Analysis of an augmented fully-mixed approach for the coupling of quasi-Newtonian fluids and porous media*

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Abstract

In this paper we introduce and analyze an augmented mixed finite element method for the coupling of quasi-Newtonian fluids and porous media. The flows are governed by a class of nonlinear Stokes and linear Darcy equations, respectively, and the transmission conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law. We apply dual-mixed formulations in both domains, and, in order to handle the nonlinearity involved in the Stokes region, we set the strain and vorticity tensors as auxiliary unknowns. In turn, since the transmission conditions become essential, they are imposed weakly, which yields the introduction of the traces of the porous media pressure and the fluid velocity as the associated Lagrange multipliers. Moreover, in order to facilitate the analysis, we augment the formulation in the fluid by incorporating a redundant Galerkin-type term arising from the quasi-Newtonian constitutive law multiplied by a suitable stabilization parameter. In this way, under a suitable and explicit choice of this parameter, a generalization of the Babuska-Brezzi theory is utilized to show the well-posedness of the continuous and discrete formulations and to derive the corresponding a-priori error estimate. In particular, the feasible finite element subspaces include PEERS and Arnold-Falk-Winther elements for the stress, velocity and vorticity in the fluid, Raviart-Thomas elements and piecewise constants for the velocity and pressure in the porous medium, together with piecewise constant Stokes strain tensor and continuous piecewise linear elements for the traces. Next, we employ classical approaches, which include linearization techniques, Clément's interpolator and Helmholtz's decomposition, together with known efficiency estimates, to derive a reliable and efficient residual-based a posteriori error estimator for the coupled problem. Finally, several numerical results confirming the good performance of the method and the properties of the a posteriori error estimator, and illustrating the capability of the corresponding adaptive algorithm to identify the singular regions of the solution, are reported.

1 Introduction

The devising of suitable numerical methods for solving the Stokes-Darcy and related coupled problems, including porous media with cracks, the incorporation of the Brinkman equation in the model, and

*This research was partially supported by BASAL project CMM, Universidad de Chile, by Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, and by CONICYT project Anillo ACT1118 (ANANUM).

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linear as well as nonlinear behaviors, has become a very active research area during the last decade (see, e.g. [6], [10], [11], [12], [13], [14], [24], [30], [32], [37], [40], [42] and the references therein). In particular, a mixed finite element method for a nonlinear Stokes-Darcy flow problem arising in industrial filtering application and involving a non-Newtonian fluid, is introduced and analized in [12]. Actually, up to the authors' knowledge, this is the first work dealing with the fully-coupled problem for non-Newtonian Stokes and Darcy flows. In fact, the fluid is modeled there by the generalized nonlinear Stokes equation in the free flow region and by the generalized nonlinear Darcy equation in the porous medium. In addition, the approach in [12] employs the primal method in the Stokes domain and the dual-mixed method in the Darcy region, which means that only the original velocity and pressure unknowns are considered in the fluid, whereas a further unknown (velocity) is added in the porous medium. The corresponding interface conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law, and since one of them becomes essential, the trace of the Darcy pressure on the interface needs also to be incorporated as an additional Lagrange multiplier. More recently, the model from [12] is recasted in [13] as a reduced matching problem on the interface by using a mortar space approach. As a consequence, a parallel algorithm for the problems in both regions is derived, which allows to solve the coupled problem utilizing existing codes for Stokes and Darcy simulations.

On the other hand, the a priori and a posteriori error analyses of a new fully-mixed finite element method for the 2D Stokes-Darcy coupled problem, in which dual-mixed formulations are employed in both domains, were developed in [25] and [26]. This approach allows, on the one hand, the introduction of further unknowns of physical interest, and on the other hand, the utilization of the same family of finite element subspaces in both media, without requiring any stabilization term. The results from [25] and [26] were then extended in [27] to the case of a two-dimensional nonlinear Stokes-Darcy coupled problem. More precisely, the model here refers to the coupling of fluid flow with nonlinear porous media flow, where the nonlinearity in the latter region is given by the corresponding permeability. The utilization of dual-mixed formulations in both regions yields the pseudostress and the velocity in the fluid, together with the velocity, the pressure and its gradient in the porous medium, as the main unknowns. In addition, since the approach in [27] leads to essential transmission conditions, these are imposed weakly and hence the traces of the porous medium pressure and the fluid velocity become the corresponding Lagrange multipliers. Similarly as in [25], the remaining unknowns of physical interest can then be computed through very simple postprocessing formulae that, at the discrete level, make no use of any numerical differentiation procedure. Since the resulting variational formulation can be written as a nonlinear twofold saddle point operator equation, the generalization of the Babuška-Brezzi theory developed in [17] is applied to prove the well-posedness of the continuous and discrete schemes. Finally, a reliable and efficient residual-based a posteriori error estimator is also derived in [27]. In spite of the many contributions available in the literature on the a posteriori error analysis for variational formulations with saddle point structure, the first results concerning nonlinear twofold saddle point problems have been obtained in [27] and [15] by properly adapting and extending some related techniques from [22] and [26]. In particular, the analysis in [15] provides an abstract error estimate that can be applied to a large class of nonlinear variational formulations showing a twofold saddle point structure.

The purpose of the present paper is to extend the analysis and results from [25] and [27] to the model problem from [12], that is to the coupling of quasi-Newtonian fluids and porous media. In other words, we now develop the a priori and a posteriori error analyses of a fully-mixed formulation for a class of nonlinear Stokes models coupled with the usual linear Darcy equation, and assuming the usual transmission conditions, that is mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law. To this end, and differently from [12] where a primal approach is employed in the fluid, we apply dual-mixed formulations in both regions (exactly as in [25] and [27]), and handle the nonlinearity in the fluid by introducing the strain and vorticity tensors as additional unknowns. In addition, since the transmission conditions become again essential, they are imposed weakly, which yields the traces of the porous media pressure and the fluid velocity on the interface as the associated Lagrange multipliers. Furthermore, we follow the same approach from [22] and [23], and enrich the equations in the fluid with a redundant Galerkin-type term arising from the quasi-Newtonian constitutive law multiplied by a suitable stabilization parameter. As a consequence, the resulting augmented variational formulation shows a twofold saddle point structure that matches a slight modification of the generalized Babuška-Brezzi theory derived in [17] (see also [16]). In this way, a suitable and explicit choice of the stabilization parameter allows to prove the well-posedness of the corresponding continuous and discrete schemes. Then, following the approach from [27] and [15], we derive a reliable and efficient residual-based a posteriori error estimator for our nonliner coupled problem. As in [27], the proof of reliability makes use of a global inf-sup condition for a linearized version of the problem, Helmholtz decompositions in both media, and local approximation properties of the Clément interpolant and Raviart-Thomas operator. In turn, inverse inequalities, the localization technique based on element-bubble and edge-bubble functions, and known results from previous works, are the main tools for proving the efficiency of the estimator.

The rest of this work is organized as follows. In Section 2 we introduce the model problem and derive the augmented fully-mixed variational formulation, which is shown to have a twofold saddle point structure. A slight modification of the generalized Babuška-Brezzi theory developed in [17] is described in Section 3. This abstract framework is then applied in Section 4 to prove the well posedness of the continuous problem. Next, in Section 5 we define the Galerkin scheme and, employing the corresponding analysis from Section 3, we derive general hypotheses on the finite element subspaces ensuring that the discrete scheme becomes well posed. A specific choice of finite element subspaces satisfying these assumptions is also described here. In Section 6 we derive the residual-based a posteriori error estimator and prove its reliability and efficiency. Finally, several numerical results illustrating the performance of the method, confirming the reliability and efficiency of the a posteriori estimator, and showing the good behavior of the associated adaptive algorithm, are reported in Section 7.

2 The continuous problem

2.1 Preliminary notations

We begin this section with several notations to be used throughout the paper. In what follows, given $n \in \{2,3\}$, $\mathrm{R}^{n\times n}$ is the space of square matrices of orden n with real entries, $\mathbb{I} := (\delta_{ij})$ is the identity matrix of $\mathrm{R}^{n\times n}$, and for any $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij})$ in $\mathrm{R}^{n\times n}$, we write as usual

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji}), \quad \operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\,\mathbb{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\,\zeta_{ij},$$

which corresponds, respectively, to the transpose, the trace, and the deviator of a tensor $\boldsymbol{\tau}$, and to the tensorial product between $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$. In turn, in what follows we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, $\mathcal{S}$ is an open or closed Lipschitz curve (resp. surface in R^3), and $r \in \mathrm{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^n, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{n\times n}, \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^n.$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote $[H]^n$ and $[H]^{n\times n}$, respectively. In addition, $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ stands for the usual duality pairings between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, and between $\mathbf{H}^{-1/2}(\mathcal{S})$ and $\mathbf{H}^{1/2}(\mathcal{S})$. Note, however, that when $\mathcal{S}$ is an open Lipschitz curve (resp. surface in R^3), $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ is also employed below to denote the duality pairings between $H_{00}^{-1/2}(\mathcal{S})$ and $H_{00}^{1/2}(\mathcal{S})$, and between $\mathbf{H}_{00}^{-1/2}(\mathcal{S})$ and $\mathbf{H}_{00}^{1/2}(\mathcal{S})$ (see Section 2.3 for details). Furthermore, with div denoting the usual divergence operator, the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \left\{\mathbf{w}\in\mathbf{L}^2(\mathcal{O}): \quad \mathrm{div}\,\mathbf{w}\in L^2(\mathcal{O})\right\},$$

is standard in the realm of mixed problems (see [7], [28]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$, where $\mathbf{div}$ stands for the action of div along each row of a tensor. The Hilbert norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div};\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. Note that if $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}\,\boldsymbol{\tau}\in\mathbf{L}^2(\mathcal{O})$.

Finally, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2.2 The model problem

In order to describe the corresponding geometry, we let Ω_{S} and Ω_{D} be bounded and simply connected polyhedral domains in R^n, $n\in\{2,3\}$, such that $\Omega_{\mathrm{S}}\cap\Omega_{\mathrm{D}} = \emptyset$ and $\partial\Omega_{\mathrm{S}}\cap\partial\Omega_{\mathrm{D}} = \Sigma\neq\emptyset$. Then, we let $\Gamma_{\mathrm{S}} := \partial\Omega_{\mathrm{S}}\backslash\bar{\Sigma}$, $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}}\backslash\bar{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega_{\mathrm{S}}\cup\Sigma\cup\Omega_{\mathrm{D}}$ and Ω_{S} (and hence inward to Ω_{D} when seen on Σ). On Σ we also consider unit tangent vectors, which are given by $\mathbf{t}=\mathbf{t}_1$ when $n=2$ (see Figure 2.1 below) and by $\{\mathbf{t}_1,\mathbf{t}_2\}$ when $n=3$. The model problem we are interested in consists of the movement of an incompressible quasi-Newtonian viscous fluid that occupies Ω_{S} and that flows towards and from Ω_{D} through Σ, where Ω_{D} is saturated with the same fluid. More precisely, the governing equations in Ω_{S}

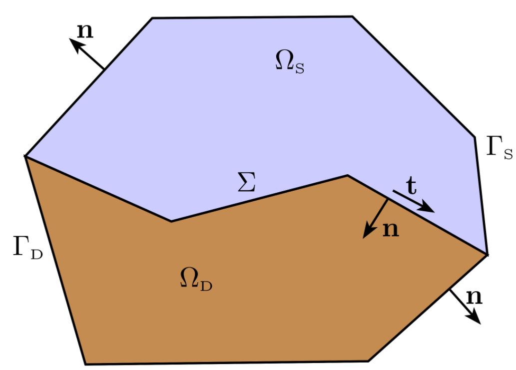

Figure 2.1: The 2D geometry of our Stokes–Darcy model

are those of the nonlinear Stokes problem written in the following stress-velocity-pressure formulation:

$$\begin{gathered}\boldsymbol{\sigma}_{\mathrm{S}} = \mu\left(|\mathbf{e}(\mathbf{u}_{\mathrm{S}})|\right)\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - p_{\mathrm{S}}\mathbb{I} \quad \text{in } \Omega_{\mathrm{S}}, \quad \mathrm{div}\,\mathbf{u}_{\mathrm{S}} = 0 \quad \text{in } \Omega_{\mathrm{S}}, \\ \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} = -\mathbf{f}_{\mathrm{S}} \quad \text{in } \Omega_{\mathrm{S}}, \quad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{S}},\end{gathered} \tag{2.1}$$

where $\boldsymbol{\sigma}_{\mathrm{S}}$ is the stress tensor, $\mathbf{u}_{\mathrm{S}}$ is the velocity, p_{S} is the pressure, $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is the nonlinear kinematic viscosity, $\mathbf{e}(\mathbf{u}_{\mathrm{S}}) := \frac{1}{2}\left\{\nabla \mathbf{u}_{\mathrm{S}} + \left(\nabla \mathbf{u}_{\mathrm{S}}\right)^{\mathtt{t}}\right\}$ is the strain tensor (or symmetric part of the velocity gradient), $|\cdot|$ is the euclidean norm of $\mathrm{R}^{n\times n}$, and $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ is a known volume force. Note here that $\boldsymbol{\sigma}_{\mathrm{S}}$ is symmetric. In turn, in Ω_{D} we consider the linearized Darcy model with Neumann boundary condition on Γ_{D}:

$$\mathbf{u}_{\mathrm{D}} = -\mathbf{K}\,\nabla p_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \operatorname{div} \mathbf{u}_{\mathrm{D}} = f_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}, \tag{2.2}$$

where $\mathbf{u}_{\mathrm{D}}$ and p_{D} denote the velocity and pressure, respectively, $f_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}})$ is a source term satisfying $\int_{\Omega_{\mathrm{D}}} f_{\mathrm{D}} = 0$, and $\mathbf{K}$ is a symmetric and positive definite tensor with entries in $L^\infty(\Omega_{\mathrm{D}})$, which describes the permeability of Ω_{D} divided by a constant approximation of the viscosity. Finally, the transmission conditions on Σ are given by

$$\begin{aligned} \mathbf{u}_{\mathrm{S}} \cdot \mathbf{n} &= \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} \qquad \text{on} \quad \Sigma, \\ \boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n} + \sum_{\ell=1}^{n-1} \kappa_\ell^{-1}(\mathbf{u}_{\mathrm{S}} \cdot \mathbf{t}_\ell)\,\mathbf{t}_\ell &= -p_{\mathrm{D}}\,\mathbf{n} \qquad \text{on} \quad \Sigma, \end{aligned} \tag{2.3}$$

where $\{\kappa_1, ..., \kappa_{n-1}\}$ is a set of positive frictional constants that can be determined experimentally.

At this point we remark that the kind of nonlinear Stokes problem given by (2.1) appears in the modeling of a large class of non-Newtonian fluids (see, e.g. [5], [31], [34], [39]). In particular, the Ladyzhenskaya law for fluids with large stresses (see [31]), also known as power law, is given by $\mu(t) := \mu_0 + \mu_1 t^{\beta-2}\ \forall\, t \in \mathrm{R}^+$, with $\mu_0 \geq 0$, $\mu_1 > 0$, and $\beta > 1$, and the Carreau law for viscoplastic flows (see, e.g. [34], [39]) reads $\mu(t) := \mu_0 + \mu_1\,(1 + t^2)^{(\beta-2)/2}\ \forall\, t \in \mathrm{R}^+$, with $\mu_0 \geq 0$, $\mu_1 > 0$, and $\beta \geq 1$.

In what follows we let $\mu_{ij} : \mathrm{R}^{n\times n} \to \mathrm{R}$ be the mapping given by $\mu_{ij}(\mathbf{r}) := \mu(|\mathbf{r}|)\, r_{ij}$ for all $\mathbf{r} := (r_{ij}) \in \mathrm{R}^{n\times n}$, for all $i, j \in \{1, ..., n\}$. Then, throughout this paper we assume that μ is of class C^1 and that there exist γ_0, $\alpha_0 > 0$ such that for all $\mathbf{r} := (r_{ij})$, $\mathbf{s} := (s_{ij}) \in \mathrm{R}^{n\times n}$, there holds

$$|\,\mu_{ij}(\mathbf{r})\,| \leq \gamma_0\,|\mathbf{r}|, \qquad \left|\frac{\partial}{\partial r_{kl}}\mu_{ij}(\mathbf{r})\right| \leq \gamma_0 \quad \forall\, i,j,k,l \in \{1,...,n\}, \tag{2.4}$$

and

$$\sum_{i,j,k,l\,=\,1}^{n} \frac{\partial}{\partial r_{kl}}\mu_{ij}(\mathbf{r})\, s_{ij}\, s_{kl} \geq \alpha_0\,|\mathbf{s}|^2. \tag{2.5}$$

It is easy to check that the Carreau law satisfies (2.4) and (2.5) for all $\mu_0 > 0$, and for all $\beta \in [1, 2]$. In particular, with $\beta = 2$ we recover the usual linear Stokes model.

2.3 Further notations

In order to derive the weak formulation of the coupled problem given by (2.1), (2.2), and (2.3), we need to introduce other notations and definitions. In fact, given $\star \in \{\mathrm{S}, \mathrm{D}\}$, u, $v \in L^2(\Omega_\star)$, $\mathbf{u}$, $\mathbf{v} \in \mathbf{L}^2(\Omega_\star)$, and $\boldsymbol{\sigma}$, $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$, we denote

$$(u, v)_\star := \int_{\Omega_\star} u\,v, \qquad (\mathbf{u}, \mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u} \cdot \mathbf{v}, \quad \text{and} \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}.$$

In addition, we let $\overline{\mathbb{L}}^2(\Omega_{\mathrm{S}})$ and $\underline{\mathbb{L}}^2(\Omega_{\mathrm{S}})$ be the subspaces of symmetric and skew-symmetric tensors of $\mathbb{L}^2(\Omega_{\mathrm{S}})$, respectively, that is

$$\overline{\mathbb{L}}^2(\Omega_{\mathrm{S}}) := \left\{ \mathbf{r}_{\mathrm{S}} \in \mathbb{L}^2(\Omega_{\mathrm{S}}) : \quad \mathbf{r}_{\mathrm{S}}^{\mathtt{t}} = \mathbf{r}_{\mathrm{S}} \right\}$$

and

$$\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) \,:=\, \Big\{ \boldsymbol{\eta}_{\mathrm{S}} \,\in\, \mathbb{L}^2(\Omega_{\mathrm{S}}): \quad \boldsymbol{\eta}_{\mathrm{S}}^{\mathtt{t}} \,=\, -\,\boldsymbol{\eta}_{\mathrm{S}} \Big\}.$$

Furthermore, we also need the space $\mathbf{H}^{1/2}_{00}(\Sigma) \,:=\, H^{1/2}_{00}(\Sigma) \times H^{1/2}_{00}(\Sigma)$, where

$$H^{1/2}_{00}(\Sigma) \,:=\, \Big\{ v|_{\Sigma}: \quad v \,\in\, H^1(\Omega_{\mathrm{S}})\,, \quad v \,=\, 0 \quad \text{on} \quad \Gamma_{\mathrm{S}} \Big\}.$$

Equivalently, if $E_{0,\mathrm{S}} : H^{1/2}(\Sigma) \to L^2(\partial\Omega_{\mathrm{S}})$ is the extension operator defined by

$$E_{0,\mathrm{S}}(\psi) \,:=\, \begin{cases} \psi & \text{on} \quad \Sigma \\ 0 & \text{on} \quad \Gamma_{\mathrm{S}} \end{cases} \qquad \forall\, \psi \,\in\, H^{1/2}(\Sigma)\,,$$

we have that

$$H^{1/2}_{00}(\Sigma) \,=\, \Big\{ \psi \,\in\, H^{1/2}(\Sigma): \quad E_{0,\mathrm{S}}(\psi) \,\in\, H^{1/2}(\partial\Omega_{\mathrm{S}}) \Big\},$$

endowed with the norm $\|\psi\|_{1/2,00,\Sigma} \,:=\, \|E_{0,\mathrm{S}}(\psi)\|_{1/2,\partial\Omega_{\mathrm{S}}}$. In turn, if $\mathbf{E}_{0,\mathrm{S}} : \mathbf{H}^{1/2}(\Sigma) \to \mathbf{L}^2(\partial\Omega_{\mathrm{S}})$ is the vector version of $E_{0,\mathrm{S}}$, we have that $\|\boldsymbol{\psi}\|_{1/2,00,\Sigma} \,:=\, \|\mathbf{E}_{0,\mathrm{S}}(\boldsymbol{\psi})\|_{1/2,\partial\Omega_{\mathrm{S}}} \;\; \forall\, \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$. The dual space of $H^{1/2}_{00}(\Sigma)$ (resp. $\mathbf{H}^{1/2}_{00}(\Sigma)$) is $H^{-1/2}_{00}(\Sigma)$ (resp. $\mathbf{H}^{-1/2}_{00}(\Sigma)$) and the corresponding duality pairing is denoted in each case by $\langle \cdot, \cdot \rangle_{\Sigma}$. In particular, note that given $\boldsymbol{\eta} \,\in\, \mathbf{H}^{-1/2}(\partial\Omega_{\mathrm{S}})$, its restriction to Σ defined by $\langle \boldsymbol{\eta}|_{\Sigma}, \boldsymbol{\psi} \rangle_{\Sigma} \,:=\, \langle \boldsymbol{\eta}, \mathbf{E}_{0,\mathrm{S}}(\boldsymbol{\psi}) \rangle_{\partial\Omega_{\mathrm{S}}} \;\; \forall\, \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$, is an element of $\mathbf{H}^{-1/2}_{00}(\Sigma)$.

2.4 The augmented fully-mixed variational formulation

We now proceed with the announced weak formulation. We begin by observing, as in [21] and [22], that, thanks to the fact that $\operatorname{tr}\mathbf{e}(\mathbf{u}_{\mathrm{S}}) \,=\, \operatorname{div}\mathbf{u}_{\mathrm{S}}$, the first two equations from (2.1), that is

$$\boldsymbol{\sigma}_{\mathrm{S}} \,=\, \mu\,(|\mathbf{e}(\mathbf{u}_{\mathrm{S}})|)\,\mathbf{e}(\mathbf{u}_{\mathrm{S}}) \,-\, p_{\mathrm{S}}\,\mathbb{I} \quad \text{and} \quad \operatorname{div}\mathbf{u}_{\mathrm{S}} \,=\, 0 \quad \text{in} \quad \Omega_{\mathrm{S}}\,,$$

are equivalent to

$$\boldsymbol{\sigma}_{\mathrm{S}} \,=\, \mu\,(|\mathbf{e}(\mathbf{u}_{\mathrm{S}})|)\,\mathbf{e}(\mathbf{u}_{\mathrm{S}}) \,-\, p_{\mathrm{S}}\,\mathbb{I} \quad \text{and} \quad p_{\mathrm{S}} \,=\, -\,\frac{1}{n}\operatorname{tr}\boldsymbol{\sigma}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,,$$

and hence, eliminating the pressure p_{S}, the Stokes problem (2.1) can be rewritten as

$$\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}} \,=\, \mu\,(|\mathbf{e}(\mathbf{u}_{\mathrm{S}})|)\,\mathbf{e}(\mathbf{u}_{\mathrm{S}}) \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \quad \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \,=\, -\,\mathbf{f}_{\mathrm{S}} \quad \text{in}\;\Omega_{\mathrm{S}}\,, \quad \mathbf{u}_{\mathrm{S}} \,=\, \mathbf{0} \quad \text{on}\;\Gamma_{\mathrm{S}}\,. \tag{2.6}$$

Moreover, in order to handle the nonlinearity defining $\boldsymbol{\sigma}_{\mathrm{S}}$, we adopt the approach from [22] (see also [23]) and introduce the additional unknowns

$$\mathbf{t}_{\mathrm{S}} \,:=\, \mathbf{e}(\mathbf{u}_{\mathrm{S}}) \quad \text{and} \quad \boldsymbol{\gamma}_{\mathrm{S}} \,:=\, \frac{1}{2}\Big\{\nabla\mathbf{u}_{\mathrm{S}} \,-\, \big(\nabla\mathbf{u}_{\mathrm{S}}\big)^{\mathtt{t}}\Big\} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \tag{2.7}$$

where $\boldsymbol{\gamma}_{\mathrm{S}}$ is the vorticity (or skew-symmetric part of the velocity gradient), so that (2.6) reduces to

$$\begin{gathered} \mathbf{t}_{\mathrm{S}} \,=\, \nabla\mathbf{u}_{\mathrm{S}} \,-\, \boldsymbol{\gamma}_{S} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \quad \boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}} \,=\, \mu\,(|\mathbf{t}_{\mathrm{S}}|)\,\mathbf{t}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \\ \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \,=\, -\,\mathbf{f}_{\mathrm{S}} \quad \text{in}\;\Omega_{\mathrm{S}}\,, \quad \mathbf{u}_{\mathrm{S}} \,=\, \mathbf{0} \quad \text{on}\;\Gamma_{\mathrm{S}}\,, \end{gathered} \tag{2.8}$$

with both $\mathbf{t}_{\mathrm{S}}$ and $\boldsymbol{\sigma}_{\mathrm{S}}$ symmetric tensors, and such that $\operatorname{tr}\mathbf{t}_{\mathrm{S}} \,=\, 0$ in Ω_{S}. Then, multiplying the first equation of (2.8) by $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$, integrating by parts the expression $(\nabla\mathbf{u}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}}$, introducing the

Dirichlet boundary condition $\mathbf{u}_{\mathrm{S}} = \mathbf{0}$ on Γ_{S}, and using that $(\mathbf{t}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} = (\mathbf{t}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}}$ (which follows from the fact that $\mathbf{t}_{\mathrm{S}} : \mathbb{I} = \operatorname{tr} \mathbf{t}_{\mathrm{S}} = 0$), we arrive at

$$(\mathbf{t}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} + (\mathbf{div}\, \boldsymbol{\tau}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\, \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Sigma} + (\boldsymbol{\tau}_{\mathrm{S}}, \gamma_{\mathrm{S}})_{\mathrm{S}} = 0 \quad \forall\, \boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div}; \Omega_{\mathrm{S}}), \tag{2.9}$$

with unknowns

$$\mathbf{t}_{\mathrm{S}} \in \overline{\mathbb{L}}_0^2(\Omega_{\mathrm{S}}), \quad \mathbf{u}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}}), \quad \boldsymbol{\varphi} := -\mathbf{u}_{\mathrm{S}}|_{\Sigma} \in \mathbf{H}_{00}^{1/2}(\Sigma), \quad \text{and} \quad \gamma_{\mathrm{S}} \in \underline{\mathbb{L}}^2(\Omega_{\mathrm{S}}), \tag{2.10}$$

where

$$\overline{\mathbb{L}}_0^2(\Omega_{\mathrm{S}}) = \left\{ \mathbf{r}_{\mathrm{S}} \in \overline{\mathbb{L}}^2(\Omega_{\mathrm{S}}) : \quad \operatorname{tr} \mathbf{r}_{\mathrm{S}} = 0 \right\}.$$

Next, multiplying the second and third equations of (2.8) by $\mathbf{r}_{\mathrm{S}} \in \overline{\mathbb{L}}_0^2(\Omega_{\mathrm{S}})$ and $\mathbf{v}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$, respectively, and imposing the symmetry of $\boldsymbol{\sigma}_{\mathrm{S}}$ in a weak sense, we obtain

$$(\mu(|\mathbf{t}_{\mathrm{S}}|)\, \mathbf{t}_{\mathrm{S}}, \mathbf{r}_{\mathrm{S}})_{\mathrm{S}} - (\mathbf{r}_{\mathrm{S}}, \boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} = 0 \qquad \forall\, \mathbf{r}_{\mathrm{S}} \in \overline{\mathbb{L}}_0^2(\Omega_{\mathrm{S}}) \tag{2.11}$$

$$(\mathbf{div}\, \boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} = -(\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \qquad \forall\, \mathbf{v}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}}), \tag{2.12}$$

and

$$(\boldsymbol{\sigma}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} = 0 \qquad \forall\, \boldsymbol{\eta}_{\mathrm{S}} \in \underline{\mathbb{L}}^2(\Omega_{\mathrm{S}}), \tag{2.13}$$

where the unknown $\boldsymbol{\sigma}_{\mathrm{S}}$ is sought in $\mathbb{H}(\mathbf{div}; \Omega_{\mathrm{S}})$. Note that the decomposition $\overline{\mathbb{L}}^2(\Omega_{\mathrm{S}}) = \overline{\mathbb{L}}_0^2(\Omega_{\mathrm{S}}) \oplus \mathrm{R}\, \mathbb{I}$ and the fact that both $\mathbf{t}_{\mathrm{S}}$ and $\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}}$ belong to $\overline{\mathbb{L}}_0^2(\Omega_{\mathrm{S}})$ guarantee that (2.11) is equivalent to requiring it for all $\mathbf{r}_{\mathrm{S}} \in \overline{\mathbb{L}}^2(\Omega_{\mathrm{S}})$.

On the other hand, we now consider the first equation of (2.2) in the form $\mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D}} = -\nabla p_{\mathrm{D}}$ in Ω_{D}, and, as suggested by the Neumann boundary condition on Γ_{D}, introduce the space

$$\mathbf{H}_0(\operatorname{div}; \Omega_{\mathrm{D}}) := \left\{ \mathbf{v}_{\mathrm{D}} \in \mathbf{H}(\operatorname{div}; \Omega_{\mathrm{D}}) : \quad \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}} \right\}.$$

Then, multiplying by $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_0(\operatorname{div}; \Omega_{\mathrm{D}})$, integrating by parts the expression $(\nabla p_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}}$, and recalling that the normal vector $\mathbf{n}$ on Σ points inwards Ω_{D}, we get

$$(\mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} - (\operatorname{div} \mathbf{v}_{\mathrm{D}}, p_{\mathrm{D}})_{\mathrm{D}} - \langle \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, \lambda \rangle_{\Sigma} = 0 \quad \forall\, \mathbf{v}_{\mathrm{D}} \in \mathbf{H}_0(\operatorname{div}; \Omega_{\mathrm{D}}), \tag{2.14}$$

with unknowns

$$\mathbf{u}_{\mathrm{D}} \in \mathbf{H}_0(\operatorname{div}; \Omega_{\mathrm{D}}), \quad p_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}}), \quad \text{and} \quad \lambda := p_{\mathrm{D}}|_{\Sigma} \in H^{1/2}(\Sigma). \tag{2.15}$$

In turn, multiplying by $q_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}})$ the second equation of (2.2) and integrating on Ω_{D}, we obtain

$$(\operatorname{div} \mathbf{u}_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}} = (f_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}} \quad \forall\, q_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}}). \tag{2.16}$$

Finally, since the transmission conditions given by (2.3) become essential (which follows from the fact that dual-mixed approaches are employed in both domains), we impose them weakly and obtain the equations

$$\begin{aligned} -\langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_{\Sigma} - \langle \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n}, \xi \rangle_{\Sigma} &= 0 \quad \forall\, \xi \in H^{1/2}(\Sigma), \\ \langle \boldsymbol{\sigma}_{\mathrm{S}}\, \mathbf{n}, \boldsymbol{\psi} \rangle_{\Sigma} - \sum_{\ell=1}^{n-1} \kappa_\ell^{-1} \langle \boldsymbol{\varphi} \cdot \mathbf{t}_\ell, \boldsymbol{\psi} \cdot \mathbf{t}_\ell \rangle_{\Sigma} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_{\Sigma} &= 0 \quad \forall\, \boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma), \end{aligned} \tag{2.17}$$

where we have replaced $\mathbf{u}_{\rm S}|_\Sigma$ and $p_{\rm D}|_\Sigma$ by $-\boldsymbol{\varphi}$ and λ, respectively. At this point we remark that, in principle, the spaces for the unknowns $\mathbf{u}_{\rm S}$ and $p_{\rm D}$ (cf. (2.10) and (2.15)) do not allow enough regularity for the pair of traces $(\boldsymbol{\varphi},\lambda)$ to live in $\mathbf{H}^{1/2}_{00}(\Sigma)\times H^{1/2}(\Sigma)$. However, it is easy to see from (2.8) and (2.2) that $\mathbf{u}_{\rm S}$ and $p_{\rm D}$ belong to $\mathbf{H}^1(\Omega_{\rm S})$ and $H^1(\Omega_{\rm D})$, respectively, which confirms the indicated space for $(\boldsymbol{\varphi},\lambda)$.

