Characterizing Efficiency on Infinite-dimensional Commodity Spaces with Ordering Cones Having Possibly Empty Interior

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Abstract

Some production models in a finance economy require infinite dimensional commodity spaces, where efficiency is defined in terms of an ordering cone having possibly empty interior. Since weak efficiency is more tractable than efficiency from a mathematical point of view, this paper characterizes the equality between efficiency and weak efficiency in infinite dimensional spaces without further assumptions, like closedness or free-disposability. This is obtained as an application of our main result that characterizes the solutions to a unified vector optimization problem in terms of its scalarization. Standard models as efficiency, weak efficiency (defined in terms of quasi-relative interior), weak strict efficiency, strict efficiency or strong solutions, are carefully described. In addition, we exhibit two particular instances and compute the efficient and weak efficient solution set in Lebesgue spaces.

Keywords. Vector optimization, Scalarization, Efficiency, Infinite dimensional commodity space, Quasi relative interior.

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1 Introduction

Multiobjective or vector optimization problems are being the focus of attention of researchers coming from mathematics, economics, and many other disciplines, in recent

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years. This is mainly because most real-life problems are modeled within this framework, and involve the optimization of several criteria simultaneously. There are various notions of solutions to vector optimization problems, among them two arise: efficient (Pareto) and weak efficient (weak Pareto) solutions. From the mathematical point of view, the efficient solution concept is less tractable than weak efficient; whereas efficiency is more important for concrete applications than weak efficiency.

The purpose of this paper is two fold: on the one hand, we characterize the coincidence of the efficient and weak efficient solutions without free-disposability assumption, where the ordering cones may have possibly empty topological interior, like the natural cones in the space of square-integrable (or simply integrable) functions. Here, the notion of quasi-relative interior, which coincides with the relative interior in finite dimensional spaces, will play an important role. Cones with empty interior arise, for instance, when considering production models in finance with an infinite dimensional commodity space; see [38] and references therein. A characterization of the equality of efficient and weak efficient solution sets within the context of production theory was given by Bonnisseau and Crettez in [4], under closedness and free disposability assumptions in finite dimensional spaces with respect to the standard non-negative cone (so that its interior is non-empty). A result ensuring that all points of the boundary of a closed production set under free-disposability are weakly efficient, was stated earlier; see for instance [3]. It will be recovered in Section 5. On the other hand, our main goal is to develop an approach allowing us, in particular, to deal with efficiency and weak efficiency, among other notions of solution, in a unified manner: this will be carried out via a non-linear scalarizing function when the preference relations may be induced by sets having possibly empty interior, or they are non-necessarily transitive.

We refer to the books [30, 28] for a theoretical treatment of vector optimization problems concerning existence and optimality conditions. Some concrete models in infinite dimensional spaces may be found in [28]; and existence results of efficient points for preference relations which are reflexive and transitive, not necessarily coming from an ordering cone, are established in [15].

The paper is structured as follows. Section 2 provides the basic definitions and preliminaries concerning quasi-interior and quasi-relative interior points of convex sets. In Section 3, the nonlinear scalarizing function to be used is revised, along with an analysis of the scalarization procedure. The specializations to the standard models: efficiency, weak efficiency (with quasi-relative interior), weak strict efficiency, strict efficiency, are described carefully in Remark 4.7. Section 5 establishes the characterization of efficiency in terms of weak efficiency (involving convex cones with possibly empty interior), as an application of previous results. A model in the space of square-integrable functions appeared in finance and the computation of the set of efficient solutions for a particular instance, are formulated in Section 6; whereas a similar model in the space of square-summable sequences is presented in Section 7. The paper ends with final conclusions in Section 8.

2 Formulation of the problem

Given a nonempty set $S \subsetneq L$, a vector function $f : M \to L$, with L being a (real) topological vector space and M is any nonempty set, we say that $\bar{x}$ is a S-minimal of f on M, iff

$$\bar{x} \in M : \; f(y) - f(\bar{x}) \notin S \;\text{ for all } y \in M, \; y \neq \bar{x}. \tag{$\mathcal{P}$}$$

The set of S-minimal solutions is denoted by $E_S = E_S(M)$. When f is real valued, $E(f, M)$ stands for the set of minima of f on M, i.e. $E(f, M) = \underset{M}{\operatorname{argmin}}\, f$.

One recognizes in ($\mathcal{P}$) a vector optimization problem. In particular, when a convex cone P is given, it induces several preferences by particularizing S. Thus we recover efficient, weak efficient, strict efficient and (Henig) proper efficient solutions of f on M, among others, in the classical sense. For more details, see [13].

In connection to ($\mathcal{P}$), given $\varepsilon \in \mathbb{R}$ and $0 \neq q \in Y$, we associate the approximate problem:

$$\text{find } \bar{x} \in M \quad f(x) - f(\bar{x}) \notin -\varepsilon q + S \quad \forall\, x \in M, \; x \neq \bar{x}, \tag{$\mathcal{P}(\varepsilon q)$}$$

where S is any set satisfying $S - \mathbb{R}_{++}q \subseteq S$, where $\mathbb{R}_{++} := \;]0, +\infty[$. We denote by $E_S(\varepsilon q)$ the solution set to ($\mathcal{P}(\varepsilon q)$). The previous inclusion is a natural condition in approximate efficiency since it yields: $\varepsilon_1 < \varepsilon_2 \implies E_S(\varepsilon_1 q) \subseteq E_S(\varepsilon_2 q)$; and $E_S = E_S(0) \subseteq E_S(\varepsilon q) \;\; \forall\, \varepsilon \in \mathbb{R}_+ \doteq [0, +\infty[$. Consequently, $E_S \subseteq \bigcap_{\varepsilon > 0} E_S(\varepsilon q)$.
In case f is a real function we denote by $E(f, M, \varepsilon)$ the set of ε-solutions, that is, $\bar{x} \in E(f, M, \varepsilon)$ if and only if $f(x) - f(\bar{x}) \geq -\varepsilon$ for all $x \in M$. Hence, $E(f, M, 0) = E(f, M)$.

There are several scalarizing functions allowing us to substitute the vector problem ($\mathcal{P}$) by a scalar one. A detailed and good account of some of those schemes may be found in the book [12] as well as in [6]. A further non-linear scalarizing function (for different purposes) was introduced by Hiriart-Urruty [26], which is called the "*oriented distance function*" and defined, given $A \subseteq L$, by

$$\Delta_A(y) := d_A(y) - d_{L \setminus A}(y). \tag{1}$$

Here, $d_A(y)$ is the distance from A to y, i.e., $d_A(y) = \inf_{x\in A} \|y-x\|$. Its main properties have been established in [36], where the main notions of solution in vector optimization are formulated in terms of some kind of minima for a certain oriented distance function. This function was successfully employed in [11] to provide existence of Lagrange multiplier in ε-Pareto efficiency by using Mordukhovich subdifferential for cones having empty interior. Note that $\Delta_{-S} = d_{-S}$, provided int $S = \emptyset$. See also the references therein.

However, the scalarizing function, which still remains useful because of its importance in the development of theoretical and algorithmic issues in vector optimization ([12, 6]), is the function $\xi_{q,S}\colon L \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\xi_{q,S}(y) := \inf\{t \in \mathbb{R}\colon y \in tq + S\}, \tag{2}$$

and $\xi_{q,S}(y) = +\infty$ whenever there is no $t \in \mathbb{R}$ such that $y \in tq+S$. Here $0 \neq q \in L$, and $S \subsetneq L$. This function was rediscovered in production theory for some specialization of S and q (see [3, 4, 31]), where it is called the "shortage function". Hence, $\xi_{q,S}(y)$ is related with the distance from y to the boundary of S in the q-direction. It measures the amount, in q unit, by which y is short of reaching bd S. Such a function was independently introduced in [32, 18, 30], although a similar function already appeared earlier in [33, Example 2, p. 139]. Other uses may be found in [30, 19, 34, 25, 35] and references therein. We refer also to the good books [20, 8, 12]. Regarding nonlinear scalarization for approximate efficiency, we refer to [22] and [23].

By using the scalarizing function (2), we characterize completely the solutions to $(\mathcal{P}(\varepsilon q))$ via solutions to its scalarization for general ordering sets S having possibly empty interior (Theorems 4.3 and 4.5). Specializations of this characterization allow us to cover situations where, for instance, Theorem 4.5 and 4.6 in [24] are not applicable, since the involved sets are not necessarily closed or have nonempty interior.

3 Basic Definitions and Preliminaries on Quasi-relative Interior

Throughout this paper, L denotes any locally convex and topological vector space. For given $A \subseteq L$, we denote by $\mathcal{C}(A)$, int A, cl A, bd A, the complementary of A, the (topological) interior, the closure and the boundary of A, respectively. Moreover, we set cone $A := \bigcup_{t\geq 0} tA = \{ta :\ t \geq 0,\ a \in A\}$, which is the smallest cone containing A, and clcone $A :=$ cl(cone A). We say A is solid iff int $A \neq \emptyset$.

In order to deal with infinite-dimensional commodity spaces (like L^p or l^p, for $1 \leq p < +\infty$), which appear in economies with production (see [38] and references therein), we will use the notion of quasi-relative interior since in such spaces the ordering cones (like L^p_+ or l^p_+) have empty interior. This allows one to substitute the interior by the quasi-relative interior in the definition of weak efficiency (see Remark 4.7).

