An exactly divergence-free finite element method for a generalized Boussinesq problem

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Abstract

We propose and analyze a mixed finite element method with exactly divergence-free velocities for the numerical simulation of a generalized Boussinesq problem, describing the motion of a non-isothermal incompressible fluid subject to a heat source. The method is based on using divergence-conforming elements of order k for the velocities, discontinuous elements of order $k-1$ for the pressure, and standard continuous elements of order k for the discretization of the temperature. The H^1-conformity of the velocities is enforced by a discontinuous Galerkin approach. The resulting numerical scheme yields exactly divergence-free velocity approximations; thus, it is provably energy-stable without the need to modify the underlying differential equations. We prove the existence and stability of discrete solutions, and derive optimal error estimates in the mesh size for small and smooth solutions.

Key words: Generalized Boussinesq equations, non-isothermal incompressible flow problems, divergence-conforming elements, discontinuous Galerkin methods

Mathematics Subject Classifications (1991): 65N12, 65N30, 76D05, 80A20

1 Introduction

The numerical simulation of incompressible non-isothermal fluid flow problems has become increasingly important for the design and analysis of devices in many branches of engineering. Relevant industrial applications include heat pipes, heat exchangers, chemical reactors, or cooling processes. Temperature-dependent flows have also become of great interest in geophysical or oceanographic flows with applications to weather and climate predictions.

The last decade has seen a significant interest in the development and analysis of efficient finite element methods for such problems. We mention here only [3, 4, 5, 10, 12, 19, 20, 21] and the references therein. In particular, in [20] a conforming method is presented and analyzed for

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approximating non-isothermal incompressible fluid flow problems. However, the analysis there hinges on technical assumptions which may be difficult to verify in practice. The work [21] studies a finite element method for time-dependent non-isothermal incompressible fluid flow problems. Here, the governing equations are discretized by the backward Euler method in time and conforming finite elements in space.

In this paper, we propose an alternative approach for the numerical approximation (in space) of a non-isothermal flow problem. As a model problem, we consider the generalized Boussinesq model analyzed theoretically in [17]: it couples the stationary incompressible Navier-Stokes equations for the fluid variables (velocity and pressure) with a convection-diffusion equation for the temperature variable. The coupling is non-linear through a temperature-dependent viscosity, and through a buoyancy term acting in direction opposite to gravity.

Following [9], we employ divergence-conforming Brezzi-Douglas-Marini (BDM) elements of order k for the approximation of the velocity, discontinuous elements of order $k-1$ for the pressure, and continuous elements of order k for the temperature. To enforce H^1-continuity of the velocities, we use an interior penalty discontinuous Galerkin (DG) technique. The resulting mixed finite element method has the distinct property that it yields exactly divergence-free velocity approximations. Thus, it exactly preserves an essential constraint of the governing equations. In addition, the method is provably energy-stable without the need for symmetrization of the convective discretization; confer also the discussion in [8, 9].

We show the existence and stability of discrete solutions by mimicking the fixed point arguments presented in [17] for the continuous problem. A crucial aspect of this argument is the construction of a suitable lifting of the temperature boundary data into the computational domain. To deal with this difficulty on the discrete level, we shall use a slightly more restrictive small data assumption. We then derive optimal error estimates for problems with small and sufficiently smooth solutions. In particular, we show that the velocity errors in the DG energy norm, the pressure errors in the L^2-norm, and the temperature errors in the H^1-norm converge of order $O(h^k)$ in the mesh size h.

The rest of the paper is structured as follows. In Section 2, we introduce a generalized Boussinesq model problem, and review the results from [17] regarding existence and uniqueness of solutions. In Section 3, we present our finite element discretization, and review the stability properties of the discrete formulation. In Section 4, we establish the existence and stability of approximate solutions under a small data assumption. In Section 5, we state and prove our a-priori error estimates. We end the paper with concluding remarks in Section 6.

We end this section by fixing some notation. To that end, let $\mathcal{O}$ be a domain in $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary $\partial\mathcal{O}$. For $r \geq 0$ and $p \in [1, \infty]$, we denote by $L^p(\mathcal{O})$ and $W^{r,p}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{L^p(\mathcal{O})}$ and $\|\cdot\|_{W^{r,p}(\mathcal{O})}$, respectively. Note that $W^{0,p}(\mathcal{O}) = L^p(\mathcal{O})$. If $p = 2$, we write $H^r(\mathcal{O})$ in place of $W^{r,2}(\mathcal{O})$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{r,\mathcal{O}}$, respectively. For $r \geq 0$, we write $|\cdot|_{r,\mathcal{O}}$ for the H^r-seminorm. The space $H^1_0(\mathcal{O})$ is the space of functions in $H^1(\mathcal{O})$ with vanishing trace on Γ, and $L^2_0(\mathcal{O})$ is the space of L^2-functions with vanishing mean value over $\mathcal{O}$. Spaces of vector-valued functions are denoted in bold face. For example, $\mathbf{H}^r(\mathcal{O}) = [H^r(\mathcal{O})]^d$ for $r \geq 0$. For simplicity, we also write $\|\cdot\|_{r,\mathcal{O}}$ and $|\cdot|_{r,\mathcal{O}}$ for the corresponding norms and seminorms on these spaces. Furthermore, we will use the vector-valued Hilbert spaces

