \in \mathbf{H}^1(e)$ for each $e \in \mathcal{E}_h(\Gamma)$. Then there exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Gamma)$ there holds*

$$h_e \left\| (\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\,\mathbf{s} + \frac{d\,\boldsymbol{\varphi}_h}{d\mathbf{s}} \right\|_{0,e}^2 \leq C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T_e}^2 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T_e}^2 + h_e \left\| \frac{d}{d\mathbf{s}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \right\|_{0,e}^2 \right\},$$

where T_e is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. See [26, Lemma 21]. □

We derive now the upper bound for the term concerning the Neumann boundary condition on Γ. To this end, and for simplicity, we assume that $\mathbf{g}$ is piecewise polynomial on Γ. Otherwise, one would proceed as in the proof of related results by adding and substracting a suitable projection of $\mathbf{g}$ onto a polynomial space (see, e.g. [26, Lemma 23]).

Lemma 4.15. *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Gamma)$ there holds*

$$h_e\,\|\mathbf{g} - \boldsymbol{\sigma}_h\,\boldsymbol{\nu}\|_{0,e}^2 \leq C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 + h_T^2\,\|\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,T}^2 \right\}, \tag{4.42}$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. Given $e \in \mathcal{E}_h(\Gamma)$, we let T be the triangle of $\mathcal{T}_h$ having e as an edge, define $\mathbf{v}_e := \mathbf{g} - \boldsymbol{\sigma}_h\,\boldsymbol{\nu}$ on e, and consider the vector version $\mathbf{L}$ of the extension operator $L : C(e) \to C(T)$. Then, applying (4.38), recalling that $\psi_e = 0$ on $\partial T\backslash e$, extending $\psi_e\,\mathbf{L}(v_e)$ by zero in $\Omega\backslash T$ so that the resulting function belongs to $\mathbf{H}^1(\Omega)$, and replacing the datum $\mathbf{g}$ by $\boldsymbol{\sigma}\,\boldsymbol{\nu}$ on Γ, we get

$$\|\mathbf{v}_e\|_{0,e}^2 \leq c_2 \int_e \psi_e\,\mathbf{v}_e \cdot \big(\mathbf{g} - \boldsymbol{\sigma}_h\,\boldsymbol{\nu}\big) = c_2\langle(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,\boldsymbol{\nu}, \psi_e\,\mathbf{L}(\mathbf{v}_e)\rangle_\Gamma\,.$$

Hence, integrating by parts in Ω, and then employing the Cauchy-Schwarz inequality, the inverse estimate (4.40), and the bound given by (4.39), we get

$$\begin{aligned}\|\mathbf{v}_e\|_{0,e}^2 \,&\le\, c_2 \left\{ \int_T \psi_e\,\mathbf{L}(\mathbf{v}_e)\cdot\mathbf{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) \,+\, \int_T (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) : \nabla\big(\psi_e\,\mathbf{L}(\mathbf{v}_e)\big) \right\} \\ &\le\, C \left\{ \|\mathbf{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,T} \,+\, h_T^{-1}\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T} \right\} \|\psi_e\,\mathbf{L}(\mathbf{v}_e)\|_{0,T} \\ &\le\, C\,h_e^{1/2} \left\{ \|\mathbf{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,T} \,+\, h_T^{-1}\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T} \right\} \|\mathbf{v}_e\|_{0,e}\,,\end{aligned}$$

which, after minor manipulations and using that $h_e \le h_T$, yields (4.42) and completes the proof. □

The proof of efficiency of $\boldsymbol{\theta}$ is completed with the following result.

Lemma 4.16. *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Gamma)$ there holds*

$$h_e\,\|\boldsymbol{\varphi}_h + \mathbf{u}_h\|_{0,e}^2 \,\le\, C \left\{ h_T^2\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, h_T^2\,\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{0,T}^2 \,+\, \|\mathbf{u}-\mathbf{u}_h\|_{0,T}^2 \,+\, h_e\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. Adding and substracting $\boldsymbol{\varphi} = -\mathbf{u}$ on Γ, and then employing the discrete trace inequality (4.41) (cf. Lemma 4.9), we obtain for each $e \in \mathcal{E}_h(\Gamma)$