Given a convex set $A \subseteq L$ and $x \in A$, $N_A(x)$ stands for the normal cone to A at x, defined by $N_A(x) := \{y^* \in L^* : \langle y^*, a - x\rangle \leq 0, \ \ \forall\, a \in A\}$, where L^* is the topological dual of L, and $\langle \cdot, \cdot \rangle$ stands for the duality product between L^* and L. We say that $x \in A$ is a (see, for instance, [5, 27]):

(a) quasi-interior point of A, denoted by $x \in \text{qi } A$, iff $\text{clcone}(A - x) = L$, or equivalently, $N_A(x) = \{0\}$;

(b) quasi-relative interior point of A, denoted by $x \in \text{qri } A$, iff $\text{clcone}(A - x)$ is a linear subspace of L, or equivalently, $N_A(x)$ is a linear subspace of L^*.

For any convex set A, we have that ([5, 27]) $\text{qi } A \subseteq \text{qri } A$ and, $\text{int } A \neq \emptyset$ implies $\text{int } A = \text{qi } A$. Similarly, if $\text{qi } A \neq \emptyset$, then $\text{qi } A = \text{qri } A$. Moreover [5], if Y is a finite dimensional space, then $\text{qi } A = \text{int } A$ and $\text{qri } A = \text{ri } A$, where $\text{ri } A$ means the relative interior of A, which is the interior with respect to its affine hull. Obviously $\text{qri } A$ is convex, $t \text{ qri } A = \text{qri } A$ for all $t > 0$, and $\text{cl}(\text{qri } A) = \text{cl } A$ provided $\text{qri } A \neq \emptyset$. In addition, ([5, Lemma 2.9]) if $x_1 \in \text{qri } A$ and $x_2 \in A$, then $tx_1 + (1-t)x_2 \in \text{qri } A$ for all $0 < t \leq 1$. Hence, if P is a convex cone, then $\text{qri } P + P = \text{qri } P$.

The (nonnegative) polar cone of a set $A \subseteq L$, is defined by

$$A^* := \{y^* \in L^* : \ \langle y^*, a\rangle \geq 0 \ \ \forall\, a \in A\}. \tag{3}$$

The next result [5], which is an useful characterization of the quasi-relative interior, applies to spaces like L^p or l^p, $1 \leq p < +\infty$.

Theorem 3.1. [5, *Theorem* 3.10] *Let* $P \subseteq L$ *be a convex cone such that* $\text{cl}(P - P) = L$. *Then:*

$$y \in \text{qri } P \quad \Longleftrightarrow \quad y \in P \ \text{and} \ \langle y^*, y\rangle > 0, \ \forall\, y^* \in P^* \setminus \{0\}.$$

Consequently, if additionally P *is closed, then*

$$y \in \text{qri } P \quad \Longleftrightarrow \quad \langle y^*, y\rangle > 0, \ \forall\, y^* \in P^* \setminus \{0\}.$$

Remark 3.2. *Let* $A \subseteq L$ *be a convex set such that* $0 \in A$. *Then, it is not difficult to check that (see also [21])*

$$\text{cone}(A - A) = \text{cone } A - \text{cone } A.$$

Proposition 3.3. *Let $P \subseteq L$ be a convex cone such that* $\mathrm{cl}(P - P) = L$. *Then*

$$\mathrm{qri}\ P = \mathrm{qi}\ P.$$

Proof. We only need to prove that $\mathrm{qri}\ P \subseteq \mathrm{qi}\ P$. Let $y \in \mathrm{qri}\ P$. Then $y \in P$ and $0 \in \mathrm{qri}(P - y)$, and therefore $\mathrm{clcone}(P - y)$ is a linear subspace. Thus by assumption and the preceding remark, we obtain

$$L = \mathrm{cl}(P - P) = \mathrm{clcone}(P - P) = \mathrm{cl}(\mathrm{clcone}(P - y) - \mathrm{clcone}(P - y)) = \mathrm{clcone}(P - y). \tag{4}$$

Hence $y \in \mathrm{qi}\ P$. □

By recalling that a nonsupport point $x \in A$ of A is such that every closed supporting hyperplane to A at x contains A, it is proven that the nonsupport points coincide with the quasi-relative interior points [5, Proposition 2.6]. Some examples where $\mathrm{int}\ A = \emptyset$ but $\mathrm{qri}\ A \neq \emptyset$ may be found in [5].

In general, we recall that every nonempty and convex subset of a separable Banach space admits quasi-relative interior points [5, Theorem 2.19]. There are only a few infinite-dimensional spaces, whose natural ordering cones have non-empty interior; among them we mention l^∞, the space of bounded variation on $\mathbb{R}$, or the space of continuous real-valued functions defined on a compact set of $\mathbb{R}^n$.

4 The Scalarizing Function Revisited and a Unified Vector Optimization Problem

As in previous section, L continues to be a locally convex and topological vector space. In this section, we recall some properties of the function $\xi_{q,A}$ defined by (2). It follows that $\xi_{q,a+A}(y) = \xi_{q,A}(y-a)$ for all $a \in L$. If A is closed, then $y \in \xi_{q,A}(y)q + A$, provided $\xi_{q,A}(y)$ is finite.

This function $\xi_{q,A}$ is a nonlinear Minkowski-type functional and has many separation properties (see [30], [20], [19], [29]) and plays an important role in many areas, including mathematical economics or finance; see [4] and [17]. In addition, the function $\xi_{q,A}$ enjoys very nice properties, which some of the them are shown in [20, 34, 35]. The next proposition collects those to be used later on without further assumptions.

Proposition 4.1. *Let $\lambda \in \mathbb{R}$, $0 \neq q \in L$ and $\emptyset \neq A \subsetneq L$. The following assertions hold:*

(a) $\{y \in L : \xi_{q,A}(y) < \lambda\} = \lambda q + A - \mathbb{R}_{++}q$ *and* $\lambda q + A \subseteq \{y \in L : \xi_{q,A}(y) \leq \lambda\}$; *thus* $\{y \in L : \xi_{q,A}(y) < 0\} = -\mathbb{R}_{++}q + A$ *and* $\{y \in L : \xi_{q,A}(y) < +\infty\} = \mathbb{R}q + A$.

(*b*) $\lambda q + \operatorname{int} A \subseteq \{y \in L : \xi_{q,A}(y) < \lambda\}$.

(*c*) $\{y \in L : \ \xi_{q,A}(y) \leq \lambda\} \subseteq \lambda q + \operatorname{cl}(A - \mathbb{R}_{++}q)$.

(*d*) $\{y \in L : \xi_{q,A}(y) = \lambda\} \subseteq \lambda q + \operatorname{cl}(A - \mathbb{R}_{++}q) \setminus (A - \mathbb{R}_{++}q)$.

More simpler expressions are obtained under the assumption $A - \mathbb{R}_{++}q \subseteq A$, which is equivalent to $A - \mathbb{R}_{+}q \subseteq A$, or equivalently $A - \mathbb{R}_{+}q = A$.
The next corollary is a consequence of the preceding proposition.

Corollary 4.2. *Let $\lambda \in \mathbb{R}$, $0 \neq q \in L$ and $\emptyset \neq A \subsetneq L$.*

(*a*) *Assume that* $\operatorname{cl} A - \mathbb{R}_{++}q \subseteq A$, *then*

$$\{y \in L : \ \ \xi_{q,A}(y) \leq \lambda\} = \lambda q + \operatorname{cl} A, \ \ \forall\, \lambda \in \mathbb{R} \ \ \textit{and} \ \ \xi_{q,A}(y) = \xi_{q,\operatorname{cl} A}(y).$$

(*b*) *Assume that* $\operatorname{int} A \neq \emptyset$ *and* $A - \mathbb{R}_{++}q \subseteq \operatorname{int} A$, *then*

$$\{y \in L : \ \ \xi_{q,A}(y) < \lambda\} = \lambda q + \operatorname{int} A, \ \ \forall\, \lambda \in \mathbb{R} \ \ \textit{and} \ \ \xi_{q,A}(y) = \xi_{q,\operatorname{int} A}(y).$$

(*c*) *If* $\operatorname{cl} A - \mathbb{R}_{++}q \subseteq \operatorname{int} A$ *then, for all* $\lambda \in \xi_{q,A}(L)$,

$$\{y \in L : \ \ \xi_{q,A}(y) = \lambda\} = \lambda q + \operatorname{bd} A.$$

Given $\bar{x} \in M$, the value $-r_0 := (\xi_{q,f(\bar{x})+S} \circ f)(\bar{x}) = \xi_{q,S}(0)$, which is independent of f, measures the distance of the origin from the boundary of S in the q-direction. Later on, we shall provide a wide class of sets for which such a value is computable. We easily obtain:

$$-\mathbb{R}_{++}q \subseteq \operatorname{clcone}(S - \mathbb{R}_{++}q); \tag{5}$$

$$\xi_{q,S}(0) \in \mathbb{R} \Longleftrightarrow [0 \in \mathbb{R}q + S \text{ and } \exists\, t \in \mathbb{R},\ 0 \notin tq + S - \mathbb{R}_{++}q]; \tag{6}$$

$$q \in (-S) \setminus (S - \mathbb{R}_{++}q) \implies -1 \leq \xi_{q,S}(0) \leq 1. \tag{7}$$

The preceding assertions can be simplified under the assumption $S - \mathbb{R}_{++}q \subseteq S$, as one can check it easily.