$$\begin{aligned}
\mathbf{H}(\mathrm{div}\,;\mathcal{O}) \;&=\; \big\{\, \mathbf{w}\in \mathbf{L}^2(\mathcal{O}) \,:\, \mathrm{div}\;\mathbf{w}\in L^2(\mathcal{O}) \,\big\}\,,\\
\mathbf{H}_0(\mathrm{div}\,;\mathcal{O}) \;&=\; \big\{\, \mathbf{w}\in \mathbf{H}(\mathrm{div}\,;\mathcal{O}) \,:\, \mathbf{w}\cdot\mathbf{n}_{\partial\mathcal{O}} = 0 \text{ on } \partial\mathcal{O} \,\big\}\,,\\
\mathbf{H}_0(\mathrm{div}^{\,0};\mathcal{O}) \;&=\; \big\{\, \mathbf{w}\in \mathbf{H}_0(\mathrm{div}\,;\mathcal{O}) \,:\, \mathrm{div}\;\mathbf{w}\equiv 0 \text{ in } \Omega \,\big\}\,,
\end{aligned} \tag{1.1}$$

with $\mathbf{n}_{\mathcal{O}}$ denoting the unit outward normal on $\partial\mathcal{O}$. These spaces are endowed with the norm

$$\|\mathbf{w}\|^2_{\mathrm{div},\mathcal{O}} = \|\mathbf{w}\|^2_{0,\mathcal{O}} + \|\mathrm{div}\;\mathbf{w}\|^2_{0,\mathcal{O}}.$$

In the subsequent analysis, we denote by $C_\infty > 0$ the embedding constant such that

$$\|\mathbf{u}\|_{1,\mathcal{O}} \le C_\infty \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{O})}, \qquad \|\theta\|_{1,\mathcal{O}} \le C_\infty \|\theta\|_{W^{1,\infty}(\mathcal{O})}, \tag{1.2}$$

for all $\mathbf{u}\in \mathbf{W}^{1,\infty}(\mathcal{O})$ and $\theta \in W^{1,\infty}(\mathcal{O})$. Finally, we shall frequently use the notation C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters.

2 Weak formulation of a generalized Boussinesq problem

In this section, we introduce a model problem, cast it into weak form, discuss the stability properties of the forms involved, and review some theoretical properties regarding existence and uniqueness of solutions.

2.1 Model problem

We consider the stationary generalized Boussinesq problem analyzed theoretically in [17]. The governing partial differential equations then are given by

$$-\mathbf{div}(\nu(\theta)\nabla\mathbf{u}) \,+\, (\mathbf{u}\cdot\nabla)\mathbf{u} \,+\, \nabla p \,-\, \mathbf{g}\,\theta \;=\; \mathbf{0} \qquad \text{in } \Omega, \tag{2.1}$$

$$\mathrm{div}\,\mathbf{u} \;=\; 0 \qquad \text{in } \Omega, \tag{2.2}$$

$$-\mathrm{div}\,(\kappa(\theta)\,\nabla\theta) \,+\, \mathbf{u}\cdot\nabla\theta \;=0 \qquad \text{in } \Omega, \tag{2.3}$$

$$\mathbf{u} \;=\; \mathbf{0} \qquad \text{on } \Gamma, \tag{2.4}$$

$$\theta \;=\; \theta_D \qquad \text{on } \Gamma. \tag{2.5}$$

Here, Ω is a polygon or polyhedron in $\mathbb{R}^d$, $d = 2,3$ with Lipschitz boundary $\Gamma = \partial\Omega$. The unknowns are the fluid velocity $\mathbf{u}$, the pressure p, and the temperature θ. The given data are the non-vanishing boundary temperature $\theta_D \in H^{1/2}(\Gamma)$, and the external force per unit mass $\mathbf{g}\in \mathbf{L}^2(\Omega)$, usually acting in direction opposite to gravity.

The functions $\nu(\cdot)$ and $\kappa(\cdot)$ are the fluid viscosity and the thermal conductivity, respectively. We assume that ν and κ are Lipschitz continuous and satisfy

$$|\nu(\theta_1) - \nu(\theta_2)| \le \nu_{\mathrm{lip}}|\theta_1-\theta_2|, \qquad |\kappa(\theta_1) - \kappa(\theta_2)| \le \kappa_{\mathrm{lip}}|\theta_1-\theta_2|, \tag{2.6}$$

for all values of θ_1, θ_2, with Lipschitz constants $\nu_{\rm lip}$, $\kappa_{\rm lip} > 0$. Moreover, we suppose that ν and κ are bounded from above and from below, that is, there are positive constants such that

$$0 \,<\, \nu_1 \,\le\, \nu(\theta) \,\le\, \nu_2, \qquad 0 \,<\, \kappa_1 \,\le\, \kappa(\theta) \,\le\, \kappa_2, \tag{2.7}$$

for all values of θ.