$$\begin{aligned}h_e\,\|\boldsymbol{\varphi}_h + \mathbf{u}_h\|_{0,e}^2 \,&\le\, 2\,h_e \left\{ \|\boldsymbol{\varphi}_h - \boldsymbol{\varphi}\|_{0,e}^2 \,+\, \|\mathbf{u}-\mathbf{u}_h\|_{0,e}^2 \right\} \\ &\le\, C \left\{ h_e\,\|\boldsymbol{\varphi}_h - \boldsymbol{\varphi}\|_{0,e}^2 \,+\, \|\mathbf{u}-\mathbf{u}_h\|_{0,T}^2 \,+\, h_T^2\,|\mathbf{u}|_{1,T}^2 \right\},\end{aligned} \tag{4.43}$$

where the last term uses that $h_e \le h_T$ and that $\mathbf{u}_h$ is piecewise constant (cf. (3.2)). Then, using that $\nabla\mathbf{u} = \mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{\gamma}$ in Ω, adding and substracting $\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h$, and employing the upper bound from Lemma 4.11, we find that

$$\begin{aligned}h_T^2\,|\mathbf{u}|_{1,T}^2 \,=\, h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{\gamma}\|_{0,T}^2 \,&\le\, 2\,h_T^2 \left\{ \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{0,T}^2 \,+\, \|\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\|_{0,T}^2 \right\} \\ &\le\, C \left\{ h_T^2\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, h_T^2\,\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{0,T}^2 \,+\, \|\mathbf{u}-\mathbf{u}_h\|_{0,T}^2 \right\}.\end{aligned} \tag{4.44}$$

Finally, (4.43) and (4.44) yield the required estimate and finish the proof. □

5 Numerical results

In this section we present some numerical results confirming the reliability and efficiency of the a posteriori error estimator $\boldsymbol{\theta}$ analyzed in Section 4, and illustrating the performance of the associated adaptive algorithm. We begin by introducing additional notations. The variable N stands for the number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$ (equivalently, the number of unknowns of (3.5)), and the individual and global errors are denoted by:

$$\mathsf{e}(\boldsymbol{\sigma}) \,:=\, \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega}\,, \qquad \mathsf{e}(\boldsymbol{\rho}) \,:=\, \|\boldsymbol{\rho}_h\|_{0,\Omega}\,, \qquad \mathsf{e}(\mathbf{u}) \,:=\, \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\,,$$

$$\mathsf{e}(\boldsymbol{\varphi}) \,:=\, \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,\Gamma}\,, \qquad \mathsf{e}(\boldsymbol{\gamma}) \,:=\, \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{0,\Omega}\,, \quad \text{and}$$

$$\mathsf{e} \,:=\, \left\{ [\mathsf{e}(\boldsymbol{\sigma})]^2 \,+\, [\mathsf{e}(\boldsymbol{\rho})]^2 \,+\, [\mathsf{e}(\mathbf{u})]^2 \,+\, [\mathsf{e}(\boldsymbol{\varphi})]^2 \,+\, [\mathsf{e}(\boldsymbol{\gamma})]^2 \right\}^{1/2},$$

where $((\boldsymbol{\sigma},\mathbf{0}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \in \mathbf{H}\times\mathbf{Q}$ and $((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \in \mathbf{H}_h\times\mathbf{Q}_h$ are the unique solutions of (2.7) and (3.5), respectively. Also, we define the effectivity index

$$\texttt{eff}(\boldsymbol{\theta}) \,:=\, \mathsf{e}/\boldsymbol{\theta}\,.$$

In turn, we let $r(\boldsymbol{\sigma})$, $r(\boldsymbol{\rho})$, $r(\mathbf{u})$, $r(\boldsymbol{\varphi})$, $r(\boldsymbol{\gamma})$, and r be the experimental rates of convergence given by