The following theorem, which appears for the first time without solidness of S, characterizes solutions to $(\mathcal{P}((r_0{+}\varepsilon)q))$ through solutions to scalar problems, depending on $\xi_{q,S}(0)$ and without additional assumptions on S.

Theorem 4.3. *Let $\emptyset \neq S \subseteq L$, $\varepsilon \geq 0$, $q \in L \setminus \{0\}$, $\bar{x} \in M$ and assume that $-r_0 \doteq \xi_{q,S}(0) \in \mathbb{R}$. The following assertions are equivalent:*

(*a*) $\bar{x} \in E_S((\varepsilon + r_0)q)$;

(*b*) $\bar{x} \in E(\xi_{q,f(\bar{x})+S} \circ f, M, \varepsilon)$ *and*

$$E(\xi_{q,f(\bar{x})+S} \circ f, M, \varepsilon) \setminus \{\bar{x}\} \subseteq \{x \in M \setminus \{\bar{x}\} : \ f(x) - f(\bar{x}) \in$$

$$[(\varepsilon - r_0)q + \mathrm{cl}(S - \mathbb{R}_{++}q)] \cap [-(\varepsilon + r_0)q + \mathcal{C}(S)]\}.$$

Proof. (*a*) $\Longrightarrow$ (*b*): Since $f(x) - f(\bar{x}) \notin -(\varepsilon + r_0)q + S$ for all $x \in M$, $x \neq \bar{x}$, then $(\xi_{q,f(\bar{x})+S} \circ f)(x) \geq -r_0 - \varepsilon = (\xi_{q,-f(\bar{x})+S} \circ f)(\bar{x}) - \varepsilon$ by Proposition 4.1(a), which implies that $\bar{x} \in E(\xi_{q,f(\bar{x})+S} \circ f, M, \varepsilon)$. On the other hand, take any $x \in M$, $x \neq \bar{x}$, such that $x \in E(\xi_{q,f(\bar{x})+S} \circ f, M, \varepsilon)$. Then, $(\xi_{q,f(\bar{x})+S} \circ f)(x') \geq (\xi_{q,f(\bar{x})+S} \circ f)(x) - \varepsilon$ for all $x' \in M$. In particular, $(\xi_{q,f(\bar{x})+S} \circ f)(x) \leq \varepsilon - r_0$, which gives $f(x) - f(\bar{x}) \in (\varepsilon - r_0)q + \mathrm{cl}(S - \mathbb{R}_{++}q)$ by Proposition 4.1(d). This, along with the fact that $f(x) - f(\bar{x}) \in -(\varepsilon + r_0)q + \mathcal{C}(S)$ yield the inclusion in (b).
(b) $\Longrightarrow$ (a): If on the contrary $\bar{x} \notin E_S((\varepsilon + r_0)q)$, then there exists $x' \in M$, $x' \neq \bar{x}$, such that $f(x') - f(\bar{x}) \in -(\varepsilon + r_0)q + S$. Thus, $(\xi_{q,f(\bar{x})+S} \circ f)(x') \leq -(\varepsilon + r_0)$. Hence $(\xi_{q,f(\bar{x})+S} \circ f)(x') = -(\varepsilon + r_0)$ since $\bar{x} \in E(\xi_{q,f(\bar{x})+S} \circ f, M, \varepsilon)$; therefore $x' \in E(\xi_{q,f(\bar{x})+S} \circ f, M) \subseteq E(\xi_{q,f(\bar{x})+S} \circ f, M, \varepsilon)$. By the inclusion assumption, we get $f(x') - f(\bar{x}) \in -(\varepsilon + r_0)q + \mathcal{C}(S)$, which contradicts a previous relation, proving that (a) holds. □

An example showing the inclusion in (b) for $\varepsilon > 0$ may be strict, is exhibited in [13].

Before going further, some remarks are in order. Our Theorem 4.3 largely extends and generalizes similar results appearing elsewhere. In particular, Theorems 4.5 and 4.6 in [24] cannot be applied. Indeed, Theorem 4.5 requires that S be closed and free-disposal: $S - P = S$, whereas Theorem 4.6 needs that S be solid and free-disposal. Moreover, our theorem provides more information.

From Theorem 4.3 a simple corollary is obtained.

Corollary 4.4. *Let* $\varepsilon \geq 0$, $\bar{x} \in M$ *and* $-r_0 \doteq \xi_{q,S}(0) \in \mathbb{R}$.

(a) *Assume* $r_0 = 0$. *If* $E(\xi_{q,f(\bar{x})+S} \circ f, M, \varepsilon) = \{\bar{x}\}$, *then* $\bar{x} \in E_S(\varepsilon q)$.

(b) *Assume that* $S - \mathbb{R}_{++}q \subseteq S$. *Then,* $E(\xi_{q,f(\bar{x})+S} \circ f, M) = \{\bar{x}\} \Longrightarrow \bar{x} \in E_S(r_0 q)$. *If, additionally,* S *is closed, then*

$$\bar{x} \in E_S(r_0 q) \Longleftrightarrow E(\xi_{q,f(\bar{x})+S} \circ f, M) = \{\bar{x}\}.$$

(c) *Assume that* $S - \mathbb{R}_{++}q \subseteq S$ *and* $r_0 > \varepsilon$. *If* $\bar{x} \in E_S(\varepsilon q)$, *then* $E(\xi_{q,-f(\bar{x})+S} \circ f, M) = \{\bar{x}\}$.

Proof. (a) and (b) follow from Theorem 4.3.
(c): Since $E_S(\varepsilon q) \subseteq E_S(r_0 q)$, $\bar{x} \in E_S(r_0 q)$. By using Theorem 4.3 (setting $\varepsilon = 0$), we obtain $\bar{x} \in E(\xi_{q,f(\bar{x})+S} \circ f, M)$ and

$$E(\xi_{q,f(\bar{x})+S} \circ f, M) \setminus \{\bar{x}\} \subseteq \{x \in M \setminus \{\bar{x}\} :\ f(x) - f(\bar{x}) \in -r_0 q + \mathrm{cl}(S) \setminus S\}.$$

If there exists $x' \in E(\xi_{q,-f(\bar{x})+S} \circ f, M)$, $x' \neq \bar{x}$, then $-\varepsilon > -r_0 = (\xi_{q,f(\bar{x})+S} \circ f)(\bar{x}) \geq (\xi_{q,f(\bar{x})+S} \circ f)(x')$. Thus, by Proposition 4.1(a), one gets $f(x') - f(\bar{x}) \in -\varepsilon q + S$, contradicting the fact that $\bar{x} \in E_S(\varepsilon q)$. $\square$

One can realize that the second part of (b) in Corollary 4.4 extends that of Theorem 4.5 in [24], which is valid for improvement closed sets S. The notion of improvement set was introduced in [9], and it will be recalled after next theorem.

When $r_0 = 0$ a more precise formulation of Theorem 4.3 is obtained.

Theorem 4.5. *Assume that* $\xi_{q,S}(0) = 0$ *and* $S - \mathbb{R}_{++}q \subseteq S$ *(which implies* $-q \in$ clcone S *by* (5)*). Let us consider problem* $(\mathcal{P}(\varepsilon q))$ *with* $\varepsilon \geq 0$*, and* $\bar{x} \in M$*. The following assertions are equivalent:*

(a) $\bar{x} \in E_S(\varepsilon q)$;

(b) $\bar{x} \in E(\xi_{q,f(\bar{x})+S} \circ f, M, \varepsilon)$ *and*

$$E(\xi_{q,f(\bar{x})+S} \circ f, M, \varepsilon) \setminus \{\bar{x}\} \subseteq \{x \in M :\ x \neq \bar{x},\ f(x) - f(\bar{x}) \in [\varepsilon q + \mathrm{cl}(S)] \cap [-\varepsilon q + \mathcal{C}(S)]\}.$$

We now provide two important sets of assumptions implying the fulfillment of the hypothesis of Theorem 4.5:

- Assumption **(B)** ([13, 14]): $0 \in \mathrm{bd}\, S$, $\mathrm{cl}(S) - \mathbb{R}_{++}q \subseteq \mathrm{int}\, S \neq \emptyset$. It implies that $-\mathbb{R}_{++}q \subseteq \mathrm{int}\, S$.
- Assumption **(C)**: S is conic, $q \in (-S) \setminus S$, and $S - \mathbb{R}_{++}q \subseteq S$; where a set S is conic if $\mathbb{R}_{++}S = S$, or equivalently, $tS = S$ for all $t > 0$.