The variational formulation of problem (2.1)–(2.5) amounts to finding $(\mathbf{u}, p, \theta) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega)$ such that $\theta|_\Gamma = \theta_D$ and

$$\begin{aligned} A_{\rm S}(\theta; \mathbf{u}, \mathbf{v}) \,+\, O_{\rm S}(\mathbf{u}; \mathbf{u}, \mathbf{v}) \,-\, B(\mathbf{v}, p) - D(\theta, \mathbf{v}) \,&=\, 0, \\ B(\mathbf{u}, q) \,&=\, 0, \\ A_{\rm T}(\theta; \theta, \psi) \,+\, O_{\rm T}(\mathbf{u}; \theta, \psi) \,&=\, 0, \end{aligned} \tag{2.8}$$

for all $(\mathbf{v}, q, \psi) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega)$. Here, the forms are given by

$$A_{\rm S}(\psi; \mathbf{u}, \mathbf{v}) = \int_\Omega \nu(\psi)\, \nabla \mathbf{u} : \nabla \mathbf{v}, \qquad O_{\rm S}(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \int_\Omega ((\mathbf{w} \cdot \nabla)\mathbf{u}) \cdot \mathbf{v}, \tag{2.9}$$

$$A_{\rm T}(\varphi; \theta, \psi) = \int_\Omega \kappa(\varphi) \nabla\theta \cdot \nabla\psi, \qquad O_{\rm T}(\mathbf{v}; \theta, \psi) = \int_\Omega (\mathbf{v} \cdot \nabla\theta)\psi, \tag{2.10}$$

$$B(\mathbf{v}, q) = \int_\Omega q \operatorname{div} \mathbf{v}, \qquad D(\theta, \mathbf{v}) = \int_\Omega \theta \mathbf{g} \cdot \mathbf{v}. \tag{2.11}$$

2.2 Stability

Next, let us discuss the stability properties of the forms appearing in (2.8).

We start by discussing boundedness of the forms. Due to the bounds (2.7), the following continuity properties hold:

$$|A_{\rm S}(\cdot; \mathbf{u}, \mathbf{v})| \le \nu_2 \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \qquad \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.12}$$

$$|A_{\rm T}(\cdot; \theta, \psi)| \le \kappa_2 \|\theta\|_{1,\Omega} \|\psi\|_{1,\Omega}, \qquad \theta, \psi \in H^1(\Omega), \tag{2.13}$$

$$|B(\mathbf{v}, q)| \le C_B \|\mathbf{v}\|_{1,\Omega} \|q\|_{0,\Omega}, \qquad \mathbf{v} \in \mathbf{H}^1(\Omega),\ q \in L^2(\Omega). \tag{2.14}$$

Moreover, from the Lipschitz continuity of ν and κ in (2.6) and Hölder's inequality it readily follows that, for $\theta_1, \theta_2 \in H^1(\Omega)$, $\mathbf{u} \in \mathbf{W}^{1,\infty}(\Omega)$, $\theta \in W^{1,\infty}(\Omega)$,

$$|A_{\rm S}(\theta_1; \mathbf{u}, \mathbf{v}) - A_{\rm S}(\theta_2; \mathbf{u}, \mathbf{v})| \le \nu_{\rm lip} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)} \|\theta_1 - \theta_2\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \qquad \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.15}$$

$$|A_{\rm T}(\theta_1; \theta, \psi) - A_{\rm T}(\theta_2; \theta, \psi)| \le \kappa_{\rm lip} \|\theta\|_{W^{1,\infty}(\Omega)} \|\theta_1 - \theta_2\|_{1,\Omega} \|\psi\|_{1,\Omega}, \qquad \psi \in H^1(\Omega). \tag{2.16}$$

The forms $O_{\rm S}$ and $O_{\rm T}$ are linear in each argument. Hölder's inequality and standard Sobolev embeddings then give the following bounds:

$$|O_{\rm S}(\mathbf{w}; \mathbf{u}, \mathbf{v})| \le C_{\rm S} \|\mathbf{w}\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \qquad \mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.17}$$

$$|O_{\rm T}(\mathbf{w}; \theta, \psi)| \le C_{\rm T} \|\mathbf{w}\|_{1,\Omega} \|\theta\|_{1,\Omega} \|\psi\|_{1,\Omega}, \qquad \mathbf{w} \in \mathbf{H}^1(\Omega),\ \theta, \psi \in H^1(\Omega). \tag{2.18}$$

Similarly, we have

$$|D(\theta, \mathbf{v})| \leq C_D \|\mathbf{g}\|_{0,\Omega} \|\theta\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \qquad \theta \in H^1(\Omega),\ \mathbf{v} \in \mathbf{H}^1(\Omega). \tag{2.19}$$