$$r(\boldsymbol{\sigma}) \,:=\, \frac{\log(\mathsf{e}(\boldsymbol{\sigma})/\mathsf{e}'(\boldsymbol{\sigma}))}{\log(h/h')}\,,\quad r(\boldsymbol{\rho}) \,:=\, \frac{\log(\mathsf{e}(\boldsymbol{\rho})/\mathsf{e}'(\boldsymbol{\rho}))}{\log(h/h')}\,,\quad r(\mathbf{u}) \,:=\, \frac{\log(\mathsf{e}(\mathbf{u})/\mathsf{e}'(\mathbf{u}))}{\log(h/h')}\,,$$
$$r(\boldsymbol{\varphi}) \,:=\, \frac{\log(\mathsf{e}(\boldsymbol{\varphi})/\mathsf{e}'(\boldsymbol{\varphi}))}{\log(h/h')}\,,\quad r(\boldsymbol{\gamma}) \,:=\, \frac{\log(\mathsf{e}(\boldsymbol{\gamma})/\mathsf{e}'(\boldsymbol{\gamma}))}{\log(h/h')}\,,\quad \text{and} \quad r \,:=\, \frac{\log\left(\mathsf{e}/\mathsf{e}'\right)}{\log(h/h')}\,,$$

where h and h' denote two consecutive meshsizes with errors e and e', respectively. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

In what follows we describe the examples to be considered, which are basically the same ones employed in [28]. In Example 1 we consider the Young modulus $E = 1$ and the Poisson ratio $\nu = 0.4999$, which yields the Lamé parameters $\lambda := \frac{E\,\nu}{(1+\nu)\,(1-2\,\nu)} = 1666.4444$ and $\mu := \frac{E}{2(1+\nu)} = 0.3333$. Then, we take the square domain $\Omega :=]-1/2,1/2[^2$ and choose $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution $\mathbf{u}$ is given by the first column of the fundamental solution at $\mathbf{x}_0 := (1,0)^{\mathtt{t}}$, that is

$$\mathbf{u}(\mathbf{x}) := \left\{ -\frac{(\lambda+3\mu)}{4\pi\mu\,(\lambda+2\,\mu)}\,\log\,\|\mathbf{x}-\mathbf{x}_0\|\,\mathbf{I} + \frac{(\lambda+\mu)}{4\pi\mu\,(\lambda+2\mu)}\,\frac{(\mathbf{x}-\mathbf{x}_0)\,(\mathbf{x}-\mathbf{x}_0)^{\mathtt{t}}}{\|\mathbf{x}-\mathbf{x}_0\|^2}\right\}\begin{pmatrix}1\\0\end{pmatrix} \qquad \forall\,\mathbf{x}\in\Omega\,.$$

In particular, $\mathbf{f} = \mathbf{0}$ and $\mathbf{u}$ is smooth in a neighborhood of $\bar{\Omega}$, whence [28, Theorem 4.2] yields an a priori rate of convergence of $O(h)$. This fact was confirmed by the numerical results provided in [28].

Next, in Example 2 we consider the same Lamé parameters from Example 1, take the L-shaped domain $\Omega :=]-1,1[^2\setminus[0,1]^2$, and choose $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution $\mathbf{u}$ is given, in polar coordinates, by

$$\mathbf{u}(\mathtt{r},\theta) = \mathtt{r}^{5/3}\sin((2\theta-\pi)/3)\begin{pmatrix}1\\1\end{pmatrix} \qquad \forall\,(\mathtt{r},\theta)\in\Omega\,.$$

Note in this case that the partial derivatives of $\mathbf{u}$, of order ≥ 2, are singular at the origin. Moreover, because of the power of $\mathtt{r}$, we find that $\mathbf{f} := -\,\mathbf{div}\,\boldsymbol{\sigma}$ belongs to $\mathbf{H}^{2/3}(\Omega)$, whence [28, Theorem 4.2] yields in this case an a priori rate of convergence of $O(h^{2/3})$. This fact was also confirmed by the numerical results provided in [28]. According to the preceding remarks, this example is utilized to illustrate the behavior of the adaptive algorithm associated with $\boldsymbol{\theta}$, which applies the following procedure from [41]:

1) Start with a coarse mesh $\mathcal{T}_h$.

2) Solve the discrete problem (3.5) for the actual mesh $\mathcal{T}_h$.

3) Compute the error indicators θ_T on each triangle $T\in\mathcal{T}_h$.

4) Evaluate stopping criterion and decide to finish or go to next step.

5) Use *blue-green* procedure to refine each $T'\in\mathcal{T}_h$ whose local error indicator $\theta_{T'}$ satisfies

$$\theta_{T'} \,\geq\, \frac{1}{2}\,\max\left\{\theta_T : \quad T\in\mathcal{T}_h\right\}.$$

6) Define resulting mesh as actual mesh $\mathcal{T}_h$, and go to step 2.