According to the whole above analysis, we find that our resulting weak formulation reduces to a nonlinear system of eight unknowns, namely

$$\begin{gathered}
\mathbf{t}_{\rm S}\,\in\,\mathbb{L}^2_0(\Omega_{\rm S})\,,\quad \mathbf{u}_{\rm S}\,\in\,\mathbf{L}^2(\Omega_{\rm S})\,,\quad \boldsymbol{\varphi}\,\in\,\mathbf{H}^{1/2}_{00}(\Sigma)\,,\quad \boldsymbol{\gamma}_{\rm S}\,\in\,\mathbb{L}^2(\Omega_{\rm S})\,,\\
\boldsymbol{\sigma}_{\rm S}\,\in\,\mathbb{H}(\mathbf{div};\Omega_{\rm S})\,,\quad \mathbf{u}_{\rm D}\,\in\,\mathbf{H}_0(\mathrm{div};\Omega_{\rm D})\,,\quad p_{\rm D}\,\in\,L^2(\Omega_{\rm D})\,,\quad\text{and}\quad \lambda\,\in\,H^{1/2}(\Sigma)\,,
\end{gathered}\tag{2.18}$$

and the eight equations given by (2.9), (2.11), (2.12), (2.13), (2.14), (2.16), and (2.17). However, it is not difficult to show that this system is not uniquely solvable since, given any solution $(\mathbf{t}_{\rm S},\mathbf{u}_{\rm S},\boldsymbol{\varphi},\boldsymbol{\gamma}_{\rm S},\boldsymbol{\sigma}_{\rm S},\mathbf{u}_{\rm D},p_{\rm D},\lambda)$ in the indicated spaces, and given any constant $c\in\mathrm{R}$, the vector defined by $(\mathbf{t}_{\rm S},\mathbf{u}_{\rm S},\boldsymbol{\varphi},\boldsymbol{\gamma}_{\rm S},\boldsymbol{\sigma}_{\rm S}\,-\,c\,\mathbb{I},\mathbf{u}_{\rm D},p_{\rm D}\,+\,c,\lambda\,+\,c)$ also becomes a solution. In order to avoid this non-uniqueness from now on we require that the Darcy pressure $p_{\rm D}$ belongs to $L^2_0(\Omega_{\rm D})$, where

$$L^2_0(\Omega_{\rm D})\,:=\,\left\{q_{\rm D}\in L^2(\Omega_{\rm D}):\quad\int_{\Omega_{\rm D}}q_{\rm D}\,=\,0\right\}.$$

Note that the decomposition $L^2(\Omega_{\rm D})=L^2_0(\Omega_{\rm D})\oplus\mathrm{R}$, the boundary conditions $\mathbf{u}_{\rm D}\cdot\mathbf{n}=0$ on $\Gamma_{\rm D}$ and $\mathbf{u}_{\rm S}=\mathbf{0}$ on $\Gamma_{\rm S}$, the first transmission condition in (2.3), and the fact that $\int_{\Omega_{\rm D}}f_{\rm D}=0$, guarantee that (2.16) is equivalent to requiring it for all $q_{\rm D}\in L^2_0(\Omega_{\rm D})$.

Now, it is quite clear that there are many different ways of ordering the equations forming the resulting nonlinear system. Throughout the rest of the paper, and for convenience of the analysis, we adopt one leading to a twofold saddle point structure. More precisely, by considering subsequently (2.11), (2.14), (2.9), (2.17), (2.16), (2.12), and (2.13), and denoting throughout the rest of the paper

$$\langle\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma}\,:=\,\sum_{\ell=1}^{n-1}\kappa_\ell^{-1}\,\langle\boldsymbol{\varphi}\cdot\mathbf{t}_\ell,\boldsymbol{\psi}\cdot\mathbf{t}_\ell\rangle_\Sigma\,,\tag{2.19}$$

we arrive at the matrix operator represented as follows, where the unknowns and corresponding test functions are displayed along columns and rows, respectively,

$$\left(\begin{array}{c|ccc|c|ccc}
 & \mathbf{t}_{\rm S} & \mathbf{u}_{\rm D} & \boldsymbol{\sigma}_{\rm S} & (\boldsymbol{\varphi},\lambda) & p_{\rm D} & \mathbf{u}_{\rm S} & \boldsymbol{\gamma}_{\rm S}\\ \hline
\mathbf{r}_{\rm S} & (\mu(|\mathbf{t}_{\rm S}|)\,\mathbf{t}_{\rm S},\mathbf{r}_{\rm S})_{\rm S} & & -\,(\mathbf{r}_{\rm S},\boldsymbol{\sigma}^{\tt d}_{\rm S})_{\rm S} & & & & \\
\mathbf{v}_{\rm D} & & (\mathbf{K}^{-1}\mathbf{u}_{\rm D},\mathbf{v}_{\rm D})_{\rm D} & & -\langle\mathbf{v}_{\rm D}\cdot\mathbf{n},\lambda\rangle_\Sigma & -\,(\mathrm{div}\,\mathbf{v}_{\rm D},p_{\rm D})_{\rm D} & & \\
\boldsymbol{\tau}_{\rm S} & (\mathbf{t}_{\rm S},\boldsymbol{\tau}^{\tt d}_{\rm S})_{\rm S} & & & \langle\boldsymbol{\tau}_{\rm S}\,\mathbf{n},\boldsymbol{\varphi}\rangle_\Sigma & & (\mathbf{div}\,\boldsymbol{\tau}_{\rm S},\mathbf{u}_{\rm S})_{\rm S} & (\boldsymbol{\tau}_{\rm S},\boldsymbol{\gamma}_{\rm S})_{\rm S}\\ \hline
(\boldsymbol{\psi},\xi) & & -\langle\mathbf{u}_{\rm D}\cdot\mathbf{n},\xi\rangle_\Sigma & \langle\boldsymbol{\sigma}_{\rm S}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma & \begin{array}{c}-\,\langle\boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_\Sigma\\ +\,\langle\boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_\Sigma\\ -\,\langle\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma}\end{array} & & & \\ \hline
q_{\rm D} & & -\,(\mathrm{div}\,\mathbf{u}_{\rm D},q_{\rm D})_{\rm D} & & & & & \\
\mathbf{v}_{\rm S} & & & (\mathbf{div}\,\boldsymbol{\sigma}_{\rm S},\mathbf{v}_{\rm S})_{\rm S} & & & & \\
\boldsymbol{\eta}_{\rm S} & & & (\boldsymbol{\sigma}_{\rm S},\boldsymbol{\eta}_{\rm S})_{\rm S} & & & &
\end{array}\right).$$

Furthermore, in order to facilitate the forthcoming analysis, and particularly, to be able to apply a generalization of the Babuška-Brezzi theory for twofold saddle point problems (see Section 3 below), we enrich the above formulation by adding the consistent equation given by

$$\rho\,(\boldsymbol{\sigma}^{\tt d}_{\rm S}\,-\,\mu(|\mathbf{t}_{\rm S}|)\,\mathbf{t}_{\rm S},\boldsymbol{\tau}^{\tt d}_{\rm S})_{\rm S}\,=\,0\qquad\forall\,\boldsymbol{\tau}_{\rm S}\,\in\,\mathbb{H}(\mathbf{div};\Omega_{\rm S})\,,\tag{2.20}$$

where ρ is a positive stabilization parameter to be choosen later. Note that (2.20), which is included from now on into the left-upper block, arises from the quasi-Newtonian constitutive law given by the second equation of (2.8). Additionally, we consider the decomposition

$$\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \,=\, \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \,\oplus\, \mathrm{R}\,\mathbb{I}\,, \tag{2.21}$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \,:=\, \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) : \quad \int_{\Omega_{\mathrm{S}}} \mathrm{tr}\,(\boldsymbol{\tau}) = 0 \right\}\,, \tag{2.22}$$

and redefine $\boldsymbol{\sigma}_{\mathrm{S}}$ and $\boldsymbol{\tau}_{\mathrm{S}}$ as $\boldsymbol{\sigma}_{\mathrm{S}} \,+\, \ell\,\mathbb{I}$ and $\boldsymbol{\tau}_{\mathrm{S}} \,+\, \jmath\,\mathbb{I}$, respectively, with

$$\boldsymbol{\sigma}_{\mathrm{S}},\, \boldsymbol{\tau}_{\mathrm{S}} \,\in\, \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \quad \text{and} \quad \ell,\, \jmath \,\in\, \mathrm{R}\,. \tag{2.23}$$

Consequently, denoting $\widetilde{\mu}(|\mathbf{t}_{\mathrm{S}}|) \,:=\, 1 - \rho\mu(|\mathbf{t}_{\mathrm{S}}|)$, the matrix operator of our variational formulation is represented now by

$$\left(\begin{array}{ccc|c|cccc}
(\mu(|\mathbf{t}_{\mathrm{S}}|)\,\mathbf{t}_{\mathrm{S}},\mathbf{r}_{\mathrm{S}})_{\mathrm{S}} & & -\,(\mathbf{r}_{\mathrm{S}},\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} & & & & & \\
 & (\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} & & -\langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\lambda\rangle_\Sigma & -\,(\mathrm{div}\,\mathbf{v}_{\mathrm{D}},p_{\mathrm{D}})_{\mathrm{D}} & & & \\
(\widetilde{\mu}(|\mathbf{t}_{\mathrm{S}}|)\,\mathbf{t}_{\mathrm{S}},\boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} & & \rho(\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}},\boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} & \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n},\boldsymbol{\varphi}\rangle_\Sigma & & (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}})_{\mathrm{S}} & (\boldsymbol{\tau}_{\mathrm{S}},\boldsymbol{\gamma}_{\mathrm{S}})_{\mathrm{S}} & \\
\hline
 & & & -\,\langle \boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_\Sigma & & & & \\
 & -\langle \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_\Sigma & \langle \boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma & +\,\langle \boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_\Sigma & & & & \ell\langle \mathbf{n},\boldsymbol{\psi}\rangle_\Sigma \\
 & & & -\,\langle \boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} & & & & \\
\hline
 & -\,(\mathrm{div}\,\mathbf{u}_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}} & & & & & & \\
 & & (\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} & & & & & \\
 & & (\boldsymbol{\sigma}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} & & & & & \\
 & & & \jmath\langle \mathbf{n},\boldsymbol{\varphi}\rangle_\Sigma & & & &
\end{array}\right)$$

with unknowns

$$\begin{gathered}
\mathbf{t}_{\mathrm{S}} \,\in\, \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}})\,, \quad \mathbf{u}_{\mathrm{D}} \,\in\, \mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{D}})\,, \quad \boldsymbol{\sigma}_{\mathrm{S}} \,\in\, \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})\,, \quad (\boldsymbol{\varphi},\lambda) \,\in\, \mathbf{H}^{1/2}_{00}(\Sigma)\times H^{1/2}(\Sigma)\,, \\
p_{\mathrm{D}} \,\in\, L^2_0(\Omega_{\mathrm{D}})\,, \quad \mathbf{u}_{\mathrm{S}} \,\in\, \mathbf{L}^2(\Omega_{\mathrm{S}})\,, \quad \boldsymbol{\gamma}_{\mathrm{S}} \,\in\, \underline{\mathbb{L}}^2(\Omega_{\mathrm{S}})\,, \quad \text{and} \quad \ell \in \mathrm{R}\,,
\end{gathered} \tag{2.24}$$

and corresponding test functions

$$\begin{gathered}
\mathbf{r}_{\mathrm{S}} \,\in\, \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}})\,, \quad \mathbf{v}_{\mathrm{D}} \,\in\, \mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{D}})\,, \quad \boldsymbol{\tau}_{\mathrm{S}} \,\in\, \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})\,, \quad (\boldsymbol{\psi},\xi) \,\in\, \mathbf{H}^{1/2}_{00}(\Sigma)\times H^{1/2}(\Sigma)\,, \\
q_{\mathrm{D}} \,\in\, L^2_0(\Omega_{\mathrm{D}})\,, \quad \mathbf{v}_{\mathrm{S}} \,\in\, \mathbf{L}^2(\Omega_{\mathrm{S}})\,, \quad \boldsymbol{\eta}_{\mathrm{S}} \,\in\, \underline{\mathbb{L}}^2(\Omega_{\mathrm{S}})\,, \quad \text{and} \quad \jmath \in \mathrm{R}\,.
\end{gathered} \tag{2.25}$$

The above structure suggests the introduction of the spaces

$$\begin{gathered}
\mathbf{X}_1 \,:=\, \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}}) \times \mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})\,, \quad \mathbf{M}_1 \,:=\, \mathbf{H}^{1/2}_{00}(\Sigma)\times H^{1/2}(\Sigma)\,, \\
\mathbb{M} \,:=\, L^2_0(\Omega_{\mathrm{D}}) \times \mathbf{L}^2(\Omega_{\mathrm{S}}) \times \underline{\mathbb{L}}^2(\Omega_{\mathrm{S}}) \times \mathrm{R}\,, \quad \text{and} \quad \mathbb{X} \,:=\, \mathbf{X}_1 \times \mathbf{M}_1\,,
\end{gathered} \tag{2.26}$$

endowed with the associated product norms, and the operators $\mathbf{A}_1 : \mathbf{X}_1 \to \mathbf{X}_1'$, $\mathbf{B}_1 : \mathbf{X}_1 \to \mathbf{M}_1'$, $\mathbf{S} : \mathbf{M}_1 \to \mathbf{M}_1'$, $\mathbb{A} : \mathbb{X} \to \mathbb{X}'$, and $\mathbb{B} : \mathbb{X} \to \mathbb{M}'$, given, respectively, by

$$\begin{gathered}
[\mathbf{A}_1(\underline{\mathbf{t}}),\underline{\mathbf{r}}] \,:=\, (\mu(|\mathbf{t}_{\mathrm{S}}|)\,\mathbf{t}_{\mathrm{S}},\mathbf{r}_{\mathrm{S}})_{\mathrm{S}} \,+\, (\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \,-\, (\mathbf{r}_{\mathrm{S}},\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} \\
+\, (\mathbf{t}_{\mathrm{S}},\boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} \,+\, \rho\,(\boldsymbol{\sigma}_{\mathrm{S}}^{\mathtt{d}} - \mu(|\mathbf{t}_{\mathrm{S}}|)\,\mathbf{t}_{\mathrm{S}},\boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}}\,,
\end{gathered} \tag{2.27}$$

$$[\mathbf{B}_1(\underline{\mathbf{r}}),\underline{\boldsymbol{\psi}}] \,:=\, -\,\langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_\Sigma \,+\, \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma\,, \tag{2.28}$$

$$[\mathbf{S}(\underline{\boldsymbol{\varphi}}),\underline{\boldsymbol{\psi}}] \,:=\, \langle \boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_\Sigma \,-\, \langle \boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_\Sigma \,+\, \langle \boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma}\,, \tag{2.29}$$

$$[\mathbb{A}(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] \,:=\, [\mathbf{A}_1(\underline{\mathbf{t}}),\underline{\mathbf{r}}] \,+\, [\mathbf{B}_1(\underline{\mathbf{t}}),\underline{\boldsymbol{\psi}}] \,+\, [\mathbf{B}_1(\underline{\mathbf{r}}),\underline{\boldsymbol{\varphi}}] \,-\, [\mathbf{S}(\underline{\boldsymbol{\varphi}}),\underline{\boldsymbol{\psi}}]\,, \tag{2.30}$$

and

$$[\mathbb{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{q}] := -(\operatorname{div}\mathbf{v}_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}} + (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + (\boldsymbol{\tau}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} + \jmath\,\langle\mathbf{n},\boldsymbol{\psi}\rangle_{\Sigma}\,, \tag{2.31}$$

for all $\underline{\mathbf{t}} := (\mathbf{t}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}},\boldsymbol{\sigma}_{\mathrm{S}}) \in \mathbf{X}_1$, $\underline{\mathbf{r}} := (\mathbf{r}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}},\boldsymbol{\tau}_{\mathrm{S}}) \in \mathbf{X}_1$, $\underline{\boldsymbol{\varphi}} := (\boldsymbol{\varphi},\lambda) \in \mathbf{M}_1$, $\underline{\boldsymbol{\psi}} := (\boldsymbol{\psi},\xi) \in \mathbf{M}_1$, and $\underline{q} := (q_{\mathrm{D}},\mathbf{v}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}},\jmath) \in \mathbb{M}$, where $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators. In addition, we let $\mathbf{B}_1' : \mathbf{M}_1 \to \mathbf{X}_1'$ and $\mathbb{B}' : \mathbb{M} \to \mathbb{X}'$ be the adjoints of $\mathbf{B}_1$ and $\mathbb{B}$, respectively, which satisfy $[\mathbf{B}_1'(\underline{\boldsymbol{\psi}}),\underline{\mathbf{r}}] = [\mathbf{B}_1(\underline{\mathbf{r}}),\underline{\boldsymbol{\psi}}]$ and $[\mathbb{B}'(\underline{q}),(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] = [\mathbb{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{q}]$ for all $\underline{\mathbf{r}} \in \mathbf{X}_1$, $\underline{\boldsymbol{\psi}} \in \mathbf{M}_1$, and $\underline{q} \in \mathbb{M}$. Then, it is clear that $\mathbb{A}$ can also be defined as the matrix operator

$$\mathbb{A}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}) := \begin{bmatrix}\mathbf{A}_1 & \mathbf{B}_1' \\ \mathbf{B}_1 & -\mathbf{S}\end{bmatrix}\begin{bmatrix}\underline{\mathbf{r}} \\ \underline{\boldsymbol{\psi}}\end{bmatrix} \in \mathbb{X}' \qquad \forall\,(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}) \in \mathbb{X}\,. \tag{2.32}$$

Next, we let $\mathbb{F} \in \mathbb{X}'$ and $\mathbb{G} \in \mathbb{M}'$ be the functionals defined by

$$[\mathbb{F},(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] := 0 \qquad \forall\,(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}) \in \mathbb{X} \quad \text{and} \quad [\mathbb{G},\underline{q}] := -(\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} - (f_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}} \qquad \forall\,\underline{q} \in \mathbb{M}\,, \tag{2.33}$$

and observe that, denoting $\underline{p} := (p_{\mathrm{D}},\mathbf{u}_{\mathrm{S}},\boldsymbol{\gamma}_{\mathrm{S}},\ell) \in \mathbb{M}$, our augmented fully-mixed variational formulation reduces to the twofold saddle point operator equation: Find $((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{p}) \in \mathbb{X}\times\mathbb{M}$ such that

$$\begin{aligned}
[\mathbb{A}(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] + [\mathbb{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{p}] &= [\mathbb{F},(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] \qquad \forall\,(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}) \in \mathbb{X}\,,\\
[\mathbb{B}(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{q}] &= [\mathbb{G},\underline{q}] \qquad \forall\,\underline{q} \in \mathbb{M}\,,
\end{aligned} \tag{2.34}$$

or, equivalently, such that

$$\begin{bmatrix}\mathbb{A} & \mathbb{B}' \\ \mathbb{B} & \mathbf{O}\end{bmatrix}\begin{bmatrix}(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}) \\ \underline{p}\end{bmatrix} = \begin{bmatrix}\mathbb{F} \\ \mathbb{G}\end{bmatrix}. \tag{2.35}$$

Certainly, (2.32) and (2.35) explain here the use of the "twofold saddle point" concept.

In the following section we adapt the approach from [17, Sections 2 and 3] to derive the necessary abstract theory for analyzing the kind of variational problems characterized by (2.35) and (2.32).

3 A modified abstract theory for twofold saddle point problems

3.1 The continuous setting

Let X_1, M_1 and M be Hilbert spaces, set $X := X_1 \times M_1$, and denote their duals by X_1', M_1', M', and $X' := X_1' \times M_1'$, respectively. Next, given a nonlinear operator $A_1 : X_1 \to X_1'$, and linear bounded operators $S : M_1 \to M_1'$, $B_1 : X_1 \to M_1'$, and $B : X \to M'$, we let $B_1' : M_1 \to X_1'$ and $B' : M \to X'$ be the corresponding adjoints, and define the nonlinear operator $A : X \to X'$ as follows

$$A(\mathbf{r},\boldsymbol{\psi}) := \begin{bmatrix}A_1 & B_1' \\ B_1 & -S\end{bmatrix}\begin{bmatrix}\mathbf{r} \\ \boldsymbol{\psi}\end{bmatrix} \in X' := X_1' \times M_1' \quad \forall\,(\mathbf{r},\boldsymbol{\psi}) \in X\,. \tag{3.1}$$

Then we are interested in the following nonlinear variational problem: Given $(F,G) \in X' \times M'$, find $((\mathbf{t},\boldsymbol{\varphi}),p) \in X \times M$ such that

$$\begin{bmatrix}A & B' \\ B & \mathbf{O}\end{bmatrix}\begin{bmatrix}(\mathbf{t},\boldsymbol{\varphi}) \\ p\end{bmatrix} = \begin{bmatrix}F \\ G\end{bmatrix} \tag{3.2}$$

or, equivalently, such that

$$\begin{aligned}
[A(\mathbf{t},\boldsymbol{\varphi}),(\mathbf{r},\boldsymbol{\psi})] + [B'(p),(\mathbf{r},\boldsymbol{\psi})] &= [F,(\mathbf{r},\boldsymbol{\psi})] \qquad \forall\,(\mathbf{r},\boldsymbol{\psi}) \in X\,,\\
[B(\mathbf{t},\boldsymbol{\varphi}),q] &= [G,q] \qquad \forall\,q \in M\,.
\end{aligned} \tag{3.3}$$

In order to prove the main theorem for the solvability of the continuous formulation (3.2), we need to recall the following auxiliary result from [17].

Lemma 3.1 *Let $\widehat{X}_1$ and $\widehat{M_1}$ be Hilbert spaces, and let $\widehat{A}_1 : \widehat{X}_1 \to \widehat{X}_1'$ be a nonlinear operator. In addition, let $\widehat{B}_1 : \widehat{X}_1 \to \widehat{M_1}'$ and $\widehat{S} : \widehat{M_1} \to \widehat{M_1}'$ be linear and bounded operators, and let $\widehat{B}_1' : \widehat{M_1} \to \widehat{X}_1'$ be the adjoint of $\widehat{B}_1$. Assume that*

i) *$\widehat{A}_1 : \widehat{X}_1 \to \widehat{X}_1'$ is Lipschitz continuous and strongly monotone, that is, there exist constants $\widehat{\gamma}$, $\widehat{\alpha} > 0$ such that*

$$\|\widehat{A}_1(\mathbf{s}) \,-\, \widehat{A}_1(\mathbf{r})\|_{\widehat{X}_1'} \,\leq\, \widehat{\gamma}\,\|\mathbf{s}-\mathbf{r}\|_{\widehat{X}_1} \quad \forall\,\mathbf{s},\,\mathbf{r} \in \widehat{X}_1$$

and

$$[\widehat{A}_1(\mathbf{s}) \,-\, \widehat{A}_1(\mathbf{r}), \mathbf{s}-\mathbf{r}] \,\geq\, \widehat{\alpha}\,\|\mathbf{s}-\mathbf{r}\|_{\widehat{X}_1}^2 \quad \forall\,\mathbf{s},\,\mathbf{r} \in \widehat{X}_1\,.$$

ii) *$\widehat{S}$ is positive semi-definite on $\widehat{M_1}$, that is,*

$$[\widehat{S}(\boldsymbol{\psi}), \boldsymbol{\psi}] \,\geq\, 0 \quad \forall\,\boldsymbol{\psi} \in \widehat{M_1}\,.$$

iii) *$\widehat{B}_1$ satisfies an inf-sup condition on $\widehat{X}_1 \times \widehat{M_1}$, that is, there exists $\widehat{\beta} > 0$ such that*

$$\sup_{\substack{\mathbf{r}\in\widehat{X}_1\\ \mathbf{r}\neq\mathbf{0}}} \frac{[\widehat{B}_1(\mathbf{r}), \boldsymbol{\psi}]}{\|\mathbf{r}\|_{\widehat{X}_1}} \,\geq\, \widehat{\beta}\,\|\boldsymbol{\psi}\|_{\widehat{M_1}} \quad \forall\,\boldsymbol{\psi} \in \widehat{M_1}\,.$$

Then, given $(\widehat{F}, \widehat{G}) \in \widehat{X}_1' \times \widehat{M_1}'$, there exists a unique $(\mathbf{t}, \boldsymbol{\varphi}) \in \widehat{X}_1 \times \widehat{M_1}$ such that

$$\begin{bmatrix} \widehat{A}_1 & \widehat{B}_1' \\ \widehat{B}_1 & -\widehat{S} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \boldsymbol{\varphi} \end{bmatrix} = \begin{bmatrix} \widehat{F} \\ \widehat{G} \end{bmatrix}\,.$$

In addition, there exists $\widehat{C} > 0$, depending only on $\widehat{\gamma}$, $\widehat{\alpha}$, $\widehat{\beta}$ and $\|\widehat{B}_1\|$, such that

$$\|(\mathbf{t}, \boldsymbol{\varphi})\|_{\widehat{X}_1\times\widehat{M_1}} \,\leq\, \widehat{C}\left\{\, \|\widehat{F}\|_{\widehat{X}_1'} \,+\, \|\widehat{G}\|_{\widehat{M_1}'} \,+\, \|\widehat{A}_1(\mathbf{0})\|_{\widehat{X}_1'} \,\right\}. \tag{3.4}$$

Proof. See [17, Lemma 2.1]. ∎

We now go back to the analysis of problem (3.2). To this end, we let V be the kernel of B, that is

$$V \,:=\, \Big\{ (\mathbf{r}, \boldsymbol{\psi}) \in X : \quad [B(\mathbf{r}, \boldsymbol{\psi}), q] \,=\, 0 \quad \forall\, q \in M \Big\}\,,$$

and denote by $\widetilde{X}_1$ and $\widetilde{M_1}$ the subspaces of X_1 and M_1, respectively, such that $V = \widetilde{X}_1 \times \widetilde{M_1}$. Note that the boundedness of B implies that both $\widetilde{X}_1$ and $\widetilde{M_1}$ are closed. Then, the following theorem provides sufficient conditions for the well-posedness of (3.2).

Theorem 3.1 *Assume that*

i) *$A_1|_{\widetilde{X}_1} : \widetilde{X}_1 \to \widetilde{X}_1'$ is Lipschitz continuous and strongly monotone, that is, there exist constants γ, $\alpha > 0$ such that*

$$\|A_1(\mathbf{s}) \,-\, A_1(\mathbf{r})\|_{\widetilde{X}_1'} \,\leq\, \gamma\,\|\mathbf{s}-\mathbf{r}\|_{X_1} \quad \forall\,\mathbf{s},\,\mathbf{r} \in \widetilde{X}_1$$

and

$$[A_1(\mathbf{s}) \,-\, A_1(\mathbf{r}), \mathbf{s}-\mathbf{r}] \,\geq\, \alpha\,\|\mathbf{s}-\mathbf{r}\|_{X_1}^2 \quad \forall\,\mathbf{s},\,\mathbf{r} \in \widetilde{X}_1\,.$$

ii) *For each pair* $(\mathbf{r}, \mathbf{r}^\perp) \in \widetilde{X}_1 \times \widetilde{X}_1^\perp$ *there holds the pseudolinear property*

$$A_1(\mathbf{r} + \mathbf{r}^\perp) = A_1(\mathbf{r}) + A_1(\mathbf{r}^\perp). \tag{3.5}$$

iii) S *is positive semi-definite on* $\widetilde{M}_1$*, that is,*

$$[S(\boldsymbol{\psi}), \boldsymbol{\psi}] \geq 0 \quad \forall\, \boldsymbol{\psi} \in \widetilde{M}_1 .$$

iv) B_1 *satisfies an inf-sup condition on* $\widetilde{X}_1 \times \widetilde{M}_1$*, that is, there exists* $\beta_1 > 0$ *such that*

$$\sup_{\substack{\mathbf{r} \in \widetilde{X}_1 \\ \mathbf{r} \neq \mathbf{0}}} \frac{[B_1(\mathbf{r}), \boldsymbol{\psi}]}{\|\mathbf{r}\|_{X_1}} \geq \beta_1 \|\boldsymbol{\psi}\|_{M_1} \quad \forall\, \boldsymbol{\psi} \in \widetilde{M}_1 .$$

v) B *satisfies an inf-sup condition on* $X \times M$*, that is, there exists* $\beta > 0$ *such that*

$$\sup_{\substack{(\mathbf{r},\boldsymbol{\psi}) \in X \\ (\mathbf{r},\boldsymbol{\psi}) \neq \mathbf{0}}} \frac{[B(\mathbf{r}, \boldsymbol{\psi}), q]}{\|(\mathbf{r}, \boldsymbol{\psi})\|_X} \geq \beta \|q\|_M \quad \forall\, q \in M .$$

Then, there exists a unique $((\mathbf{t}, \boldsymbol{\varphi}), p) \in X \times M$ *solution of* (3.2)*. Moreover, there exists* $C > 0$*, depending only on* α*,* γ*,* β_1*,* β*,* $\|S\|$*, and* $\|B_1\|$ *such that*

$$\|((\mathbf{t}, \boldsymbol{\varphi}), p)\|_{X \times M} \leq C \Big\{ \|F\|_{X'} + \|G\|_{M'} \Big\}. \tag{3.6}$$

Proof. We adapt the proof of [17, Theorem 2.1] to the present situation. We begin by recalling from [28, Chapter I, Lemma 4.1] that the inf-sup condition satisfied by B (cf. v)) implies that $B : V^\perp \to M'$ and $B' : M \to V^o$ are isomorphisms and that

$$\|B^{-1}\|, \|(B')^{-1}\| \leq \frac{1}{\beta}. \tag{3.7}$$

As usual, V^o stands here for the set of functionals in X' that vanish on V. Hence, we now let $(\mathbf{t}^\perp, \boldsymbol{\varphi}^\perp) := B^{-1}(G) \in V^\perp$, and observe, thanks to (3.7), that

$$\|(\mathbf{t}^\perp, \boldsymbol{\varphi}^\perp)\|_X \leq \frac{1}{\beta} \|G\|_{M'} . \tag{3.8}$$

Next, we let $F_1 \in X_1'$ and $G_1 \in M_1'$ be such that $F = (F_1, G_1)$, and introduce the functionals

$$\widehat{F}_1 := F_1 - A_1(\mathbf{t}^\perp) - B_1'(\boldsymbol{\varphi}^\perp) \in X_1' \quad \text{and} \quad \widehat{G}_1 := G_1 - B_1(\mathbf{t}^\perp) + S(\boldsymbol{\varphi}^\perp) \in M_1' . \tag{3.9}$$

Then, having in mind hypotheses i), iii), and iv), a straightforward application of Lemma 3.1 yields the existence of a unique $(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\varphi}}) \in V := \widetilde{X}_1 \times \widetilde{M}_1$ such that

$$\begin{aligned} [A_1(\widetilde{\mathbf{t}}), \mathbf{r}] + [B_1'(\widetilde{\boldsymbol{\varphi}}), \mathbf{r}] &= [\widehat{F}_1, \mathbf{r}] \qquad \forall\, \mathbf{r} \in \widetilde{X}_1 , \\ [B_1(\widetilde{\mathbf{t}}), \boldsymbol{\psi}] - [S(\widetilde{\boldsymbol{\varphi}}), \boldsymbol{\psi}] &= [\widehat{G}_1, \boldsymbol{\psi}] \qquad \forall\, \boldsymbol{\psi} \in \widetilde{M}_1 , \end{aligned} \tag{3.10}$$

and there exists $\widetilde{C} > 0$, depending only on γ, α, β_1 and $\|B_1\|$, such that

$$\|(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\varphi}})\|_{X_1 \times M_1} \leq \widetilde{C} \Big\{ \|\widehat{F}_1\|_{\widetilde{X}_1'} + \|\widehat{G}_1\|_{\widetilde{M}_1'} \Big\}. \tag{3.11}$$

Note that we have also used here, which is a consequence of ii), that $A_1(\mathbf{0}) = \mathbf{0}$. It follows from (3.10) that the pair of functionals $(\widehat{F}_1 - A_1(\widetilde{\mathbf{t}}) - B_1'(\widetilde{\boldsymbol{\varphi}}), \widehat{G}_1 - B_1(\widetilde{\mathbf{t}}) + S(\widetilde{\boldsymbol{\varphi}}))$ belongs to $\widetilde{X}_1^o \times \widetilde{M}_1^o =: V^o$, and hence, according to the above mentioned property of B' and the bound (3.7) again, there exists a unique $p \in M$ such that

$$B'(p) \,=\, (\widehat{F}_1 - A_1(\widetilde{\mathbf{t}}) - B_1'(\widetilde{\boldsymbol{\varphi}}), \widehat{G}_1 - B_1(\widetilde{\mathbf{t}}) + S(\widetilde{\boldsymbol{\varphi}}))\,, \tag{3.12}$$

and

$$\|p\|_M \,\leq\, \frac{1}{\beta}\left\{ \|\widehat{F}_1 - A_1(\widetilde{\mathbf{t}}) - B_1'(\widetilde{\boldsymbol{\varphi}})\|_{X_1'} \,+\, \|\widehat{G}_1 - B_1(\widetilde{\mathbf{t}}) + S(\widetilde{\boldsymbol{\varphi}})\|_{M_1'} \right\}. \tag{3.13}$$

Next, replacing $\widehat{F}_1$ and $\widehat{G}_1$ from (3.9) into (3.12), and using the pseudolinear property (3.5) and the linearity of B_1', B_1, and S, we find that

$$B'(p) \,=\, (F_1 - A_1(\mathbf{t}^\perp + \widetilde{\mathbf{t}}) - B_1'(\boldsymbol{\varphi}^\perp + \widetilde{\boldsymbol{\varphi}}), G_1 - B_1(\mathbf{t}^\perp + \widetilde{\mathbf{t}}) + S(\boldsymbol{\varphi}^\perp + \widetilde{\boldsymbol{\varphi}}))\,,$$

which, in terms of the operator A (cf. (3.1)) and the functional $F = (F_1, G_1)$, can be rewritten as

$$A(\mathbf{t}^\perp + \widetilde{\mathbf{t}}, \boldsymbol{\varphi}^\perp + \widetilde{\boldsymbol{\varphi}}) \,+\, B'(p) \,=\, F\,. \tag{3.14}$$

In turn, since $B(\mathbf{t}^\perp, \boldsymbol{\varphi}^\perp) = G$ and $(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\varphi}})$ belongs to V, we easily see that

$$B(\mathbf{t}^\perp + \widetilde{\mathbf{t}}, \boldsymbol{\varphi}^\perp + \widetilde{\boldsymbol{\varphi}}) \,=\, G\,, \tag{3.15}$$

and therefore, it becomes clear from (3.14) and (3.15) that the pair $((\mathbf{t}^\perp + \widetilde{\mathbf{t}}, \boldsymbol{\varphi}^\perp + \widetilde{\boldsymbol{\varphi}}), p) \in X \times M$ constitutes a solution of (3.2). The corresponding bound (3.6) follows from (3.8), (3.9), (3.11), and (3.13), by employing also the properties of the operators involved. We omit details.