Assumption **(B)** was introduced in [13] and clearly it implies those imposed in Theorem 4.5; the fact that $\xi_{q,S}(0) = 0$ is a consequence of Corollary 4.2(c). More precisely, the preceding theorem was obtained in [13] under Assumption **(B)**, and it was employed to derive complete scalarizations for problems $(\mathcal{P}(\varepsilon q))$; whereas Lagrange optimality conditions, both in the convex and non convex cases, were established in [14]. We recall that a set S is improvement with respect to a convex cone P iff $0 \notin S$ and $S - P = S$ (free-disposability). Such a notion was introduced originally in [9] when $K = \mathbb{R}^n_+$. In case int $P \neq \emptyset$, it is easy to check that, if $S - P = S$, then S satisfies $\mathrm{cl}(S) - \mathbb{R}_{++}q \subseteq \mathrm{int}\, S$ for each $q \in \mathrm{int}\, P$ (more details may be found in [13]). Observe that the requirement $0 \in \text{bd}\, S$ in Assumption **(B)** is not restrictive, since one can find $y_0 \in L$ satisfying $0 \in \text{bd}(S - y_0)$ once $S \neq L$. Thus, every improvement set satisfies Assumption **(B)**, provided int $P \neq \emptyset$. However, there are sets that are not free-disposal but satisfy Assumption **(B)**, for instance $S = -P \setminus \{0\}$ with P being non pointed.

On the other hand, Lemma 4.6 below asserts that the hypothesis of our Theorem 4.5 is implied by Assumption **(C)**. This is new in the literature, since **(C)** considers an important class of (not necessarily solid) sets which includes the standard models described in Remark 4.7.

We point out that there is no relationship between Assumptions **(B)** and **(C)**.

Lemma 4.6. *Let $\emptyset \neq S \subseteq L$ be a conic set. Then,*

$$\xi_{q,S}(0) = \begin{cases} 0 & \text{if,} \quad q \in (-S) \setminus S, \\ -\infty & \text{if,} \quad q \in S, \\ +\infty & \text{if,} \quad q \notin S \cup (-S),\ 0 \notin S, \\ 0 & \text{if,} \quad q \notin S \cup (-S),\ 0 \in S, \end{cases}$$

Proof. It is straightforward once one notes that: $q \in S$ if and only if $0 \in -tq + S$ for all $t > 0$, so that $\xi_{q,S}(0) \leq -t$; $q \notin S$ if and only if $0 \notin -tq + S$ for all $t > 0$, and therefore $q \notin S \cup (-S)$ if and only if $0 \notin tq + S$ for all $t \in \mathbb{R}$, $t \neq 0$. □

The choice for q in the standard models (strong efficiency, weak efficiency, efficiency, weak strict efficiency, strict efficiency) so that Theorem 4.5 is applicable, appears in the next remark. Note that Theorems 4.5 and 4.6 of [24] are not applicable to any case described below, since the sets S are not necessarily closed or solid.

Remark 4.7. *Let us consider $\{0\} \neq P \neq L$ to be a convex cone satisfying $P \setminus (-P) \neq \emptyset$, and set $l(P) := P \cap (-P)$. The assumptions of Theorem 4.5 are satisfied under any of the following circumstances:*

(*i*) (*weak efficiency*) $S = -\text{qri}\ P \neq \emptyset$ *with* $q \in \text{qri}\ P$ *and* P *being such that* cl P *is not a subspace;* $\xi_{q,S}(0) = 0$. *Here, we set* $E_W = E_S$ *and obtain*

$$\bar{x} \in E_W \Longleftrightarrow \begin{cases} \bar{x} \in E(\xi_{q,f(\bar{x})-\text{qri}\ P} \circ f, M) \quad \text{and} \\ E(\xi_{q,f(\bar{x})-\text{qri}\ P} \circ f, M) \setminus \{\bar{x}\} \\ \subseteq \Big\{x \in M \setminus \{\bar{x}\} :\ f(x) - f(\bar{x}) \in -(\text{cl}\ P \setminus \text{qri}\ P)\Big\}. \end{cases} \tag{8}$$

(*ii*) (*efficiency*) $S = (-P) \setminus P$: *we choose* $q \in P \setminus (-P)$; $\xi_{q,S}(0) = 0$. *We set* $E = E_S$ *and obtain*

$$\bar{x} \in E \Longleftrightarrow \begin{cases} \bar{x} \in E(\xi_{q,f(\bar{x})-P\setminus l(P)} \circ f, M) \ \text{ and} \\ E(\xi_{q,f(\bar{x})-P\setminus l(P)} \circ f, M) \setminus \{\bar{x}\} \\ \subseteq \Big\{ x \in M \setminus \{\bar{x}\} : \ f(x) - f(\bar{x}) \in -(\text{cl } P \setminus P) \cup l(P) \Big\}. \end{cases} \tag{9}$$

(*iii*) (*weak strict efficiency*) $S = -P \setminus \{0\}$, *with* P *being such that* cl P *is not a subspace and* qri $P \neq \emptyset$*: we take* $q \in$ qri P*;* $\xi_{q,S}(0) = 0$*. In this case we set* $E_{1W} = E_S$ *and obtain*

$$\bar{x} \in E_{1W} \Longleftrightarrow \begin{cases} \bar{x} \in E(\xi_{q,f(\bar{x})-P\setminus\{0\}} \circ f, M) \ \text{ and} \\ E(\xi_{q,f(\bar{x})-P\setminus\{0\}} \circ f, M) \setminus \{\bar{x}\} \\ \subseteq \Big\{ x \in M \setminus \{\bar{x}\} : \ f(x) - f(\bar{x}) \in -(\text{cl } P \setminus P) \cup \{0\} \Big\}. \end{cases} \tag{10}$$

(*iv*) (*strict efficiency*) $S = -P$*: in this case we choose* $q \in P \setminus (-P)$*;* $\xi_{q,S}(0) = 0$*. We set* $E_1 = E_S$ *and obtain*

$$\bar{x} \in E_1 \Longleftrightarrow \begin{cases} \bar{x} \in E(\xi_{q,f(\bar{x})-P} \circ f, M) \ \text{ and} \\ E(\xi_{q,f(\bar{x})-P} \circ f, M) \setminus \{\bar{x}\} \\ \subseteq \Big\{ x \in M \setminus \{\bar{x}\} : \ f(x) - f(\bar{x}) \in -(\text{cl } P \setminus P) \Big\}. \end{cases} \tag{11}$$

(*v*) (*strong solutions*) $S = \mathcal{C}(P)$ *with* $q \in P \setminus (-P)$*;* $\xi_{q,S}(0) = 0$*.*

We point out that for any $q \in$ qri P, one gets

$$\xi_{q,\bar{y}-P}(y) = \xi_{q,\bar{y}-P\setminus\{0\}}(y) = \xi_{q,\bar{y}-P\setminus l(P)}(y) = \xi_{q,\bar{y}-\text{qri } P}(y), \ \ \forall\ \bar{y}, y \in L. \tag{12}$$

Moreover,

$$E_1 \subseteq E_{1W} \subseteq E \subseteq E_W, \tag{13}$$

provided cl P is not a subspace, since in such a case we get

$$\text{qri } P \subseteq P \setminus (-P).$$

Furthermore, the specializations described in Remark 4.7 encompasses the concrete situations $P = L^p_+(\Omega)$, $P = l^p_+$, and in these cases such expressions may be simplified because of the closednes and pointedness ($l(P) = \{0\}$) of those cones..

We are now in a position to establish our main result on complete scalarizations which is valid, in particular, for efficient, weak efficient and weak strict efficient solutions as described in Remark 4.7, provided P is a (not necessarily solid or pointed) closed and convex cone. This is new under Assumption (**C**). The analogue under Assumption (**B**) was established in [13].

Theorem 4.8. *Suppose that* $\emptyset \neq S \subseteq Y$, *$S$ is conic satisfying* $S - \mathbb{R}_{++}q \subseteq S$ *with* $q \in (-S) \setminus S$.

(*a*) *If* $0 \in \mathcal{C}(S)$ *and* $S + \text{cl } S \setminus S \subseteq S$, *then*

(*a*1) $x \in E_S \Longleftrightarrow [x \in E(\xi_{q,f(x)+S} \circ f, M) \text{ and } E(\xi_{q,f(x)+S} \circ f, M) \subseteq E_S]$;

(*a*2) $E_S = \bigcup_{x \in E_S} E(\xi_{q,f(x)+S} \circ f, M)$.

(*b*) *Let* $\bar{x} \in M$ *and* $\varepsilon \geq 0$, *then:*

$$x \in E_S(\varepsilon q) \Longrightarrow x \in E(\xi_{q,f(x)+S} \circ f, M, \varepsilon) \Longrightarrow \bar{x} \in E_S(\delta q) \ \forall\, \delta > \varepsilon \Longrightarrow \bar{x} \in E_{\text{int } S}(\varepsilon q),$$

where the last implication holds provided $\text{int } S \neq \emptyset$.