The numerical results shown below were obtained using a MATLAB code. In Tables 5.1 and 5.2 we display the convergence history of our mixed finite element scheme (3.5) as applied to Example 1 for a finite sequence of quasi-uniform triangulations of $\overline{\Omega}$. While this example was already considered in [28, Section 6], the novelty now is certainly the computation of the effectivity indexes. Indeed, we notice from the last column of Table 5.2 that the effectivity indexes $\texttt{eff}(\boldsymbol{\theta})$ remain always in a neighborhood of 0.15, which illustrates the reliability and efficiency of $\boldsymbol{\theta}$ in the case of a regular solution. In turn, as previously observed in [28, Section 6], it is clear from the experimental rates of convergence shown in these tables that the $O(h)$ predicted by [28, Theorem 4.2] when $\delta = 1$ is attained in all the unknowns of this example.

Next, in Tables 5.3 up to 5.6 we provide the convergence history of the quasi-uniform and adaptive refinements, as applied to Example 2. We notice in the quasi-uniform case that $r(\boldsymbol{\sigma})$ oscillates around 2/3, whence, being $\texttt{e}(\boldsymbol{\sigma})$ the dominant component of the total error $\texttt{e}$, this oscillation is also reflected in the global rate of convergence r. In addition, it is clear from these tables that the total errors of the adaptive scheme decrease faster than those obtained by the quasi-uniform one, which is confirmed by the global experimental rates of convergence provided in Table 5.6. This fact becomes also evident from Figure 5.1, mainly from $N \cong 1E+04$ on, where we display $\texttt{e}$ vs. N for both refinements. Furthermore, it is quite straightforward from the values of r in Table 5.6 that the adaptive method is able to recover the quasi-optimal rate of convergence $O(h)$ for $\texttt{e}$. In turn, the reliability and efficiency of $\boldsymbol{\theta}$ is clearly confirmed by the effectivity indexes from Table 5.6 (most of them around 0.30) for this example with a non-smooth solution. Intermediate meshes obtained with the adaptive refinement are displayed in Figure 5.2. As expected, the method is able to recognize the origin as a singularity of the solution of this example. Finally, in order to illustrate the accurateness of the proposed mixed method and the adaptive scheme induced by $\boldsymbol{\theta}$, several components of the approximate (left) and exact (right) solutions of Example 2 are displayed in Figures 5.3 up to 5.5. Note here that the values of $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}_h$ on Γ are depicted along a straight line beginning at the point $(-1,-1)$ and then continuing counterclockwise.

h	N	$\texttt{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\texttt{e}(\mathbf{u})$	$r(\mathbf{u})$
1/8	1044	3.364E−02	–	1.087E−02	–
1/12	2284	2.159E−02	1.094	7.206E−03	1.014
1/16	4004	1.595E−02	1.052	5.396E−03	1.005
1/24	8884	1.051E−02	1.029	3.594E−03	1.002
1/32	15684	7.845E−03	1.017	2.695E−03	1.001
1/48	35044	5.208E−03	1.010	1.796E−03	1.000
1/64	62084	3.899E−03	1.007	1.347E−03	1.000
1/96	139204	2.595E−03	1.004	8.980E−04	1.000
1/128	247044	1.944E−03	1.003	6.735E−04	1.000
1/192	554884	1.295E−03	1.002	4.490E−04	1.000
1/256	985604	9.711E−04	1.001	3.367E−04	1.000

Table 5.1: Convergence history for $\boldsymbol{\sigma}$ and $\mathbf{u}$ (Example 1)

N	$\mathtt{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathtt{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
1044	3.642E−02	–	2.387E−02	–	5.609E−02	–	0.2271
2284	1.605E−02	2.021	1.234E−02	1.628	3.046E−02	1.506	0.1916
4004	9.025E−03	2.000	7.851E−03	1.570	2.066E−02	1.350	0.1765
8884	4.126E−03	1.930	4.220E−03	1.531	1.258E−02	1.223	0.1642
15684	2.404E−03	1.877	2.731E−03	1.513	9.058E−03	1.142	0.1592
35044	1.140E−03	1.841	1.482E−03	1.508	5.818E−03	1.092	0.1548
62084	6.765E−04	1.814	9.610E−04	1.506	4.289E−03	1.060	0.1529
139204	3.267E−04	1.795	5.220E−04	1.505	2.814E−03	1.040	0.1512
247044	1.957E−04	1.782	3.386E−04	1.505	2.095E−03	1.026	0.1504
554884	9.536E−05	1.773	1.840E−04	1.504	1.386E−03	1.018	0.1497
985604	5.737E−05	1.766	1.194E−04	1.503	1.036E−03	1.012	0.1494