Next, we review the positivity properties of the forms in (2.9) and (2.10). By the Poincaré inequality and the bounds (2.7), the elliptic forms A_{S} and A_{T} are coercive:

$$|A_{\mathrm{S}}(\cdot; \mathbf{v}, \mathbf{v})| \geq \alpha_{\mathrm{S}} \|\mathbf{v}\|_{1,\Omega}^2, \qquad \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{2.20}$$

$$|A_{\mathrm{T}}(\cdot; \psi, \psi)| \geq \alpha_{\mathrm{T}} \|\psi\|_{1,\Omega}^2, \qquad \psi \in H_0^1(\Omega). \tag{2.21}$$

To discuss the convective form O_{S} and O_{T}, we introduce the kernel

$$\mathbf{X} = \left\{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) \,:\, B(\mathbf{v}, q) = 0 \ \forall\, q \in L_0^2(\Omega) \right\} = \left\{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) \,:\, \operatorname{div} \mathbf{v} \equiv 0 \text{ in } \Omega \right\}. \tag{2.22}$$

Clearly, $\mathbf{X} \subset \mathbf{H}_0(\operatorname{div}^0; \Omega)$. Then, integration by parts shows that,

$$O_{\mathrm{S}}(\mathbf{w}; \mathbf{v}, \mathbf{v}) = 0, \qquad \mathbf{w} \in \mathbf{X},\ \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.23}$$

$$O_{\mathrm{T}}(\mathbf{w}; \psi, \psi) = 0, \qquad \mathbf{w} \in \mathbf{X},\ \psi \in H^1(\Omega). \tag{2.24}$$

Finally, the bilinear form B satisfies the continuous inf-sup condition

$$\sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \setminus \{\mathbf{0}\}} \frac{B(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,\Omega}} \geq \beta \|q\|_{0,\Omega}, \quad \forall\, q \in L_0^2(\Omega), \tag{2.25}$$

with an inf-sup constant $\beta > 0$ only depending on Ω; see [13], for instance.

2.3 Results concerning existence and uniqueness

In this section, we review some results regarding the existence and uniqueness of solutions of (2.8). To that end, it is enough to study the reduced problem of (2.8) on the kernel $\mathbf{X}$. in (2.22). It consists in finding $(\mathbf{u}, \theta) \in \mathbf{X} \times H^1(\Omega)$ such that $\theta|_\Gamma = \theta_D$ and

$$\begin{aligned} A_{\mathrm{S}}(\theta; \mathbf{u}, \mathbf{v}) \,+\, O_{\mathrm{S}}(\mathbf{u}; \mathbf{u}, \mathbf{v}) \,-\, D(\theta, \mathbf{v}) \,&=\, 0, \\ A_{\mathrm{T}}(\theta; \theta, \psi) \,+\, O_{\mathrm{T}}(\mathbf{u}; \theta, \psi) \,&=\, 0, \end{aligned} \tag{2.26}$$

for all $(\mathbf{v}, \psi) \in \mathbf{X} \times H_0^1(\Omega)$.

The following equivalence property is standard; see [13].

Lemma 2.1 *If $(\mathbf{u}, p, \theta) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega)$ is a solution of (2.8), then $\mathbf{u} \in \mathbf{X}$ and $(\mathbf{u}, p)$ is also a solution of (2.26). Conversely, if $(\mathbf{u}, \theta) \in \mathbf{X} \times H^1(\Omega)$ is a solution of (2.26), then there exists a unique pressure $p \in L_0^2(\Omega)$ such that $(\mathbf{u}, p, \theta)$ is a solution of (2.8).*

The following existence result for the reduced problem (2.26) is proved in [17, Theorem 2.1]. To state it, we write the temperature θ as

$$\theta = \theta_0 + \theta_1, \tag{2.27}$$

where $\theta_0 \in H_0^1(\Omega)$ and θ_1 is such that

$$\theta_1 \in H^1(\Omega), \qquad \theta_1|_\Gamma = \theta_D. \tag{2.28}$$

Theorem 2.2 *Assume* (2.6) *and* (2.7). *Then, for any* $\mathbf{g} \in \mathbf{L}^2(\Omega)$, *there is a lifting* $\theta_1 \in H^1(\Omega)$ *of* $\theta_D \in H^{1/2}(\Gamma)$ *satisfying* (2.28) *such that the reduced problem* (2.26) *has a solution* $(\mathbf{u}, \theta = \theta_0 + \theta_1) \in \mathbf{H}_0^1(\Omega) \times H^1(\Omega)$. *Furthermore, there exist constants* C_u *and* C_θ *only depending on* $\|\mathbf{g}\|_{0,\Omega}$, *and the stability constants in Section 2.2, such that*

$$\|\mathbf{u}\|_{1,\Omega} \leq C_u \|\theta_1\|_{1,\Omega}, \qquad \|\theta\|_{1,\Omega} \leq C_\theta \|\theta_1\|_{1,\Omega}. \tag{2.29}$$

The work [17, Section 7] also establishes the uniqueness of small solutions to problem (2.26), albeit under additional smoothness assumptions on the domain. Here, we restrict ourselves to proving the following (more straightforward) uniqueness result, whose proof is motivated by a similar argument in [10] for Stokes-Oldroyd problems.