Proof. (*a*1): One implication is straightforward. For the other we need only to check the inclusion due to Theorem 4.5: take any $x' \in E(\xi_{q,f(x)+S} \circ f, M)$, thus $(\xi_{q,f(x)+S} \circ f)(y) \geq (\xi_{q,f(x)+S} \circ f)(x') = (\xi_{q,f(x)+S} \circ f)(x) = 0$ for all $y \in M$. Hence, by the same theorem and the fact that $0 \in \mathcal{C}(S)$, we obtain for all $y \in M$,

$$f(y) - f(x') = f(y) - f(x) + f(x) - f(x') \in \mathcal{C}(S) - \text{cl } S \setminus S \subseteq \mathcal{C}(S),$$

proving that $x' \in E_S$, since $S + (\text{cl } S \setminus S) \subseteq S \Longleftrightarrow \mathcal{C}(-S) + (\text{cl } S \setminus S) \subseteq \mathcal{C}(-S)$. This completes the proof.
(*a*2): It is a consequence of (*a*1).
(*b*): It is straightforward. □

5 Characterizing when the Efficient and Weakly Efficient Solution Sets Coincide

We now consider, in the setting of production theory, a production set $\emptyset \neq Y \subseteq L$, which is not necessarily closed or convex. Here, L is as before, and it is equipped with a proper and convex cone P (by proper we mean $\{0\} \neq P \neq L$), satisfying $\text{qri } P \neq \emptyset$. We denote

$$E_W(Y, P) := \{\bar{y} \in Y : \ y - \bar{y} \notin -\text{qri } P, \ \ \forall\, y \in Y\}. \tag{14}$$

and

$$E(Y, P) := \{\bar{y} \in Y : \ y - \bar{y} \notin -P \setminus l(P), \ \ \forall\, y \in Y\}, \tag{15}$$

where, as before, $l(P) \doteq P \cap (-P)$. As a direct consequence, we obtain

$$E_W(Y, P) = Y \setminus (Y + \text{qri } P) \subseteq Y \cap \text{bd } Y. \tag{16}$$

$$E(Y,P) = Y \setminus (Y + [P \setminus (-P)]). \tag{17}$$

As for (13), we get

$$\text{cl } P \text{ is not a subspace} \Longrightarrow E(Y,P) \subseteq E_W(Y,P). \tag{18}$$

It is worth emphasize that, in case int $P \neq \emptyset$, the set Y is closed and satisfies the free disposability condition: $Y + P = Y$ (which implies that $Y + \text{int } P = \text{int } Y$, see [7] for instance), one obtains $E_W(Y,P) = \text{bd } Y$, a well known fact.

Our notion of weakly efficient point when quasi-relative interior is considered, is termed quasi minimal point in [2].

From Remark 4.7 ($S = -\text{qri } P$, $S = -P \setminus l(P) = (-P) \setminus P$, f to be the identity, and $q \in \text{qri } P$), it follows that

$$\bar{y} \in E_W(Y,P) \Longleftrightarrow \begin{cases} \bar{y} \in E(\xi_{q,\bar{y}-\text{qri } P}, Y) \\ E(\xi_{q,\bar{y}-\text{qri } P}, Y) \setminus \{\bar{y}\} \subseteq \Big\{ y \in Y \setminus \{\bar{y}\} :\ y - \bar{y} \in -(\text{cl } P \setminus \text{qri } P) \Big\} \end{cases}$$

$$\bar{y} \in E(Y,P) \Longleftrightarrow \begin{cases} \bar{y} \in E(\xi_{q,\bar{y}-P\setminus l(P)}, Y) \\ E(\xi_{q,\bar{y}-P\setminus l(P)}, Y) \setminus \{\bar{y}\} \subseteq \Big\{ y \in Y \setminus \{\bar{y}\} :\ y - \bar{y} \in -(\overline{P} \setminus P) \cup l(P) \Big\}. \end{cases}$$

Let us introduce the following assumption originated in [4]:

Assumption (H): for all $y, y' \in Y \setminus (Y + \text{qri } P)$, such that $y - y' \in P$, $y' - y \notin P$, one has $\frac{1}{2}(y + y') \in Y + \text{qri } P$.

A simple condition implying the validity of Assumption (**H**) is the strict convexity on the set Y satisfying int $Y \neq \emptyset$. We say that a convex set $Y \subseteq L$ with int $Y \neq \emptyset$, is strictly convex iff $u, v \in Y$, $u \neq v$, then $\frac{1}{2}(u + v) \in \text{int } Y$. This condition was considered in [1] under free-disposability.

Proposition 5.1. *Assume that $Y \subseteq L$ is strictly convex such that* int $Y \neq \emptyset$. *Let $P \subseteq L$ be a convex cone such that* cl P *is not a subspace and* qri $P \neq \emptyset$. *Then, Assumption* (**H**) *holds, and so by Theorem 5.2 below, $E_W(Y,P) = E(Y,P)$.*

Proof. Let $y, y' \in E_W(Y,P) = Y \setminus (Y + \text{qri } P)$. Then $y, y' \in Y \cap \text{bd } Y$ by (16). By hipothesis, $\frac{1}{2}(y + y') \in \text{int } Y$, which implies that $\frac{1}{2}(y + y') \notin E_W(Y,P)$. This means that there exists $y_0 \in Y$ such that

$$y_0 - \frac{1}{2}(y + y') \in -\text{qri } P.$$

Thus, $\frac{1}{2}(y + y') \in Y + \text{qri } P$, proving the fulfillment of Assumption (**H**). □

Theorem 5.2. *Let P be a convex cone such that* cl P *is not a subspace and* $q \in \text{qri } P$. *Assume that Assumption* **(H)** *fulfills and* $\bar{y} \in E_W(Y,P)$. *Then,*

$$E(\xi_{q,\bar{y}-P\setminus l(P)}, Y) \setminus \{\bar{y}\} \subseteq \Big\{ y \in Y \setminus \{\bar{y}\} : \; y - \bar{y} \in -(\text{cl } P \setminus P) \cup l(P) \Big\}, \tag{19}$$

and therefore $\bar{y} \in E(Y,P)$.

Proof. By Theorem 4.5 and (12), we obtain

$$\bar{y} \in E(\xi_{q,\bar{y}-\text{qri } P}, Y) = E(\xi_{q,\bar{y}-P\setminus l(P)}, Y)$$

and therefore

$$E(\xi_{q,\bar{y}-P\setminus l(P)}, Y) \setminus \{\bar{y}\} \subseteq \Big\{ y \in Y \setminus \{\bar{y}\} : \; y - \bar{y} \in -(\text{cl } P \setminus \text{qri } P) \Big\}.$$

Thus, any $y \in E(\xi_{q,\bar{y}-P\setminus l(P)}, Y) \setminus \{\bar{y}\}$ satisfies

$$y - \bar{y} \in -(\text{cl } P \setminus \text{qri } P). \tag{20}$$

We know from (16) that

$$\bar{y} \in Y \setminus (Y + \text{qri } P). \tag{21}$$

We distinguish two cases:

(a) $y - \bar{y} \notin -P$: in such a case $y - \bar{y} \in -(\text{cl } P \setminus P)$, and therefore y belongs to the set on the right-hand side of (19).

(b) $y - \bar{y} \in -P$: in this situation, if $y \in Y + \text{qri } P$ then $\bar{y} \in y + P \subseteq Y + \text{qri } P + P \subseteq Y + \text{qri } P$, reaching a contradiction to (21). If on the contrary $y \in Y \setminus (Y + \text{qri } P)$, by assuming that $y - \bar{y} \notin P$ we can use Assumption **(H)** to get $y' \doteq \frac{1}{2}(y + \bar{y}) \in Y + \text{qri } P$. This implies that $\bar{y} - y' = \frac{1}{2}(\bar{y} - y) \in P$, and therefore

$$\bar{y} \in y' + P \subseteq Y + \text{qri } P + P \subseteq Y + \text{qri } P,$$

giving a contradiction to (21) again. Hence, $y - \bar{y} \in P$, that is, $y - \bar{y} \in P \cap (-P) = l(P)$, proving that y belongs to the set on the right-hand side of (19). This completes the proof of (19).

An application of Theorem 4.5 allows us to conclude that $\bar{y} \in E(Y,P)$. □

The next result extends and generalizes that due to Bonnisseau and Crettez [4, Theorem 1]: no closedness on Y, or pointedness on P, or non-emptiness of the interior of P, or free-disposability, is required, apart from the infinite dimensional setting. Under the same assumptions on Y but in infinite dimension was proved in [35, Proposition 6.1]. We need only quasi-relative interior.

An earlier version without closedness or free-disposability was obtained in [16] in finite dimension with ordering cones having non-empty interior. Existence of efficient points for preference relations which are reflexive and transitive may be found in [15].