Table 5.2: Convergence history for $\boldsymbol{\varphi}$, $\boldsymbol{\gamma}$, $\mathtt{e}$, and effectivity index (Example 1)

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h	N	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$
1/1	73	3.107E+03	–	3.966E+03	–
1/3	713	8.641E+02	1.165	1.615E+02	2.914
1/5	1878	6.384E+02	0.593	5.721E+01	2.031
1/7	3898	4.873E+02	0.803	2.527E+01	2.429
1/9	6218	3.920E+02	0.866	1.593E+01	1.835
1/11	9408	3.599E+02	0.425	1.042E+01	2.114
1/13	13333	3.172E+02	0.756	7.628E+00	1.869
1/15	17318	2.958E+02	0.487	5.740E+00	1.988
1/17	22338	2.633E+02	0.931	4.638E+00	1.702
1/20	31364	2.427E+02	0.501	3.404E+00	1.904
1/25	49273	2.049E+02	0.759	2.034E+00	2.308
1/35	94938	1.720E+02	0.520	1.126E+00	1.758
1/42	137179	1.583E+02	0.457	7.667E−01	2.106
1/50	191774	1.294E+02	1.152	5.588E−01	1.814
1/56	242234	1.152E+02	1.026	4.354E−01	2.202
1/63	308798	1.077E+02	0.571	3.342E−01	2.246
1/70	381354	9.951E+01	0.753	2.785E−01	1.731
1/80	497729	9.789E+01	0.127	2.201E−01	1.763
1/90	624949	8.678E+01	1.022	1.745E−01	1.970
1/100	768804	8.348E+01	0.368	1.427E−01	1.913
1/120	1124779	7.142E+01	0.855	9.926E−02	1.989
1/140	1518284	6.433E+01	0.678	6.918E−02	2.342

Table 5.3: Convergence history for $\boldsymbol{\sigma}$ and $\mathbf{u}$ (quasi-uniform scheme, EXAMPLE 2)

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N	$\mathtt{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathtt{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
73	2.020E+04	–	7.724E+03	–	2.220E+04	–	0.8936
713	5.520E+02	3.277	2.970E+02	2.966	1.080E+03	2.752	0.3984
1878	1.955E+02	2.032	1.054E+02	2.028	6.783E+02	0.910	0.3591
3898	9.219E+01	2.234	5.730E+01	1.811	4.999E+02	0.907	0.3496
6218	5.910E+01	1.769	3.552E+01	1.903	3.983E+02	0.904	0.3287
9408	3.866E+01	2.114	2.645E+01	1.469	3.631E+02	0.461	0.3610
13333	2.924E+01	1.673	1.982E+01	1.728	3.193E+02	0.770	0.3488
17318	2.284E+01	1.726	1.546E+01	1.734	2.972E+02	0.501	0.3636
22338	1.896E+01	1.485	1.335E+01	1.175	2.644E+02	0.935	0.3585
31364	1.249E+01	2.568	1.043E+01	1.518	2.433E+02	0.511	0.4102
49273	8.879E+00	1.530	7.007E+00	1.783	2.052E+02	0.763	0.3686
94938	5.090E+00	1.654	4.420E+00	1.369	1.721E+02	0.522	0.3812
137179	3.418E+00	2.184	3.478E+00	1.314	1.583E+02	0.458	0.4279
191774	2.687E+00	1.380	2.692E+00	1.470	1.295E+02	1.153	0.3675
242234	2.324E+00	1.280	2.335E+00	1.254	1.153E+02	1.026	0.3607
308798	2.106E+00	0.838	2.006E+00	1.290	1.078E+02	0.572	0.3805
381354	1.487E+00	3.300	1.734E+00	1.383	9.954E+01	0.754	0.3749
497729	1.202E+00	1.595	1.490E+00	1.137	9.790E+01	0.124	0.3842
624949	9.922E−01	1.629	1.264E+00	1.394	8.680E+01	1.022	0.3922
768804	8.616E−01	1.340	1.137E+00	1.009	8.349E+01	0.369	0.3949
1124779	6.768E−01	1.324	9.039E−01	1.257	7.143E+01	0.855	0.3939
1518284	5.304E−01	1.580	7.456E−01	1.249	6.434E+01	0.678	0.4148