Theorem 2.3 *Let* $(\mathbf{u}, \theta) \in \left[\mathbf{X} \cap \mathbf{W}^{1,\infty}(\Omega)\right] \times W^{1,\infty}(\Omega)$ *be a solution to problem* (2.26), *and assume that there exists a sufficiently small constant* $M > 0$ *such that*

$$\max\{\|\mathbf{g}\|_{0,\Omega}\,,\ \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)},\ \|\theta\|_{W^{1,\infty}(\Omega)}\} \leq M. \tag{2.30}$$

Then, the solution is unique. (A precise condition on M *can be found in* (2.42).*)*

Proof. Let $(\mathbf{u}, \theta)$ and $(\mathbf{u}^\star, \theta^\star)$ be two solutions of problem (2.26), both satisfying assumption (2.30). By subtracting the two corresponding variational formulations from each other, it follows that

$$[A_{\mathrm{S}}(\theta; \mathbf{u}, \mathbf{v}) - A_{\mathrm{S}}(\theta^\star; \mathbf{u}^\star, \mathbf{v})] + [O_{\mathrm{S}}(\mathbf{u}; \mathbf{u}, \mathbf{v}) - O_{\mathrm{S}}(\mathbf{u}^\star; \mathbf{u}^\star, \mathbf{v})] - D(\theta - \theta^\star, \mathbf{v}) = 0, \tag{2.31}$$

and

$$[A_{\mathrm{T}}(\theta; \theta, \psi) - A_{\mathrm{T}}(\theta^\star; \theta^\star, \psi)] \,+\, [O_{\mathrm{T}}(\mathbf{u}; \theta, \psi) - O_{\mathrm{T}}(\mathbf{u}^\star; \theta^\star, \psi)] = 0, \tag{2.32}$$

for all $\mathbf{v} \in \mathbf{X}$ and $\psi \in H_0^1(\Omega)$.

In (2.31), we write

$$\begin{aligned} [A_{\mathrm{S}}(\theta; \mathbf{u}, \mathbf{v}) - A_{\mathrm{S}}(\theta^\star; \mathbf{u}^\star, \mathbf{v})] &= A_{\mathrm{S}}(\theta; \mathbf{u} - \mathbf{u}^\star, \mathbf{v}) + [A_{\mathrm{S}}(\theta; \mathbf{u}^\star, \mathbf{v}) - A_{\mathrm{S}}(\theta^\star; \mathbf{u}^\star, \mathbf{v})], \\ [O_{\mathrm{S}}(\mathbf{u}; \mathbf{u}, \mathbf{v}) - O_{\mathrm{S}}(\mathbf{u}^\star; \mathbf{u}^\star, \mathbf{v})] &= O_{\mathrm{S}}(\mathbf{u}; \mathbf{u} - \mathbf{u}^\star, \mathbf{v}) + O_{\mathrm{S}}(\mathbf{u} - \mathbf{u}^\star; \mathbf{u}^\star, \mathbf{v}). \end{aligned} \tag{2.33}$$

Similarly, in (2.32),

$$\begin{aligned} [A_{\mathrm{T}}(\theta; \theta, \psi) - A_{\mathrm{T}}(\theta^\star; \theta^\star, \psi)] &= A_{\mathrm{T}}(\theta; \theta - \theta^\star, \psi) + [A_{\mathrm{T}}(\theta; \theta^\star, \psi) - A_{\mathrm{T}}(\theta^\star; \theta^\star, \psi)], \\ [O_{\mathrm{T}}(\mathbf{u}; \theta, \psi) - O_{\mathrm{T}}(\mathbf{u}^\star; \theta^\star, \psi)] &= O_{\mathrm{T}}(\mathbf{u}; \theta - \theta^\star, \psi) + O_{\mathrm{T}}(\mathbf{u} - \mathbf{u}^\star; \theta^\star, \psi). \end{aligned} \tag{2.34}$$

Then, by choosing the test function $\mathbf{v} = \mathbf{u} - \mathbf{u}^\star \in \mathbf{X}$ in (2.31), and using (2.33), the coercivity property (2.20), and the fact that $O_{\mathrm{S}}(\mathbf{u}; \mathbf{u} - \mathbf{u}^\star, \mathbf{u} - \mathbf{u}^\star) = 0$, see (2.23), we obtain

$$\begin{aligned} \alpha_{\mathrm{S}} \|\mathbf{u} - \mathbf{u}^\star\|_{1,\Omega}^2 \leq\ & |A_{\mathrm{S}}(\theta; \mathbf{u}^\star, \mathbf{u} - \mathbf{u}^\star) - A_{\mathrm{S}}(\theta^\star; \mathbf{u}^\star, \mathbf{u} - \mathbf{u}^\star)| \\ & + |O_{\mathrm{S}}(\mathbf{u} - \mathbf{u}^\star; \mathbf{u}^\star, \mathbf{u} - \mathbf{u}^\star)| + |D(\theta - \theta^\star, \mathbf{u} - \mathbf{u}^\star)|. \end{aligned} \tag{2.35}$$