For the uniqueness, let $((\mathbf{t}, \boldsymbol{\varphi}), \bar{p}) \in X \times M$ be another solution of (3.2), that is

$$A(\mathbf{t}, \boldsymbol{\varphi}) \,+\, B'(\bar{p}) \,=\, F \quad \text{and} \quad B(\mathbf{t}, \boldsymbol{\varphi}) \,=\, G\,.$$

It follows that $(\mathbf{t}, \boldsymbol{\varphi}) - (\mathbf{t}^\perp, \boldsymbol{\varphi}^\perp) \in V$, and hence, using again the pseudolinear property (3.5), we find that $A_1(\mathbf{t}) = A_1(\mathbf{t} - \mathbf{t}^\perp) + A_1(\mathbf{t}^\perp)$. As a consequence, there holds

$$A(\mathbf{t}, \boldsymbol{\varphi}) \,=\, A\big((\mathbf{t}, \boldsymbol{\varphi}) - (\mathbf{t}^\perp, \boldsymbol{\varphi}^\perp)\big) \,+\, A(\mathbf{t}^\perp, \boldsymbol{\varphi}^\perp)\,,$$

which, combined with the fact that $A(\mathbf{t}, \boldsymbol{\varphi}) - F$ belongs to V^o, yields

$$[A\big((\mathbf{t}, \boldsymbol{\varphi}) - (\mathbf{t}^\perp, \boldsymbol{\varphi}^\perp)\big), (\mathbf{r}, \boldsymbol{\psi})] \,=\, [F - A(\mathbf{t}^\perp, \boldsymbol{\varphi}^\perp), (\mathbf{r}, \boldsymbol{\psi})] \qquad \forall\, (\mathbf{r}, \boldsymbol{\psi}) \in V\,,$$

that is

$$\begin{aligned} [A_1(\mathbf{t} - \mathbf{t}^\perp), \mathbf{r}] \,+\, [B_1'(\boldsymbol{\varphi} - \boldsymbol{\varphi}^\perp), \mathbf{r}] &= [\widehat{F}_1, \mathbf{r}] \qquad \forall\, \mathbf{r} \in \widetilde{X}_1\,, \\ [B_1(\mathbf{t} - \mathbf{t}^\perp), \boldsymbol{\psi}] \,-\, [S(\boldsymbol{\varphi} - \boldsymbol{\varphi}^\perp), \boldsymbol{\psi}] &= [\widehat{G}_1, \boldsymbol{\psi}] \qquad \forall\, \boldsymbol{\psi} \in \widetilde{M}_1\,. \end{aligned}$$

This shows that $(\mathbf{t} - \mathbf{t}^\perp, \boldsymbol{\varphi} - \boldsymbol{\varphi}^\perp)$ is a solution of (3.10), and hence, because of the unique solvability of that problem, we deduce that $(\mathbf{t} - \mathbf{t}^\perp, \boldsymbol{\varphi} - \boldsymbol{\varphi}^\perp) = (\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\varphi}})$, that is $(\mathbf{t}, \boldsymbol{\varphi}) = (\widetilde{\mathbf{t}} + \mathbf{t}^\perp, \widetilde{\boldsymbol{\varphi}} + \boldsymbol{\varphi}^\perp)$. Finally, since $B'(p) = B'(\bar{p}) = F - A(\mathbf{t}, \boldsymbol{\varphi}) \in V^o$ and $B' : M \to V^o$ is an isomorphism, we conclude that $p = \bar{p}$, which finishes the proof.

■

Before going on with the analysis, we now describe a particular case providing sufficient conditions for the pseudolinear property (3.5). More precisely, let us assume that X_1 can be decomposed as the product space $X_1^\ell \times X_1^r$ in such a way that

I) B does not depend on the variables from X_1^ℓ.

II) A_1 is linear in $\{\mathbf{0}\} \times X_1^r$, where $\mathbf{0}$ denotes the null vector of X_1^ℓ.

III) for each $\mathbf{t} := (\mathbf{t}^\ell, \mathbf{t}^r) \in X_1^\ell \times X_1^r =: X_1$ there holds

$$A_1(\mathbf{t}) = A_1(\mathbf{t}^\ell, \mathbf{0}) + A_1(\mathbf{0}, \mathbf{t}^r),$$

where $\mathbf{0}$ denotes, respectively, the null vectors of X_1^r and X_1^ℓ.

Then, recalling that $V = \widetilde{X}_1 \times \widetilde{M}_1$, we deduce from I) that $\widetilde{X}_1 = X_1^\ell \times \widetilde{X}_1^r$, where $\widetilde{X}_1^r$ is a subspace of X_1^r. In addition, it follows from the above that $\widetilde{X}_1^\perp = \{\mathbf{0}\} \times \big(\widetilde{X}_1^r\big)^\perp \subseteq \{\mathbf{0}\} \times X_1^r$, where $\mathbf{0}$ denotes again the null vector of X_1^ℓ. Consequently, given $\mathbf{t} := (\mathbf{t}^\ell, \mathbf{t}^r) \in \widetilde{X}_1$ and $\mathbf{t}^\perp := (\mathbf{0}, \mathbf{t}^{\perp,r}) \in \widetilde{X}_1^\perp$, we use II) and III) to find that

$$\begin{aligned} A_1(\mathbf{t}) + A_1(\mathbf{t}^\perp) &= A_1(\mathbf{t}^\ell, \mathbf{0}) + A_1(\mathbf{0}, \mathbf{t}^r) + A_1(\mathbf{0}, \mathbf{t}^{\perp,r}) = A_1(\mathbf{t}^\ell, \mathbf{0}) + A_1(\mathbf{0}, \mathbf{t}^r + \mathbf{t}^{\perp,r}) \\ &= A_1(\mathbf{t}^\ell, \mathbf{t}^r + \mathbf{t}^{\perp,r}) = A_1(\mathbf{t} + \mathbf{t}^\perp), \end{aligned}$$

which shows that (3.5) holds. In particular, we prove below in Section 4 that our formulation from Section 2 does satisfy the assumptions I), II), and III).

On the other hand, if A_1 is linear, Theorem 3.1 reduces to the following.

Theorem 3.2 *Assume that*

i) $A_1 : \widetilde{X}_1 \to \widetilde{X}_1'$ *is linear, bounded and* $\widetilde{X}_1$*-elliptic, that is, there exist* γ, $\alpha > 0$ *such that*

$$\|A_1(\mathbf{r})\|_{\widetilde{X}_1'} \le \gamma \|\mathbf{r}\|_{X_1} \qquad \forall\, \mathbf{r} \in \widetilde{X}_1,$$

and

$$[A_1(\mathbf{r}), \mathbf{r}] \ge \alpha \|\mathbf{r}\|_{X_1}^2 \qquad \forall\, \mathbf{r} \in \widetilde{X}_1.$$

ii) *The conditions* iii) - v) *from Theorem* 3.1 *are satisfied.*

Then, there exists a unique $((\mathbf{t}, \boldsymbol{\varphi}), p) \in X \times M$ *solution of* (3.2)*. Moreover, there exists* $C > 0$*, depending only on* α, γ, β_1, β, $\|S\|$, *and* $\|B_1\|$ *such that*

$$\|((\mathbf{t}, \boldsymbol{\varphi}), p)\|_{X \times M} \le C \Big\{ \|F\|_{X'} + \|G\|_{M'} \Big\}. \tag{3.16}$$

Proof. It suffices to observe that the linearity, boundedness and ellipticity of A_1 imply that this operator is Lipschitz continuous and strongly monotone in $\widetilde{X}_1$. In addition, it is clear that A_1 satisfies (3.5). Thus, the proof follows from a straightforward application of Theorem 3.1.

■

It is important to remark at this point that (3.16) is equivalent to the global inf-sup condition

$$\|((\mathbf{s}, \boldsymbol{\phi}), \rho)\|_{X \times M} \le C \sup_{\substack{((\mathbf{r}, \boldsymbol{\psi}), q) \in X \times M \\ ((\mathbf{r}, \boldsymbol{\psi}), q) \neq \mathbf{0}}} \frac{[A(\mathbf{s}, \boldsymbol{\phi}), (\mathbf{r}, \boldsymbol{\psi})] + [B'(\rho), (\mathbf{r}, \boldsymbol{\psi})] + [B(\mathbf{s}, \boldsymbol{\phi}), q]}{\|((\mathbf{r}, \boldsymbol{\psi}), q)\|_{X \times M}} \tag{3.17}$$

for all $((\mathbf{s}, \boldsymbol{\phi}), \rho) \in X \times M$.

3.2 The discrete setting

We now turn our attention to the Galerkin scheme of problem (3.2). To this end, we let $X_{1,h}$, $M_{1,h}$, and M_h be finite-dimensional subspaces of X_1, M_1, and M, respectively. Here, the subindex h, which identifies the finite dimensional subspaces, is taken in a numerable family $\mathrm{I} := \{h_j\}_{j\in\mathbb{N}}$ such that $h_j \geq h_{j+1}$ for all $j \in \mathbb{N}$. Then, defining $X_h := X_{1,h} \times M_{1,h}$, the Galerkin scheme reduces to: Find $((\mathbf{t}_h, \boldsymbol{\varphi}_h), p_h) \in X_h \times M_h$ such that

$$\begin{array}{rcll} [A(\mathbf{t}_h, \boldsymbol{\varphi}_h), (\mathbf{r}, \boldsymbol{\psi})] + [B'(p_h), (\mathbf{r}, \boldsymbol{\psi})] & = & [F, (\mathbf{r}, \boldsymbol{\psi})] & \forall\, (\mathbf{r}, \boldsymbol{\psi}) \in X_h\,, \\ [B(\mathbf{t}_h, \boldsymbol{\varphi}_{\boldsymbol{h}}), q] & = & [G, q] & \forall\, q \in M_h\,. \end{array} \tag{3.18}$$

Next, we let V_h be the discrete kernel of B, that is,

$$V_h := \Big\{ (\mathbf{r}_h, \boldsymbol{\psi}_h) \in X_h : \quad [B(\mathbf{r}_h, \boldsymbol{\psi}_h), q] = 0 \qquad \forall\, q \in M_h \Big\},$$

and let $\widetilde{X}_{1,h}$ and $\widetilde{M}_{1,h}$ be subspaces of $X_{1,h}$ and $M_{1,h}$, respectively, such that $V_h = \widetilde{X}_{1,h} \times \widetilde{M}_{1,h}$.

The following theorem establishes the well posedness of (3.18).

Theorem 3.3 *Assume that*

i) $A_1|_{\widetilde{X}_{1,h}} : \widetilde{X}_{1,h} \to \widetilde{X}'_{1,h}$ *is Lipschitz continuous and strongly monotone, that is, there exist constants* γ_h, $\alpha_h > 0$ *such that*

$$\| A_1(\mathbf{s}_h) - A_1(\mathbf{r}_h) \|_{\widetilde{X}'_{1,h}} \leq \gamma_h\, \|\mathbf{s}_h - \mathbf{r}_h\|_{X_1} \quad \forall\, \mathbf{s}_h,\, \mathbf{r}_h \in \widetilde{X}_{1,h}$$

and

$$[A_1(\mathbf{s}_h) - A_1(\mathbf{r}_h), \mathbf{s}_h - \mathbf{r}_h] \geq \alpha_h\, \|\mathbf{s}_h - \mathbf{r}_h\|^2_{X_1} \quad \forall\, \mathbf{s}_h,\, \mathbf{r}_h \in \widetilde{X}_{1,h}\,.$$

ii) *For each pair* $(\mathbf{r}_h, \mathbf{r}_h^{\perp}) \in \widetilde{X}_{1,h} \times \widetilde{X}_{1,h}^{\perp}$ *there holds the discrete pseudolinear property*

$$[A_1(\mathbf{r}_h + \mathbf{r}_h^{\perp}), \mathbf{s}_h] = [A_1(\mathbf{r}_h), \mathbf{s}_h] + [A_1(\mathbf{r}_h^{\perp}), \mathbf{s}_h] \quad \forall\, \mathbf{s}_h \in X_{1,h}\,, \tag{3.19}$$

where $\widetilde{X}_{1,h}^{\perp}$ *is the orthogonal of* $\widetilde{X}_{1,h}$ *within* $X_{1,h}$.

iii) S *is positive semi-definite on* $\widetilde{M}_{1,h}$, *that is,*

$$[S(\boldsymbol{\psi}_h), \boldsymbol{\psi}_h] \geq 0 \quad \forall\, \boldsymbol{\psi}_h \in \widetilde{M}_{1,h}\,.$$

iv) B_1 *satisfies an inf-sup condition on* $\widetilde{X}_{1,h} \times \widetilde{M}_{1,h}$, *that is, there exists* $\beta_{1,h} > 0$ *such that*

$$\sup_{\substack{\mathbf{r}_h \in \widetilde{X}_{1,h} \\ \mathbf{r}_h \neq \mathbf{0}}} \frac{[B_1(\mathbf{r}_h), \boldsymbol{\psi}_h]}{\|\mathbf{r}_h\|_{X_1}} \geq \beta_{1,h}\, \|\boldsymbol{\psi}_h\|_{M_1} \quad \forall\, \boldsymbol{\psi}_h \in \widetilde{M}_{1,h}\,.$$

v) B *satisfies an inf-sup condition on* $X_h \times M_h$, *that is, there exists* $\beta_h > 0$ *such that*

$$\sup_{\substack{(\mathbf{r}_h, \boldsymbol{\psi}_h) \in X_h \\ (\mathbf{r}_h, \boldsymbol{\psi}_h) \neq \mathbf{0}}} \frac{[B(\mathbf{r}_h, \boldsymbol{\psi}_h), q_h]}{\|(\mathbf{r}_h, \boldsymbol{\psi}_h)\|_X} \geq \beta_h\, \|q_h\|_M \quad \forall\, q_h \in M_h\,.$$

Then, there exists a unique $((\mathbf{t}_h, \boldsymbol{\varphi}_h), p_h) \in X_h \times M_h$ *solution of* (3.18). *Moreover, there exists* $C_h > 0$, *depending only on* α_h, γ_h, $\beta_{1,h}$, β_h, $\|S\|$, *and* $\|B_1\|$ *such that*

$$\|((\mathbf{t}_h, \boldsymbol{\varphi}_h), p_h)\|_{X\times M} \;\le\; C_h \left\{ \|F|_{X_h}\|_{X'_h} \;+\; \|G|_{M_h}\|_{M'_h} \right\}. \tag{3.20}$$

Proof. It follows analogously to the proof of Theorem 3.1 by adapting now the proof of [17, Theorem 3.1]. In particular, the discrete inf-sup condition satisfied by B (cf. v)) and [28, Chapter I, Lemma 4.1] imply that the discrete counterparts of B and B', namely $B_h : V_h^{\perp}\cap X_h \to M'_h$ and $B'_h : M_h \to V_h^{o}\cap X'_h$, respectively, are isomorphisms with

$$\|B_h^{-1}\|\,,\ \|(B'_h)^{-1}\| \;\le\; \frac{1}{\beta_h}\,. \tag{3.21}$$

The rest of the proof makes use also of the discrete version of Lemma 3.1. We omit further details.

∎

It is interesting to observe at this point that the same sufficient conditions introduced above for the pseudolinear property (3.5) yield now the verification of (3.19). In fact, decomposing the space $X_{1,h} \;=\; X^{\ell}_{1,h} \times X^{r}_{1,h}$, with $X^{\ell}_{1,h} \;\subseteq\; X^{\ell}_{1}$ and $X^{r}_{1,h} \;\subseteq\; X^{r}_{1}$, and assuming I), II), and III), we easily see that B does not depend on the variables from $X^{\ell}_{1,h}$, $A_1|_{X_{1,h}}$ is linear in $\big\{\mathbf{0}\big\}\times X^{r}_{1,h}$, and for each $\mathbf{t}_h :=(\mathbf{t}^{\ell}_h, \mathbf{t}^{r}_h) \in X^{\ell}_{1,h}\times X^{r}_{1,h} =\!:\; X_{1,h}$ there holds $[A_1(\mathbf{t}_h), \mathbf{s}_h] \;=\; [A_1(\mathbf{t}^{\ell}_h, \mathbf{0}), \mathbf{s}_h] + [A_1(\mathbf{0}, \mathbf{t}^{r}_h), \mathbf{s}_h] \quad \forall\, \mathbf{s}_h \in X_{1,h}$. Consequently, we find that $\widetilde{X}_{1,h} \;=\; X^{\ell}_{1,h} \times \widetilde{X}^{r}_{1,h}$, where $\widetilde{X}^{r}_{1,h}$ is a subspace of $X^{r}_{1,h}$, and also that $\widetilde{X}^{\perp}_{1,h} \subseteq \big\{\mathbf{0}\big\} \times X^{r}_{1,h}$, whence the discrete pseudolinear property (3.19) follows similarly to the proof of (3.5) from the assumptions indicated. Further details are omitted.

On the other hand, the linear version of Theorem 3.3 is established as follows.

Theorem 3.4 *Assume that*

i) $A_1|_{X_{1,h}} : X_{1,h} \to X'_{1,h}$ *is linear, bounded and* $\widetilde{X}_{1,h}$*-elliptic, that is, there exist* γ_h, $\alpha_h > 0$ *such that*

$$\|A_1(\mathbf{r}_h)\|_{X'_{1,h}} \;\le\; \gamma_h\, \|\mathbf{r}_h\|_{X_1} \qquad \forall\, \mathbf{r}_h \in \widetilde{X}_{1,h}\,,$$

and

$$[A_1(\mathbf{r}_h), \mathbf{r}_h] \;\ge\; \alpha_h\, \|\mathbf{r}_h\|^2_{X_1} \qquad \forall\, \mathbf{r}_h \in \widetilde{X}_{1,h}\,.$$

ii) *The conditions* iii) - v) *from Theorem* 3.3 *are satisfied.*

Then, there exists a unique $((\mathbf{t}_h, \boldsymbol{\varphi}_h), p_h) \in X_h \times M_h$ *solution of* (3.18). *Moreover, there exists* $C_h > 0$, *depending only on* α_h, γ_h, $\beta_{1,h}$, β_h, $\|S\|$, *and* $\|B_1\|$ *such that*

$$\|((\mathbf{t}_h, \boldsymbol{\varphi}_h), p_h)\|_{X\times M} \;\le\; C_h \left\{ \|F|_{X_h}\|_{X'_h} \;+\; \|G|_{M_h}\|_{M'_h} \right\}. \tag{3.22}$$

Proof. It reduces to verify the hypotheses of Theorem 3.3. We omit details.

∎

As for the continuous case, we notice here that (3.22) is equivalent to the global inf-sup condition

$$\|((\mathbf{s}_h, \boldsymbol{\phi}_h), \rho_h)\|_{X\times M} \le C_h \sup_{\substack{((\mathbf{r},\boldsymbol{\psi}),q)\in X_h\times M_h\\ ((\mathbf{r},\boldsymbol{\psi}),q)\neq \mathbf{0}}} \frac{[A(\mathbf{s}_h, \boldsymbol{\phi}_h), (\mathbf{r}, \boldsymbol{\psi})] + [B'(\rho_h), (\mathbf{r}, \boldsymbol{\psi})] + [B(\mathbf{s}_h, \boldsymbol{\phi}_h), q]}{\|((\mathbf{r}, \boldsymbol{\psi}), q)\|_{X\times M}} \tag{3.23}$$

for all $((\mathbf{s}_h, \boldsymbol{\phi}_h), \rho_h) \in X_h \times M_h$.

It is important to remark now that, from the point of view of the stability of the Galerkin schemes, one actually should require that in Theorems 3.3 and 3.4 all the constants α_h, γ_h, $\beta_{1,h}$, and β_h, and hence C_h in (3.20) and (3.22), be independent of h. Indeed, these theorems are usually stated by assuming the existence of uniform lower bounds for α_h, $\beta_{1,h}$, and β_h, and a uniform upper bound for γ_h. Needless to say, the derivation of these uniform bounds (equivalently, the obtention of constants not depending on the meshsize h) becomes precisely the core issue of the numerical analysis of any particular Galerkin scheme of the form (3.18).

We now aim to provide the error estimates for the abstract Galerkin scheme (3.18). For this purpose, and in order to simplify the corresponding analysis, we proceed as in [17] and introduce a differentiability hypothesis on the nonlinear operator A_1. In addition, we suppose that A_1 is Lipschitz-continuous in the whole space X_1, and adopt slightly more general strong monotonicity properties involving separately the continuous and discrete spaces. More precisely, throughout the rest of the section we assume the following hypotheses:

(A.1) there exist constants γ, $\alpha > 0$, independent of h, such that

$$\|A_1(\mathbf{s}) \,-\, A_1(\mathbf{r})\|_{X_1'} \,\le\, \gamma\,\|\mathbf{s}-\mathbf{r}\|_{X_1} \quad \forall\,\mathbf{s},\,\mathbf{r}\in X_1\,, \tag{3.24}$$

$$[A_1(\mathbf{t}+\mathbf{s}) \,-\, A_1(\mathbf{t}+\mathbf{r}),\mathbf{s}-\mathbf{r}] \,\ge\, \alpha\,\|\mathbf{s}-\mathbf{r}\|_{X_1}^2 \quad \forall\,\mathbf{t}\in X_1,\ \forall\,\mathbf{s},\,\mathbf{r}\in \widetilde{X}_1\,, \tag{3.25}$$

and

$$[A_1(\mathbf{t}_h+\mathbf{s}_h) \,-\, A_1(\mathbf{t}_h+\mathbf{r}_h),\mathbf{s}_h-\mathbf{r}_h] \,\ge\, \alpha\,\|\mathbf{s}_h-\mathbf{r}_h\|_{X_1}^2 \quad \forall\,\mathbf{t}_h\in X_{1,h},\ \forall\,\mathbf{s}_h,\,\mathbf{r}_h\in \widetilde{X}_{1,h}\,. \tag{3.26}$$

(A.2) $A_1 : X_1 \to X_1'$ has a hemi-continuous first order Gâteaux derivative $\mathcal{D}A_1 : X_1 \to \mathcal{L}(X_1, X_1')$, which means that for any $\mathbf{s}$, $\mathbf{r}\in X_1$, the mapping $R \ni \mu \to \mathcal{D}A_1(\mathbf{s}+\mu\,\mathbf{r})(\mathbf{r})(\cdot) \in X_1'$ is continuous.

Note here that the discrete strong monotonicity condition (3.26) does not follow in general from the continuous one (3.25) since the component $\widetilde{X}_{1,h}$ of the discrete kernel V_h is not necessarily contained in the corresponding component $\widetilde{X}_1$ of V. This is the reason why we have to impose them separately. Then, we have the following result.

Lemma 3.2 *For any $\mathbf{s}\in X_1$, the Gâteaux derivative $\mathcal{D}A_1(\mathbf{s})$ constitutes a bounded bilinear form on $X_1\times X_1$ that becomes elliptic on $\widetilde{X}_1 \cup \widetilde{X}_{1,h}$, with boundedness and ellipticity constants given by γ and α, respectively.*

Proof. Given $\mathbf{s}\in X_1$, the Gâteaux derivative $\mathcal{D}A_1(\mathbf{s})$ is the operator in $\mathcal{L}(X_1, X_1')$ (equivalently, the bilinear form on $X_1\times X_1$) defined by

$$\mathcal{D}A_1(\mathbf{s})(\mathbf{r},\widehat{\mathbf{r}}) \,:=\, \lim_{\epsilon\to 0} \frac{[A_1(\mathbf{s}\,+\,\epsilon\,\mathbf{r}),\widehat{\mathbf{r}}] \,-\, [A_1(\mathbf{s}),\widehat{\mathbf{r}}]}{\epsilon} \qquad \forall\,\mathbf{r},\,\widehat{\mathbf{r}}\in X_1\,.$$

The rest of the proof follows as in [17, Lemma 3.1] by employing the assumptions **(A.1)** and **(A.2)** in the above definition. We omit further details and refer the reader to [17].

∎

Our next goal is to provide the Cea estimate for the Galerkin scheme (3.18). To this end, we now let $P : X\times M \to (X\times M)' := X'\times M'$ be the nonlinear operator obtained after adding the equations on the left hand side of (3.3), that is

$$[P(\vec{\mathbf{t}}),\vec{\mathbf{r}}] \,:=\, [A(\mathbf{t},\boldsymbol{\varphi}),(\mathbf{r},\boldsymbol{\psi})] \,+\, [B'(p),(\mathbf{r},\boldsymbol{\psi})] \,+\, [B(\mathbf{t},\boldsymbol{\varphi}),q]$$

for all $\vec{\mathbf{t}} := ((\mathbf{t}, \boldsymbol{\varphi}), p)$, $\vec{\mathbf{r}} := ((\mathbf{r}, \boldsymbol{\psi}), q) \in X \times M$, or, equivalently, using (3.1),

$$[P(\vec{\mathbf{t}}), \vec{\mathbf{r}}] := [A_1(\mathbf{t}), \mathbf{r}] + [B_1'(\boldsymbol{\varphi}), \mathbf{r}] + [B_1(\mathbf{t}), \boldsymbol{\psi}] - [S(\boldsymbol{\varphi}), \boldsymbol{\psi}] + [B'(p), (\mathbf{r}, \boldsymbol{\psi})] + [B(\mathbf{t}, \boldsymbol{\varphi}), q] \tag{3.27}$$

for all $\vec{\mathbf{t}} := ((\mathbf{t}, \boldsymbol{\varphi}), p)$, $\vec{\mathbf{r}} := ((\mathbf{r}, \boldsymbol{\psi}), q) \in X \times M$. Then, it is easy to see that, given $\vec{\mathbf{s}} := ((\mathbf{s}, \boldsymbol{\phi}), \rho) \in X \times M$, the Gâteaux derivative of P at $\vec{\mathbf{s}}$ is obtained by replacing $[A_1(\mathbf{t}), \mathbf{r}]$ in (3.27) by $\mathcal{D}A_1(\mathbf{s})(\mathbf{t}, \mathbf{r})$. In this way we arrive at

$$\mathcal{D}P(\vec{\mathbf{s}})(\vec{\mathbf{t}}, \vec{\mathbf{r}}) := \mathcal{D}A_1(\mathbf{s})(\mathbf{t}, \mathbf{r}) + [B_1'(\boldsymbol{\varphi}), \mathbf{r}] + [B_1(\mathbf{t}), \boldsymbol{\psi}] - [S(\boldsymbol{\varphi}), \boldsymbol{\psi}] + [B'(p), (\mathbf{r}, \boldsymbol{\psi})] + [B(\mathbf{t}, \boldsymbol{\varphi}), q] \tag{3.28}$$

for all $\vec{\mathbf{t}} := ((\mathbf{t}, \boldsymbol{\varphi}), p)$, $\vec{\mathbf{r}} := ((\mathbf{r}, \boldsymbol{\psi}), q) \in X \times M$, which, according to Lemma 3.2, becomes a bounded bilinear form on $\big(X \times M\big) \times \big(X \times M\big)$. Moreover, assuming for a moment the conditions iii) - v) of Theorem 3.3 with constants independent of h, and having in mind Lemma 3.2 again, we deduce that $\mathcal{D}P(\vec{\mathbf{s}})(\cdot, \cdot)$ satisfies the hypotheses of the linear version given by Theorem 3.4 with constants independent of h and $\vec{\mathbf{s}}$ as well. It follows, in virtue of (3.22) (equivalently (3.23)), that there exists $\widetilde{C} > 0$, independent of h, such that

$$\|\vec{\mathbf{s}}_h\|_{X \times M} \le \widetilde{C} \sup_{\substack{\vec{\mathbf{r}}_h \in X_h \times M_h \\ \vec{\mathbf{r}}_h \neq \mathbf{0}}} \frac{\mathcal{D}P(\vec{\mathbf{s}})(\vec{\mathbf{s}}_h, \vec{\mathbf{r}}_h)}{\|\vec{\mathbf{r}}_h\|_{X \times M}} \qquad \forall\, \vec{\mathbf{s}}_h \in X_h \times M_h\,. \tag{3.29}$$

We are now in a position to establish the announced a priori error estimate.

Theorem 3.5 *Assume that the hypotheses of Theorems* 3.1 *and* 3.3 *hold, and let* $\vec{\mathbf{t}} := ((\mathbf{t}, \boldsymbol{\varphi}), p) \in X \times M$ *and* $\vec{\mathbf{t}}_h := ((\mathbf{t}_h, \boldsymbol{\varphi}_h), p_h) \in X_h \times M_h$ *be the unique solutions of* (3.2) *and* (3.18)*, respectively. Then, there exists* $C > 0$*, independent of* h*, such that*

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{X \times M} \le C \inf_{\vec{\mathbf{s}}_h \in X_h \times M_h} \|\vec{\mathbf{t}} - \vec{\mathbf{s}}_h\|_{X \times M}\,. \tag{3.30}$$

Proof. We proceed as in the proof of [17, Theorem 3.3]. Hence, given $\vec{\mathbf{s}} \in X \times M$ and $\vec{\mathbf{s}}_h \in X_h \times M_h$, we apply (3.29) to $\vec{\mathbf{t}}_h - \vec{\mathbf{s}}_h$ and obtain

$$\|\vec{\mathbf{t}}_h - \vec{\mathbf{s}}_h\|_{X \times M} \le \widetilde{C} \sup_{\substack{\vec{\mathbf{r}}_h \in X_h \times M_h \\ \vec{\mathbf{r}}_h \neq \mathbf{0}}} \frac{\mathcal{D}P(\vec{\mathbf{s}})(\vec{\mathbf{t}}_h - \vec{\mathbf{s}}_h, \vec{\mathbf{r}}_h)}{\|\vec{\mathbf{r}}_h\|_{X \times M}}\,. \tag{3.31}$$

In turn, since the hemi-continuity of $\mathcal{D}A_1$ (cf. **(A.2)**) implies the same property for $\mathcal{D}P$, we deduce the existence of $\mu_0 \in (0,1)$ such that

$$\begin{aligned} [P(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}_h] - [P(\vec{\mathbf{s}}_h), \vec{\mathbf{r}}_h] &= \int_0^1 \mathcal{D}P(\mu\, \vec{\mathbf{t}}_h + (1-\mu)\, \vec{\mathbf{s}}_h)(\vec{\mathbf{t}}_h - \vec{\mathbf{s}}_h, \vec{\mathbf{r}}_h)\, d\mu \\ &= \mathcal{D}P(\mu_0\, \vec{\mathbf{t}}_h + (1-\mu_0)\, \vec{\mathbf{s}}_h)(\vec{\mathbf{t}}_h - \vec{\mathbf{s}}_h, \vec{\mathbf{r}}_h)\,, \end{aligned} \tag{3.32}$$

and hence, using in particular $\vec{\mathbf{s}} = \mu_0\, \vec{\mathbf{t}}_h + (1-\mu_0)\, \vec{\mathbf{s}}_h$ in (3.31), we find that

$$\|\vec{\mathbf{t}}_h - \vec{\mathbf{s}}_h\|_{X \times M} \le \widetilde{C} \sup_{\substack{\vec{\mathbf{r}}_h \in X_h \times M_h \\ \vec{\mathbf{r}}_h \neq \mathbf{0}}} \frac{[P(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}_h] - [P(\vec{\mathbf{s}}_h), \vec{\mathbf{r}}_h]}{\|\vec{\mathbf{r}}_h\|_{X \times M}}\,. \tag{3.33}$$

Next, since (3.2) and (3.18) yield $[P(\vec{\mathbf{t}}), \vec{\mathbf{r}}_h] = [P(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}_h] \quad \forall\, \vec{\mathbf{r}}_h \in X_h \times M_h$, and since **(A.1)** implies that P is also Lipschitz-continuous, say with a constant $\widetilde{\gamma}$, we obtain from (3.33) that

$$\|\vec{\mathbf{t}}_h - \vec{\mathbf{s}}_h\|_{X \times M} \le \widetilde{C}\, \widetilde{\gamma}\, \|\vec{\mathbf{t}} - \vec{\mathbf{s}}_h\|_{X \times M} \qquad \forall\, \vec{\mathbf{s}}_h \in X_h \times M_h\,. \tag{3.34}$$

Finally, it is easy to see that (3.34) and the triangle inequality give (3.30) and complete the proof. ∎

4 Analysis of the continuous problem

We now go back to the augmented fully mixed variational formulation introduced in Section 2.4 and apply Theorem 3.1 to prove the well posedness of (2.34). In fact, we begin by observing from the definition of $\mathbb{B}$ (cf. (2.31)) that the kernel of this operator reduces to

$$\mathbb{V} := \Big\{ (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{X} : \quad [\mathbb{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{q}] = 0 \quad \forall\, \underline{q} \in \mathbb{M} \Big\} = \widetilde{\mathbf{X}}_1 \times \widetilde{\mathbf{M}}_1 ,$$

where

$$\widetilde{\mathbf{X}}_1 = \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}}) \times \widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_{\mathrm{D}}) \times \widetilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}}) \quad \text{and} \quad \widetilde{\mathbf{M}}_1 = \widetilde{\mathbf{H}}^{1/2}_{00}(\Sigma) \times H^{1/2}(\Sigma) ,$$

with

$$\widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_{\mathrm{D}}) := \Big\{ \mathbf{v}_{\mathrm{D}} \in \mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{D}}) : \quad \mathrm{div}(\mathbf{v}_{\mathrm{D}}) \in \mathbb{P}_0(\Omega_{\mathrm{D}}) \Big\} ,$$

$$\widetilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}}) = \Big\{ \boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) : \quad \boldsymbol{\tau}_{\mathrm{S}} = \boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{t}} \quad \text{and} \quad \mathbf{div}\, \boldsymbol{\tau}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}} \Big\} ,$$

and

$$\widetilde{\mathbf{H}}^{1/2}_{00}(\Sigma) := \Big\{ \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma) : \quad \langle \mathbf{n}, \boldsymbol{\psi} \rangle_{\Sigma} = 0 \Big\} .$$

The following lemma shows that $\mathbf{A}_1$ verifies the assumptions (3.24) and (3.25) (cf. **(A.1)**), which imply, in particular, that $\mathbf{A}_1$ satisfies the hypothesis i) in Theorem 3.1.