Table 5.4: Convergence history for $\boldsymbol{\varphi}$, $\boldsymbol{\gamma}$, $\mathtt{e}$, and effectivity index (quasi-uniform scheme, EXAMPLE 2)

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N	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$
73	3.107E+03	–	3.966E+03	–
224	1.852E+03	0.923	9.058E+02	2.634
676	1.135E+03	0.887	2.911E+02	2.056
1264	8.404E+02	0.959	1.351E+02	2.452
2495	5.810E+02	1.086	5.425E+01	2.685
4286	4.479E+02	0.962	2.962E+01	2.237
6636	3.496E+02	1.133	1.905E+01	2.020
9471	2.752E+02	1.346	1.310E+01	2.106
15034	2.187E+02	0.994	8.178E+00	2.039
24695	1.730E+02	0.945	4.884E+00	2.078
34829	1.377E+02	1.329	3.172E+00	2.511
60717	1.063E+02	0.931	1.918E+00	1.810
94831	8.485E+01	1.010	1.198E+00	2.112
136714	6.844E+01	1.175	8.121E−01	2.125
241371	5.276E+01	0.915	4.664E−01	1.951
368348	4.248E+01	1.025	3.090E−01	1.948
525936	3.511E+01	1.070	2.143E−01	2.055
955896	2.628E+01	0.970	1.164E−01	2.042
1453383	2.106E+01	1.057	7.816E−02	1.902

Table 5.5: Convergence history for $\boldsymbol{\sigma}$ and $\mathbf{u}$ (adaptive scheme Example 2)

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N	$\mathtt{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathtt{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
73	2.220E+04	–	7.724E+03	–	2.220E+04	–	0.8936
224	3.345E+03	3.205	1.556E+03	2.858	4.226E+03	2.959	0.5302
676	9.327E+02	2.312	5.313E+02	1.946	1.589E+03	1.771	0.4089
1264	4.642E+02	2.230	3.424E+02	1.404	1.028E+03	1.391	0.3802
2495	2.054E+02	2.398	1.495E+02	2.437	6.364E+02	1.411	0.3557
4286	1.192E+02	2.011	1.073E+02	1.225	4.766E+02	1.069	0.3402
6636	8.668E+01	1.457	8.520E+01	1.057	3.706E+02	1.151	0.3430
9471	5.086E+01	2.998	5.997E+01	1.974	2.865E+02	1.447	0.3211
15034	3.557E+01	1.548	4.537E+01	1.208	2.263E+02	1.020	0.3203
24695	2.223E+01	1.893	3.464E+01	1.088	1.779E+02	0.970	0.3248
34829	1.693E+01	1.586	2.752E+01	1.338	1.414E+02	1.334	0.3066
60717	1.081E+01	1.615	2.049E+01	1.062	1.088E+02	0.945	0.3103
94831	7.247E+00	1.793	1.602E+01	1.102	8.666E+01	1.020	0.3117
136714	5.505E+00	1.503	1.306E+01	1.130	6.989E+01	1.175	0.2990
241371	3.940E+00	1.177	9.571E+00	1.093	5.377E+01	0.923	0.3057
368348	2.654E+00	1.870	7.464E+00	1.176	4.322E+01	1.034	0.3066
525936	1.779E+00	2.247	6.778E+00	0.542	3.580E+01	1.057	0.2994
955896	1.241E+00	1.205	4.713E+00	1.216	2.673E+01	0.979	0.3029
1453383	8.444E−01	1.838	3.852E+00	0.963	2.143E+01	1.055	0.3022

Table 5.6: Convergence history for $\boldsymbol{\varphi}$, $\boldsymbol{\gamma}$, $\mathtt{e}$, and effectivity index (adaptive scheme, Example 2)