Corollary 5.3. *Let $P \subseteq L$ be a convex cone such that* cl P *is not a subspace and* qri $P \neq \emptyset$*, and* $\emptyset \neq Y \subseteq L$ *Then,*

$$E_W(Y,P) = E(Y,P) \text{ if, and only if } Y \text{ satisfies Assumption } (\mathbf{H}). \tag{22}$$

Proof. The "if" part is a consequence of the preceding theorem. The "only if" part is as follows. Take $y, y' \in Y \setminus (Y + \text{qri } P) = E_W(Y,P)$, such that $y - y' \in P$, $y' - y \notin P$. The equality $E_W(Y,P) = E(Y,P)$ entails $y - y' \notin (-P) \setminus P$ and $y' - y \notin (-P) \setminus P$, yielding a contradiction. □

6 Applications in L^p

We now consider the typical situation in $L = L^p$. More precisely, given $1 \leq p < +\infty$, a nonempty, bounded and open set Ω in $\mathbb{R}^n$, $L^p(\Omega;\mathbb{R})$ denotes the set of measurable functions (with respect to Lebesgue measure) such that $\int_\Omega |u|^p = \int_\Omega |u(t)|^p dt < +\infty$. It is equipped with the norm $\|u\|_p := \left(\int_\Omega |u|^p\right)^{1/p}$. One can check that the pointed closed and convex cone

$$P = L^p_+(\Omega;\mathbb{R}) := \{y \in L^p(\Omega;\mathbb{R}) : \ y \geq 0 \ \text{ a. e. in } \Omega\}$$

has empty interior. For simplicity we use L^p, L^p_+ and L^p_{++}, instead of $L^p(\Omega;\mathbb{R})$, $L^p_+(\Omega;\mathbb{R})$ and qri L^p_+ respectively. It is easy to show that $L^p_+ - L^p_+ = L^p = L^p_+ - L^p_{++}$, and therefore, by Proposition 3.3, we get [5, Examples 3.11]

$$L^p_{++} = \text{qi } L^p_+ = \text{qri } L^p_+ = \{u \in L^p : \ u(x) > 0, \ \ \text{a. e. in } \Omega\}.$$

In what follows $|A|$ denotes the Lebesgue measure of A. The following proposition is easily obtained.

Proposition 6.1. *Let $1 \leq p < +\infty$, $\Omega \subseteq \mathbb{R}^n$ be nonempty, open and bounded set. If $\emptyset \neq Y \subseteq L^p$, then*

$$E_W(Y, L^p_+) \setminus E(Y, L^p_+) =$$

$$= \Big\{\bar{y} \in E_W(Y, L^p_+) : \ \exists\, y \in Y, \ \exists\, \Omega' \subseteq \Omega, \ |\Omega'| > 0; \ y = \bar{y}, \ a.\ e.\ \Omega';$$

$$\bar{y} > y \ a.\ e.\ \Omega \setminus \Omega', \ |\Omega \setminus \Omega'| > 0\Big\}.$$

Proof. Let $\bar{y} \in E_W(Y, L^p_+) = Y \setminus (Y + L^p_{++})$. If on the contrary $\bar{y} \notin E(Y, L^p_+) = Y \setminus (Y + (L^p_+ \setminus \{0\}))$, there exists $y \in Y$ such that $\bar{y} - y \in L^p_+ \setminus \{0\}$. On the other hand, $y - \bar{y} \notin -L^p_{++}$. On combining the last two relations, we get the desired result. □

We now present an instance which may appear in production models within a finance economy.

A Model in Finance

Let us consider a model with a single firm and goods (projects), where the state of nature is represented by $\Omega \subseteq \mathbb{R}^n$. The commodity space is L^2, where each element is a good (firm). It may be interpreted as a random variable with finite variance: given $x \in L^2$, $x(t)$ represents the benefit of the project corresponding to the state of nature t. We assume that the preference relation is given by the closed and convex cone L^2_+. The production (opportunity) set is given by $Y \subseteq L^2$. The standard assumptions our model must satisfy are the following (see [10, 37]):

(a) $0 \in Y$: it means possibility of inaction;

(b) Y is convex;

(c) $-L^2_+ \subseteq Y$: sometimes it is written as $Y - L^2_+ = Y$ (free disposal): if a project is possible, then so is every other project having lower profit;

(d) $Y \cap L^2_+$ is bounded: it asserts that the production possibilities of the economy as a whole are bounded, i. e., only limited profits by the firm are obtained.

Thus, the problem consists in finding an efficient solution with respect to the ordering cone $P = -L^2_+$. As a concrete instance, take

$$Y \doteq \{x \in L^2 : \ \|x\|_2 \leq 1\} - L^2_+.$$

Clearly it satisfies (a), (b), (c) and (d). Actually, in this case, the free disposability assumption becomes $Y - L^2_+ = Y$.

In the next instance we are refering to efficient solutions with respect to the cone $P = L^p_+$, that is, we are looking for minimal elements contrary to maximal elements as described in the above model.

Example 6.2. *Let us consider $p \in \mathbb{N}$, $p \geq 2$, Y_1 to be the unit ball in L^p, that is,*

$$Y_1 \doteq \{x \in L^p : \ \|x\|_p \leq 1\},$$

and $Y = Y_1 + L^p_+$. We are interested in the efficient solutions with respect to the cone $P = L^p_+$. Actually, the free disposability assumption, for the present case, reads as $L^p_+ \subseteq Y$, which is equivalent to $Y + L^p_+ = Y$; whereas the boundedness assumption refers to the set $Y \cap (-L^p_+)$.
We shall prove that

$$E_W(Y_1, L^p_+) = E(Y_1, L^p_+) = E(Y, L^p_+) \subseteq E_W(Y, L^p_+), \tag{23}$$

and

$$E_W(Y_1, L^p_+) = \Big\{z \in L^p : \ \|z\|_p = 1, \ z \leq 0 \text{ a. e. in } \Omega\Big\}, \tag{24}$$

whereas

$$E_W(Y, L^p_+) = \bigcup_{x\in L^p_+} \Big\{z \in L^p : \ \|z - x\|_p = 1, \ z \leq x, \ \textit{a. e. in} \ \ \Omega;$$

$$x = 0, \ \textit{a. e. in } \Omega' \subseteq \Omega, \ |\Omega'| > 0, \ x = z, \ \textit{a. e. in } \Omega \setminus \Omega'\Big\} \tag{25}$$

$$= \Big\{z \in L^p : \ \exists \ \Omega' \subseteq \Omega, \ |\Omega'| > 0, \int_{\Omega'} |z|^p = 1; \ z \leq 0, \ \textit{a. e. } \Omega'; \ z \geq 0 \ \textit{a. e. } \Omega \setminus \Omega'\Big\}.$$

The first equality in (23) *follows from Proposition 5.1 because of the strict convexity of* Y_1*, and the inclusion comes from* (18)*. For the second equality, one inclusion easily follows from* (17) *once we notice that* $Y_1 \subseteq Y$ *and*

$$Y + (L^p_+ \setminus \{0\}) = Y_1 + L^p_+ + (L^p_+ \setminus \{0\}) = Y_1 + (L^p_+ \setminus \{0\}).$$

Let us prove the opposite inclusion. From above, we have

$$Y + (L^p_+ \setminus \{0\}) = Y_1 + (L^p_+ \setminus \{0\}) = \bigcup_{y\in L^p_+\setminus\{0\}} \Big\{z \in L^p : \ \|z - y\|_p \leq 1\Big\}, \tag{26}$$

which implies that

$$E(Y, L^p_+) = Y \setminus (Y + (L^p_+ \setminus \{0\})$$

$$= \Big(\bigcup_{x\in L^p_+} \Big\{z \in L^p : \ \|z - x\|_p \leq 1\Big\}\Big) \cap \bigcap_{y\in L^p_+\setminus\{0\}} \Big\{z \in L^p : \ \|z - y\|_p > 1\Big\}. \tag{27}$$

Let $z \in E(Y, L^p_+)$*. Then* $z \in L^p$ *and* $\|z - x\|_p \leq 1$ *for some* $x \in L^p_+$*, and obviously* $x \notin L^p_+ \setminus \{0\}$*. Thus* $x = 0$ *a. e. in* Ω*. Hence from* (27)*, we obtain*

$$E(Y, L^p_+) \subseteq \Big\{z \in L^p : \ \|z\|_p \leq 1\Big\} \cap \bigcap_{y\in L^p_+\setminus\{0\}} \Big\{z \in L^p : \ \|z - y\|_p > 1\Big\}.$$

$$= Y_1 \setminus (Y_1 + (L^p_+ \setminus \{0\}) = E(Y_1, L^p_+),$$

which completes the second equality in (23)*.*

We now compute (25) *and* (24) *can be deduced from it.*

Since

$$Y + L^p_{++} = Y_1 + L^p_+ + L^p_{++} = Y_1 + L^p_{++} = \bigcup_{y\in L^p_{++}} \Big\{z \in L^p : \ \|z - y\|_p \leq 1\Big\}, \tag{28}$$

we get

$$Y \setminus (Y + L^p_{++}) = \Big(\bigcup_{x \in L^p_+} \Big\{ z \in L^p : \ \|z - x\|_p \le 1 \Big\} \Big) \cap \bigcap_{y \in L^p_{++}} \Big\{ z \in L^p : \ \|z - y\|_p > 1 \Big\}. \tag{29}$$

Let $z \in Y \setminus (Y + L^p_{++})$. *Then* $z \in L^p$ *and* $\|z - x\|_p \le 1$ *for some* $x \in L^p_+$, *and obviously* $x \notin L^p_{++}$. *Since* $\delta x + y \in L^p_{++}$ *for all* $y \in L^p_{++}$ *and all* $\delta > 0$, *we also get* $\|z - \delta x - y\|_p > 1$ *for all* $y \in L^p_{++}$, *and so* $z - \delta x \notin L^p_{++} \cup \{0\}$. *Set*

$$\Omega^\delta_- \doteq \{t \in \Omega : \ z(t) - \delta x(t) < 0\}, \ \Omega^\delta_+ \doteq \{t \in \Omega : \ z(t) - \delta x(t) > 0\} \quad \textit{and}$$

$$\Omega^\delta_0 \doteq \Omega \setminus (\Omega^\delta_- \cup \Omega^\delta_+) = \{t \in \Omega : \ z(t) = \delta x(t)\}.$$