Lemma 4.1 *Let $\mathbf{A}_1 : \mathbf{X}_1 \to \mathbf{X}_1'$ be the nonlinear operator defined by* (2.27)*. Then there exists a constant $\gamma > 0$ such that*

$$\|\mathbf{A}_1(\underline{\mathbf{r}}) - \mathbf{A}_1(\underline{\mathbf{s}})\|_{\mathbf{X}_1'} \le \gamma \|\underline{\mathbf{r}} - \underline{\mathbf{s}}\|_{\mathbf{X}_1} \qquad \forall\, \underline{\mathbf{r}},\, \underline{\mathbf{s}} \in \mathbf{X}_1 . \tag{4.1}$$

Furthermore, assume that the parameter ρ lies in $\left(0, \dfrac{\alpha_0}{\gamma_0^2}\right)$, where α_0 and γ_0 are the positive constants from (2.4) *and* (2.5)*. Then, there exists a constant $\alpha > 0$ such that*

$$[\mathbf{A}_1(\underline{\mathbf{t}} + \underline{\mathbf{r}}) - \mathbf{A}_1(\underline{\mathbf{t}} + \underline{\bar{\mathbf{r}}}), \underline{\mathbf{r}} - \underline{\bar{\mathbf{r}}}] \ge \alpha \, \|\underline{\mathbf{r}} - \underline{\bar{\mathbf{r}}}\|^2_{\mathbf{X}_1} \qquad \forall\, \underline{\mathbf{t}} \in X_1 , \quad \forall\, \underline{\mathbf{r}},\, \underline{\bar{\mathbf{r}}} \in \widetilde{\mathbf{X}}_1 . \tag{4.2}$$

Proof. We begin by observing from (2.27) that $\mathbf{A}_1 : \mathbf{X}_1 \to \mathbf{X}_1'$ can be decomposed as

$$[\mathbf{A}_1(\underline{\mathbf{r}}), \underline{\bar{\mathbf{r}}}] = [\mathbf{A}_{1\mathrm{S}}(\mathbf{r}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}), (\bar{\mathbf{r}}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}_{\mathrm{S}})] + [\mathbf{A}_{1\mathrm{D}}(\mathbf{v}_{\mathrm{D}}), \bar{\mathbf{v}}_{\mathrm{D}})] \quad \forall\, \underline{\mathbf{r}} = (\mathbf{r}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\tau}_{\mathrm{S}}),\, \underline{\bar{\mathbf{r}}} = (\bar{\mathbf{r}}_{\mathrm{S}}, \bar{\mathbf{v}}_{\mathrm{D}}, \bar{\boldsymbol{\tau}}_{\mathrm{S}}) \in \mathbf{X}_1,$$

where $\mathbf{A}_{1\mathrm{S}} : \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}}) \times \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \to \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}})' \times \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})'$ and $\mathbf{A}_{1\mathrm{D}} : \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}) \to \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})'$ are the nonlinear and linear operators, respectively, given by

$$[\mathbf{A}_{1\mathrm{S}}(\mathbf{r}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}), (\bar{\mathbf{r}}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}_{\mathrm{S}})] := (\mu(|\mathbf{r}_{\mathrm{S}}|)\, \mathbf{r}_{\mathrm{S}}, \bar{\mathbf{r}}_{\mathrm{S}})_{\mathrm{S}} - (\bar{\mathbf{r}}_{\mathrm{S}}, \boldsymbol{\tau}^{d}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{r}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}^{\mathtt{d}}_{\mathrm{S}})_{\mathrm{S}} + \rho\, (\boldsymbol{\tau}^{\mathtt{d}}_{\mathrm{S}} - \mu(|\mathbf{r}_{\mathrm{S}}|)\, \mathbf{r}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}^{\mathtt{d}}_{\mathrm{S}})_{\mathrm{S}} \tag{4.3}$$

and

$$[\mathbf{A}_{1\mathrm{D}}(\mathbf{v}_{\mathrm{D}}), \bar{\mathbf{v}}_{\mathrm{D}})] := (\mathbf{K}^{-1}\mathbf{v}_{\mathrm{D}}, \bar{\mathbf{v}}_{\mathrm{D}})_{\mathrm{D}} . \tag{4.4}$$

Next, we recall from [22, Lemma 3.1] that there exists $\bar{\gamma} > 0$ such that

$$\|\mathbf{A}_{1\mathrm{S}}(\mathbf{r}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}) - \mathbf{A}_{1\mathrm{S}}(\bar{\mathbf{r}}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}_{\mathrm{S}})\|_{\overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}})' \times \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})'} \le \bar{\gamma}\, \|(\mathbf{r}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}) - (\bar{\mathbf{r}}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}_{\mathrm{S}})\|_{\overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}}) \times \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})}$$

for all $(\mathbf{r}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}), (\bar{\mathbf{r}}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}_{\mathrm{S}}) \in \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}}) \times \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$, and hence, thanks also to the boundedness of $\mathbf{A}_{1\mathrm{D}}$, we conclude (4.1), that is the Lipschitz continuity of $\mathbf{A}_1$. On the other hand, it was proved in [22, Lemma 3.2] that, under the present assumption on ρ and having in mind that $\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}} = 0 \quad \forall\, \boldsymbol{\tau}_{\mathrm{S}} \in \widetilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}})$, there exists $\bar{\alpha} > 0$ such that

$$
\begin{aligned}
&[\mathbf{A}_{1\mathrm{S}}((\widetilde{\mathbf{r}}_S, \widetilde{\boldsymbol{\tau}}_S) + (\mathbf{r}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}})) - \mathbf{A}_{1\mathrm{S}}((\widetilde{\mathbf{r}}_S, \widetilde{\boldsymbol{\tau}}_S) + (\bar{\mathbf{r}}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}_{\mathrm{S}})), (\mathbf{r}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}) - (\bar{\mathbf{r}}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}_{\mathrm{S}})] \\
&\qquad \geq\ \bar{\alpha}\, \|(\mathbf{r}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}) - (\bar{\mathbf{r}}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}_{\mathrm{S}})\|^2_{\overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}}) \times \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})}\,,
\end{aligned}
\tag{4.5}
$$

for all $(\widetilde{\mathbf{r}}_S, \widetilde{\boldsymbol{\tau}}_S) \in \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}}) \times \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$ and for all $(\mathbf{r}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}), (\bar{\mathbf{r}}_{\mathrm{S}}, \bar{\boldsymbol{\tau}}_{\mathrm{S}}) \in \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}}) \times \widetilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}})$. At this point we remark that both [22, Lemma 3.1] and [22, Lemma 3.2] follow from [22, Lemma 2.1], which is actually the key result making use of the Gâteaux derivative of $\mathbf{A}_{1\mathrm{S}}$. In turn, it was established in [25, Lemma 3.2] that there exists $c > 0$ such that

$$
\|\mathbf{v}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \ \geq\ c\, \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}} \qquad \forall \mathbf{v}_{\mathrm{D}} \in \widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_{\mathrm{D}})\,,
\tag{4.6}
$$

which, together with the fact that $\mathbf{K}$ is positive definite, imply the strong coerciveness property for $\mathbf{A}_{1\mathrm{D}} : \widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_{\mathrm{D}}) \to \widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_{\mathrm{D}})'$. In this way, (4.5) and (4.6) yield (4.2) and complete the proof. ∎

As previously announced, note that the assumption i) required by Theorem 3.1 follows from (4.1), using that $\|\mathbf{A}_1(\underline{\mathbf{r}}) - \mathbf{A}_1(\underline{\mathbf{s}})\|_{\tilde{X}_1'} \leq \|\mathbf{A}_1(\underline{\mathbf{r}}) - \mathbf{A}_1(\underline{\mathbf{s}})\|_{\mathbf{X}_1'} \ \leq\ \gamma\, \|\underline{\mathbf{r}} - \underline{\mathbf{s}}\|_{X_1}$, and from (4.2) (with $\underline{\mathbf{t}} = \mathbf{0}$).

We continue the analysis with the inf-sup conditions for $\mathbf{B}_1$ and $\mathbb{B}$ (cf. iv) and v) in Theorem 3.1).

Lemma 4.2 *There exists a constant $\beta_1 > 0$ such that*

$$
\sup_{\substack{\underline{\mathbf{r}} \in \tilde{\mathbf{X}}_1 \\ \underline{\mathbf{r}} \neq \mathbf{0}}} \frac{[\mathbf{B}_1(\underline{\mathbf{r}}), \underline{\boldsymbol{\psi}}]}{\|\underline{\mathbf{r}}\|_{\mathbf{X}_1}} \ \geq\ \beta_1\, \|\underline{\boldsymbol{\psi}}\|_{\mathbf{M}_1} \quad \forall\, \underline{\boldsymbol{\psi}} \in \widetilde{\mathbf{M}}_1\,.
\tag{4.7}
$$

Proof. These results are very similar to the ones provided in [25, Lemma 3.8]. Indeed, because of the diagonal character of $\mathbf{B}_1$ (cf. (2.28)), one first realizes that (4.7) is equivalent to finding positive constants $\tilde{\beta}_{\mathrm{S}}$ and $\tilde{\beta}_{\mathrm{D}}$ such that

$$
\sup_{\boldsymbol{\tau}_{\mathrm{S}} \in \tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}}) \backslash \mathbf{0}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S}}\, \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}}} \ \geq\ \tilde{\beta}_{\mathrm{S}}\, \|\boldsymbol{\psi}\|_{1/2,\Sigma} \quad \forall\, \boldsymbol{\psi} \ \in\ \tilde{\mathbf{H}}^{1/2}_{00}(\Sigma)\,,
\tag{4.8}
$$

and

$$
\sup_{\mathbf{v}_{\mathrm{D}} \in \tilde{\mathbf{H}}(\mathrm{div};\Omega_{\mathrm{D}}) \backslash \mathbf{0}} \frac{\langle \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, \xi \rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \ \geq\ \tilde{\beta}_{\mathrm{D}}\, \|\xi\|_{1/2,\Sigma} \qquad \forall\, \xi \ \in\ H^{1/2}(\Sigma)\,.
\tag{4.9}
$$

The proof of (4.9) can be found in [27, Lemma 3.3] (see also [25, Lemma 3.8]), whereas for (4.8) we need to slightly modify the corresponding arguments given there. In fact, given $\boldsymbol{\chi} \in \mathbf{H}^{-1/2}_{00}(\Sigma)$ we let $\boldsymbol{\tau}$ be the $\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$–component of $\mathbf{e}(\mathbf{z}) \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$, where $\mathbf{z} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$ is the unique solution of the boundary value problem:

$$
\mathbf{div}\, \mathbf{e}(\mathbf{z}) \,=\, \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{z} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}\,, \qquad \mathbf{e}(\mathbf{z})\, \mathbf{n} \,=\, \boldsymbol{\chi} \quad \text{on} \quad \Sigma\,.
\tag{4.10}
$$

In other words, $\boldsymbol{\tau} \,:=\, \mathbf{e}(\mathbf{z}) \,-\, c\,\mathbb{I}$, where $c \,:=\, \dfrac{1}{n\,|\Omega_{\mathrm{S}}|} \displaystyle\int_{\Omega_{\mathrm{S}}} \mathrm{tr}\, \mathbf{e}(\mathbf{z})$ (cf. (2.21)), which implies that $\boldsymbol{\tau} \in \tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}})$ and $\boldsymbol{\tau}\, \mathbf{n} \,=\, \boldsymbol{\chi} - c\,\mathbf{n} \quad \text{on} \quad \Sigma$. It follows that $\langle \boldsymbol{\tau}\, \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma \,=\, \langle \boldsymbol{\chi}, \boldsymbol{\psi} \rangle_\Sigma$ for each $\boldsymbol{\psi} \in \tilde{\mathbf{H}}^{1/2}_{00}(\Sigma)$, which proves the surjectivity of the operator $\boldsymbol{\tau} \,\to\, \boldsymbol{\tau}\, \mathbf{n}$ from $\tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}})$ to $\big(\tilde{\mathbf{H}}^{1/2}_{00}(\Sigma)\big)'$, that is (4.8). ∎

Lemma 4.3 *There exists a constant $\beta > 0$ such that*

$$\sup_{\substack{(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})\in\mathbb{X}\\ (\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})\neq\mathbf{0}}} \frac{[\mathbb{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{q}]}{\|(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})\|_{\mathbb{X}}} \geq \beta\,\|\underline{q}\|_{\mathbb{M}} \qquad \forall\,\underline{q}\in\mathbb{M}\,. \tag{4.11}$$

Proof. Analogously to the proof of Lemma 4.2, and because of the structure of $\mathbb{B}$ (cf. (2.31)), we find that (4.11) is equivalent to the following three independent inequalities

$$\sup_{\boldsymbol{\tau}_{\mathrm{S}}\in\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})\backslash\mathbf{0}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + (\boldsymbol{\tau}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}}} \geq \beta_{\mathrm{S}}\,\|(\mathbf{v}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}})\| \qquad \forall\,(\mathbf{v}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}}) \in \mathbf{L}^2(\Omega_{\mathrm{S}})\times\mathbb{L}^2(\Omega_{\mathrm{S}})\,, \tag{4.12}$$

$$\sup_{\mathbf{v}_{\mathrm{D}}\in\mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{D}})\backslash\mathbf{0}} \frac{(\mathrm{div}\,\mathbf{v}_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \geq \beta_{\mathrm{D}}\,\|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \qquad \forall\,q_{\mathrm{D}} \in L^2_0(\Omega_{\mathrm{D}})\,, \tag{4.13}$$

and

$$\sup_{\boldsymbol{\psi}\in\mathbf{H}^{1/2}_{00}(\Sigma)\backslash\mathbf{0}} \frac{\jmath\,\langle\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma}{\|\boldsymbol{\psi}\|_{1/2,\Sigma}} \geq \beta_\Sigma\,|\jmath| \qquad \forall\,\jmath\in\mathbb{R}\,, \tag{4.14}$$

with β_{S}, β_{D}, $\beta_\Sigma > 0$. Actually, except for the term $(\boldsymbol{\tau}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}}$ appearing in (4.12), the statement of the present lemma coincides with the one provided in [25, Lemma 3.6]. Hence, for the derivation of (4.13) and (4.14) we simply refer to the proof of that result, whereas the proof of (4.12), being a slight modification of [25, eq. (3.4)], can be found in several places (see, e.g. [20, Lemma 3.4]). In particular, we recall that the proof of (4.14) relies on the existence of a fixed element $\psi_0 \in \mathbf{H}^{1/2}(\Sigma)$ such that $\langle\mathbf{n},\boldsymbol{\psi}_0\rangle_\Sigma \neq 0$ (see [25, Section 3.2] for details).

■

We now check that the assumptions I), II) and III) specified in Section 3 are satisfied by our variational formulation (2.34). For this purpose we decompose $\mathbf{X}_1$ (cf. (2.26)) as $\mathbf{X}_1^\ell\times\mathbf{X}_1^r$, where $\mathbf{X}_1^\ell := \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}})$ and $\mathbf{X}_1^r := \mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{D}})\times\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$. Then, it is easy to see from (2.28) that $\mathbf{B}_1$ does not depend on the variable from $\mathbf{X}_1^\ell$. In addition, it is clear from (2.27) that for each $\underline{\mathbf{t}} := (\mathbf{0},\mathbf{u}_{\mathrm{D}},\boldsymbol{\sigma}_{\mathrm{S}})$, $\underline{\mathbf{r}} := (\mathbf{r}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}},\boldsymbol{\tau}_{\mathrm{S}})\in\mathbf{X}_1$ there holds

$$[\mathbf{A}_1(\underline{\mathbf{t}}),\underline{\mathbf{r}}] := (\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} - (\mathbf{r}_{\mathrm{S}},\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S}})_{\mathrm{S}} + \rho\,(\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S}},\boldsymbol{\tau}^{\mathtt{d}}_{\mathrm{S}})_{\mathrm{S}}\,,$$

which shows that $\mathbf{A}_1$ is linear in $\{\mathbf{0}\}\times\mathbf{X}_1^r$. Similarly, from the definition of $\mathbf{A}_1$ we also find that for each $\underline{\mathbf{t}} := (\underline{\mathbf{t}}^\ell,\underline{\mathbf{t}}^r) := (\mathbf{t}_{\mathrm{S}},(\mathbf{u}_{\mathrm{D}},\boldsymbol{\sigma}_{\mathrm{S}}))\in\mathbf{X}_1 := \mathbf{X}_1^\ell\times\mathbf{X}_1^r$ and for each $\underline{\mathbf{r}} := (\mathbf{r}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}},\boldsymbol{\tau}_{\mathrm{S}})\in\mathbf{X}_1$ there holds

$$\begin{aligned}[\mathbf{A}_1(\underline{\mathbf{t}}^\ell,\mathbf{0}),\underline{\mathbf{r}}] + [\mathbf{A}_1(\mathbf{0},\underline{\mathbf{t}}^r),\underline{\mathbf{r}}] &= (\mu(|\mathbf{t}_{\mathrm{S}}|)\,\mathbf{t}_{\mathrm{S}},\mathbf{r}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{t}_{\mathrm{S}},\boldsymbol{\tau}^{\mathtt{d}}_{\mathrm{S}})_{\mathrm{S}} - \rho\,(\mu(|\mathbf{t}_{\mathrm{S}}|)\,\mathbf{t}_{\mathrm{S}},\boldsymbol{\tau}^{\mathtt{d}}_{\mathrm{S}})_{\mathrm{S}}\\ &+ (\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} - (\mathbf{r}_{\mathrm{S}},\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S}})_{\mathrm{S}} + \rho\,(\boldsymbol{\sigma}^{\mathtt{d}}_{\mathrm{S}},\boldsymbol{\tau}^{\mathtt{d}}_{\mathrm{S}})_{\mathrm{S}} = [\mathbf{A}_1(\underline{\mathbf{t}}),\underline{\mathbf{r}}]\,,\end{aligned}$$

which proves that $\mathbf{A}_1(\underline{\mathbf{t}}) = \mathbf{A}_1(\underline{\mathbf{t}}^\ell,\mathbf{0}) + \mathbf{A}_1(\mathbf{0},\underline{\mathbf{t}}^r)$. It follows from the previous analysis that $\mathbf{A}_1$ satisfies the pseudolinear property (3.5), which confirms the verification of the hypothesis ii) of Theorem 3.1.

On the other hand, it is quite straightforward from (2.19) and (2.29) that for each $\underline{\boldsymbol{\psi}} := (\boldsymbol{\psi},\xi)\in\mathbf{M}_1$ there holds

$$[\mathbf{S}(\underline{\boldsymbol{\psi}}),\underline{\boldsymbol{\psi}}] = \sum_{\ell=1}^{n-1}\kappa_\ell^{-1}\,\|\boldsymbol{\psi}\cdot\mathbf{t}_\ell\|^2_{0,\Sigma} \geq 0\,, \tag{4.15}$$

which shows the positive definiteness of $\mathbf{S}$, thus verifying the hypothesis iii) of Theorem 3.1.

We are now ready to establish the main result concerning the existence and uniqueness of solution of the problem (2.34).

Theorem 4.1 *Assume that the parameter ρ appearing in the definition of the non linear operator $\mathbf{A}_1$ (cf. (2.27)) lies in $\left(0, \frac{\alpha_0}{\gamma_0^2}\right)$, where γ_0 and α_0 are the positive constants from (2.4) and (2.5). Then, there exists a unique $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) \in \mathbb{X} \times \mathbb{M}$ solution of (2.34). Moreover, there exists $C > 0$, depending only on α, γ, β_1, β, $\|S\|$, and $\|\mathbf{B}_1\|$, such that*

$$\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p})\|_{\mathbb{X}\times\mathbb{M}} \leq C \left\{ \|\mathbb{F}\|_{\mathbb{X}'} + \|\mathbb{G}\|_{\mathbb{M}'} \right\}. \tag{4.16}$$

Proof. Thanks to the analysis developed in this section, the proof follows from a direct application of Theorem 3.1.

∎

We end this section with the converse of the derivation of the variational formulation (2.34).

Theorem 4.2 *Let $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) \in \mathbb{X}\times\mathbb{M}$ be the unique solution of the variational formulation (2.34) with $\mathbb{F}$ and $\mathbb{G}$ given by (2.33), and define $p_{\mathrm{S}} := -\frac{1}{2}\mathrm{tr}\,(\boldsymbol{\sigma}_{\mathrm{S}})$. Then $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$, $p_{\mathrm{D}} \in H^1(\Omega_{\mathrm{D}})$, $\boldsymbol{\varphi} = -\mathbf{u}_{\mathrm{S}}$ on Σ, $\lambda = p_{\mathrm{D}}$ on Σ, and we have a solution of the system (2.8), (2.2), and (2.3).*

Proof. It basically follows by applying integration by parts backwardly in (2.34), and using suitable test functions. We omit further details.

∎

5 The mixed finite element scheme

In this section we introduce the Galerkin scheme of problem (2.34) and analyze its well-posedness by establishing suitable assumptions on the finite element subspaces involved. Then, we provide specific examples of these subspaces satisfying the required hypotheses.

5.1 Preliminaries

We begin by selecting a set of arbitrary discrete spaces, namely

$$\begin{gathered}\mathbf{H}_h(\Omega_{\mathrm{S}}) \subseteq \mathbf{H}(\mathrm{div};\Omega_{\mathrm{S}}), \qquad \mathbb{L}^2_h(\Omega_{\mathrm{S}}) \subseteq \mathbb{L}^2(\Omega_{\mathrm{S}}), \qquad L_h(\Omega_{\mathrm{S}}) \subseteq L^2(\Omega_{\mathrm{S}}), \qquad \Lambda^{\mathrm{S}}_h(\Sigma) \subseteq H^{1/2}_{00}(\Sigma), \\ \mathbf{H}_h(\Omega_{\mathrm{D}}) \subseteq \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}), \qquad L_h(\Omega_{\mathrm{D}}) \subseteq L^2(\Omega_{\mathrm{D}}), \qquad \text{and} \quad \Lambda^{\mathrm{D}}_h(\Sigma) \subseteq H^{1/2}(\Sigma).\end{gathered} \tag{5.1}$$

Then, we define the subspaces

$$\begin{aligned}
\mathbb{H}_h(\Omega_{\mathrm{S}}) &:= \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) : \quad \boldsymbol{c}^{\mathrm{t}}\boldsymbol{\tau} \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \quad \forall \boldsymbol{c} \in \mathbb{R}^n \right\}, \\
\mathbb{H}_{0,h}(\Omega_{\mathrm{S}}) &:= \mathbb{H}_h(\Omega_{\mathrm{S}}) \cap \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}), \\
\mathbf{H}_{0,h}(\Omega_{\mathrm{D}}) &:= \mathbf{H}_h(\Omega_{\mathrm{D}}) \cap \mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{D}}), \\
\mathbf{L}_h(\Omega_{\mathrm{S}}) &:= [L_h(\Omega_{\mathrm{S}})]^n, \\
\mathbb{L}_h(\Omega_{\mathrm{S}}) &:= [L_h(\Omega_{\mathrm{S}})]^{n\times n}, \\
\mathbb{L}_{0,h}(\Omega_{\mathrm{S}}) &:= \mathbb{L}_h(\Omega_{\mathrm{S}}) \cap \overline{\mathbb{L}}^2_0(\Omega_{\mathrm{S}}), \quad \text{and} \\
\boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma) &:= [\Lambda^{\mathrm{S}}_h(\Sigma)]^n.
\end{aligned} \tag{5.2}$$

In addition, in order to deal with the mean value condition for the Darcy pressure p_{D}, we define

$$L_{0,h}(\Omega_{\mathrm{D}}) := L_h(\Omega_{\mathrm{D}}) \cap L_0^2(\Omega_{\mathrm{D}}) . \tag{5.3}$$

Then, the global unknowns and corresponding finite element subspaces are as follows:

$$\begin{array}{rcccl} \underline{\mathbf{t}}_h & := & (\mathbf{t}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\sigma}_{\mathrm{S},h}) & \in \mathbf{X}_{1,h} & := \mathbb{L}_{0,h}(\Omega_{\mathrm{S}}) \times \mathbf{H}_{0,h}(\Omega_{\mathrm{D}}) \times \mathbb{H}_{0,h}(\Omega_{\mathrm{S}}) , \\ \underline{\boldsymbol{\varphi}}_h & := & (\boldsymbol{\varphi}_h, \lambda_h) & \in \mathbf{M}_{1,h} & := \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) \times \Lambda_h^{\mathrm{D}}(\Sigma) , \\ \underline{p}_h & := & (p_{\mathrm{D},h}, \mathbf{u}_{\mathrm{S},h}, \boldsymbol{\gamma}_{\mathrm{S},h}, \ell_h) & \in \mathbb{M}_h & := L_{0,h}(\Omega_{\mathrm{D}}) \times \mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbb{L}_h^2(\Omega_{\mathrm{S}}) \times \mathbb{R} . \end{array} \tag{5.4}$$

In this way, setting $\mathbb{X}_h := \mathbf{X}_{1,h} \times \mathbf{M}_{1,h}$, the Galerkin scheme for (2.34) reduces to: Find $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ such that

$$\begin{array}{rcll} [\mathbb{A}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbb{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{p}_h] & = & [\mathbb{F}, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] & \forall\, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{X}_h , \\ [\mathbb{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{q}_h] & = & [\mathbb{G}, \underline{q}_h] & \forall\, \underline{q}_h \in \mathbb{M}_h . \end{array} \tag{5.5}$$

5.2 The main results

We now adapt the analysis from Section 4 to the discrete case and follow very closely the approach from [25, Section 4.1] to establish general hypotheses on the finite element subspaces (5.1) ensuring, by means of the abstract theory developed in Section 3.2, the well-posedness of (5.5). We begin by observing that in order to have meaningful spaces $\mathbb{H}_{0,h}(\Omega_{\mathrm{S}})$ and $L_{0,h}(\Omega_{\mathrm{D}})$ (cf. (5.2) and (5.3)), we need to be able to eliminate multiples of the identity matrix $\mathbb{I}$ from $\mathbb{H}_h(\Omega_{\mathrm{S}})$ and the constant polynomials from $L_h(\Omega_{\mathrm{D}})$. This request is certainly satisfied if we assume the following:

(H.0) $[P_0(\Omega_{\mathrm{S}})]^2 \subseteq \mathbf{H}_h(\Omega_{\mathrm{S}})$ and $P_0(\Omega_{\mathrm{D}}) \subseteq L_h(\Omega_{\mathrm{D}})$,

where $P_0(\Omega_{\mathrm{S}})$ and $P_0(\Omega_{\mathrm{D}})$ are the spaces of constant polynomials on Ω_{S} and Ω_{D}, respectively. In particular, it follows that $\mathbb{I} \in \mathbb{H}_h(\Omega_{\mathrm{S}})$ for all h, and hence there holds the decomposition:

$$\mathbb{H}_h(\Omega_{\mathrm{S}}) = \mathbb{H}_{0,h}(\Omega_{\mathrm{S}}) \oplus P_0(\Omega_{\mathrm{S}})\mathbb{I} . \tag{5.6}$$

Next, according to the same diagonal argument utilized in the proof of Lemma 4.3 (see also [25, Lemma 3.6]) we deduce that $\mathbb{B}$ satisfies the discrete inf-sup condition uniformly on $\mathbb{X}_h \times \mathbb{M}_h$ if and only if there exist $\widetilde{\beta}_{\mathrm{S}}$, $\widetilde{\beta}_{\mathrm{D}}$, $\widetilde{\beta}_{\Sigma} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_{0,h}(\Omega_{\mathrm{S}}) \\ \boldsymbol{\tau}_{\mathrm{S},h} \neq \mathbf{0}}} \frac{(\mathbf{div}\, \boldsymbol{\tau}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h})_{\mathrm{S}} + (\boldsymbol{\tau}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \geq \widetilde{\beta}_{\mathrm{S}} \, \|(\mathbf{v}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})\| \quad \forall\, (\mathbf{v}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h}) \in \mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbb{L}_h^2(\Omega_{\mathrm{S}}) , \tag{5.7}$$

$$\sup_{\substack{\mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_{0,h}(\Omega_{\mathrm{D}}) \\ \mathbf{v}_{\mathrm{D},h} \neq \mathbf{0}}} \frac{(\mathrm{div}\, \mathbf{v}_{\mathrm{D},h}, q_{\mathrm{D},h})_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \geq \widetilde{\beta}_{\mathrm{D}} \, \|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} \quad \forall\, q_{\mathrm{D},h} \in L_{0,h}(\Omega_{\mathrm{D}}) , \tag{5.8}$$

$$\sup_{\substack{\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) \\ \boldsymbol{\psi}_h \neq \mathbf{0}}} \frac{\jmath \, \langle \mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\psi}_h\|_{1/2,\Sigma}} \geq \widetilde{\beta}_\Sigma \, |\jmath| \quad \forall\, \jmath \in \mathbb{R} . \tag{5.9}$$

However, since $\mathbf{div}\, \mathbb{H}_h(\Omega_{\mathrm{S}}) = \mathbf{div}\, \mathbb{H}_{0,h}(\Omega_{\mathrm{S}})$ (cf. (5.6)) and $(\mathbb{I}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}} = 0$ (because of the symmetry of $\mathbb{I}$ and the skew-symmetry of $\boldsymbol{\eta}_{\mathrm{S},h}$), we find that the supremum in (5.7) remains the same if taken on $\mathbb{H}_h(\Omega_S)$ instead of $\mathbb{H}_{0,h}(\Omega_S)$. Notice also that a sufficient condition for (5.9) is the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$ such that $\boldsymbol{\psi}_0 \in \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma)$ for all h and $\langle \mathbf{n}, \boldsymbol{\psi}_0\rangle_\Sigma \neq 0$. Consequently, we now introduce the following hypotheses summarizing the above analysis:

(H.1) There exist $\widetilde{\beta}_{\mathrm{S}}$, $\widetilde{\beta}_{\mathrm{D}} > 0$, independent of h, and there exists $\boldsymbol{\psi}_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$, such that

$$\sup_{\substack{\boldsymbol{\tau}_{\mathrm{S},h}\in\mathbb{H}_h(\Omega_{\mathrm{S}})\\ \boldsymbol{\tau}_{\mathrm{S},h}\neq\mathbf{0}}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h})_{\mathrm{S}} + (\boldsymbol{\tau}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}}}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \geq \widetilde{\beta}_{\mathrm{S}}\, \|(\mathbf{v}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})\| \quad \forall\, (\mathbf{v}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h}) \in \mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbb{L}_h^2(\Omega_{\mathrm{S}})\,, \tag{5.10}$$

$$\sup_{\substack{\mathbf{v}_{\mathrm{D},h}\in\mathbf{H}_{0,h}(\Omega_{\mathrm{D}})\\ \mathbf{v}_{\mathrm{D},h}\neq\mathbf{0}}} \frac{(\operatorname{div}\mathbf{v}_{\mathrm{D},h}, q_{\mathrm{D},h})_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D},h}\|_{\operatorname{div};\Omega_{\mathrm{D}}}} \geq \widetilde{\beta}_{\mathrm{D}}\, \|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} \quad \forall\, q_{\mathrm{D},h} \in L_{0,h}(\Omega_{\mathrm{D}})\,, \tag{5.11}$$

$$\boldsymbol{\psi}_0 \in \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) \quad \forall h \quad \text{and} \quad \langle \mathbf{n}, \boldsymbol{\psi}_0\rangle_\Sigma \neq 0\,. \tag{5.12}$$

On the other hand, we now look at the discrete kernel of $\mathbb{B}$, which is defined by

$$\mathbb{V}_h := \Big\{ (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{X}_h : \quad \mathbb{B}((\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{q}_h) = 0 \quad \forall \underline{q}_h \in \mathbb{M}_h \Big\}\,.$$

In addition, in order to have a more explicit definition of $\mathbb{V}_h$ we introduce the following assumption:

(H.2) $\operatorname{div}\,\mathbf{H}_h(\Omega_{\mathrm{S}}) \subseteq L_h(\Omega_{\mathrm{S}})$ and $\operatorname{div}\,\mathbf{H}_h(\Omega_{\mathrm{D}}) \subseteq L_h(\Omega_{\mathrm{D}})$.