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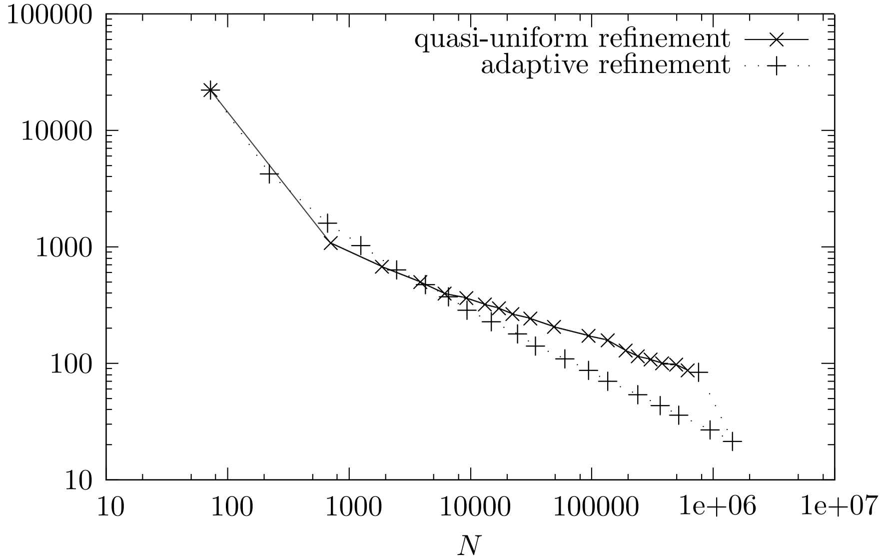

Figure 5.1: Example 2, total error e vs. N for the quasi-uniform and adaptive schemes