Analogously, by taking $\psi = \theta - \theta^\star \in H_0^1(\Omega)$ in (2.32), and using (2.34), the coercivity (2.21) for A_T, and the fact that $O_{\mathrm{T}}(\mathbf{u}; \theta - \theta^\star, \theta - \theta^\star) = 0$, cf. (2.24), we find that

$$\alpha_{\mathrm{T}} \|\theta - \theta^\star\|_{1,\Omega}^2 \leq |A_{\mathrm{T}}(\theta; \theta^\star, \theta - \theta^\star) - A_{\mathrm{T}}(\theta^\star; \theta^\star, \theta - \theta^\star)| + |O_{\mathrm{T}}(\mathbf{u} - \mathbf{u}^\star; \theta^\star, \theta - \theta^\star)|. \tag{2.36}$$

From (2.15) and (2.16) and since $\|\mathbf{u}^\star\|_{\mathbf{W}^{1,\infty}(\Omega)} \leq M$ and $\|\theta^\star\|_{W^{1,\infty}(\Omega)} \leq M$ by assumption (2.30), the right-hand sides in (2.35) and (2.36) can be bounded by

$$|A_\mathrm{S}(\theta;\mathbf{u}^\star,\mathbf{u}-\mathbf{u}^\star) - A_\mathrm{S}(\theta^\star;\mathbf{u}^\star,\mathbf{u}-\mathbf{u}^\star)| \leq \nu_\mathrm{lip} M\|\theta-\theta^\star\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega}, \tag{2.37}$$

and

$$|A_\mathrm{T}(\theta;\theta^\star,\theta-\theta^\star) - A_\mathrm{T}(\theta^\star;\theta^\star,\theta-\theta^\star)| \leq \kappa_\mathrm{lip} M\|\theta-\theta^\star\|_{1,\Omega}^2,$$

respectively. Hence, by using these inequalities in (2.35) and (2.36), respectively, and the continuity of O_S, O_T, D, we find that

$$\begin{aligned}\alpha_\mathrm{S}\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega}^2 \leq &\nu_\mathrm{lip} M\|\theta-\theta^\star\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega} + C_\mathrm{S}\|\mathbf{u}^\star\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega}^2\\ &+ C_D\|\mathbf{g}\|_{0,\Omega}\|\theta-\theta^\star\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega},\end{aligned} \tag{2.38}$$

as well as

$$\alpha_\mathrm{T}\|\theta-\theta^\star\|_{1,\Omega}^2 \leq \kappa_\mathrm{lip} M\|\theta-\theta^\star\|_{1,\Omega}^2 + C_\mathrm{T}\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega}\|\theta^\star\|_{1,\Omega}\|\theta-\theta^\star\|_{1,\Omega}. \tag{2.39}$$

We continue bounding the right-hand sides of (2.38) and (2.39) by applying the embedding estimate (1.2), assumption (2.30), and the inequality $|ab| \leq \frac{a^2}{2} + \frac{b^2}{2}$. This results in

$$\begin{aligned}\alpha_\mathrm{S}\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega}^2 \leq &M(C_D+\nu_\mathrm{lip})\|\theta-\theta^\star\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega} + C_\mathrm{S}C_\infty M\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega}^2\\ \leq &M(C_\mathrm{S}C_\infty + \frac{C_D}{2} + \frac{\nu_\mathrm{lip}}{2})\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega}^2 + \frac{M}{2}(C_D+\nu_\mathrm{lip})\|\theta-\theta^\star\|_{1,\Omega}^2,\end{aligned} \tag{2.40}$$

respectively,

$$\begin{aligned}\alpha_\mathrm{T}\|\theta-\theta^\star\|_{1,\Omega}^2 \leq &\kappa_\mathrm{lip} M\|\theta-\theta^\star\|_{1,\Omega}^2 + C_\mathrm{T}C_\infty M\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega}\|\theta-\theta^\star\|_{1,\Omega}\\ \leq &M\Big(\kappa_\mathrm{lip} + \frac{C_\mathrm{T}C_\infty}{2}\Big)\|\theta-\theta^\star\|_{1,\Omega}^2 + \frac{M}{2}C_\mathrm{T}C_\infty\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega}^2.\end{aligned} \tag{2.41}$$