It follows from the definition of $\mathbb{B}$ (cf. (2.31)) and **(H.2)** that $\mathbb{V}_h := \widetilde{\mathbf{X}}_{1,h} \times \widetilde{\mathbf{M}}_{1,h}$, where

$$\widetilde{\mathbf{X}}_{1,h} := \mathbb{L}_{0,h}(\Omega_{\mathrm{S}}) \times \widetilde{\mathbf{H}}_{0,h}(\Omega_{\mathrm{D}}) \times \widetilde{\mathbb{H}}_{0,h}(\Omega_{\mathrm{S}}) \quad \text{and} \quad \widetilde{\mathbf{M}}_{1,h} := \boldsymbol{\Lambda}_{0,h}^{\mathrm{S}}(\Sigma) \times \Lambda_h^{\mathrm{D}}(\Sigma)\,,$$

with

$$\widetilde{\mathbf{H}}_{0,h}(\Omega_{\mathrm{D}}) := \Big\{ \mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_{0,h}(\Omega_{\mathrm{D}}) : \quad \operatorname{div}\mathbf{v}_{\mathrm{D},h} \in \mathbb{P}_0(\Omega_{\mathrm{D}}) \Big\}\,,$$

$$\widetilde{\mathbb{H}}_{0,h}(\Omega_{\mathrm{S}}) := \Big\{ \boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_{0,h}(\Omega_{\mathrm{S}}) : \quad (\boldsymbol{\tau}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}} = 0 \quad \forall\, \boldsymbol{\eta}_{\mathrm{S},h} \in \mathbb{L}_h^2(\Omega_{\mathrm{S}}) \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S},h} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}} \Big\}\,,$$

and

$$\boldsymbol{\Lambda}_{0,h}^{\mathrm{S}}(\Sigma) := \Big\{ \boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) : \quad \langle \mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma = 0 \Big\}\,.$$

Then, applying the same diagonal argument employed in the proof of Lemma 4.2 (see also [25, Lemma 3.8]), we find that $\mathbf{B}_1$ satisfies the discrete inf-sup condition uniformly on $\widetilde{\mathbf{X}}_{1,h} \times \widetilde{\mathbf{M}}_{1,h}$ if and only if there exist $\widehat{\beta}_{\mathrm{S}}$, $\widehat{\beta}_{\mathrm{D}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{\mathrm{S},h}\in\widetilde{\mathbb{H}}_{0,h}(\Omega_{\mathrm{S}})\\ \boldsymbol{\tau}_{\mathrm{S},h}\neq\mathbf{0}}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\,\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \geq \widehat{\beta}_{\mathrm{S}}\, \|\boldsymbol{\psi}_h\|_{1/2,\Sigma} \qquad \forall \boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_{0,h}^{\mathrm{S}}(\Sigma), \tag{5.13}$$

$$\sup_{\substack{\mathbf{v}_{\mathrm{D},h}\in\widetilde{\mathbf{H}}_{0,h}(\Omega_{\mathrm{D}})\\ \mathbf{v}_{\mathrm{D},h}\neq\mathbf{0}}} \frac{\langle \mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D},h}\|_{\operatorname{div};\Omega_{\mathrm{D}}}} \geq \widehat{\beta}_{\mathrm{D}}\, \|\xi_h\|_{1/2,\Sigma} \qquad \forall \xi_h \in \Lambda_h^{\mathrm{D}}(\Sigma). \tag{5.14}$$

In addition, the characterization of the elements of $\mathbf{\Lambda}^{\mathrm{S}}_{0,h}(\Sigma)$ yields the supremum in (5.13) to remain unchanged if, instead of $\widetilde{\mathbb{H}}_{0,h}(\Omega_{\mathrm{S}})$, it is taken on

$$\widetilde{\mathbb{H}}_h(\Omega_{\mathrm{S}}) := \Big\{ \boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_h(\Omega_{\mathrm{S}}) : \quad (\boldsymbol{\tau}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}} = 0 \quad \forall\, \boldsymbol{\eta}_{\mathrm{S},h} \in \mathbb{L}^2_h(\Omega_{\mathrm{S}}) \quad \text{and} \quad \mathbf{div}\, \boldsymbol{\tau}_{\mathrm{S},h} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}} \Big\}. \tag{5.15}$$

In this way, we now add the following hypothesis:

(H.3) There exist $\widehat{\beta}_{\mathrm{S}}$, $\widehat{\beta}_{\mathrm{D}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{\mathrm{S},h} \in \widetilde{\mathbb{H}}_h(\Omega_{\mathrm{S}}) \\ \boldsymbol{\tau}_{\mathrm{S},h} \neq \mathbf{0}}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\, \mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div};\Omega_{\mathrm{S}}}} \geq \widehat{\beta}_{\mathrm{S}}\, \|\boldsymbol{\psi}_h\|_{1/2,\Sigma} \qquad \forall\, \boldsymbol{\psi}_h \in \mathbf{\Lambda}^{\mathrm{S}}_{0,h}(\Sigma), \tag{5.16}$$

$$\sup_{\substack{\mathbf{v}_{\mathrm{D},h} \in \widetilde{\mathbf{H}}_{0,h}(\Omega_{\mathrm{D}}) \\ \mathbf{v}_{\mathrm{D},h} \neq \mathbf{0}}} \frac{\langle \mathbf{v}_{\mathrm{D},h} \cdot \mathbf{n}, \xi_h \rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \geq \widehat{\beta}_{\mathrm{D}}\, \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \in \Lambda^{\mathrm{D}}_h(\Sigma). \tag{5.17}$$

From now on we assume that the arbitrary finite element subspaces introduced in (5.1) satisfy the above derived hypotheses **(H.0)**, **(H.1)**, **(H.2)** and **(H.3)**. Hence, we are in a position to prove that the assumptions required by Theorem 3.3 are satisfied. We begin with the following lemma which yields the hypothesis i) of that theorem and the assumption (3.26) (cf. **(A.1)**) as well.

Lemma 5.1 *Let $\gamma > 0$ be the same constant provided by Lemma 4.1. Then*

$$\|\mathbf{A}_1(\underline{\mathbf{r}}_h) - \mathbf{A}_1(\underline{\mathbf{s}}_h)\|_{\widetilde{\mathbf{X}}'_{1,h}} \leq \gamma\, \|\underline{\mathbf{r}}_h - \underline{\mathbf{s}}_h\|_{\mathbf{X}_1} \quad \forall\, \underline{\mathbf{r}}_h, \underline{\mathbf{s}}_h \in \widetilde{\mathbf{X}}_{1,h}. \tag{5.18}$$

Furthermore, assume that the parameter ρ lies in $\left(0, \dfrac{\alpha_0}{\gamma_0^2}\right)$, where α_0 and γ_0 are the positive constants from (2.4) and (2.5), and let $\alpha > 0$ be the same constant provided by Lemma 4.1. Then

$$[\mathbf{A}_1(\underline{\mathbf{t}}_h + \underline{\mathbf{r}}_h) - \mathbf{A}_1(\underline{\mathbf{t}}_h + \underline{\mathbf{s}}_h), \underline{\mathbf{r}}_h - \underline{\mathbf{s}}_h] \geq \alpha\, \|\underline{\mathbf{r}}_h - \underline{\mathbf{s}}_h\|^2_{\mathbf{X}_1} \qquad \forall\, \underline{\mathbf{t}}_h \in X_{1,h}\,, \quad \forall\, \underline{\mathbf{r}}_h, \underline{\mathbf{s}}_h \in \widetilde{\mathbf{X}}_{1,h}\,. \tag{5.19}$$

Proof. It is clear that (5.18) follows straightforwardly from (4.1) by noting that

$$\|\mathbf{A}_1(\underline{\mathbf{r}}_h) - \mathbf{A}_1(\underline{\mathbf{s}}_h)\|_{\widetilde{\mathbf{X}}'_{1,h}} \;\leq\; \|\mathbf{A}_1(\underline{\mathbf{r}}_h) - \mathbf{A}_1(\underline{\mathbf{s}}_h)\|_{\mathbf{X}'_1}\,.$$

In turn, similarly as for the continuous case, the discrete strong monotonicity (5.19) follows from the corresponding property of the operator $\mathbf{A}_{1\mathrm{S}}|_{\mathbb{L}_{0,h}(\Omega_{\mathrm{S}}) \times \widetilde{\mathbb{H}}_{0,h}(\Omega_{\mathrm{S}})} : \mathbb{L}_{0,h}(\Omega_{\mathrm{S}}) \times \widetilde{\mathbb{H}}_{0,h}(\Omega_{\mathrm{S}}) \to \mathbb{L}_{0,h}(\Omega_{\mathrm{S}})' \times \widetilde{\mathbb{H}}_{0,h}(\Omega_{\mathrm{S}})'$, which makes use now of the fact that $\mathbf{div}\, \boldsymbol{\tau}_{\mathrm{S},h} = \mathbf{0} \quad \forall\, \boldsymbol{\tau}_{\mathrm{S},h} \in \widetilde{\mathbb{H}}_{0,h}(\Omega_{\mathrm{S}})$, and also from the strong coercivenes of $\mathbf{A}_{1\mathrm{D}}|_{\widetilde{\mathbf{H}}_{0,h}(\Omega_{\mathrm{D}})} : \widetilde{\mathbf{H}}_{0,h}(\Omega_{\mathrm{D}}) \to \widetilde{\mathbf{H}}_{0,h}(\Omega_{\mathrm{D}})'$. We omit further details and refer to the proofs of Lemma 4.1 and [22, Lemmas 3.1 and 3.2].

■

As stated in advance, we note here that the hypothesis i) in Theorem 3.3 is given by (5.18) and (5.19) (with $\underline{\mathbf{t}}_h = \mathbf{0}$), whereas (5.19) is precisely (3.26) (cf. **(A.1)**). We observe next, according to (4.15), that for each $\underline{\boldsymbol{\psi}}_h := (\boldsymbol{\psi}_h, \xi_h) \in \mathbf{M}_{1,h} \subseteq \mathbf{M}_1$ there holds

$$[\mathbf{S}(\underline{\boldsymbol{\psi}}_h), \underline{\boldsymbol{\psi}}_h] \;=\; \sum_{\ell=1}^{n-1} \kappa_\ell^{-1}\, \|\boldsymbol{\psi}_h \cdot \mathbf{t}_\ell\|^2_{0,\Sigma} \;\geq\; 0\,, \tag{5.20}$$

which yields the hypothesis iii) of Theorem 3.3. The analysis is continued with the discrete inf-sup conditions for $\mathbf{B}_1$ and $\mathbb{B}$ (cf. iv) and v) in Theorem 3.3).

Lemma 5.2 *There exists a constant $\widehat{\beta}_1 > 0$, independent of h, such that*

$$\sup_{\substack{\underline{\mathbf{r}}_h \in \widetilde{\mathbf{X}}_1 \\ \underline{\mathbf{r}}_h \neq \mathbf{0}}} \frac{[\mathbf{B}_1(\underline{\mathbf{r}}_h), \underline{\boldsymbol{\psi}}_h]}{\|\underline{\mathbf{r}}_h\|_{\mathbf{X}_1}} \geq \widehat{\beta}_1 \, \|\underline{\boldsymbol{\psi}}_h\|_{\mathbf{M}_1} \qquad \forall \, \underline{\boldsymbol{\psi}}_h \in \widetilde{\mathbf{M}}_{1,h} \, .$$

Proof. It follows directly from **(H.3)**. ∎

Lemma 5.3 *There exists a constant $\widehat{\beta} > 0$, independent of h, such that*

$$\sup_{\substack{(\underline{\mathbf{r}}_h \underline{\boldsymbol{\psi}}_h) \in \mathbb{X}_h \\ (\underline{\mathbf{r}}_h \underline{\boldsymbol{\psi}}_h) \neq \mathbf{0}}} \frac{[\mathbb{B}(\underline{\mathbf{r}}_h \underline{\boldsymbol{\psi}}_h), \underline{q}_h]}{\|(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)\|_{\mathbb{X}}} \geq \widehat{\beta} \, \|\underline{q}_h\|_{\mathbb{M}} \qquad \forall \, \underline{q}_h \in \mathbb{M}_h \, .$$

Proof. It follows directly from **(H.1)**. ∎

The following theorem establishes the well posedness of (5.5).

Theorem 5.1 *Assume that the hypotheses* **(H.0)**, **(H.1)**, **(H.2)** *and* **(H.3)** *hold, and that ρ lives in $\left(0, \frac{\alpha_0}{\gamma_0^2}\right)$. Then, the Galerkin scheme* (5.5) *has a unique solution $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h) \in \mathbb{X}_h \times \mathbb{M}_h$, and there exists $C > 0$, depending only on α, γ, $\widehat{\beta}_1$, $\widehat{\beta}$, $\|S\|$ and $\|\mathbf{B}_1\|$, such that*

$$\|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}} \leq C \left\{ \|\mathbb{F}|_{\mathbb{X}_h}\|_{\mathbb{X}_h'} + \|\mathbb{G}|_{\mathbb{M}_h}\|_{\mathbb{M}_h'} \right\} . \tag{5.21}$$

Proof. According to the previous analysis, the proof follows from a direct application of Theorem 3.3. ∎

We end this section with the corresponding Cea a priori error estimate. To this end, we first recall from Section 2.2 that μ is assumed to be of class C^1, which yields the assumption **(A.2)**, that is the hemi-continuity of the Gâteaux derivative $\mathcal{D}A_1 : X_1 \to \mathcal{L}(X_1, X_1')$. Consequently, we have the following result.

Theorem 5.2 *Assume that the hypotheses* **(H.0)**, **(H.1)**, **(H.2)** *and* **(H.3)** *hold, and that ρ lives in $\left(0, \frac{\alpha_0}{\gamma_0^2}\right)$. Let $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) \in \mathbb{X} \times \mathbb{M}$ and $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ be the unique solutions of the continuous and discrete formulations* (2.34) *and* (5.5), *respectively. Then, there exists $C > 0$, independent of h, such that*

$$\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}} \leq C \inf_{((\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{q}_h) \in \mathbb{X}_h \times \mathbb{M}_h} \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) - ((\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{q}_h)\|_{\mathbb{X}\times\mathbb{M}} \tag{5.22}$$

Proof. It is a straightforward application of Theorem 3.5. ∎

5.3 Particular choices of finite element subspaces

We now specify concrete examples of finite element subspaces in 2D and 3D satisfying the hypotheses introduced in the previous section. To this end, we let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D} formed by shape-regular triangles (in R^2) or tetrahedra (in R^3), and assume that they match in Σ so that $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. We also let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$). Furthermore, given an integer $k \geq 0$ and a subset S of $\mathbb{R}^n$, we denote by $P_k(S)$ the space of polynomials defined on S of total degree at most k. Note that, according to the notation described in Section 1, $\mathbf{P}_k(S)$ and $\mathbb{P}_k(S)$ stand for $[P_k(S)]^n$ and $[P_k(S)]^{n\times n}$, respectively. In addition, we let b_T be the element-bubble function defined as the unique polynomial in $P_{n+1}(T)$ vanishing on ∂T with $\int_T b_T = 1$, and denote by $\mathbf{x} := (x_1, x_2, \cdots, x_n)$ a generic vector of R^n. Then, we define for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ the local Raviart-Thomas and bubble spaces of order 0, respectively, by (see, e.g. [7], [38])

$$\mathrm{RT}_0(T) := \mathbf{P}_0(T) \oplus P_0(T)\,\mathbf{x}\,,$$

and

$$B_0(T) := \begin{cases} P_0(T)\left(\frac{\partial b_T}{\partial x_2}\,,\ -\frac{\partial b_T}{\partial x_1}\right) & \text{in} \quad \mathrm{R}^2\,, \\ \nabla \times \big(b_T\,\mathbf{P}_0(T)\big) & \text{in} \quad \mathrm{R}^3\,. \end{cases}$$

5.3.1 PEERS + Raviart-Thomas in 2D

We specify the discrete spaces in (5.1) as follows:

$$\begin{aligned}
\mathbf{H}_h(\Omega_{\mathrm{S}}) &:= \Big\{\tau \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{S}}): \quad \tau|_T \in \mathrm{RT}_0(T) \oplus B_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}\Big\}\,, \\
\mathbf{H}_h(\Omega_{\mathrm{D}}) &:= \Big\{\tau \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}): \quad \tau|_T \in \mathrm{RT}_0(T) \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}\Big\}\,, \\
L_h(\Omega_{\mathrm{S}}) &:= \Big\{v \in L^2(\Omega_{\mathrm{S}}): \quad v|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}\Big\}\,, \\
L_h(\Omega_{\mathrm{D}}) &:= \Big\{q \in L^2(\Omega_{\mathrm{D}}): \quad q|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}\Big\}\,, \quad \text{and} \\
\mathbb{L}_h^2(\Omega_{\mathrm{S}}) &:= \Big\{\boldsymbol{\eta} := \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}: \quad \eta \in C(\bar{\Omega}_{\mathrm{S}})\,, \quad \eta|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}\Big\}\,.
\end{aligned} \tag{5.23}$$

Note here that the product space $\mathbb{H}_h(\Omega_{\mathrm{S}}) \times \mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbb{L}_h^2(\Omega_{\mathrm{S}})$, with $\mathbb{H}_h(\Omega_{\mathrm{S}})$ and $\mathbf{L}_h(\Omega_{\mathrm{S}})$ defined according to (5.2), constitutes the classical PEERS originally introduced in [1] for a mixed finite element aproximation of the linear elasticity problem with Dirichlet boundary conditions (see also [33]). In turn, $\mathbf{H}_h(\Omega_{\mathrm{D}}) \times L_h(\Omega_{\mathrm{D}})$ is the Raviart-Thomas stable element of lowest order for the mixed formulation of the Poisson problem (see, e.g. [7], [36]). These facts are particularly important for the rest of the analysis since, as we will make it clear below, all the discrete inf-sup conditions that are required in the hypotheses indicated in Section 5.2, either are already available in the literature or can be derived from related results provided there.

Next, in order to define the spaces on the interface Σ, thus completing the list in (5.1), we follow the simplest approach suggested in [25] and [35]. To this end, we assume, without loss of generality, that the number of edges e of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h, and denote the resulting edges still by e. Since Σ_h is inherited from the interior triangulations, it is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded) and, therefore, so is Σ_{2h}. Certainly, if the number of edges of Σ_h were odd, we simply reduce it to the even case by joining any pair of two adjacent elements, and then construct Σ_{2h} from this partition. Hence, denoting by x_0 and x_1 the extreme points of Σ, we define

$$\begin{aligned}
\Lambda_h^{\mathrm{S}}(\Sigma) &:= \Big\{\psi \in C(\Sigma): \quad \psi|_e \in P_1(e) \quad \forall\, e \in \Sigma_{2h}\,, \quad \psi(x_0) = \psi(x_1) = 0\Big\}\,, \quad \text{and} \\
\Lambda_h^{\mathrm{D}}(\Sigma) &= \Big\{\xi \in C(\Sigma): \quad \xi|_e \in P_1(e) \quad \forall\, e \in \Sigma_{2h}\Big\}\,.
\end{aligned} \tag{5.24}$$

Our analysis below will also utilize the finite element subspaces of $H_{00}^{-1/2}(\Sigma)$ and $\mathbf{H}_{00}^{-1/2}(\Sigma)$ given by

$$\Phi_h(\Sigma) := \Big\{\phi_h \in L^2(\Sigma): \quad \phi_h|_e \in P_0(e) \quad \forall\,\text{edge } e \in \Sigma_h\Big\}\,, \quad \text{and}$$

$$\mathbf{\Phi}_h(\Sigma) := \Big\{ \boldsymbol{\phi}_h \in \mathbf{L}^2(\Sigma) : \quad \boldsymbol{\phi}_h|_e \in \mathbf{P}_0(e) \quad \forall \,\text{edge } e \in \Sigma_h \Big\}.$$

In what follows we establish from (5.23), (5.24), and the accompanying definitions (5.2) and (5.4), that the hypotheses **(H.0) - (H.3)** are satisfied. In fact, the verification of **(H.0)** and **(H.2)** is quite straightforward from the definitions given in (5.23). Now, the discrete inf-sup conditions (5.10) and (5.11) are proved in [33, Theorem 4.5] and [7, Chapter IV, Section IV.1.2], respectively. Alternatively, one can also look at [1, Lemma 4.4] and [36, Chapter 7, Section 7.2.2]. In turn, the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}_{00}(\Sigma)$ verifying (5.12) follows as in [25, Section 3.2] (see also [27, Section 3.2]). In fact, we pick one interior corner point of Σ and define a function v that is continuous, linear on each side of Σ, equal to one in the chosen vertex, and zero on all other ones. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors on the two sides of Σ that meet at the corner point, then $\boldsymbol{\psi}_0 := v\,(\mathbf{n}_1 + \mathbf{n}_2)$ satisfies that property. If the interface Σ were a line segment (without interior corners), we pick v as the continuous linear function on Σ, equal to one in any interior point and zero in the extreme points, and define $\boldsymbol{\psi}_0 := v\,\mathbf{n}$. We have thus verified the assumptions required by **(H.1)**.

On the other hand, concerning the discrete inf-sup conditions yielding **(H.3)**, we first recall from the analyses in [25] and [35], that the existence of a stable discrete lifting of the normal traces of $\widetilde{\mathbf{H}}_{0,h}(\Omega_{\mathrm{D}})$ implies that a sufficient condition for (5.17) is the existence of $\widehat{\beta}_{\mathrm{D}} > 0$, independent of h, such that

$$\sup_{\substack{\phi_h \in \Phi_h(\Sigma)\\ \phi_h \neq \mathbf{0}}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \geq \widehat{\beta}_{\mathrm{D}}\, \|\xi_h\|_{1/2,\Sigma} \qquad \forall \xi_h \in \Lambda^{\mathrm{D}}_h(\Sigma)\,. \tag{5.25}$$

In fact, a detailed proof of (5.17), whose main ingredients were the explicit construction of such a lifting and then the demonstration of (5.25), was first provided in [25, Lemmas 4.2, 5.1 and 5.2] under the assumption of quasi-uniformity around the interface Σ. This result was improved recently in [35, Sections 4 and 5] where it was shown for the 2D case without any requirement on the meshes. In turn, in order to proceed similarly with (5.16), we need to introduce suitable changes into the arguments from [25] and [35]. The reason for it is rather technical and has to do with the fact that the tensors $\boldsymbol{\tau}_{\mathrm{S},h} \in \widetilde{\mathbb{H}}_h(\Omega_{\mathrm{S}})$ (cf. (5.15)), space where the supremum in (5.16) is taken, must also satisfy the discrete symmetry condition $(\boldsymbol{\tau}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h})_{\mathrm{S}} = 0 \quad \forall \boldsymbol{\eta}_{\mathrm{S},h} \in \mathbb{L}^2_h(\Omega_{\mathrm{S}})$. More precisely, since the Raviart-Thomas or related projection operators do not preserve any kind of symmetry, the way in which the lifting was built in [25] is not applicable to construct a stable discrete lifting of the normal traces of $\widetilde{\mathbb{H}}_h(\Omega_{\mathrm{S}})$. Instead of it, we now proceed a bit differently and still show, using results from [20], [25], and [35], that a sufficient condition for (5.16) is the analogue of (5.25), that is the existence of $\widehat{\beta}_{\mathrm{S}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\phi}_h \in \mathbf{\Phi}_h(\Sigma)\\ \boldsymbol{\phi}_h \neq \mathbf{0}}} \frac{\langle \boldsymbol{\phi}_h, \boldsymbol{\psi}_h\rangle_\Sigma}{\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}} \geq \widehat{\beta}_{\mathrm{S}}\, \|\boldsymbol{\psi}_h\|_{1/2,\Sigma} \qquad \forall \boldsymbol{\psi}_h \in \mathbf{\Lambda}^{\mathrm{S}}_{0,h}(\Sigma)\,. \tag{5.26}$$

In fact, given $\boldsymbol{\phi}_h \in \mathbf{\Phi}_h(\Sigma)$, we let $\big(\widetilde{\boldsymbol{\sigma}}_h, (\widetilde{\mathbf{u}}_h, \widetilde{\gamma}_h, \widetilde{\boldsymbol{\varphi}}_h)\big) \in \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \big(\mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbb{L}^2_h(\Omega_{\mathrm{S}}) \times \mathbf{\Lambda}^{\mathrm{S}}_h(\Sigma)\big)$ be the unique solution of the Galerkin scheme:

$$\begin{aligned}
(\widetilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h)_{\mathrm{S}} + (\mathbf{div}\,\boldsymbol{\tau}_h, \mathbf{u}_h)_{\mathrm{S}} + (\boldsymbol{\tau}_h, \widetilde{\gamma}_h)_{\mathrm{S}} + \langle \boldsymbol{\tau}_h\,\mathbf{n}, \widetilde{\boldsymbol{\varphi}}_h\rangle_\Sigma &= 0\,,\\
(\mathbf{div}\,\widetilde{\boldsymbol{\sigma}}_h, \mathbf{v}_h)_{\mathrm{S}} + (\widetilde{\boldsymbol{\sigma}}_h, \boldsymbol{\eta}_h)_{\mathrm{S}} + \langle \widetilde{\boldsymbol{\sigma}}_h\,\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma &= \langle \boldsymbol{\phi}_h, \boldsymbol{\psi}_h\rangle_\Sigma\,,
\end{aligned} \tag{5.27}$$

for all $\big(\boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\eta}_h, \boldsymbol{\psi}_h)\big) \in \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \big(\mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbb{L}^2_h(\Omega_{\mathrm{S}}) \times \mathbf{\Lambda}^{\mathrm{S}}_h(\Sigma)\big)$. Note that (5.27) actually corresponds to the PEERS-based mixed finite element approximation of a particular linear elasticity problem in Ω_{S} (see, e.g. (4.10)) with homogeneous Dirichlet boundary condition on Γ_{S} and Neumann boundary condition given by $\boldsymbol{\phi}_h$ on Σ. Moreover, the well-posedness of (5.27) is proved, modulus minor changes, by combining [20, Section 4.3] with [35, Theorem 5.1] and [25, Lemma 5.2]. In particular, the associated stability result insures the existence of $\widetilde{C} > 0$, independent of h, such that

$$\|\big(\widetilde{\boldsymbol{\sigma}}_h, (\widetilde{\mathbf{u}}_h, \widetilde{\gamma}_h, \widetilde{\boldsymbol{\varphi}}_h)\big)\| \,\le\, \widetilde{C}\,\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}\,. \tag{5.28}$$

Therefore, since the second equation in (5.27) establishes that $\widetilde{\boldsymbol{\sigma}}_h$ belongs to $\widetilde{\mathbb{H}}_h(\Omega_{\mathrm{S}})$ and that $\langle\widetilde{\boldsymbol{\sigma}}_h\,\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma \,=\, \langle\boldsymbol{\phi}_h, \boldsymbol{\psi}_h\rangle_\Sigma \;\; \forall\,\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma)$, we deduce, using also (5.28), that

$$\frac{|\,\langle\boldsymbol{\phi}_h, \boldsymbol{\psi}_h\rangle_\Sigma\,|}{\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}} \,=\, \frac{|\,\langle\widetilde{\boldsymbol{\sigma}}_h\,\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma\,|}{\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}} \,\le\, \frac{1}{\widetilde{C}}\,\frac{|\,\langle\widetilde{\boldsymbol{\sigma}}_h\,\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma\,|}{\|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{div};\Omega_{\mathrm{S}}}}\,,$$

which implies that

$$\sup_{\substack{\boldsymbol{\phi}_h\in\boldsymbol{\Phi}_h(\Sigma)\\ \boldsymbol{\phi}_h\neq\mathbf{0}}} \frac{\langle\boldsymbol{\phi}_h, \boldsymbol{\psi}_h\rangle_\Sigma}{\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}} \,\le\, \frac{1}{\widetilde{C}} \sup_{\substack{\boldsymbol{\tau}_h\in\widetilde{\mathbb{H}}_h(\Omega_{\mathrm{S}})\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_h\,\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma}{\|\boldsymbol{\tau}_h\|_{\mathbf{div};\Omega_{\mathrm{S}}}} \qquad \forall\,\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma)\,. \tag{5.29}$$

Thus, it is quite clear from (5.29) that the discrete inf-sup condition (5.16) is a straightforward consequence of (5.26). Moreover, since the latter has already been proved in [25, Lemma 5.2], we conclude in this way the full verification of the hypothesis **(H.3)**.

Thanks to the previous results and analyses, we can establish the following theorems.

Theorem 5.3 *Assume that the stabilization parameter* ρ *lives in* $\left(0, \frac{\alpha_0}{\gamma_0^2}\right)$, *and let* $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) \in \mathbb{X}\times\mathbb{M}$ *be the unique solution of the continuous formulation* (2.34). *In addition, let* $\mathbb{X}_h := \mathbf{X}_{1,h}\times\mathbf{M}_{1,h}$ *and* $\mathbb{M}_h$ *be the finite element subspaces defined by* (5.4) *in terms of the specific discrete spaces given by* (5.23) *and* (5.24). *Then, the Galerkin scheme* (5.5) *has a unique solution* $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h) \in \mathbb{X}_h\times\mathbb{M}_h$ *and there exist* C_1, $C_2 > 0$, *independent of* h, *such that*

$$\|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}} \,\le\, C_1\left\{\,\|\mathbb{F}|_{\mathbb{X}_h}\|_{\mathbb{X}_h'} \,+\, \|\mathbb{G}|_{\mathbb{M}_h}\|_{\mathbb{M}_h'}\,\right\}, \tag{5.30}$$

and

$$\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}} \,\le\, C_2 \inf_{((\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{q}_h)\in\mathbb{X}_h\times\mathbb{M}_h} \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) - ((\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{q}_h)\|_{\mathbb{X}\times\mathbb{M}}\,. \tag{5.31}$$

Proof. Having verified the hypotheses **(H.0)**, **(H.1)**, **(H.2)** and **(H.3)**, the proof is a straightforward application of Theorems 5.1 and 5.2.

■

The following theorem provides the theoretical rate of convergence of the Galerkin scheme (5.5), under suitable regularity assumptions on the exact solution.

Theorem 5.4 *Let* $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) \in \mathbb{X}\times\mathbb{M}$ *and* $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h) \in \mathbb{X}_h\times\mathbb{M}_h$ *be the unique solutions of the continuous and discrete formulations* (2.34) *and* (5.5), *respectively. Assume that there exists* $\delta\in(0,1]$ *such that* $\mathbf{t}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$, $\operatorname{div}\mathbf{u}_{\mathrm{D}} \in H^\delta(\Omega_{\mathrm{D}})$, $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$, *and* $\boldsymbol{\gamma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$. *Then,* $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1+\delta}(\Omega_{\mathrm{S}})$, $p_{\mathrm{D}} \in H^{1+\delta}(\Omega_{\mathrm{D}})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\lambda \in H^{1/2+\delta}(\Sigma)$, *and there exists* $C > 0$, *independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned} \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) \,-\, ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}} \,\le\, C\,h^\delta \Big\{ &\|\mathbf{t}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} \,+\, \|\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} \,+\, \|\operatorname{div}\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} \\ +\, \|\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} \,+\, \|\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} \,+\, \|\boldsymbol{\gamma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} \,+\, &\|\mathbf{u}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}} \,+\, \|p_{\mathrm{D}}\|_{1+\delta,\Omega_{\mathrm{D}}}\Big\}\,. \end{aligned} \tag{5.32}$$

Proof. We first recall from Theorem 4.2 that $\nabla \mathbf{u}_S = \mathbf{t}_S + \boldsymbol{\gamma}_S$ and $\nabla p_D = -\mathbf{K}^{-1}\mathbf{u}_D$, which implies that $\mathbf{u}_S \in \mathbf{H}^{1+\delta}(\Omega_S)$, and $p_D \in H^{1+\delta}(\Omega_D)$, whence $\boldsymbol{\varphi} = -\mathbf{u}_S|_\Sigma \in \mathbf{H}^{1/2+\delta}(\Sigma)$ and $\lambda = p_D|_\Sigma \in H^{1/2+\delta}(\Sigma)$. The rest of the proof follows from the Cea estimate (5.31), the approximation properties of the subspaces involved (see, e.g. [4], [7] and [29]), and the fact that, thanks to the trace theorems in Ω_S and Ω_D, respectively, there holds

$$\|\boldsymbol{\varphi}\|_{1/2+\delta,\Sigma} \le c\,\|\mathbf{u}_S\|_{1+\delta,\Omega_D} \quad \text{and} \quad \|\lambda\|_{1/2+\delta,\Sigma} \le c\,\|p_D\|_{1+\delta,\Omega_D}.$$

∎

5.3.2 PEERS + Raviart-Thomas in 3D

We now introduce the 3D version of the spaces defined in Section 5.3.1 (cf. (5.23)). More precisely, we set

$$\begin{aligned}
\mathbf{H}_h(\Omega_S) &:= \Big\{\tau \in \mathbf{H}(\mathrm{div};\Omega_S): \quad \tau|_T \in \mathrm{RT}_0(T)\oplus B_0(T) \quad \forall\, T\in \mathcal{T}_h^S\Big\},\\
\mathbf{H}_h(\Omega_D) &:= \Big\{\tau \in \mathbf{H}(\mathrm{div};\Omega_D): \quad \tau|_T \in \mathrm{RT}_0(T) \qquad \forall\, T\in \mathcal{T}_h^D\Big\},\\
L_h(\Omega_S) &:= \Big\{v \in L^2(\Omega_S): \quad v|_T \in P_0(T) \quad \forall\, T\in \mathcal{T}_h^S\Big\},\\
L_h(\Omega_D) &:= \Big\{q \in L^2(\Omega_D): \quad q|_T \in P_0(T) \quad \forall\, T\in \mathcal{T}_h^D\Big\}, \quad \text{and}\\
\mathbb{L}_h^2(\Omega_S) &:= \Big\{\boldsymbol{\eta} \in \mathbb{L}^2(\Omega_S): \quad \boldsymbol{\eta}\in \mathbb{C}(\bar{\Omega}_S), \quad \boldsymbol{\eta}|_T \in \mathbb{P}_1(T) \quad \forall\, T\in \mathcal{T}_h^S\Big\}.
\end{aligned} \tag{5.33}$$

Actually, except for the fact that the vectors and tensors live now in R^3 and $\mathrm{R}^{3\times 3}$, respectively, the above definitions look pretty much as those in (5.23).