Define for fixed $\lambda > 0$ *and* $\varepsilon > 0$,

$$y(t) = \begin{cases} -\lambda(z(t) - \delta x(t)), & \text{if} \quad t \in \Omega^\delta_-; \\ z(t) - \delta x(t), & \text{if} \quad t \in \Omega^\delta_+; \\ \varepsilon, & \text{if} \quad t \in \Omega^\delta_0. \end{cases}$$

Then $y \in L^p_{++}$, *and therefore* $\|z - \delta x - y\|_p > 1$ *reduces to*

$$(1 + \lambda)^p \int_{\Omega^\delta_-} |z - \delta x|^p + \varepsilon^p |\Omega^\delta_0| > 1.$$

Letting $\lambda \to 0$ *and* $\varepsilon \to 0$, *the previous inequality yields*

$$\int_{\Omega^\delta_-} |z - \delta x|^p \ge 1,$$

and so $|\Omega^\delta_-| > 0$ *for all* $\delta \in \ ]0, 1]$. *We also obtain*

$$0 < \delta_1 < \delta_2 \le 1 \Longrightarrow \Omega^0_- \subseteq \Omega^{\delta_1}_- \subseteq \Omega^{\delta_2}_- \subseteq \Omega^1_-. \tag{30}$$

Thus

$$1 \ge \int_\Omega |x - z|^p \ge \int_{\Omega^\delta_-} |z - x|^p \ge \int_{\Omega^\delta_-} |z - \delta x|^p \ge 1,$$

which implies that

$$\int_\Omega |z - x|^p = \int_{\Omega^\delta_-} |z - x|^p = \int_{\Omega^\delta_-} |z - \delta x|^p = 1, \quad \forall \ \delta \in \ ]0, 1]. \tag{31}$$

Hence

$$\int_{\Omega^1_+} |z - x|^p = 0,$$

yielding $|\Omega^1_+| = 0$ *(it may occur that* $\Omega^1_+ = \emptyset$*), which means* $z \le x$ *a. e. in* Ω. *For* $0 < \delta < 1$, *one obtains*

$$1 = \int_{\Omega^\delta_-} (x - z)^p = \int_{\Omega^\delta_-} [\delta x - z + (1 - \delta) x]^p =$$

$$= \int_{\Omega_-^\delta} \Big[(\delta x - z)^p + \sum_{k=1}^{p-1} \left(\begin{array}{c} p \\ k \end{array} \right) (\delta x - z)^k (1-\delta)^{p-k} x^{p-k} + (1-\delta)^p x^p \Big] \geq$$

$$\geq \int_{\Omega_-^\delta} (\delta x - z)^p = 1,$$

where $\left(\begin{array}{c} p \\ k \end{array} \right) \doteq \dfrac{p!}{k!(p-k)!}$. *Then,*

$$x = 0 \quad a.\ e.\ in\ \Omega_-^\delta \quad for\ all\ 0 < \delta < 1. \tag{32}$$

Moreover, for $0 < \delta < 1$,

$$1 = \int_\Omega |z - x|^p = \int_{\Omega_-^1} |z - x|^p = \int_{\Omega_-^\delta} |z - x|^p + \int_{\Omega_-^1 \setminus \Omega_-^\delta} |z - x|^p = 1 + \int_{\Omega_-^1 \setminus \Omega_-^\delta} |z - x|^p.$$

This gives $|\Omega_-^1 \setminus \Omega_-^\delta| = 0$, *which together with* (32) *and the fact that* $|\Omega_+^1| = 0$ *yield* $x = 0$ *a. e. in* $\Omega_-^1 \cup \Omega_+^1$.

This proves that any z *in* $E_W(Y, L_+^p)$ *belongs to the set on the right-hand side of* (25) *by taking* $\Omega' \doteq \Omega_-^1 \cup \Omega_+^1$, *and so* $\Omega \setminus \Omega' = \Omega_0^1$.

To prove the reverse implication, take any z *in the right-hand side of* (25). *Then, for each* $y \in L_{++}^p$ *we obtain*

$$\int_\Omega |z - y|^p = \int_{\Omega'} |z - y|^p + \int_{\Omega \setminus \Omega'} |z - y|^p \geq \int_{\Omega'} (y - z)^p =$$

$$= \int_{\Omega'} \Big(y^p + \sum_{k=1}^{p-1} \left(\begin{array}{c} p \\ k \end{array} \right) y^k (-z)^{p-k} + (-z)^p \Big).$$

Since $y > 0$ *and* $z \leq 0$ *a.e. in* Ω', *we obtain*

$$\int_\Omega |z - y|^p \geq \int_{\Omega'} y^p + \int_{\Omega'} |z|^p > 1.$$

Hence, z *belongs to the right-hand side of* (29), *proving the equality in* (25).

In order to check (24), *simply note that*

$$E_W(Y_1, L_+^p) = Y_1 \setminus (Y_1 + L_{++}^p) = \bigcap_{y \in L_{++}^p} \Big\{ z \in L^p : \ \|z\|_p \leq 1, \ \|z - y\|_p > 1 \Big\}, \tag{33}$$

and then follows the same reasoning as above.

7 Applications in l^p

Here, l^p is the set of sequences $x = (x_i)_{i \in \mathbb{N}}$ such that $\|x\|^p = \Big(\sum_{i \in \mathbb{N}} |x_i|^p \Big)^{1/p} < +\infty$. As above we also get [5, p. 29]

$$l_{++}^p = \text{qi}\ l_+^p = \text{qri}\ l_+^p = \{x = (x_i)_{i \in \mathbb{N}} \in l^p : \ x_i > 0, \ \ \forall\ i \in \mathbb{N}\}.$$

In the following $|I|$ means the cardinality of the set I.

Example 7.1. *Let us consider $p \in \mathbb{N}$, $p \geq 2$, Y_1 to be the unit ball in l^p, that is,*

$$Y_1 \doteq \{x \in l^p : \ \|x\|_p \leq 1\},$$

and $Y = Y_1 + l^p_+$. We shall prove that

$$E_W(Y_1, l^p_+) = E(Y_1, l^p_+) = E(Y, l^p_+) \subseteq E_W(Y, l^p_+), \tag{34}$$

and

$$E_W(Y_1, l^p_+) = \Big\{z \in l^p : \ \|z\|_p = 1, \ z_i \leq 0 \ \forall \ i \in \mathbb{N}\Big\}, \tag{35}$$

whereas

$$E_W(Y, l^p_+) = \bigcup_{x \in l^p_+} \Big\{z \in l^p : \|z - x\|_p = 1, \ z_i < x_i = 0, \ \forall \ i \in I'; \ z_i = x_i \ \forall \ i \in \mathbb{N} \setminus I'\Big\}. \tag{36}$$

$$= \Big\{z \in l^p : \ \exists \ \emptyset \neq I' \subseteq \mathbb{N}, \ \sum_{i \in I'} |z_i|^p = 1; \ z_i \leq 0, \ \forall \ i \in I'; \ z_i \geq 0 \ \forall \ i \in \mathbb{N} \setminus I'\Big\}.$$

The proof of (34) *is similar to that in the preceding example.*

We proceed as in the previous example by computing only (36)*, since* (35) *can be derived from it.*

Since

$$Y + l^p_{++} = Y_1 + l^p_+ + l^p_{++} = Y_1 + l^p_{++} = \bigcup_{y \in l^p_{++}} \Big\{z \in l^p : \ \|z - y\|_p \leq 1\Big\}, \tag{37}$$

we get

$$Y \setminus (Y + l^p_{++}) = \Big(\bigcup_{x \in l^p_+} \Big\{z \in l^p : \ \|z - x\|_p \leq 1\Big\}\Big) \cap \bigcap_{y \in l^p_{++}} \Big\{z \in l^p : \ \|z - y\|_p > 1\Big\}. \tag{38}$$

Following the reasoning in the preceding example, given $0 < \delta \leq 1$, $\lambda > 0$ and $1 > \varepsilon > 0$, we consider

$$y_i = \begin{cases} -\lambda(z_i - \delta x_i), & \text{if} \quad i \in I^\delta_-; \\ z_i - \delta x_i, & \text{if} \quad i \in I^\delta_+; \\ \dfrac{\varepsilon^{1/p}}{2^{i/p}}, & \text{if} \quad i \in I^\delta_0, \end{cases}$$

where

$$I^\delta_- \doteq \{i \in \mathbb{N} : \ z_i - \delta x_i < 0\}, \ I^\delta_+ \doteq \{i \in \mathbb{N} : \ z_i - \delta x_i > 0\},$$
$$I^\delta_0 \doteq \mathbb{N} \setminus (I^\delta_- \cup I^\delta_+) = \{i \in \mathbb{N} : \ z_i = \delta x_i\}.$$

Then $y \in l^p_{++}$. As before, we deduce that

$$(1 + \lambda)^p \sum_{i \in I^\delta_-} |z_i - \delta x_i|^p + \varepsilon \sum_{i \in I^\delta_0} \frac{1}{2^i} > 1.$$