Finally, adding up (2.40) and (2.41), and bringing all the terms to the left-hand side of the resulting inequality, we conclude that

$$\Big(\alpha_\mathrm{S} - M(C_\mathrm{S}C_\infty + K)\Big)\|\mathbf{u}-\mathbf{u}^\star\|_{1,\Omega}^2 + \Big(\alpha_\mathrm{T} - M(\kappa_\mathrm{lip}+K)\Big)\|\theta-\theta^\star\|_{1,\Omega}^2 \leq 0,$$

with $K := (C_\mathrm{T}C_\infty + C_D + \nu_\mathrm{lip})/2$. Thus, if M satisfies

$$M < \min\Big\{\frac{\alpha_\mathrm{S}}{C_\mathrm{S}C_\infty+K}, \frac{\alpha_\mathrm{T}}{\kappa_\mathrm{lip}+K}\Big\}, \tag{2.42}$$

then $\theta = \theta^\star$ and $\mathbf{u} = \mathbf{u}^\star$. This completes the proof. □

3 Finite element discretization

In this section, we introduce our finite element method for approximating problem (2.1)–(2.5), review the discrete stability properties of the forms involved, and discuss the reduced version of the discrete variational problem.

3.1 Preliminaries

We consider a family of regular and shape-regular triangulations $\mathcal{T}_h$ of mesh size h that partition the domain Ω into simplices $\{K\}$ (i.e., triangles for $d = 2$ and tetrahedra for $d = 3$). For each K we denote by $\mathbf{n}_K$ the unit outward normal vector on the boundary ∂K, and by h_K the elemental diameter. As usual, we define the mesh size by $h = \max_{K \in \mathcal{T}_h} h_K$. We denote by $\mathcal{E}_I(\mathcal{T}_h)$ the set of all interior edges (faces) of $\mathcal{T}_h$, by $\mathcal{E}_B(\mathcal{T}_h)$ the set of all boundary edges (faces), and define $\mathcal{E}_h(\mathcal{T}_h) = \mathcal{E}_I(\mathcal{T}_h) \cup \mathcal{E}_B(\mathcal{T}_h)$. The $(d-1)$-dimensional diameter of an edge (face) e is denoted by h_e.

We will use standard average and jump operators. To define them, let K^+ and K^- be two adjacent elements of $\mathcal{T}_h$, and $e = \partial K^+ \cap \partial K^- \in \mathcal{E}_I(\mathcal{T}_h)$. Let $\mathbf{u}$ and τ be a piecewise smooth vector-valued, respectively matrix-valued function, and let us denote by $\mathbf{u}^\pm$, $\tau^\pm$ the traces of $\mathbf{u}$, τ on e, taken from within the interior of $K^\pm$. Then, we define the jump of $\mathbf{u}$, respectively the mean value of τ at $\mathbf{x} \in e$ by

$$[\![\mathbf{u}]\!] = \mathbf{u}^+ \otimes \mathbf{n}_{K^+} + \mathbf{u}^- \otimes \mathbf{n}_{K^-}, \qquad \{\!\{\tau\}\!\} = \frac{1}{2}(\tau^+ + \tau^-), \tag{3.1}$$

where for $\mathbf{u} = (u_1, ..., u_d)$ and $\mathbf{n} = (n_1, ..., n_d)$, we denote by $\mathbf{u} \otimes \mathbf{n}$ the tensor product matrix $[\mathbf{u} \otimes \mathbf{n}]_{i,j} = u_i n_j$, $1 \leq i, j \leq d$. For a boundary edge (face) $e = \partial K^+ \cap \Gamma$, we set $[\![\mathbf{u}]\!] = \mathbf{u}^+ \otimes \mathbf{n}$, with $\mathbf{n}$ denoting the unit outward normal vector on Γ, and $\{\!\{\tau\}\!\} = \tau^+$.

3.2 Exactly divergence-free finite element approximation

For an approximation order $k \geq 1$ and a mesh $\mathcal{T}_h$ on Ω, we consider the discrete spaces

$$\begin{aligned}
\mathbf{V}_h &= \left\{ \mathbf{v} \in H_0(\mathrm{div}\,;\Omega) \,:\, \mathbf{v}|_K \in [\mathbb{P}_k(K)]^d,\ K \in \mathcal{T}_h \right\}, \\
Q_h &= \left\{ q \in L^2_0(\Omega) \,:\, q|_K \in \mathbb{P}_{k-1}(K),\ K \in \mathcal{T}_h \right\}, \\
\Psi_h &= \left\{ \psi \in \mathcal{C}(\overline{\Omega}) \,:\, \psi|_K \in \mathbb{P}_k(K),\ K \in \mathcal{T}_h \right\}, \\
\Psi_{h,0} &= \Psi_h \cap H^1_0(\Omega).
\end{aligned} \tag{3.2}$$

Here, the space $\mathbb{P}_k(K)$ denotes the usual space of polynomials of total degree less or equal than k on element K. The space $\mathbf{V}_h$ is non-conforming in $\mathbf{H}^1_0(\Omega)$, while Q_h and Ψ_h are conforming in $L^2_0(\Omega)$ and $H^1(\Omega)$, respectively. In fact, the space $\mathbf{V}_h$ is the space of divergence-conforming Brezzi-Douglas-Marini (BDM) elements; see [7].