Next, in order to complete the list of spaces in (5.1), we need to define those living on the interface Σ. To this end, and for reasons that will become clear below, we introduce an independent triangulation $\Sigma_{\widehat{h}}$ of the interface Σ by triangles K of diameter $\widehat{h}_K$ and define $\widehat{h}_\Sigma := \max\{\widehat{h}_K: \quad K\in\Sigma_{\widehat{h}}\}$. The above is certainly in addition to Σ_h, the usual partition of Σ inherited from $\mathcal{T}_h^S$ (or $\mathcal{T}_h^D$), also formed by triangles K of diameter h_K, and for which we set $h_\Sigma := \max\{h_K: \quad K\in\Sigma_h\}$. Hence, denoting by $\partial\Sigma$ the polygonal boundary of Σ, we define

$$\begin{aligned}
\Lambda_{\widehat{h}}^S(\Sigma) &:= \Big\{\psi \in C(\Sigma): \quad \psi|_K \in P_1(K) \quad \forall\, K\in\Sigma_{\widehat{h}}, \quad \psi = 0 \text{ on } \partial\Sigma\Big\},\\
\Lambda_{\widehat{h}}^D(\Sigma) &= \Big\{\xi \in C(\Sigma): \quad \xi|_K \in P_1(K) \quad \forall\, K\in\Sigma_{\widehat{h}}\Big\}, \quad \text{and}\\
\boldsymbol{\Lambda}_{\widehat{h}}^S(\Sigma) &:= [\Lambda_{\widehat{h}}^S(\Sigma)]^3,
\end{aligned} \tag{5.34}$$

which, from now on, replace the spaces $\Lambda_h^S(\Sigma)$, $\Lambda_h^D(\Sigma)$, and $\boldsymbol{\Lambda}_h^S(\Sigma)$ specified in (5.1) and (5.2).

In what follows we show that the hypotheses **(H.0) - (H.3)** are satisfied. Indeed, as in the 2D case, the verification of **(H.0)** and **(H.2)** is also quite straightforward from the definitions given in (5.38). Furthermore, the proofs of the discrete inf-sup conditions (5.10) and (5.11) can also be found in [33, Theorem 4.5] and [7, Chapter IV, Section IV.1.2], respectively. In addition, the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$ verifying (5.12) is derived similarly to the procedure described in Section 5.3.1. The assumptions required by **(H.1)** are then satisfied.

Now, concerning the discrete inf-sup conditions (5.17) and (5.16), we first remark that the same approaches yielding the corresponding sufficiency of (5.25) and (5.26) in the 2D case, which are based on the results from [25], [35], and [20], can also be applied to the present three-dimensional situation.

In this case, however, the 3D analogue of [35, Theorem 5.1], being still an open problem, can not be employed. Therefore, in order to construct the stable discrete lifting of the normal traces of $\widetilde{\mathbf{H}}_{0,h}(\Omega_{\mathrm{D}})$ and prove the well-posedness of the Galerkin scheme (5.27), we need to employ some inverse inequalities on Σ, which requires quasi-uniform meshes in a neighborhood of this interface. Furthermore, it can be proved (see, e.g. the second part of the proof of [19, Lemma 7.5]) that there exists $C_0 \in]0,1[$ such that for each pair $(h_\Sigma, \widehat{h}_\Sigma)$ verifying $h_\Sigma \le C_0\, \widehat{h}_\Sigma$, the 3D versions of (5.25) and (5.26) are satisfied. Note that this restriction on the meshsizes explains the need of having introduced the independent partition $\Sigma_{\widehat{h}}$ of Σ. We have thus confirmed the hypotheses from **(H.3)**.

We are now in a position to state the following main results. Their proofs, being basically the same of Theorems 5.3 and 5.4, are omitted.

Theorem 5.5 *Assume that the stabilization parameter ρ lives in $\left(0, \frac{\alpha_0}{\gamma_0^2}\right)$, and that the meshes $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ are quasi-uniform around the interface Σ. In addition, let $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) \in \mathbb{X} \times \mathbb{M}$ be the unique solution of the continuous formulation (2.34), and let $\mathbb{X}_h := \mathbf{X}_{1,h} \times \mathbf{M}_{1,h}$ and $\mathbb{M}_h$ be the finite element subspaces defined by (5.4) in terms of the specific discrete spaces given by (5.33) and (5.34). Then, whenever the pair $(h_\Sigma, \widehat{h}_\Sigma)$ verifies $h_\Sigma \le C_0\, \widehat{h}_\Sigma$, the Galerkin scheme (5.5) has a unique solution $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ and there exist C_1, $C_2 > 0$, independent of h, h_Σ, and $\widehat{h}_\Sigma$, such that*

$$\|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}} \le C_1 \left\{ \|\mathbb{F}|_{\mathbb{X}_h}\|_{\mathbb{X}_h'} + \|\mathbb{G}|_{\mathbb{M}_h}\|_{\mathbb{M}_h'} \right\}, \tag{5.35}$$

and

$$\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}} \le C_2 \inf_{((\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{q}_h) \in \mathbb{X}_h \times \mathbb{M}_h} \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) - ((\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{q}_h)\|_{\mathbb{X}\times\mathbb{M}}. \tag{5.36}$$

Theorem 5.6 *Let $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) \in \mathbb{X} \times \mathbb{M}$ and $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ be the unique solutions of the continuous and discrete formulations (2.34) and (5.5), respectively. Assume that there exists $\delta \in (0,1]$ such that $\mathbf{t}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$, $\operatorname{div} \mathbf{u}_{\mathrm{D}} \in H^\delta(\Omega_{\mathrm{D}})$, $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{div}\, \boldsymbol{\sigma}_{\mathrm{S}} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$, and $\boldsymbol{\gamma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$. Then, $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1+\delta}(\Omega_{\mathrm{S}})$, $p_{\mathrm{D}} \in H^{1+\delta}(\Omega_{\mathrm{D}})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\lambda \in H^{1/2+\delta}(\Sigma)$, and there exists $C > 0$, independent of h, h_Σ, $\widehat{h}_\Sigma$, and the continuous and discrete solutions, such that whenever the pair $(h_\Sigma, \widehat{h}_\Sigma)$ verifies $h_\Sigma \le C_0\, \widehat{h}_\Sigma$, there holds*

$$\begin{aligned} \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) \,-\, ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}} \le C\, h^\delta \Big\{ &\|\mathbf{t}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|\operatorname{div} \mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} \\ + \|\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{div}\, \boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\boldsymbol{\gamma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} &+ \|\mathbf{u}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}} + \|p_{\mathrm{D}}\|_{1+\delta,\Omega_{\mathrm{D}}} \Big\}. \end{aligned} \tag{5.37}$$

5.3.3 AFW + BDM in 3D

Alternatively, for the 3D case we can also introduce the following discrete spaces in (5.1):

$$\begin{aligned} \mathbf{H}_h(\Omega_{\mathrm{S}}) &:= \left\{ \tau \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{S}}) : \quad \tau|_T \in \mathbf{P}_1(T) \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \right\}, \\ \mathbf{H}_h(\Omega_{\mathrm{D}}) &:= \left\{ \tau \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}) : \quad \tau|_T \in \mathbf{P}_1(T) \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}} \right\}, \\ L_h(\Omega_{\mathrm{S}}) &:= \left\{ v \in L^2(\Omega_{\mathrm{S}}) : \quad v|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \right\}, \\ L_h(\Omega_{\mathrm{D}}) &:= \left\{ q \in L^2(\Omega_{\mathrm{D}}) : \quad q|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}} \right\}, \quad \text{and} \\ \underline{\mathbb{L}}_h^2(\Omega_{\mathrm{S}}) &:= \left\{ \boldsymbol{\eta} \in \underline{\mathbb{L}}^2(\Omega_{\mathrm{S}}) : \quad \boldsymbol{\eta}|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \right\}. \end{aligned} \tag{5.38}$$

We remark, according to the complementary definitions given in (5.2), that the product space $\mathbb{H}_h(\Omega_S) \times \mathbf{L}_h(\Omega_S) \times \mathbb{L}_h^2(\Omega_S)$ constitutes now the lowest order mixed finite element approximation of the linear elasticity problem introduced recently by Arnold Falk and Winther (AFW) (see [2], [3]). In turn, $\mathbf{H}_h(\Omega_D) \times L_h(\Omega_D)$ is the BDM space of lowest order for the mixed formulation of the corresponding Poisson problem (see, e.g. [7], [36]).

In what follows we refer to the verification of the hypotheses **(H.0) - (H.3)**. Indeed, as in the previous 2D and 3D cases, **(H.0)** and **(H.2)** follow straightforwardly from the definitions given in (5.38). Furthermore, the proofs of the discrete inf-sup conditions (5.10) and (5.11) can be found now in [2, Section 11.7, Theorem 11.9] and again in [7, Chapter IV, Section IV.1.2], respectively. In addition, the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$ verifying (5.12) is derived similarly to the procedure described in Section 5.3.1. The assumptions required by **(H.1)** are then satisfied. Next, concerning the discrete inf-sup conditions (5.17) and (5.16), we just remark that the corresponding proofs follow as in Section 5.3.2 by introducing again the independent partition $\Sigma_{\widehat{h}}$ and then defining the spaces given by (5.34). The rest of the analysis is as in the previous section and the main results are basically the same as those provided by Theorems 5.5 and 5.6, but now with the specific discrete spaces given by (5.38) and (5.34) . We omit further details.

6 The a-posteriori error analysis

In this section we restrict ourselves to the two-dimensional case and derive a reliable and efficient residual-based a-posteriori error estimate for our mixed finite element scheme (5.5) with the discrete spaces introduced in Section 5.3.1. The extension to 3D should be quite straightforward. Most of the analysis employed in the proofs makes extensive use of the estimates derived in [22] and [26]. We begin with some notations. For each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we let $\mathcal{E}(T)$ be the set of edges of T, and we denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, subdivided as follows:

$$\mathcal{E}_h = \mathcal{E}_h(\Gamma_S) \cup \mathcal{E}_h(\Omega_S) \cup \mathcal{E}_h(\Omega_D) \cup \mathcal{E}_h(\Sigma),$$

where $\mathcal{E}_h(\Gamma_S) := \{ e \in \mathcal{E}_h : e \subseteq \Gamma_S \}$, $\mathcal{E}_h(\Omega_\star) := \{ e \in \mathcal{E}_h : e \subseteq \Omega_\star \}$ for each $\star \in \{\mathrm{S}, \mathrm{D}\}$, and $\mathcal{E}_h(\Sigma) := \{ e \in \mathcal{E}_h : e \subseteq \Sigma \}$. Note that $\mathcal{E}_h(\Sigma)$ is the set of edges defining the partition Σ_h. Analogously, we let $\mathcal{E}_{2h}(\Sigma)$ be the set of *double* edges defining the partition Σ_{2h}. In what follows, h_e stands for the diameter of a given edge $e \in \mathcal{E}_h \cup \mathcal{E}_{2h}(\Sigma)$. Now, let $\star \in \{\mathrm{S}, \mathrm{D}\}$ and let $q \in [L^2(\Omega_\star)]^m$, with $m \in \{1, 2\}$, such that $q|_T \in [C(T)]^m$ for each $T \in \mathcal{T}_h^\star$. Then, given $e \in \mathcal{E}_h(\Omega_\star)$, we denote by $[q]$ the jump of q across e, that is $[q] := (q|_{T'})|_e - (q|_{T''})|_e$, where T' and T'' are the triangles of $\mathcal{T}_h^\star$ having e as an edge. Also, we fix a unit normal vector $\mathbf{n}_e := (n_1, n_2)^{\mathtt{t}}$ to the edge e (its particular orientation is not relevant) and let $\mathbf{t}_e := (-n_2, n_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Hence, given $\mathbf{v} \in \mathbf{L}^2(\Omega_\star)$ and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$ such that $\mathbf{v}|_T \in [C(T)]^2$ and $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2}$, respectively, for each $T \in \mathcal{T}_h^\star$, we let $[\mathbf{v} \cdot \mathbf{t}_e]$ and $[\boldsymbol{\tau}\, \mathbf{t}_e]$ be the tangential jumps of $\mathbf{v}$ and $\boldsymbol{\tau}$, across e, that is $[\mathbf{v} \cdot \mathbf{t}_e] := \{(\mathbf{v}|_{T'})|_e - (\mathbf{v}|_{T''})|_e\} \cdot \mathbf{t}_e$ and $[\boldsymbol{\tau}\, \mathbf{t}_e] := \{(\boldsymbol{\tau}|_{T'})|_e - (\boldsymbol{\tau}|_{T''})|_e\}\, \mathbf{t}_e$, respectively. From now on, when no confusion arises, we will simply write $\mathbf{t}$ and $\mathbf{n}$ instead of $\mathbf{t}_e$ and $\mathbf{n}_e$, respectively. Finally, for sufficiently smooth scalar, vector and tensors fields q, $\mathbf{v} := (v_1, v_2)^{\mathtt{t}}$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, we let

$$\mathbf{curl}\, \mathbf{v} := \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix}, \qquad \mathrm{curl}\, q := \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1} \right)^{\mathtt{t}},$$

$$\mathrm{rot}\, \mathbf{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \text{and} \quad \mathbf{rot}\, \boldsymbol{\tau} := \left(\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \right)^{\mathtt{t}}.$$

In what follows, $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) = ((\mathbf{t}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\sigma}_{\mathrm{S}}), (\boldsymbol{\varphi}, \lambda)), (p_{\mathrm{D}}, \mathbf{u}_{\mathrm{S}}, \boldsymbol{\gamma}_{\mathrm{S}}, \ell)) \in \mathbb{X} \times \mathbb{M}$ and $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h) = ((\mathbf{t}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\sigma}_{\mathrm{S},h}), (\boldsymbol{\varphi}_h, \lambda_h)), (p_{\mathrm{D},h}, \mathbf{u}_{\mathrm{S},h}, \boldsymbol{\gamma}_{\mathrm{S},h}, \ell_h)) \in \mathbb{X}_h \times \mathbb{M}_h$ denote the solutions of (2.34) and (5.5), respectively. Also, we let $\kappa = \kappa_1$ the only constant appearing in the second transmission condition in (2.3). Then, we introduce the global a posteriori error estimator:

$$\Theta := \left\{ \sum_{T \in \mathcal{T}_h^{\mathrm{S}}} \Theta_{\mathrm{S},T}^2 + \sum_{T \in \mathcal{T}_h^{\mathrm{D}}} \Theta_{\mathrm{D},T}^2 \right\}^{1/2}, \tag{6.1}$$

where, for each $T \in \mathcal{T}_h^{\mathrm{S}}$:

$$\begin{aligned}
\Theta_{\mathrm{S},T}^2 &:= \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2 \|\mathbf{rot}\,(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h})\|_{0,T}^2 + h_T^2 \|\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T}^2 \\
&+ \|\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h}\|_{0,T}^2 + \|\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{t}}\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_{\mathrm{S}})} h_e \|[(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h})\mathbf{t}]\|_{0,e}^2 \\
&+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \|(\boldsymbol{\sigma}_{\mathrm{S},h} + \ell_h \mathbb{I})\mathbf{n} + \lambda_h \mathbf{n} - \kappa^{-1}(\boldsymbol{\varphi}_h \cdot \mathbf{t})\,\mathbf{t}\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e \|(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h})\mathbf{t}\|_{0,e}^2 \\
&+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} \left\{ h_e \left\| (\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h})\mathbf{t} + \frac{d\boldsymbol{\varphi}_h}{d\mathbf{t}} \right\|_{0,e}^2 + h_e \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \right\},
\end{aligned} \tag{6.2}$$

and for each $T \in \mathcal{T}_h^{\mathrm{D}}$:

$$\begin{aligned}
\Theta_{\mathrm{D},T}^2 &:= \|f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_T^2 \|\mathrm{rot}\,(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 + h_T^2 \|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \\
&+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_{\mathrm{D}})} h_e \|[\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{t}]\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \|\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} + \boldsymbol{\varphi}_h \cdot \mathbf{n}\|_{0,e}^2 \\
&+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} \left\{ h_e \left\| \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{t} + \frac{d\lambda_h}{d\mathbf{t}} \right\|_{0,e}^2 + h_e \|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \right\}.
\end{aligned} \tag{6.3}$$

6.1 Reliability of the a posteriori error estimator

The main result of this section is stated as follows.

Theorem 6.1 *There exists $C_{\mathtt{rel}} > 0$, independent of h, such that*

$$\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X} \times \mathbb{M}} \le C_{\mathtt{rel}}\, \Theta. \tag{6.4}$$

We follow the general approach from [15] (see also [22] and [27]). Indeed, we begin by recalling, thanks to the hypothesis on μ (cf. (2.4) and (2.5)), and Lemmas 4.1 and 5.1, that $\mathbf{A}_1$ satisfies the assumptions **(A.1)** and **(A.2)**. Hence, a straightforward application of Lemma 3.2 implies that the Gâteaux derivative of $\mathbf{A}_1$ at any $\underline{\mathbf{r}} \in \mathbf{X}_1$, say $\mathcal{D}A_1(\underline{\mathbf{r}})$, is a uniformly bounded and uniformly elliptic bilinear form on $\mathbf{X}_1 \times \mathbf{X}_1$. Therefore, as a consequence of the continuous dependence result provided by Theorem 3.2 (cf. (3.16)), we find that the linear operator obtained by adding the two equations of the left hand side of (2.34), after replacing $\mathbf{A}_1$ by $\mathcal{D}A_1(\underline{\mathbf{r}})$, satisfies a global inf-sup condition. Furthermore, thanks to the mean value theorem applied to the continuous operator $\mathbf{A}_1$, there exists a convex combination of $\underline{\mathbf{t}}$ and $\underline{\mathbf{t}}_h$, say $\tilde{\underline{\mathbf{s}}} \in \mathbf{X}_1$, such that

$$[\mathcal{D}\mathbf{A}_1(\tilde{\underline{\mathbf{s}}})(\underline{\mathbf{t}} - \underline{\mathbf{t}}_h), \underline{\mathbf{r}}] = [\mathbf{A}_1(\underline{\mathbf{t}}) - \mathbf{A}_1(\underline{\mathbf{t}}_h), \underline{\mathbf{r}}] \quad \forall \underline{\mathbf{r}} \in \mathbf{X}_1. \tag{6.5}$$

Then, applying the above mentioned inf-sup estimate (with $\underline{\mathbf{r}} = \underline{\tilde{\mathbf{s}}}$) to the error $((\underline{\mathbf{t}}-\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}}_h), \underline{p}-\underline{p}_h)$, we find that

$$\|((\underline{\mathbf{t}} - \underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}} - \underline{\boldsymbol{\varphi}}_h), \underline{p} - \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}} \,\le\, C \sup_{\substack{((\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{q})\in\mathbb{X}\times\mathbb{M} \\ ((\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{q})\neq\mathbf{0}}} \frac{|[R, ((\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{q})]|}{\|((\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{q})\|_{\mathbb{X}\times\mathbb{M}}}\,, \tag{6.6}$$

where, acording to (6.5) and (2.34), the residual operator $R : \mathbb{X} \times \mathbb{M} \to \mathbb{R}$ is given by

$$\begin{aligned} [R, ((\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{q})] \,:=\, & [R_1, (\boldsymbol{\tau}_{\mathrm{S}}, j)] \,+\, [R_2, \mathbf{v}_{\mathrm{D}}] \,+\, [R_3, \boldsymbol{\psi}] \,+\, [R_4, \xi] \\ + \,\, & [R_5, \mathbf{v}_{\mathrm{S}}] \,+\, [R_6, q_{\mathrm{D}}] \,+\, [R_7, \mathbf{r}_{\mathrm{S}}] \,+\, [R_8, \boldsymbol{\eta}_{\mathrm{S}}]\,, \end{aligned} \tag{6.7}$$

for each $\underline{\mathbf{r}} \,:=\, (\mathbf{r}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\tau}_{\mathrm{S}}) \in \mathbf{X}_1$, $\underline{\boldsymbol{\psi}} := (\boldsymbol{\psi}, \xi) \in \mathbf{M}_1$, $\underline{q} \,:=\, (q_{\mathrm{D}}, \mathbf{v}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}}, j) \in \mathbb{M}$, with

$$\begin{aligned} [R_1, (\boldsymbol{\tau}_{\mathrm{S}}, j)] \,:=\, & -(\mathbf{t}_{\mathrm{S},h}, \boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} \,-\, (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S},h})_{\mathrm{S}} \,-\, \rho\,(\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h}, \boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}} \\ & -(\boldsymbol{\tau}_{\mathrm{S}}, \boldsymbol{\gamma}_{\mathrm{S},h})_{\mathrm{S}} - \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma \,-\, j\,\langle \mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma\,, \\ [R_2, \mathbf{v}_{\mathrm{D}}] \,:=\, & -(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \,+\, (\mathrm{div}\,\mathbf{v}_{\mathrm{D}}, p_{\mathrm{D},h})_{\mathrm{D}} \,+\, \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma\,, \end{aligned}$$

$$\begin{aligned} [R_3, \boldsymbol{\psi}] &:= \,-\,\langle \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma \,-\, \ell_h\,\langle \mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma \,+\, \kappa^{-1}\langle \boldsymbol{\varphi}_h\cdot\mathbf{t}, \boldsymbol{\psi}\cdot\mathbf{t}\rangle_\Sigma \,-\, \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma\,, \\ [R_4, \xi] &:= \langle \boldsymbol{\varphi}_h\cdot\mathbf{n}, \xi\rangle_\Sigma + \langle \mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n}, \xi\rangle_\Sigma\,, \\ [R_5, \mathbf{v}_{\mathrm{S}}] &:= \,-\,(\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}}\,, \\ [R_6, q_{\mathrm{D}}] &:= \,-\,(f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}, q_{\mathrm{D}})_{\mathrm{D}}\,, \\ [R_7, \mathbf{r}_{\mathrm{S}}] &:= \,-\,(\mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}}, \mathbf{r}_{\mathrm{S}})_{\mathrm{S}}\,, \\ [R_8, \boldsymbol{\eta}_{\mathrm{S}}] &:= \,-\,(\boldsymbol{\sigma}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S}})_S\,. \end{aligned}$$

Hence, the supremum in (6.6) can be bounded in terms of R_i, with $i \in \{1, \ldots, 8\}$, which yields

$$\begin{aligned} \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}} \,\le\, C\,\Big\{ & \|R_1\|_{(\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})\times\mathbb{R})'} \,+\, \|R_2\|_{\mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{D}})'} \,+\, \|R_3\|_{\mathbf{H}_{00}^{-1/2}(\Sigma)} \\ + \|R_4\|_{H^{-1/2}(\Sigma)} \,+\, \|R_5\|_{\mathbf{L}^2(\Omega_{\mathrm{S}})'} \,+\, & \|R_6\|_{L_0^2(\Omega_{\mathrm{D}})'} \,+\, \|R_7\|_{\mathbb{L}_0^2(\Omega_{\mathrm{S}})'} \,+\, \|R_8\|_{\underline{\mathbb{L}}^2(\Omega_{\mathrm{S}})'} \Big\}\,. \end{aligned} \tag{6.8}$$

Throughout the rest of the section we provide suitable upper bounds for each one of the terms on the right hand side of (6.8). We begin with R_1 by observing from its definition, and having in mind that $(\mathbf{t}_{\mathrm{S},h}, \mathbb{I})_{\mathrm{S}} = \mathrm{tr}\,\mathbf{t}_{\mathrm{S},h} = 0$, $(\mathbb{I}, \boldsymbol{\gamma}_{\mathrm{S},h})_{\mathrm{S}} = 0$, and $\mathrm{div}\,\mathbb{I} = 0$, that

$$[R_1, (\boldsymbol{\tau}_{\mathrm{S}}, j)] \,=\, [\widetilde{R}_1, \boldsymbol{\tau}_{\mathrm{S}} + j\,\mathbb{I}] \,-\, \rho\,(\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h}, \boldsymbol{\tau}_{\mathrm{S}}^{\mathtt{d}})_{\mathrm{S}}\,,$$

where $\widetilde{R}_1 : \mathbb{H}(\mathbf{div}; \Omega_{\mathrm{S}}) \to \mathbb{R}$ is given by

$$[\widetilde{R}_1, \widetilde{\tau}_{\mathrm{S}}] \,:=\, -\,(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h}, \widetilde{\tau}_{\mathrm{S}})_{\mathrm{S}} \,-\, (\mathbf{div}\,\widetilde{\tau}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S},h})_{\mathrm{S}} \,-\, \langle \widetilde{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma \qquad \forall\,\widetilde{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div}; \Omega_{\mathrm{S}})\,.$$

It follows, using the triangle and Cauchy Schwarz inequalities, that

$$\|R_1\|_{(\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})\times\mathbb{R})'} \,\le\, \|\widetilde{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})'} \,+\, \rho\,\|\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}\,, \tag{6.9}$$

and hence it just remains to bound $\|\widetilde{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'}$. Moreover, since the functionals $\widetilde{R}_1$ and R_2 share the same "*structure*" with $\mathbf{K}^{-1}\mathbf{u}_{D,h}$ and $\mathbf{t}_{S,h}+\boldsymbol{\gamma}_{S,h}$ playing parallel roles, the upper bounds of their norms are derived by following the same approach. More precisely, one proceeds as in [26] by using integration by parts on each element of the triangulations, by employing continuous and discrete Helmholtz decompositions of $\mathbb{H}(\mathbf{div};\Omega_S)$ and $\mathbf{H}_0(\mathrm{div};\Omega_D)$, and by applying the approximation properties of the Clément and Raviart-Thomas interpolation operators in both domains (cf. [9], [38]). In this way, and as a consequence of the analysis developed in [26], we deduce that the estimate for $\|\widetilde{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'}$ is obtained from [26, Lemma 3.8] after replacing $\boldsymbol{\sigma}^{\mathrm{d}}_{S,h}$ there by $\mathbf{t}_{S,h}+\boldsymbol{\gamma}_{S,h}$, whereas the estimate for $\|R_2\|_{\mathbf{H}_0(\mathrm{div};\Omega_D)'}$ is exactly the one given by [26, Lemma 3.9]. The corresponding results are stated as follows.

Lemma 6.1 *There exists $C_1>0$, independent of h, such that*

$$\|\widetilde{R}_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'}\;\le\;C_1\left\{\sum_{T\in\mathcal{T}_h^S}\widetilde{\Theta}^2_{S,T}\right\}^{1/2}, \tag{6.10}$$

where, for each $T\in\mathcal{T}_h^S$:

$$\begin{aligned}\widetilde{\Theta}^2_{S,T}\;:=\;&h_T^2\,\|\mathbf{rot}\,(\mathbf{t}_{S,h}+\boldsymbol{\gamma}_{S,h})\|^2_{0,T}\;+\;h_T^2\,\|\mathbf{t}_{S,h}+\boldsymbol{\gamma}_{S,h}\|^2_{0,T}\\ &+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_S)}h_e\,\|[(\mathbf{t}_{S,h}+\boldsymbol{\gamma}_{S,h})\,\mathbf{t}]\|^2_{0,e}\;+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_S)}h_e\,\|(\mathbf{t}_{S,h}+\boldsymbol{\gamma}_{S,h})\,\mathbf{t}\|^2_{0,e}\\ &+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)}\left\{h_e\left\|(\mathbf{t}_{S,h}+\boldsymbol{\gamma}_{S,h})\,\mathbf{t}+\frac{d\boldsymbol{\varphi}_h}{d\,\mathbf{t}}\right\|^2_{0,e}+\;h_e\,\|\mathbf{u}_{S,h}+\boldsymbol{\varphi}_h\|^2_{0,e}\right\}.\end{aligned} \tag{6.11}$$

Lemma 6.2 *There exists $C_2>0$, independent of h, such that*

$$\|R_2\|_{\mathbf{H}_0(\mathrm{div};\Omega_D)'}\;\le\;C_2\left\{\sum_{T\in\mathcal{T}_h^D}\widehat{\Theta}^2_{D,T}\right\}^{1/2}, \tag{6.12}$$

where, for each $T\in\mathcal{T}_h^D$:

$$\begin{aligned}\widehat{\Theta}^2_{D,T}\;:=\;&h_T^2\,\|\mathrm{rot}\,(\mathbf{K}^{-1}\mathbf{u}_{D,h})\|^2_{0,T}\;+\;h_T^2\,\|\mathbf{K}^{-1}\mathbf{u}_{D,h}\|^2_{0,T}\;+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_D)}h_e\;\|[\mathbf{K}^{-1}\mathbf{u}_{D,h}\cdot\mathbf{t}]\|^2_{0,e}\\ &+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)}\left\{h_e\left\|\mathbf{K}^{-1}\mathbf{u}_{D,h}\cdot\mathbf{t}\;+\;\frac{d\lambda_h}{d\,\mathbf{t}}\right\|^2_{0,e}+\;h_e\,\|p_{D,h}-\lambda_h\|^2_{0,e}\right\}.\end{aligned} \tag{6.13}$$

Next, we observe that the upper bounds of $\|R_3\|_{\mathbf{H}_{00}^{-1/2}(\Sigma)}$ and $\|R_4\|_{H^{-1/2}(\Sigma)}$ are also derived in [26]. In fact, noting first that R_3 can be re-written as

$$[R_3,\boldsymbol{\psi}]\;:=\;-\langle\big(\boldsymbol{\sigma}_{S,h}\;+\;\ell_h\,\mathbb{I}\big)\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma\;+\;\kappa^{-1}\langle\boldsymbol{\varphi}_h\cdot\mathbf{t},\boldsymbol{\psi}\cdot\mathbf{t}\rangle_\Sigma\;-\;\langle\boldsymbol{\psi}\cdot\mathbf{n},\lambda_h\rangle_\Sigma\qquad\forall\,\boldsymbol{\psi}\in\mathbf{H}_{00}^{1/2}(\Sigma)\,,$$

we can establish the estimates provided by the following lemma, which are based on the technical result given by [8, Theorem 2] and the fact that both Σ_h and $\Sigma_{2\,h}$ are of bounded variation.

Lemma 6.3 *There exist $C_3,\ C_4 > 0$, independent of h, such that*

$$\|R_3\|_{\mathbf{H}_{00}^{-1/2}(\Sigma)} \,\le\, C_3 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \, \| \big(\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \ell_h\,\mathbb{I}\big) \,+\, \lambda_h\mathbf{n} \,-\, \kappa^{-1}\,(\boldsymbol{\varphi}_h\cdot\mathbf{t})\|_{0,e}^2 \right\}^{1/2} \tag{6.14}$$

and

$$\|R_4\|_{H^{-1/2}(\Sigma)} \,\le\, C_4 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \, \|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 \right\}^{1/2} . \tag{6.15}$$

Proof. See [26, Lemma 3.2] for details.

■

Finally, for estimating the rest of the norms appearing on the right hand side of (6.8), we simply use Cauchy-Schwarz's inequality and the fact that R_8 can be redefined as

$$[R_8, \boldsymbol{\eta}_{\mathrm{S}}] \,:=\, -\,\frac{1}{2}\,(\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{t}}, \boldsymbol{\eta}_{\mathrm{S}})_S \qquad \forall\,\boldsymbol{\eta}_{\mathrm{S}} \,\in\, \mathbb{L}^2(\Omega_{\mathrm{S}})\,.$$

In this way, we arrive at the following lemma.