Letting $\lambda \to 0$ and $\varepsilon \to 0$, the previous inequality yields

$$\sum_{i\in\mathbb{N}} |z_i - x_i|^p = \sum_{i\in I_-^\delta} |z_i - x_i|^p = \sum_{i\in I_-^\delta} |z_i - \delta x_i|^p = 1, \quad \forall\ \delta \in\ ]0,1], \tag{39}$$

which implies that $I_-^\delta \neq \emptyset$ and $I_+^1 = \emptyset$. Similar to (32), we also obtain

$$x_i = 0 \ \ \forall\ i \in I_-^\delta, \quad \textit{and all } 0 < \delta < 1. \tag{40}$$

In addition, from the equalities $(1 < \delta < 1)$

$$1 = \sum_{i\in\mathbb{N}} |z_i - x_i|^p = \sum_{i\in I_-^1} |z_i - x_i|^p = \sum_{i\in I_-^\delta} |z_i - x_i|^p + \sum_{i\in I_-^1\setminus I_-^\delta} |z_i - x_i|^p = 1 + \sum_{i\in I_-^1\setminus I_-^\delta} |z_i - x_i|^p,$$

we get $|I_-^1 \setminus I_-^\delta| = 0$, and so $I_-^1 = I_-^\delta \neq \emptyset$, leading to $x_i = 0$ for all $i \in I_-^1$. This proves that any z in $E_W(Y, l_+^p)$ belongs to the set in the right-hand side of (36) by taking $I' \doteq I_-^1$; since $I_+^1 = \emptyset$, $z_i = x_i$ for all $i \in \mathbb{N} \setminus I' = I_0^1$.

The reverse inclusion is proved in a similar way to the previous example.

8 Conclusions

Motivated by some models in economies with production, which require infinite-dimensional commodity spaces like L^p or l^p and where the ordering cones have empty interior, we develop further a nonlinear scalarization approach without any convexity assumptions, started in [13]. This is used to characterize the coincidence of the efficient and the weakly efficient solution sets. Here, the notion of weakly efficient solution involves the quasi-relative interior instead of interior. It is known that from a mathematical point of view to compute the weakly efficient solutions is easier than the efficient ones, but in application, the latter notion has a real meaning. A couple of models are presented for which the efficient and weakly efficient solution sets are computed.

Our unified scalarization approach may be considered as an alternative procedure to that developed in [11] which use a different nonlinear scalarization function. So the next step is looking for optimality conditions in our setting.

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References

[1] Aliprantis, C. D.; Brown, D. J.; Burkinshaw, O., "Existence and optimality of competitive equilibria", Berlin, Heidelberg, New York, Springer-Verlag, 1989.

[2] Bao, T. Q.; Mordukhovich, B. S., Relative Pareto minimizers for multiobjective problems: existence and optimality conditions, *Math. Program. A*, **122** (2010), 301–347.

[3] Bonnisseau, J.-M; Cornet, B., Existence of equilibria when firms follows bounded losses pricing rules, *J. Math. Econ.*, **17** (1988), 119–147.

[4] Bonnisseau, J.-M; Crettez, B., On the characterization of efficient production vectors, *Economic Theory*, **31** (2007), 213–223.

[5] Borwein, J.M.; Lewis, A. S., Partially finite convex programming, Part I: Quasi relative interiors and duality theory, *Math. Program. A*, **57** (1992), 15–48.

[6] Bot, R. I.; Grad, S. M., Duality for vector optimization via a general scalarization, *Optimization*, **60** (2011), 1269–1290.

[7] Breckner W.W.; Kassay G., A systematization of convexity concepts for sets and functions, *J. Convex Anal.* **4** (1997), 109–127.

[8] Chen, G-Y.; Huang, X.; Yang, X., "Vector optimization. Set-valued and variational analysis". Lecture Notes in Economics and Mathematical Systems, 541. Springer-Verlag, Berlin, 2005.

[9] Chicco, M.; Mignanego, F.; Pusillo, L.; Tijs, S., Vector optimization problems via improvements sets, *J. Optim. Theory Appl.*, **150** (2011), 516–529.

[10] Debreu, G, "Theory of value". John Wiley, New York, 1959.

[11] Durea, M.; Dutta, J.; Tammer, C., Lagrange multipliers for ε-Pareto solutions in vector optimization with nonsolid cones in Banach spaces, *J. Optim. Theory and Appl.*, **145** (2010), 196–211.

[12] Eichfelder, G., "Adaptive Scalarization Methods in Multiobjective Optimization", Springer-Verlag, Berlin, 2008.

[13] Flores-Bazán, F.; Hernández, E., A unified vector optimization problem: complete scalarizations and applications, *Optimization*, **60** (2011), 1399–1419.

[14] Flores-Bazán, F.; Hernández, E., Optimality conditions for a unified vector optimization problem with not necessarily preordering relations, *J. Global Optim.*, **56** (2013), 299–325.

[15] Flores-Bazán, F.; Hernández, E.; Novo, V. Characterizing efficiency without linear structure: a unified approach. *J. Global Optim.*, **41**, no. 1 (2008), 43-60.

[16] Flores-Bazán, F.; Laengle, S.; Loyola, G., Characterizing the efficient points without closedness or free-disposability, *Central European Journal of Operational Research*, **21** (2013), 401–410.

[17] Föllmer, H.; Schied, A., Convex measures of risk and trading constraints. *Finance Stoch.*, **6** (2002), 429–447.

[18] Gerstewitz (Tammer), Ch.; Iwanow, E., Dualität für nichtkonvexe Vektoroptimierungsprobleme. (German) Wiss. Z. Tech. Hochsch. Ilmenau 31 (1985), 61–81.

[19] Gerth (Tammer), C.; Weidner, P., Nonconvex separation theorems and some applications in vector optimization, *J. Optim. Theory Appl.*, **67** (1990), 297–320.

[20] Göpfert, A.; Riahi, H.; Tammer, Ch.; Zalinescu, C., "Variational Methods in Partially Ordered Spaces", CMS Books in Mathematics, Springer-Verlag, New york, 2003.

[21] A. Grad, Quasi-relative interior-type constraint qualifications ensuring strong Lagrange duality for optimization problems with cone and affine constraints, *J. Math. Anal. Appl.*, **361** (2010), 86–95.

[22] Gutiérrez, C.; Jiménez, B.; Novo, V., A Unified Approach and Optimality Conditions for Approximate Solutions of Vector Optimization Problems, *SIAM J. Optim.,* **17** (2006), 688-710.

[23] Gutiérrez, C.; Jiménez, B.; Novo, V., Optimality conditions via scalarization for a new ε-efficiency concept in vector optimization problems, *European J. Oper. Res.,* **201** (2010), 11–22.

[24] Gutiérrez, C.; Jiménez, B.; Novo, V., Improvements sets and vector optimization, *European J. Oper. Res.,* **223** (2012), 304–311.

[25] Hernández, E.; Rodríguez-Marín, L., Nonconvex scalarization in set optimization with set-valued maps, *J. Math. Anal. Appl.,* **325** (2007), 1–18.

[26] Hiriart-Urruty, J.-B., Tangent cones, generalized gradients and mathematical programming in Banach spaces, *Math. Oper. Res.*, **4** (1979), 79–97.

[27] Limber, M. A.; Goodrich, R. K., Quasi interiors, Lagrange multipliers, and L^p spectral estimation with lattice bounds, *J. Optim. Theory Appl.*, **78** (1993), 143–161.

[28] Jahn, J., "Vector optimization. Theory, applications and extensions", Springer-Verlag, Berlin, 2004,

[29] Li, S. J.; Yang, X. Q.; Chen, G.Y., Nonconvex vector optimization of set-valued mappings, *J. Math. Anal. Appl.,* **283** (2003), 337–350.

[30] Luc, D.T., "Theory of Vector Optimization", *Lecture Notes in Economics and Mathematical Systems,* Vol. 319, Springer-Verlag, New york, Berlin, 1989.

[31] Luenberger, D. G., New optimality principles for economics efficiency and equilibrium, *J. Optim. Theory Appl.*, **75** (1992), 221–264.

[32] Pascoletti, A.; Serafini, P., Scalarizing vector optimization problems, *J. Optim. Theory Appl.*, **42** (1984), 499–524.

[33] Rubinov, A.M., Sublinear operator and theirs applications, *Russian Math. Surveys,* **32** (1977), 113–175.

[34] Rubinov, A.M.; Gasimov, R.N., Scalarization and nonlinear scalar duality for vector optimization with preferences that are not necessarely a preorder relation, *J. Global Optim.,* **29** (2004), 455–477.

[35] Tammer, C.; Zalinescu, C., Lipschitz properties of the scalarization function and applications, *Optimization*, **59** (2010), 305–319.

[36] Zaffaroni, A., Degrees of efficiency and degrees of minimality, *SIAM J. Control Optim.*, **42** (2003), 1071–1086.

[37] Magill, M.; Quinzii, M., "Theory of incomplete markets", Vol. 1, The MIT Press, Cambridge, Massachusetts, London, England, 1996.

[38] Zame, William R., Competitive equilibria in production economies with an infinite-dimensional commodity space, *Econometrica*, **55** (1987), 1075–1108.