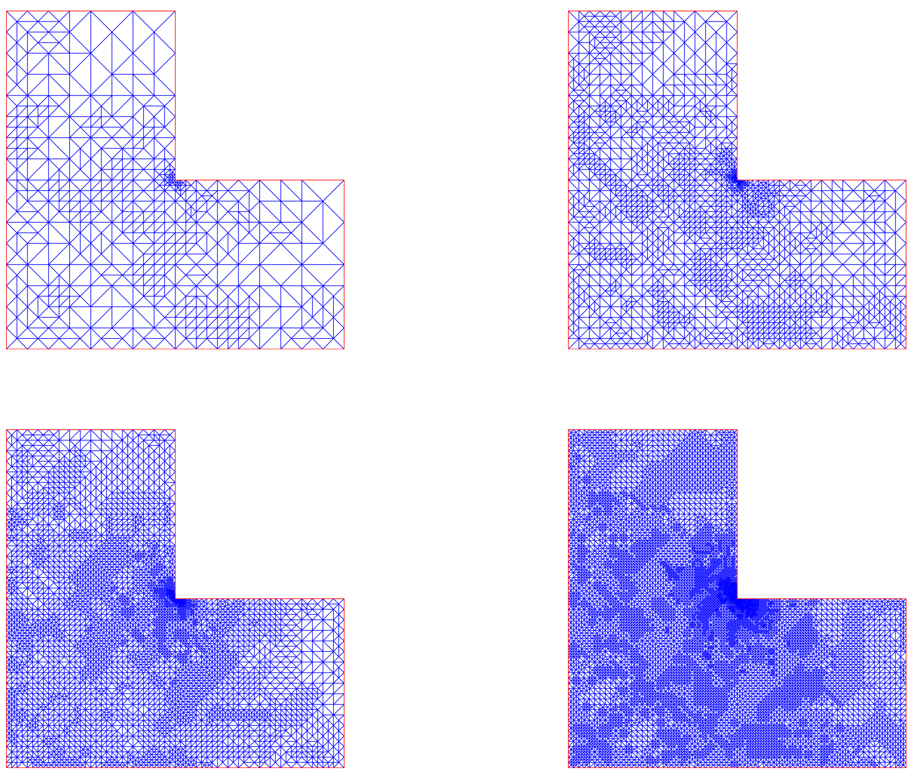

Figure 5.2: Example 2: adapted meshes for $N \in \{6636, 24695, 60717, 136714\}$

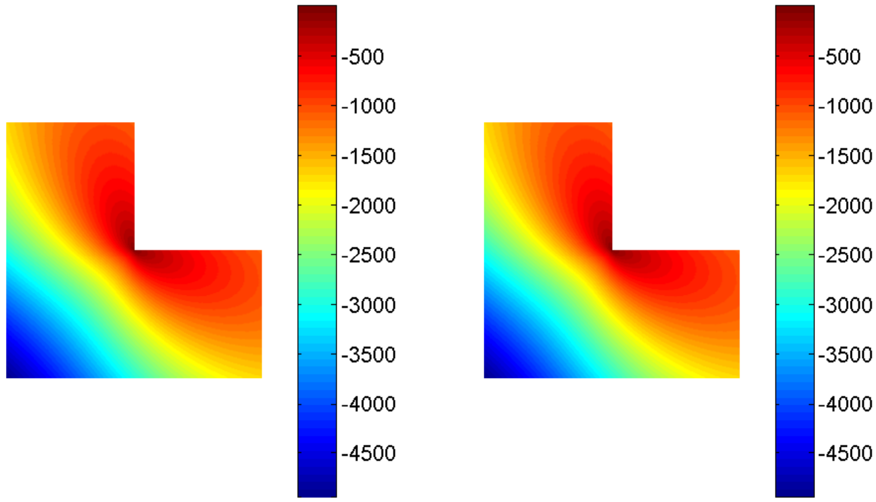

Figure 5.3: Approximate and exact σ_{11} ($N = 1453383$, Example 2)

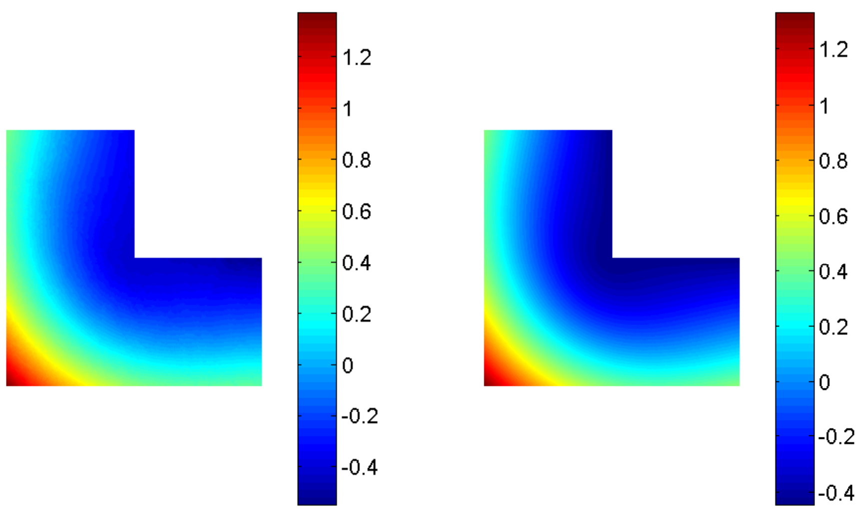

Figure 5.4: Approximate and exact u_2 ($N = 1453383$, Example 2)

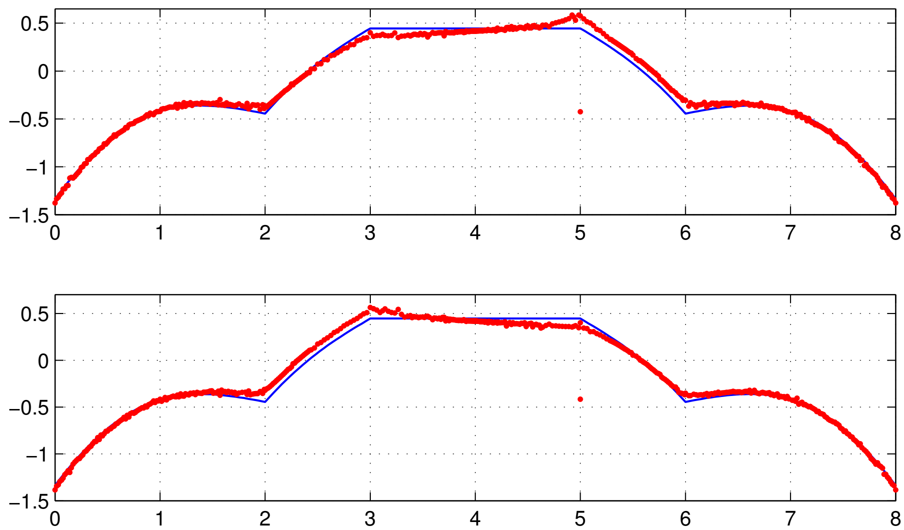

Figure 5.5: Approximate (red) and exact (blue) φ ($N = 1453383$, Example 2)

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