Lemma 6.4 *There hold*

$$\|R_5\|_{\mathbf{L}^2(\Omega_{\mathrm{S}})'} \,\le\, \|\mathbf{f}_{\mathrm{S}} \,+\, \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}\,, \tag{6.16}$$

$$\|R_6\|_{L_0^2(\Omega_{\mathrm{D}})'} \,\le\, \|f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,, \tag{6.17}$$

$$\|R_7\|_{\mathbb{L}_0^2(\Omega_{\mathrm{S}})'} \,\le\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}} \,-\, \mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}\,, \tag{6.18}$$

and

$$\|R_8\|_{\mathbb{L}^2(\Omega_{\mathrm{S}})'} \,\le\, \frac{1}{2}\,\|\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{t}}\|_{0,\Omega_{\mathrm{S}}}\,. \tag{6.19}$$

We end this section by observing that the reliability estimate (6.4) (cf. Theorem 6.1) is a direct consequence of (6.8), (6.9), and Lemmas 6.1, 6.2, 6.3 and 6.4, by using when it corresponds the obvious identities $\displaystyle\int_{\Omega_{\mathrm{S}}} = \sum_{T\in\mathcal{T}_h^{\mathrm{S}}}\int_T$ and $\displaystyle\int_{\Omega_{\mathrm{D}}} = \sum_{T\in\mathcal{T}_h^{\mathrm{D}}}\int_T$.

6.2 Efficiency of the a posteriori error estimator

The main result of this section is stated as follows.

Theorem 6.2 *There exists $C_{\mathtt{eff}} > 0$, independent of h, such that*

$$C_{\mathtt{eff}}\,\Theta \,+\, \mathrm{h.o.t.} \,\le\, \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{p}) \,-\, ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{p}_h)\|_{\mathbb{X}\times\mathbb{M}}\,, \tag{6.20}$$

where h.o.t. *stands, eventually, for one or several terms of higher order.*

In what follows we prove Theorem 6.2 by providing suitable upper bounds depending of local errors for each one of the 17 terms defining $\Theta_{\mathrm{S},T}^2$ (cf. (6.2)) and $\Theta_{\mathrm{D},T}^2$ (cf. (6.3)). To this respect, it is important to remark that most of the required efficiency estimates in this case are already available in the literature, and that the main tools employed in their proofs include Helmholtz decompositions, inverse inequalities, and the localization technique based on element-bubble and edge-bubble functions (cf. [15], [18], [22], [26]).

We begin with the zero order terms appearing in the definition of $\Theta_{\mathrm{S},T}^2$ and $\Theta_{\mathrm{D},T}^2$.

Lemma 6.5 *There hold*

$$\|\mathbf{f}_\mathrm{S} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T} \le \|\boldsymbol{\sigma}_\mathrm{S} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div},T} \qquad \forall\, T \in \mathcal{T}_h^S\,,$$

$$\|f_\mathrm{D} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T} \le \|\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div};T} \qquad \forall\, T \in \mathcal{T}_h^D\,,$$

and

$$\|\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{t}}\|_{0,\Omega_\mathrm{S}} \le 2\,\|\boldsymbol{\sigma}_\mathrm{S} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,\Omega_\mathrm{S}}$$

Proof. It suffices to recall, as established by Theorem 4.2, that $\mathbf{f}_\mathrm{S} = -\,\mathbf{div}\,\boldsymbol{\sigma}_\mathrm{S}$ in Ω_S, $f_\mathrm{D} = \mathrm{div}\,\mathbf{u}_\mathrm{D}$ in Ω_D, and $\boldsymbol{\sigma}_\mathrm{S} = \boldsymbol{\sigma}_\mathrm{S}^{\mathtt{t}}$ in Ω_S.

∎

We now bound the component of $\Theta_{\mathrm{S},T}^2$ involving the nonlinear operator.

Lemma 6.6 *There exists $C > 0$, independent of h, such that*

$$\|\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h}\|_{0,T}^2 \le C\left\{\|\boldsymbol{\sigma}_\mathrm{S} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 + \|\mathbf{t}_\mathrm{S} - \mathbf{t}_{\mathrm{S},h}\|_{0,T}^2\right\}$$

Proof. We know from [22, Lemma 2.1] that there exists $\bar{\gamma}_0 > 0$, independent of h, such that

$$\|\mu(|\mathbf{t}_\mathrm{S}|)\,\mathbf{t}_\mathrm{S} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h}\|_{0,T} \le \bar{\gamma}_0\,\|\mathbf{t}_\mathrm{S} - \mathbf{t}_{\mathrm{S},h}|\|_{0,T} \qquad \forall\, T \in \mathcal{T}_h^\mathrm{S}\,.$$

Hence, adding and substracting $\boldsymbol{\sigma}_\mathrm{S}^{\mathtt{d}} = \mu(|\mathbf{t}_\mathrm{S}|)\,\mathbf{t}_\mathrm{S}$ in Ω_S (cf. Theorem 4.2), we find that

$$\begin{aligned}\|\boldsymbol{\sigma}_{\mathrm{S},h}^{\mathtt{d}} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h}\|_{0,T} &\le \left\{\|(\boldsymbol{\sigma}_\mathrm{S} - \boldsymbol{\sigma}_{\mathrm{S},h})^{\mathtt{d}}\|_{0,T} + \|\mu(|\mathbf{t}_\mathrm{S}|)\,\mathbf{t}_\mathrm{S} - \mu(|\mathbf{t}_{\mathrm{S},h}|)\,\mathbf{t}_{\mathrm{S},h}\|_{0,T}\right\}\\ &\le \left\{\|\boldsymbol{\sigma}_\mathrm{S} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T} + \bar{\gamma}_0\,\|\mathbf{t}_\mathrm{S} - \mathbf{t}_{\mathrm{S},h}|\|_{0,T}\right\},\end{aligned}$$

which yields the result.

∎

We continue with the terms involving only $\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h}$ in the definition of $\Theta_{\mathrm{S},T}$.

Lemma 6.7 *There exist C_1, C_2, C_3, $C_4 > 0$, independent of h, such that*

$$\begin{aligned}
h_T^2\,\|\mathbf{rot}\,(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h})\|_{0,T}^2 &\le C_1\left\{\|\mathbf{t}_\mathrm{S} - \mathbf{t}_{\mathrm{S},h}\|_{0,T}^2 + \|\boldsymbol{\gamma}_\mathrm{S} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T}^2\right\} \quad \forall\, T \in \mathcal{T}_h^S\,,\\
h_T^2\,\|\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T}^2 &\le C_2\left\{\|\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2\,\|\mathbf{t}_\mathrm{S} - \mathbf{t}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2\,\|\boldsymbol{\gamma}_\mathrm{S} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T}^2\right\} \; \forall\, T \in \mathcal{T}_h^S\,,\\
h_e\,\|[(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h})\mathbf{t}]\|_{0,e}^2 &\le C_3\left\{\|\mathbf{t}_\mathrm{S} - \mathbf{t}_{\mathrm{S},h}\|_{0,w_e}^2 + \|\boldsymbol{\gamma}_\mathrm{S} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,w_e}^2\right\} \quad \forall\, e \in \mathcal{E}_h(\Omega_\mathrm{S})\,,\\
h_e\,\|(\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h})\mathbf{t}\|_{0,e}^2 &\le C_4\left\{\|\mathbf{t}_\mathrm{S} - \mathbf{t}_{\mathrm{S},h}\|_{0,T_e}^2 + \|\boldsymbol{\gamma}_\mathrm{S} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T_e}^2\right\} \quad \forall\, e \in \mathcal{E}_h(\Gamma_\mathrm{S})\,,
\end{aligned}$$

where $w_e := \cup\{T \in \mathcal{T}_h^S : e \in \mathcal{E}(T)\}$ for all $e \in \mathcal{E}_h(\Omega_\mathrm{S})$, and T_e is the triangle of $\mathcal{T}_h^S$ having e as an edge for all $e \in \mathcal{E}_h(\Gamma_\mathrm{S})$.

Proof. See [15, Lemmas 5.6 and 5.7] for details.

∎

The following four lemmas provide upper bounds for the remaining terms defining $\Theta_{\mathrm{S},T}$.

Lemma 6.8 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}(\Sigma)$ there holds*

$$h_e \, \|(\boldsymbol{\sigma}_{\mathrm{S},h} \,+\, \ell_h \, \mathbb{I})\mathbf{n} + \lambda_h \mathbf{n} \,-\, \kappa^{-1}(\boldsymbol{\varphi}_h \cdot \mathbf{t})\,\mathbf{t}\|_{0,e}^2 \,\leq\, C \Big\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \,+\, |\ell - \ell_h| \\ +\;\; h_T^2 \|\mathbf{div}\,(\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,T}^2 \,+\, h_e \,\|\lambda - \lambda_h\|_{0,e}^2 \,+\, h_e \,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \Big\},$$

where T is the triangle of $\mathcal{T}_h^S$ having e as an edge.

Proof. It suffices to apply [26, Lemma 3.16] by replacing there $\boldsymbol{\sigma}_{\mathrm{S}}$ and $\boldsymbol{\sigma}_{\mathrm{S},h}$ by $(\boldsymbol{\sigma}_{\mathrm{S}} + \ell\,\mathbb{I})$ and $(\boldsymbol{\sigma}_{\mathrm{S},h} + \ell_h\,\mathbb{I})$, respectively.

∎

Lemma 6.9 *There exists $C > 0$, independent of h, such that*

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\| (\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h})\mathbf{t} \,+\, \frac{d\,\boldsymbol{\varphi}_h}{d\mathbf{t}} \right\|_{0,e}^2 \\ \leq\, C \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \Big(\|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,T_e}^2 \,+\, \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T_e}^2 \Big) \,+\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Sigma}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^S$ having e as an edge.

Proof. It follows from the proof of [18, Lemma 20] by replacing there $\mathcal{C}^{-1}\,\boldsymbol{\sigma}$ and $\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h$ by $\mathbf{t}_{\mathrm{S}}$ and $\mathbf{t}_{\mathrm{S},h}$, respectively.

∎

Note that the estimate given by the previous lemma is of a nonlocal character. Actually, it will be the only one with this property in the efficiency analysis of the terms defining $\Theta_{\mathrm{S},T}$. However, under an additional regularity assumption on $\boldsymbol{\varphi}$, one can obtain the following local bound.

Lemma 6.10 *Assume that $\boldsymbol{\varphi}|_e \in \mathbf{H}^1(e)$ for each $e \in \mathcal{E}_h(\Sigma)$. Then, there exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e \left\| (\mathbf{t}_{\mathrm{S},h} + \boldsymbol{\gamma}_{\mathrm{S},h})\mathbf{t} \,+\, \frac{d\,\boldsymbol{\varphi}_h}{d\mathbf{t}} \right\|_{0,e}^2 \\ \leq\, C \left\{ \|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,T_e}^2 \,+\, \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T_e}^2 \,+\, \left\| \frac{d}{d\mathbf{t}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \right\|_{0,e}^2 \right\},$$

where T_e is the triangle of $\mathcal{T}_h^S$ having e as an edge.

Proof. See [18, Lemma 21] for details.

∎

Lemma 6.11 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e \,\|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \,\leq\, C \Big\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2\,\|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2\,\|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_e\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \Big\},$$

where T is the triangle of $\mathcal{T}_h^S$ having e as an edge.

Proof. Similarly to Lemma 6.9, it follows from the proof of [18, Lemma 22] by replacing there $\mathcal{C}^{-1}\boldsymbol{\sigma}$ and $\mathcal{C}^{-1}\boldsymbol{\sigma}_h$ by $\mathbf{t}_{\mathrm{S}}$ and $\mathbf{t}_{\mathrm{S},h}$, respectively. ∎

The estimates for the remaining terms defining $\Theta_{\mathrm{D},T}$ are given by the following four lemmas.

Lemma 6.12 *Assume that $\mathbf{K}^{-1}$ is piecewise polynomial on $\mathcal{T}_h^D$. Then, there exist C_1, C_2, C_3, $C_4 > 0$, independent of h, such that*

$$h_T^2\,\|\mathrm{rot}\,(\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 \,\le C_1\,\|\mathbf{u}_{\mathrm{D}} \,-\, \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \qquad \forall\, T \,\in\, \mathcal{T}_h^D\,,$$

$$h_T^2\,\|\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \,\le C_2\Big\{\,\|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 \,+\, h_T^2\,\|\mathbf{u}_{\mathrm{D}} \,-\, \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2\Big\} \qquad \forall\, T \,\in\, \mathcal{T}_h^D\,,$$

$$h_e\,\|[\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}]\|_{0,e}^2 \,\le C_3\,\|\mathbf{u}_{\mathrm{D}} \,-\, \mathbf{u}_{\mathrm{D},h}\|_{0,w_e}^2 \qquad \forall\, e \,\in\, \mathcal{E}_h(\Omega_{\mathrm{D}})\,,$$

$$h_e\,\|p_{\mathrm{D},h} - \,-\,\lambda_h\|_{0,e}^2 \,\le C_4\Big\{\|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T_e}^2 + h_T^2\,\|\mathbf{u}_{\mathrm{D}} \,-\, \mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2 + h_e\,\|\lambda - \lambda_h\|_{0,e}^2\Big\}\ \forall\, e \in \mathcal{E}_h(\Sigma)\,,$$

where $w_e := \cup\big\{T \in \mathcal{T}_h^D : \, e \in \mathcal{E}(T)\,\big\}$ for all $e \in \mathcal{E}_h(\Omega_{\mathrm{D}})$, and T_e is the triangle of $\mathcal{T}_h^D$ having e as an edge for all $e \in \mathcal{E}_h(\Sigma)$.

Proof. See [26, Lemma 3.13] for details. ∎

Lemma 6.13 *There exists $C > 0$, independent of h, such that for each $e \,\in\, \mathcal{E}_h(\Sigma)$ there holds*

$$h_e\,\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} \,+\, \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 \,\le\, C\Big\{\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \,+\, h_T^2\,\|\mathrm{div}(\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 \,+\, h_e\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2\Big\},$$

where T is the triangle of $\mathcal{T}_h^D$ having e as an edge.

Proof. See [26, Lemma 3.15] for details. ∎

We end the efficiency analysis of $\Theta_{\mathrm{D},T}$ with the analogue of Lemmas 6.9 and 6.10

Lemma 6.14 *Assume that $\mathbf{K}^{-1}$ is piecewise polynomial on $\mathcal{T}_h^D$. Then, there exists $C > 0$, independent of h, such that*

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\left\|\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} \,+\, \frac{d\lambda_h}{d\mathbf{t}}\right\|_{0,e}^2 \,\le\, C\,\left\{\sum_{e\in\mathcal{E}_h(\Sigma)}\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2 \,+\, \|\lambda \,-\, \lambda_h\|_{1/2,\Sigma}^2\right\}\,,$$

where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^D$ having e as an edge.

Proof. See [26, Lemma 3.13] for details. ∎

Similarly to Lemma 6.10, we now assume an additional regularity assumption on λ to derive, instead of the previous estimate, a local upper bound.

Lemma 6.15 *Assume that $\mathbf{K}^{-1}$ is piecewise polynomial on $\mathcal{T}_h^D$, and that $\lambda|_e \in H^1(e)$ for each $e \in \mathcal{E}_h(\Sigma)$. Then, there exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e\,\left\|\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} \,+\, \frac{d\lambda_h}{d\mathbf{t}}\right\|_{0,e}^2 \,\le\, C\,\left\{\,\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2 \,+\, h_e\,\left\|\frac{d}{d\mathbf{t}}\,(\lambda - \lambda_h)\right\|_{0,e}^2\,\right\}\,,$$

where T_e is the triangle of $\mathcal{T}_h^D$ having e as an edge.

Proof. See [26, Lemma 3.14] for details. Actually, as stated there, it follows by adapting the "*elasticity version*" given by [18, Lemma 21] to the present case.

■

We remark that if $\mathbf{K}^{-1}$ were not piecewise polynomial then higher order terms arising from suitable local polynomial approximations would appear in the corresponding efficiency estimates from the previous lemmas. This fact explains the expression "h.o.t." in (6.20).

Consequently, the global efficiency estimate of Θ, that is the proof of Theorem 6.2, follows straightforwardly from Lemmas 6.5 up to 6.15.

7 Numerical results

We begin this section by observing that, while the decomposition (2.21) was necessary for the analysis of the continuous and discrete formulations, the actual implementation of the latter can abstain from it. In fact, it is easy to see that redefining $\boldsymbol{\sigma}_{\mathrm{S},h} + \ell_h \mathbb{I}$, with $\boldsymbol{\sigma}_{\mathrm{S},h} \in \mathbb{H}_{0,h}(\Omega_{\mathrm{S}})$ and $\ell_h \in \mathrm{R}$, as simply $\boldsymbol{\sigma}_{\mathrm{S},h} \in \mathbb{H}_h(\Omega_{\mathrm{S}})$, and proceeding analogously with the test tensor $\boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_h(\Omega_{\mathrm{S}})$, the Galerkin scheme (5.4) - (5.5) can be stated, equivalently, as finding

$$\begin{aligned}
\underline{\mathbf{t}}_h &:= (\mathbf{t}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\sigma}_{\mathrm{S},h}) &\in \mathbf{X}_{1,h} &:= \mathbb{L}_{0,h}(\Omega_{\mathrm{S}}) \times \mathbf{H}_{0,h}(\Omega_{\mathrm{D}}) \times \mathbb{H}_h(\Omega_{\mathrm{S}})\,,\\
\underline{\boldsymbol{\varphi}}_h &:= (\boldsymbol{\varphi}_h, \lambda_h) &\in \mathbf{M}_{1,h} &:= \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) \times \Lambda_h^{\mathrm{D}}(\Sigma)\,,\\
\underline{p}_h &:= (p_{\mathrm{D},h}, \mathbf{u}_{\mathrm{S},h}, \boldsymbol{\gamma}_{\mathrm{S},h}) &\in \mathbb{M}_h &:= L_{0,h}(\Omega_{\mathrm{D}}) \times \mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbb{L}_h^2(\Omega_{\mathrm{S}})\,,
\end{aligned} \tag{7.1}$$

such that

$$\begin{aligned}
[\mathbb{A}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbb{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{p}_h] &= [\mathbb{F}, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \quad \forall\, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{X}_h := \mathbf{X}_{1,h} \times \mathbf{M}_{1,h}\,,\\
[\mathbb{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{q}_h] &= [\mathbb{G}, \underline{q}_h] \quad \forall\, \underline{q}_h \in \mathbb{M}_h\,.
\end{aligned} \tag{7.2}$$

In addition, the mean value condition required by the elements in $L_{0,h}(\Omega_{\mathrm{D}})$ can be certainly imposed through a suitable discrete Lagrange multiplier.

Throughout the rest of the section we present numerical examples illustrating the performance of the discrete system (7.1) - (7.2), confirming the reliability and efficiency of the a posteriori error estimator Θ derived in Section 6, and showing the behavior of the associated adaptive algorithm. We consider the specific finite element subspaces defined in Sections 5.3.1 and 5.3.2. In addition, all the nonlinear algebraic systems arising from (7.2) are solved by the Newton method with a tolerance of 1E-6 and taking as initial iteration the solution of the associated linear problem with $\mu \equiv 1$.

In what follows, N stands for the number of degrees of freedom defining $\mathbb{X}_h \times \mathbb{M}_h$. Furthermore, the individual and total errors are defined by:

$$\mathsf{e}(\mathbf{t}_{\mathrm{S}}) := \|\mathbf{t}_{\mathrm{S}} - \mathbf{t}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}\,, \qquad \mathsf{e}(\mathbf{u}_{\mathrm{D}}) := \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div};\Omega_{\mathrm{D}}}\,, \qquad \mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}) := \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div};\Omega_{\mathrm{S}}}\,,$$

$$\mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Sigma}\,, \qquad \mathsf{e}(\lambda) := \|\lambda - \lambda_h\|_{1/2,\Sigma}\,, \qquad \mathsf{e}(p_{\mathrm{D}}) := \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,,$$

$$\mathsf{e}(\mathbf{u}_{\mathrm{S}}) := \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}\,, \quad \mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}}) := \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}}\,, \quad \mathsf{e}(\underline{\mathbf{t}}) := \Big\{\mathsf{e}(\mathbf{t}_{\mathrm{S}})^2 + \mathsf{e}(\mathbf{u}_{\mathrm{D}})^2 + \mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}})^2\Big\}^{1/2},$$

$$\mathsf{e}(\underline{\boldsymbol{\varphi}}) := \Big\{\mathsf{e}(\boldsymbol{\varphi})^2 + \mathsf{e}(\lambda)^2\Big\}^{1/2}, \quad \mathsf{e}(\underline{p}) := \Big\{\mathsf{e}(p_{\mathrm{D}})^2 + \mathsf{e}(\mathbf{u}_{\mathrm{S}})^2 + \mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}})^2\Big\}^{1/2},$$

and

$$\mathsf{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p}) := \Big\{\mathsf{e}(\underline{\mathbf{t}})^2 + \mathsf{e}(\underline{\boldsymbol{\varphi}})^2 + \mathsf{e}(\underline{p})^2\Big\}^{1/2}.$$

In turn, the effectivity index with respect to Θ is defined by

$$\mathrm{eff}(\Theta) \,:=\, \mathsf{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})/\Theta\,,$$

and the individual and global experimental rates of convergence are given by

$$r(\%) \,:=\, \frac{\log(\mathsf{e}(\%)/\mathsf{e}'(\%))}{\log(h/h')} \qquad \text{for each} \quad \% \,\in\, \big\{\mathbf{t}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\sigma}_{\mathrm{S}}, \boldsymbol{\varphi}, \lambda, p_{\mathrm{D}}, \mathbf{u}_{\mathrm{S}}, \boldsymbol{\gamma}_{\mathrm{S}}, \underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p}\big\}\,,$$

and

$$r(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p}) \,:=\, \frac{\log(\mathsf{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})/\mathsf{e}'(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p}))}{\log(h/h')}\,,$$

where h and h' denote two consecutive meshsizes with errors e and e'. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered here are described below. Examples 1 (in 2D) and 2 (in 3D) are employed to illustrate the performance of the Galerkin scheme and to confirm the reliability and efficiency of the a posteriori error estimator Θ (in the case of Example 1) when a sequence of quasi-uniform meshes is considered. Then, Example 3 (in 2D) is utilized to show the behavior of the associated adaptive algorithm, which applies the following procedure from [41]:

1) Start with a coarse mesh $\mathcal{T}_h \,:=\, \mathcal{T}_h^{\mathrm{D}} \cup \mathcal{T}_h^{\mathrm{S}}$.

2) Solve the discrete problem (7.2) for the current mesh $\mathcal{T}_h$.

3) Compute $\Theta_T := \Theta_{\star,T}$ for each triangle $T \,\in\, \mathcal{T}_h^{\star}$, $\star \,\in\, \{\mathrm{D},\mathrm{S}\}$.

4) Check the stopping criterion and decide whether to finish or go to next step.

5) Use *blue-green* refinement on those $T' \in \mathcal{T}_h$ whose indicator $\Theta_{T'}$ satisfies

$$\Theta_{T'} \,\geq\, \frac{1}{2}\max_{T\in\mathcal{T}_h}\{\Theta_T : \quad T \in \mathcal{T}_h\}\,.$$

6) Define resulting meshes as current meshes $\mathcal{T}_h^{\mathrm{D}}$ and $\mathcal{T}_h^{\mathrm{S}}$, and go to step 2.

For each example we consider the parameters $\kappa_1 \,=\, \cdots \,=\, \kappa_{n-1} \,=\, 1$, $\mathbf{K} \,=\, \mathbb{I}$, and the nonlinear function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ given by the Carreau law for viscoplastic flows, that is

$$\mu(t) \,:=\, \mu_0 \,+\, \mu_1\,(1 + t^2)^{(\beta-2)/2} \qquad \forall\, t \in \mathrm{R}^+\,,$$

with $\mu_0 \,=\, \mu_1 \,=\, 0.5$ and $\beta \,=\, 1.5$. It is easy to check in this case that the assumptions (2.4) and (2.5) are satisfied with

$$\gamma_0 \,=\, \mu_0 \,+\, \mu_1\left\{\frac{|\beta-2|}{2} \,+\, 1\right\} \quad \text{and} \quad \alpha_0 \,=\, \mu_0\,.$$

Hence, for the implementation of our augmented scheme (7.2) we use the parameter $\rho \,=\, \frac{\alpha_0}{2\,\gamma_0^2}$, which certainly satisfies the required hypothesis $\rho \,\in\, \Big(0, \frac{\alpha_0}{\gamma_0^2}\Big)$.

In Example 1 we consider the regions $\Omega_{\mathrm{S}} := (0,1)\times(0.5,1)$ and $\Omega_{\mathrm{D}} := (0,1)\times(0,0.5)$, and choose the data $\mathbf{f}_{\mathrm{S}}$ and f_{D} so that the exact solution is given by the smooth functions

$$\mathbf{u}_{\mathrm{S}}(\mathbf{x}) = \begin{pmatrix} u_{\mathrm{S},1}(\mathbf{x}) \\ u_{\mathrm{S},2}(\mathbf{x}) \end{pmatrix} \quad \forall\, \mathbf{x} := (x_1,x_2) \in \Omega_{\mathrm{S}},$$

with

$$u_{\mathrm{S},1}(\mathbf{x}) := -\,x_1 \sin(2\pi x_1)\,(x_1-1)\,(x_2-1)\,\exp(x_1x_2)\,(2-x_1+x_1x_2),$$

$$u_{\mathrm{S},2}(\mathbf{x}) := (x_2-1)^2\exp(x_1x_2)\Big(2x_1\sin(2\pi x_1) - \sin(2\pi x_1) - 2\pi x_1\cos(2\pi x_1) + 2\pi x_1^2\cos(2\pi x_1) - x_1x_2\sin(2\pi x_1) + x_2x_1^2\sin(2\pi x_1)\Big),$$

$$p_{\mathrm{S}}(\mathbf{x}) = -\,\pi\cos(\pi x_1/2)\Big(x_2 + 0.5 - 2\cos(\pi\,(x_2+0.5)/2)^2\Big)/4 \quad \forall\, \mathbf{x} := (x_1,x_2) \in \Omega_{\mathrm{S}},$$

and

$$p_{\mathrm{D}}(\mathbf{x}) = x_1\,x_2\,(1-x_1)\sin(2\pi x_1)\sin(\pi x_2) \quad \forall\, \mathbf{x} := (x_1,x_2) \in \Omega_{\mathrm{D}}.$$

In Example 2 we consider the regions $\Omega_{\mathrm{S}} := (0,1)^2\times(0.5,1)$ and $\Omega_{\mathrm{D}} := (0,1)^2\times(0,0.5)$, and choose the data $\mathbf{f}_{\mathrm{S}}$ and f_{D} so that the exact solution is given by the smooth functions

$$\mathbf{u}_{\mathrm{S}}(\mathbf{x}) = \nabla\times\begin{pmatrix} x_1^2(1-x_1)^2x_2^2(1-x_2)^2(1-x_3)^2\sin(\pi x_1) \\ x_1^2(1-x_1)^2x_2^2(1-x_2)^2(1-x_3)^2\sin(\pi x_2) \\ x_1^2(1-x_1)^2x_2^2(1-x_2)^2(1-x_3)^2\sin(\pi x_3) \end{pmatrix} \quad \forall\, \mathbf{x} := (x_1,x_2,x_3) \in \Omega_{\mathrm{S}},$$

$$p_{\mathrm{S}}(\mathbf{x}) = (x_1^3 + x_2^3)\exp(x_3) \quad \forall\, \mathbf{x} := (x_1,x_2,x_3) \in \Omega_{\mathrm{S}},$$

and

$$p_{\mathrm{D}}(\mathbf{x}) = x_1\,x_2\,x_3\,(1-x_1)\,(1-x_2)\sin(2\pi x_1)\sin(2\pi x_2)\sin(\pi x_3) \quad \forall\, \mathbf{x} := (x_1,x_2,x_3) \in \Omega_{\mathrm{D}}.$$

Finally, in Example 3 we consider $\Omega_{\mathrm{D}} := (-1,0)^2$ and let Ω_{S} be the L-shaped domain given by $(-1,1)^2\setminus\bar{\Omega}_{\mathrm{D}}$, which yields a porous medium partially surrounded by a fluid. Then we choose the data $\mathbf{f}_{\mathrm{S}}$ and f_{D} so that the exact solution is given by

$$\mathbf{u}_{\mathrm{S}}(\mathbf{x}) = \operatorname{curl}\Big(3\,(x_1^2+x_2^2)^{4/3}\,(x_1^2-1)^2\,(x_2^2-1)^2\Big) \quad \forall\, \mathbf{x} := (x_1,x_2) \in \Omega_{\mathrm{S}},$$

$$p_{\mathrm{S}}(\mathbf{x}) = \frac{1}{100\,(x_1^2+x_2^2)+0.1} \quad \forall\, \mathbf{x} := (x_1,x_2) \in \Omega_{\mathrm{S}},$$

and

$$p_{\mathrm{D}}(\mathbf{x}) = \left(\frac{x_1+1}{10}\right)^2\sin^3(2\pi\,(x_2+0.5)) \quad \forall\, \mathbf{x} := (x_1,x_2) \in \Omega_{\mathrm{D}}.$$

Note that the partial derivatives of $\mathbf{u}_{\mathrm{S}}$ are singular at the origin and that p_{S} has high gradients around that point.

The numerical results shown below were obtained using a MATLAB code. In Tables 7.1, 7.2, 7.3 and 7.4 we summarize the convergence history of our augmented fully-mixed scheme (7.1) - (7.2) as applied to Examples 1 and 2, for sequences of quasi-uniform triangulations of the domains. The number of Newton iterations required in Example 1, for the tolerance given, ranges between 9 and 12. We observe there, looking at the corresponding experimental rates of convergence, that the $O(h)$

Table 7.1: EXAMPLE 1, quasi-uniform scheme

h	N	$\mathtt{e}(\mathbf{t}_\mathrm{S})$	$r(\mathbf{t}_\mathrm{S})$	$\mathtt{e}(\mathbf{u}_\mathrm{D})$	$r(\mathbf{u}_\mathrm{D})$	$\mathtt{e}(\boldsymbol{\sigma}_\mathrm{S})$	$r(\boldsymbol{\sigma}_\mathrm{S})$	$\mathtt{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathtt{e}(\lambda)$	$r(\lambda)$
1/8 s	897	3.995E−01	–	4.961E−01	–	1.186E−00	–	5.183E−01	–	6.928E−02	–
1/10	1380	3.039E−01	1.227	3.979E−01	0.988	9.426E−01	1.029	3.513E−01	1.743	4.796E−02	1.648
1/12	1967	2.470E−01	1.137	3.322E−01	0.991	7.843E−01	1.008	2.570E−01	1.714	3.486E−02	1.750
1/14	2658	2.083E−01	1.104	2.851E−01	0.993	6.730E−01	0.993	1.980E−01	1.693	2.671E−02	1.727
1/16	3453	1.805E−01	1.075	2.496E−01	0.994	5.891E−01	0.997	1.585E−01	1.663	2.133E−02	1.687
1/18	4352	1.593E−01	1.057	2.220E−01	0.995	5.236E−01	1.000	1.307E−01	1.639	1.755E−02	1.652
1/20	5355	1.427E−01	1.043	2.000E−01	0.996	4.713E−01	0.999	1.103E−01	1.616	1.479E−02	1.624
1/22	6462	1.293E−01	1.035	1.818E−01	0.997	4.285E−01	0.998	9.468E−02	1.598	1.270E−02	1.603
1/24	7673	1.183E−01	1.028	1.667E−01	0.997	3.929E−01	0.999	8.248E−02	1.585	1.106E−02	1.587
1/26	8988	1.090E−01	1.023	1.539E−01	0.998	3.627E−01	0.999	7.272E−02	1.574	9.751E−03	1.574
1/28	10407	1.010E−01	1.020	1.429E−01	0.998	3.368E−01	0.999	6.475E−02	1.565	8.684E−03	1.563
1/30	11930	9.420E−02	1.017	1.334E−01	0.998	3.144E−01	0.999	5.815E−02	1.558	7.801E−03	1.555
1/32	13557	8.823E−02	1.015	1.251E−01	0.998	2.947E−01	0.999	5.261E−02	1.552	7.059E−03	1.548
1/34	15288	8.297E−02	1.013	1.177E−01	0.999	2.774E−01	0.999	4.790E−02	1.547	6.429E−03	1.542
1/36	17123	7.832E−02	1.011	1.112E−01	0.999	2.620E−01	0.999	4.389E−02	1.532	5.889E−03	1.535
1/40	21105	7.041E−02	1.010	1.001E−01	0.999	2.358E−01	0.999	3.733E−02	1.536	5.012E−03	1.531
1/48	30317	5.860E−02	1.007	8.342E−01	0.999	1.965E−01	1.000	2.825E−02	1.529	3.797E−03	1.522
1/56	41193	5.019E−02	1.005	7.151E−02	0.999	1.685E−01	1.000	2.234E−02	1.521	3.006E−03	1.515
1/64	53733	4.389E−02	1.004	6.257E−02	1.000	1.474E−01	1.000	1.825E−02	1.516	2.457E−03	1.510
1/80	83805	3.508E−02	1.003	5.006E−02	1.000	1.179E−01	1.000	1.295E−02	1.536	1.753E−03	1.512
1/96	120533	2.923E−02	1.002	4.172E−02	1.000	9.828E−02	1.000	9.821E−03	1.519	1.332E−03	1.506
1/112	163917	2.504E−02	1.002	3.576E−02	1.000	8.424E−02	1.000	7.771E−03	1.519	1.057E−03	1.504
1/128	213957	2.191E−02	1.001	3.129E−02	1.000	7.371E−02	1.000	6.343E−03	1.521	8.644E−04	1.503
1/144	270653	1.947E−02	1.000	2.782E−02	1.000	6.552E−02	1.000	5.511E−03	1.193	7.344E−04	1.384
1/160	334005	1.753E−02	1.001	2.503E−02	1.000	5.897E−02	1.000	4.739E−03	1.432	6.298E−04	1.458
1/250	853893	1.095E−02	1.054	1.565E−02	1.053	3.686E−02	1.053	2.515E−03	1.420	3.280E−04	1.462

predicted by Theorems 5.4 and 5.6 with $\delta = 1$ is attained in all the unknowns for both examples. In addition, we notice from Table 7.2 that the effectivity index $\mathrm{eff}(\Theta)$ for Example 1 remains always in a neighborhood of 0.58, which illustrates the reliability and efficiency of Θ in the case of a regular solution. Some components of the approximate (left) and exact (right) solutions for Example 1, which illustrate the accurateness of the mixed finite element scheme, are displayed in Figures 7.1 and 7.2.

Then, in Tables 7.5, 7.6, 7.7, and 7.8 we provide the convergence history of the quasi-uniform and adaptive schemes, as applied to Example 3. The number of Newton iterations required in this case for the tolerance given, ranges between 14 and 16. We notice that the errors of the adaptive procedure decrease faster than those obtained by the quasi-uniform one, which is confirmed by the global experimental rates of convergence provided there. This fact, which is clearly emphasized from about $N = 10000$ on, is also illustrated by Figure 7.3 where we display the total errors $\mathtt{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})$ vs. the number of degrees of freedom N for both refinements. Moreover, as shown by the values of $r(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})$, the adaptive method is able to keep the quasi-optimal rate of convergence $\mathcal{O}(h)$ for the total error. Furthermore, the effectivity indexes remain bounded from above and below, which confirms the reliability and efficiency of Θ in this case of non-smooth solution. Intermediate meshes obtained with the adaptive refinement are displayed in Figure 7.4. We remark from there that the method is able to recognize the origin as a singularity of the solution of this example. Moreover, the additional refinement around the points $(x_1, x_2) = (\pm 1, 0)$ and $(x_1, x_2) = (0, \pm 1)$ indicates the presence of large errors in those neighborhoods as well. Finally, some components of the approximate (left) and exact (right) solutions for Example 3 are displayed in Figures 7.5 and 7.6.

Table 7.2: Example 1, quasi-uniform scheme (... cont)

N	$\mathsf{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$r(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathsf{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, p)$	$r(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, p)$	eff(Θ)
897	5.951E−03	–	3.695E−02	–	3.111E−01	–	1.478E−00	–	0.6134
1380	4.603E−03	1.152	2.752E−02	1.321	2.236E−01	1.479	1.147E−00	1.135	0.5962
1967	3.807E−03	1.042	2.238E−02	1.133	1.742E−01	1.369	9.405E−01	1.089	0.5880
2658	3.256E−03	1.013	1.900E−02	1.062	1.421E−01	1.321	7.987E−01	1.060	0.5850
3453	2.847E−03	1.004	1.655E−02	1.034	1.200E−01	1.265	6.944E−01	1.048	0.5830
4352	2.531E−03	1.001	1.468E−02	1.021	1.039E−01	1.226	6.142E−01	1.041	0.5815
5355	2.277E−03	1.000	1.319E−02	1.014	9.162E−02	1.193	5.508E−01	1.034	0.5804
6462	2.070E−03	1.000	1.198E−02	1.010	8.195E−02	1.171	4.994E−01	1.028	0.5795
7673	1.898E−03	1.000	1.098E−02	1.007	7.412E−02	1.153	4.568E−01	1.025	0.5788
8988	1.752E−03	1.000	1.013E−02	1.005	6.766E−02	1.140	4.209E−01	1.022	0.5783
10407	1.627E−03	1.000	9.401E−03	1.004	6.223E−02	1.129	3.903E−01	1.020	0.5778
11930	1.519E−03	1.000	8.772E−03	1.004	5.760E−02	1.120	3.638E−01	1.018	0.5774
13557	1.424E−03	1.000	8.222E−03	1.003	5.361E−02	1.112	3.407E−01	1.017	0.5771
15288	1.340E−03	1.000	7.738E−03	1.003	5.014E−02	1.105	3.203E−01	1.016	0.5768
17123	1.266E−03	1.000	7.307E−03	1.002	4.708E−02	1.100	3.023E−01	1.014	0.5766
21105	1.139E−03	1.000	6.575E−03	1.002	4.197E−02	1.091	2.717E−01	1.013	0.5762
30317	9.492E−04	1.000	5.478E−03	1.001	3.447E−02	1.079	2.260E−01	1.011	0.5757
41193	8.136E−04	1.000	4.695E−03	1.001	2.925E−02	1.067	1.934E−01	1.009	0.5753
53733	7.119E−04	1.000	4.107E−03	1.001	2.539E−02	1.058	1.690E−01	1.008	0.5750
83805	5.695E−04	1.000	3.286E−03	1.000	2.010E−02	1.048	1.350E−01	1.007	0.5747
120533	4.746E−04	1.000	2.738E−03	1.000	1.663E−02	1.040	1.124E−01	1.005	0.5745
163917	4.068E−04	1.000	2.347E−03	1.000	1.418E−02	1.033	9.629E−02	1.005	0.5743
213957	3.560E−04	1.000	2.053E−03	1.000	1.236E−02	1.029	8.421E−02	1.004	0.5742
270653	3.164E−04	1.000	1.825E−03	1.000	1.095E−02	1.032	7.484E−02	1.002	0.5742
334005	2.848E−04	1.000	1.643E−03	1.000	9.826E−03	1.024	6.733E−02	1.003	0.5742
853893	1.780E−04	1.053	1.027E−03	1.072	6.089E−03	1.053	4.204E−02	1.055	0.5742

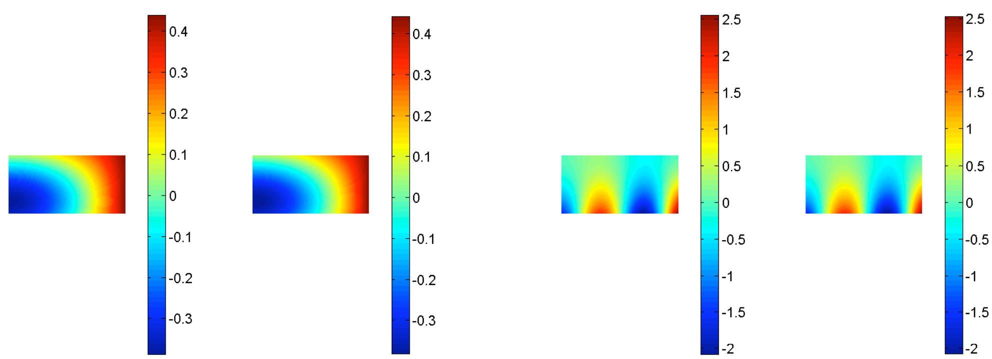

Figure 7.1: Example 1, p_{S} and $\sigma_{S,12}$ for $N = 120533$

Table 7.3: Example 2, quasi-uniform scheme

h	N	$\mathsf{e}(\mathbf{t}_{\mathrm{S}})$	$r(\mathbf{t}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathsf{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathsf{e}(\lambda)$	$r(\lambda)$
1/4	6086	1.368E−01	–	2.242E−01	–	5.213E−01	–	1.970E−02	–	1.048E−02	–
1/8	46884	7.737E−02	0.822	1.167E−01	0.942	2.617E−01	0.994	1.037E−02	0.926	1.552E−02	–
1/12	156386	5.338E−02	0.915	7.864E−02	0.974	1.738E−01	1.009	5.613E−03	1.514	8.433E−03	1.504
1/16	368576	4.065E−02	0.947	5.919E−02	0.988	1.299E−01	1.012	3.583E−03	1.560	4.867E−03	1.911
1/20	717438	3.278E−02	0.964	4.743E−02	0.993	1.037E−01	1.011	2.526E−03	1.567	3.200E−03	1.879
1/24	1236956	2.745E−02	0.974	3.956E−02	0.995	8.622E−02	1.010	1.898E−03	1.569	2.305E−03	1.800

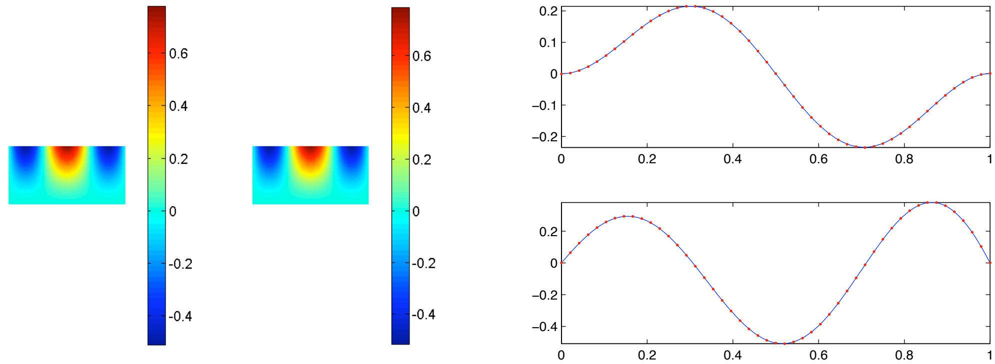

Figure 7.2: EXAMPLE 1, $u_{\mathrm{D},1}$ and $\boldsymbol{\varphi}$ for $N = 120533$

Table 7.4: EXAMPLE 2, quasi-uniform scheme (... cont)

N	$\mathsf{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$r(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathsf{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})$	$r(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})$
6086	1.930E−03	–	8.682E−03	–	5.638E−02	–	5.870E−01	–
46884	9.663E−04	0.998	2.849E−03	1.607	2.100E−02	1.425	2.981E−01	1.135
156386	5.849E−04	1.238	1.403E−03	1.747	1.154E−02	1.476	1.987E−01	1.089
368576	4.287E−04	1.080	8.348E−04	1.804	7.470E−03	1.513	1.487E−01	1.060
717438	3.414E−04	1.021	5.536E−04	1.841	5.304E−03	1.535	1.188E−01	1.048
1236956	2.840E−04	1.008	3.942E−04	1.862	4.000E−03	1.548	9.888E−02	1.041

Table 7.5: EXAMPLE 3, quasi-uniform scheme

h	N	$\mathsf{e}(\mathbf{t}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{D}})$	$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathsf{e}(\boldsymbol{\varphi})$	$\mathsf{e}(\lambda)$
1	93	5.864E−00	8.654E−01	2.699E+01	2.837E−00	1.252E−00
1/3	1094	5.295E−00	3.403E−01	1.647E+01	2.349E−00	2.221E−01
1/5	2929	3.793E−00	1.989E−01	1.306E+01	1.462E−00	1.074E−01
1/7	5797	2.700E−00	1.400E−01	1.235E+01	1.050E−00	5.071E−02
1/9	9115	2.175E−00	1.061E−01	1.125E+01	7.521E−01	4.127E−02
1/11	13958	1.774E−00	8.698E−02	9.682E−00	8.081E−01	3.203E−02
1/13	19752	1.445E−00	7.681E−02	7.542E−00	5.152E−01	2.219E−02
1/15	26116	1.264E−00	6.488E−02	6.206E−00	4.463E−01	1.674E−02
1/17	33265	1.096E−00	5.634E−02	6.407E−00	3.857E−01	1.440E−02
1/19	41839	9.756E−01	5.103E−02	5.001E−00	3.486E−01	1.094E−02
1/21	51029	9.057E−01	4.628E−02	4.528E−00	2.929E−01	9.071E−03
1/25	74062	7.374E−01	3.777E−02	4.019E−00	2.228E−01	6.148E−03
1/35	142283	5.262E−01	2.780E−02	3.502E−00	1.453E−01	4.032E−03
1/45	237444	4.022E−01	2.093E−02	2.848E−00	1.035E−01	2.481E−03
1/55	355451	3.271E−01	1.725E−02	2.563E−00	7.612E−02	1.774E−03
1/65	496414	2.746E−01	1.464E−02	2.202E−00	5.915E−02	1.365E−03

Table 7.6: EXAMPLE 3, quasi-uniform scheme (... cont)

N	$\mathtt{e}(p_{\mathrm{D}})$	$\mathtt{e}(\mathbf{u}_{\mathrm{S}})$	$\mathtt{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathtt{e}(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}},\underline{p})$	$r(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}},\underline{p})$	eff(Θ)
93	1.429E−01	1.658E−00	3.794E−00	2.811E+01	–	0.8954
1094	4.436E−02	6.076E−01	4.110E−00	1.796E+01	0.408	0.7313
2929	2.109E−02	3.951E−01	3.125E−00	1.403E+01	0.482	0.6901
5797	8.068E−03	2.837E−01	2.273E−00	1.289E+01	0.252	0.7602
9115	5.318E−03	2.283E−01	1.892E−00	1.164E+01	0.405	0.7796
13958	3.245E−03	1.839E−01	1.544E−00	9.999E−00	0.758	0.7862
19752	2.790E−03	1.543E−01	1.267E−00	7.802E−00	1.485	0.7651
26116	1.879E−03	1.345E−01	1.100E−00	6.445E−00	1.335	0.7398
33265	1.713E−03	1.173E−01	9.702E−01	6.584E−00	-0.171	0.7856
41839	1.003E−03	1.055E−01	8.605E−01	5.180E−00	2.157	0.7459
51029	9.881E−04	9.693E−02	7.996E−01	4.697E−00	0.979	0.7340
74062	6.138E−04	7.974E−02	6.569E−01	4.145E−00	0.716	0.7539
142283	4.032E−04	5.707E−02	4.734E−01	3.577E−00	0.438	0.8041
237444	2.385E−04	4.396E−02	3.622E−01	2.901E−00	0.833	0.8146
355451	1.686E−04	3.593E−02	2.954E−01	2.602E−00	0.543	0.8386
496414	1.220E−04	3.024E−02	2.490E−01	2.234E−00	0.913	0.8423

Table 7.7: EXAMPLE 3, adaptive scheme

N	$\mathtt{e}(\mathbf{t}_{\mathrm{S}})$	$\mathtt{e}(\mathbf{u}_{\mathrm{D}})$	$\mathtt{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathtt{e}(\boldsymbol{\varphi})$	$\mathtt{e}(\lambda)$
93	5.864E−00	8.654E−01	2.699E+01	2.837E−00	1.252E−00
270	6.406E−00	5.470E−01	2.793E+01	2.563E−00	9.664E−01
953	5.613E−00	3.342E−01	1.784E+01	2.367E−00	2.584E−01
2535	3.583E−00	3.613E−01	1.204E+01	2.834E−00	4.107E−01
4083	2.840E−00	2.882E−01	1.110E+01	1.729E−00	1.446E−01
6004	2.327E−00	2.560E−01	1.200E+01	1.647E−00	9.231E−02
9051	1.962E−00	2.208E−01	8.152E−00	1.293E−00	5.260E−02
11558	1.741E−00	2.169E−01	6.718E−00	9.672E−01	5.639E−02
24615	1.040E−00	1.662E−01	4.289E−00	4.600E−01	1.991E−02
43104	8.326E−01	1.247E−01	3.364E−00	2.463E−01	1.204E−02
80989	5.661E−01	1.244E−01	2.377E−00	1.855E−01	8.597E−03
126407	4.640E−01	1.157E−01	1.891E−00	9.514E−02	4.911E−03
280099	3.025E−01	1.013E−01	1.297E−00	6.864E−02	3.212E−03
468314	2.374E−01	7.989E−02	9.729E−01	3.606E−02	1.880E−03

Table 7.8: EXAMPLE 3, adaptive scheme (... cont)

N	$\mathtt{e}(p_{\mathrm{D}})$	$\mathtt{e}(\mathbf{u}_{\mathrm{S}})$	$\mathtt{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathtt{e}(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}},\underline{p})$	$r(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}},\underline{p})$	eff(Θ)
93	1.429E−01	1.658E−00	3.794E−00	2.811E+01	–	0.8954
270	8.918E−02	1.337E−00	5.003E−00	2.925E+01	-0.075	0.8519
953	5.446E−02	8.060E−01	4.181E−00	1.933E+01	0.657	0.6708
2535	4.477E−02	5.839E−01	3.033E−00	1.325E+01	0.772	0.6491
4083	2.040E−02	4.184E−01	2.446E−00	1.186E+01	0.467	0.7015
6004	1.399E−02	3.653E−01	1.997E−00	1.250E+01	-0.275	0.7789
9051	9.291E−03	2.506E−01	1.663E−00	8.652E−00	1.794	0.6887
11558	9.665E−03	2.398E−01	1.560E−00	7.187E−00	1.518	0.6671
24615	3.348E−03	1.528E−01	9.590E−01	4.545E−00	1.212	0.6450
43104	1.889E−03	1.162E−01	7.695E−01	3.562E−00	0.870	0.6311
80989	1.100E−03	8.508E−02	5.202E−01	2.509E−00	1.111	0.6343
126407	7.986E−04	6.563E−02	4.251E−01	2.000E−00	1.020	0.6179
280099	5.810E−04	4.468E−02	2.773E−01	1.367E−00	0.957	0.6303
468314	4.024E−04	3.327E−02	2.188E−01	1.029E−00	1.103	0.6098

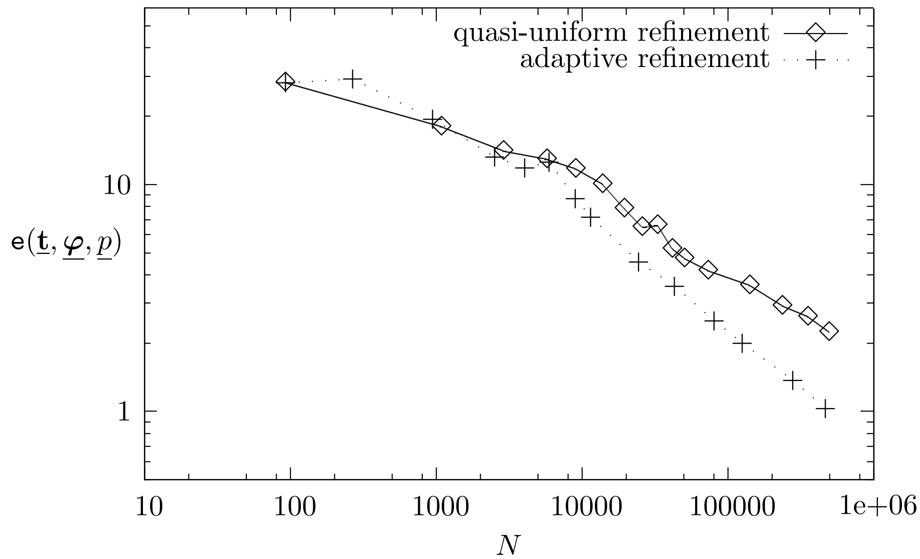

Figure 7.3: Example 3, $\mathsf{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})$ vs. N for the quasi-uniform and adaptive schemes

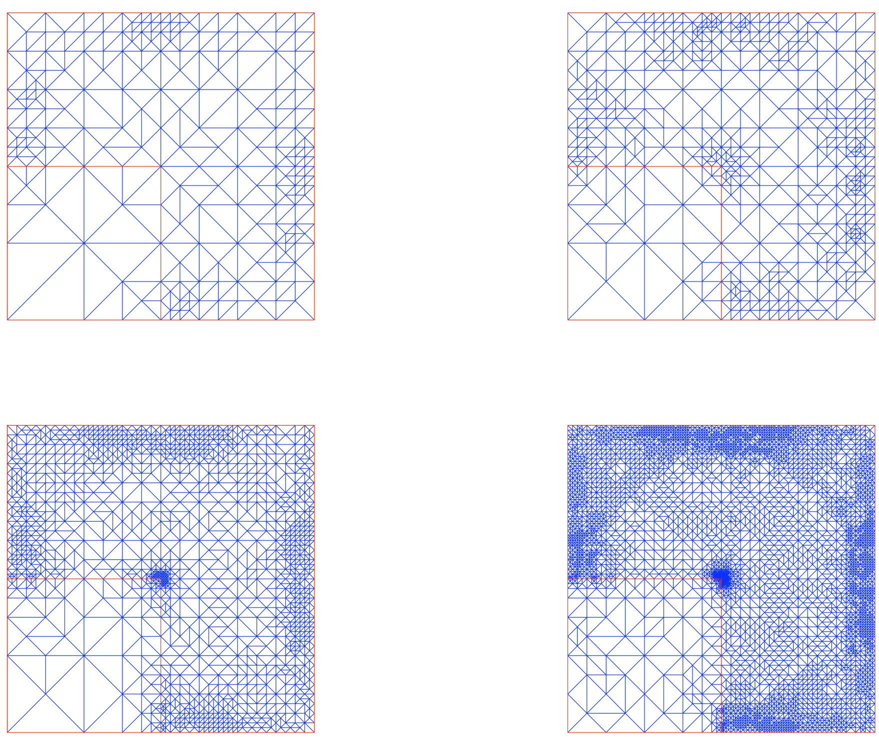

Figure 7.4: Example 3, adapted meshes with 4083, 9051, 24615 and 80989 degrees of freedom

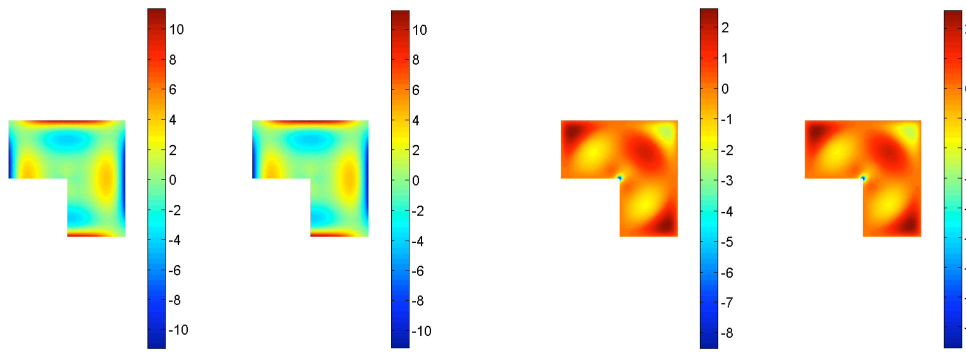

Figure 7.5: EXAMPLE 3, $t_{\mathrm{S},21}$ and $\sigma_{\mathrm{S},22}$ for adaptive scheme with $N = 280099$

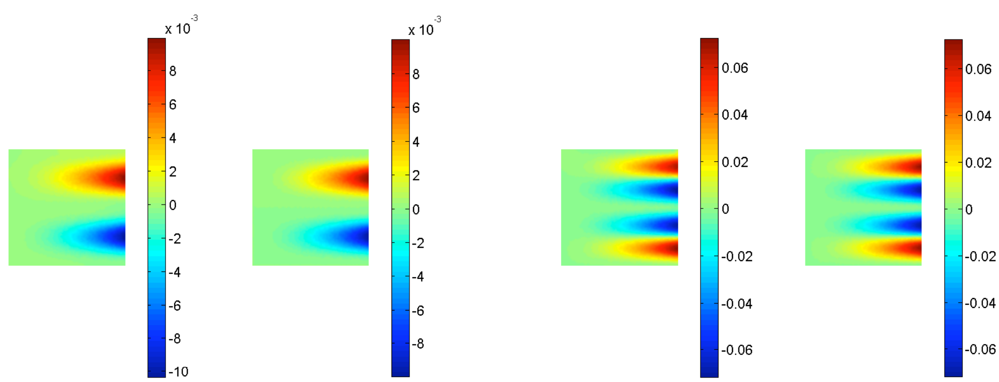

Figure 7.6: EXAMPLE 3, p_{D} and $u_{\mathrm{D},2}$ for adaptive scheme with $N = 280099$

References

[1] D.N. Arnold, F. Brezzi, and J. Douglas, *PEERS: A new mixed finite element method for plane elasticity.* Japan Journal of Applied Mathematics, vol. 1, pp. 347-367, (1984).

[2] D.N. Arnold, R.S. Falk, and R. Winther, *Finite element exterior calculus, homological tecniques, and applications.* Acta Numerica, vol. 15, pp. 1-155, (2006).

[3] D.N. Arnold, R.S. Falk, and R. Winther, *Mixed finite element methods for linear elasticity with weakly imposed symmetry.* Mathematics of Computation, vol. 76, pp. 1699-1723, (2007).

[4] I. Babuška and A.K. Aziz, *Survey lectures on the mathematical foundations of the finite element method.* In: The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A.K. (editor), Academic Press, New York, (1972).

[5] J. Baranger, K. Najib, K. and D. Sandri, *Numerical analysis of a three-fields model for a quasi-Newtonian flow.* Computer Methods in Applied Mechanics and Engineering, vol. 109, 3-4, pp. 281-292, (1993).

[6] C. Bernardi, F. Hecht, and O. Pironneau, *Coupling Darcy and Stokes equations for porous media with cracks.* ESAIM: Mathematical Modelling and Numerical Analysis, vol. 39, 1, pp. 7-35, (2005).

[7] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. Springer-Verlag, 1991.

[8] C. Carstensen, *A posteriori error estimate for the mixed finite element method.* Mathematics of Computation, vol. 66, 218, pp. 465-476, (1997).

[9] P. Clément, *Approximation by finite element functions using local regularisation.* RAIRO Modélisation Mathématique et Analyse Numérique, vol. 9, pp. 77-84, (1975).

[10] M. Discacciati, Domain Decomposition Methods for the Coupling of Surface and Groundwater Flows. Ph.D. Thesis, Ecole Polytechnique Fédérale de Lausanne, 2004.

[11] M. Discacciati, E. Miglio, and A. Quarteroni, *Mathematical and numerical models for coupling surface and groundwater flows.* Applied Numerical Mathematics, vol. 43, pp. 57-74, (2002).

[12] V.J. Ervin, E.W. Jenkins, and S. Sun, *Coupled generalized nonlinear Stokes flow with flow through a porous medium.* SIAM Journal on Numerical Analysis, vol. 47, 2, pp. 929-952, (2009).

[13] V.J. Ervin, E.W. Jenkins and S. Sun, *Coupling nonlinear Stokes and Darcy flow using mortar finite elements.* Applied Numerical Mathematics, vol. 61, 11, pp. 1198-1222, (2011).

[14] J. Galvis and M. Sarkis, *Non-matching mortar discretization analysis for the coupling Stokes-Darcy equations.* Electronic Transactions on Numerical Analysis, vol. 26, pp. 350-384, (2007).

[15] A.I. Garralda-Guillém, G.N. Gatica, A. Márquez and M. Ruiz-Galán, *A posteriori error analysis of twofold saddle point variational formulations for nonlinear boundary value problems.* IMA Journal of Numerical Analysis (2013). http://dx.doi.org/10.1093/imanum/drt006

[16] G.N. Gatica, *Solvability and Galerkin approximations of a class of nonlinear operator equations.* Zeitschrift für Analysis und ihre Anwendungen, vol. 21, 3, pp. 761-781, (2002).

[17] G.N. Gatica, N. Heuer, and S. Meddahi, *On the numerical analysis of nonlinear twofold saddle point problems*. IMA Journal of Numerical Analysis, vol. 23, 2, pp. 301-330, (2003).

[18] G.N. Gatica, G.C. Hsiao, and S. Meddahi, *A residual-based a posteriori error estimator for a two-dimensional fluid-solid interaction problem*. Numerische Mathematik, vol. 114, 1, pp. 63-106, (2009).

[19] G.N. Gatica, G.C. Hsiao, and S. Meddahi, *A coupled mixed finite element method for the interaction problem between an electromagnetic field and an elastic body*. SIAM Journal on Numerical Analysis, vol. 48, 4, pp. 1338-1368, (2010).

[20] G.N. Gatica, A. Márquez and S. Meddahi, *A new dual-mixed finite element method for the plane linear elasticity problem with pure traction boundary conditions*. Computer Methods in Applied Mechanics and Engineering, vol. 197, 9-12, pp. 11151130, (2008).

[21] G.N. Gatica, A. Márquez and M.A. Sánchez, *Analysis of a velocity-pressure-pseudostress formulation for the stationary Stokes equations*. Computer Methods in Applied Mechanics and Engineering, vol. 199, 17-20, pp. 1064-1079, (2010).

[22] G.N. Gatica, A. Márquez and M.A. Sánchez, *A priori and a posteriori error analyses of a velocity-pseudostress formulation for a class of quasi-Newtonian Stokes flows*. Computer Methods in Applied Mechanics and Engineering, vol. 200, 17-20, pp. 1619-1636, (2011).

[23] G.N. Gatica, A. Márquez and W. Rudolph, *A priori and a posteriori error analyses of augmented twofold saddle point formulations for nonlinear elasticity problems*. Computer Methods in Applied Mechanics and Engineering, vol. 264, 1, pp. 23-48, (2013).

[24] G.N. Gatica, S. Meddahi, and R. Oyarzúa, *A conforming mixed finite-element method for the coupling of fluid flow with porous media flow*. IMA Journal of Numerical Analysis, vol. 29, 1, pp. 86-108, (2009).

[25] G.N. Gatica, R. Oyarzúa and F.-J. Sayas, *Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem*. Mathematics of Computation, vol. 80, 276, pp. 1911-1948, (2011).

[26] G.N. Gatica, R. Oyarzúa and F.-J. Sayas, *A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes-Darcy coupled problem*. Computer Methods in Applied Mechanics and Engineering, vol. 200, 21-22, pp. 1877-1891, (2011).

[27] G.N. Gatica, R. Oyarzúa and F.-J. Sayas, *A twofold saddle point approach for the coupling of fluid flow with nonlinear porous media flow*. IMA Journal of Numerical Analysis, vol. 32, 3, pp. 845-887, (2012).

[28] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms. Springer Verlag, 1986.

[29] R. Hiptmair, *Finite elements in computational electromagnetism*. Acta Numerica, vol. 11, pp. 237-339, (2002).

[30] T. Karper, K.-A. Mardal, and R. Winther, *Unified finite element discretizations of coupled Darcy-Stokes flow*. Numerical Methods for Partial Differential Equations, vol. 25, 2, pp. 311-326, (2009).

[31] O. LADYZHENSKAYA, *New equations for the description of the viscous incompressible fluids and solvability in the large for the boundary value problems of them.* In: Boundary Value Problems of Mathematical Physics V, Providence, RI: AMS, (1970).

[32] W.J. LAYTON, F. SCHIEWECK, AND I. YOTOV, *Coupling fluid flow with porous media flow.* SIAM Journal on Numerical Analysis, vol. 40, 6, pp. 2195-2218, (2003).

[33] M. LONSING AND R. VERFÜRTH, *On the stability of BDMS and PEERS elements.* Numerische Mathematik, vol. 99, 1, pp. 131-140, (2004).

[34] A.F.D. LOULA AND J.N.C. GUERREIRO, *Finite element analysis of nonlinear creeping flows.* Computer Methods in Applied Mechanics and Engineering, vol. 79, pp. 87-109, (1990).

[35] A. MÁRQUEZ, S. MEDDAHI, AND F.-J. SAYAS, *Strong coupling of finite element methods for the Stokes-Darcy problem.* `http://arxiv.org/abs/1203.4717`

[36] A. QUARTERONI AND A. VALLI, Numerical Approximation of Partial Differential Equations. Springer Series in Computational Mathematics, 23. Springer-Verlag, Berlin, Heidelberg, 1994.

[37] B. RIVIERE AND I. YOTOV, *Locally conservative coupling of Stokes and Darcy flows.* SIAM Journal on Numerical Analysis, vol. 42, 5, pp. 1959-1977, (2005).

[38] J.E. ROBERTS AND J.-M. THOMAS, Mixed and Hybrid Methods. In: Handbook of Numerical Analysis, edited by P.G. Ciarlet and J.L. Lions, vol. II, Finite Element Methods (Part 1), 1991, North-Holland, Amsterdam.

[39] D. SANDRI, *Sur l'approximation numérique des écoulements quasi-Newtoniens dont la viscosité suit la loi puissance ou la loi de Carreau.* Mathematical Modelling and Numerical Analysis, vol. 27, 2, pp. 131-155, (1993).

[40] J.M. URQUIZA, D. N'DRI, A. GARON, AND M.C. DELFOUR, *Coupling Stokes and Darcy equations.* Applied Numerical Mathematics, vol. 58, 5, pp. 525-538, (2008).

[41] R. VERFÜRTH, A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques. Wiley-Teubner (Chichester), 1996.

[42] X. XIE, J. XU AND G. XUE, *Uniformly stable finite element methods for Darcy-Stokes-Brinkman models.* Journal of Computational Mathematics, vol. 26, 3, pp. 437-455, (2008).