Consistent with our choice (3.2) for the discrete spaces, we need to introduce discontinuous versions of A_{S} and O_{S}, respectively. For the discrete vector Laplacian, we take the interior penalty form [1, 2] given by

$$\begin{aligned}
A^h_{\mathrm{S}}(\psi; \mathbf{u}, \mathbf{v}) = &\int_\Omega \nu(\psi)\, \nabla_h \mathbf{u} : \nabla_h \mathbf{v} - \sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \int_e \{\!\{\nu(\psi) \nabla_h \mathbf{u}\}\!\} : [\![\mathbf{v}]\!] \\
&- \sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \int_e \{\!\{\nu(\psi) \nabla_h \mathbf{v}\}\!\} : [\![\mathbf{u}]\!] + \sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \frac{a_0}{h_e} \int_e \nu(\psi) [\![\mathbf{u}]\!] : [\![\mathbf{v}]\!].
\end{aligned} \tag{3.3}$$

Here, $a_0 > 0$ is the interior penalty parameter, and we denote by ∇_h the broken gradient operator. As discussed in [9], other choices for A_{S}^h are equally feasible (such as LDG or BR methods), provided that the stability properties in Section 3.3 below hold.

For the convection term, we take the standard upwind form [16] defined by

$$O_{\mathrm{S}}^h(\mathbf{w};\mathbf{u},\mathbf{v}) = \int_\Omega (\mathbf{w}\cdot\nabla_h)\mathbf{u}\cdot\mathbf{v} + \sum_{K\in\mathcal{T}_h}\int_{\partial K\setminus\Gamma}\frac{1}{2}(\mathbf{w}\cdot\mathbf{n}_K - |\mathbf{w}\cdot\mathbf{n}_K|)(\mathbf{u}^e-\mathbf{u})\cdot\mathbf{v}, \tag{3.4}$$

where $\mathbf{u}^e$ is the trace of $\mathbf{u}$ taken from within the exterior of K. The remaining forms are the same as in the continuous case.

Next, we introduce an approximation $\theta_{D,h}$ to the boundary datum θ_D, which belongs to the trace space

$$\theta_{D,h}\in\Lambda_h = \{\,\xi\in C(\overline{\Gamma})\,:\,\xi|_e\in\mathbb{P}_k(e),\ e\in\mathcal{E}_B(\mathcal{T}_h)\,\}. \tag{3.5}$$

Then the discrete formulation for problem (2.1)–(2.5) is to find $(\mathbf{u}_h,p_h,\theta_h)\in\mathbf{V}_h\times Q_h\times\Psi_h$ such that $\theta_h|_\Gamma=\theta_{D,h}$ and

$$\begin{aligned} A_{\mathrm{S}}^h(\theta_h;\mathbf{u}_h,\mathbf{v}) + O_{\mathrm{S}}^h(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}) - B(\mathbf{v},p_h) - D(\theta,\mathbf{v}) &= 0,\\ B(\mathbf{u}_h,q) &= 0,\\ A_{\mathrm{T}}(\theta_h;\theta_h,\psi) + O_{\mathrm{T}}(\mathbf{u}_h;\theta_h,\psi) &= 0, \end{aligned} \tag{3.6}$$

for all $(\mathbf{v},q,\psi)\in\mathbf{V}_h\times Q_h\times\Psi_{h,0}$.

A key feature of the method (3.6) is that the discrete velocity $\mathbf{u}_h$ is exactly divergence-free. To discuss this property, we introduce the discrete kernel of B

$$\mathbf{X}_h = \{\,\mathbf{v}\in\mathbf{V}_h\,:\,B(\mathbf{v},q)=0\ \forall\,q\in Q_h\,\}. \tag{3.7}$$

Since $\mathbf{V}_h\subset\mathbf{H}_0(\mathrm{div}\,;\Omega)$ and $\mathrm{div}\,\mathbf{V}_h\subseteq Q_h$, it can be readily seen that

$$\mathbf{X}_h = \{\,\mathbf{v}\in\mathbf{V}_h\,:\,\mathrm{div}\,\mathbf{v}\equiv 0 \text{ in } \Omega\,\};$$

we refer to [9] for details. Hence, $\mathbf{X}_h\subset\mathbf{H}_0(\mathrm{div}^0;\Omega)$. In particular, the following result holds.

Lemma 3.1 *An approximate velocity* $\mathbf{u}_h\in\mathbf{V}_h$ *obtained by* (3.6) *is exactly divergence-free, i.e., it satisfies* $\mathrm{div}\,\mathbf{u}_h\equiv 0$ *in* Ω.

An important consequence of Lemma 3.1 is the provable energy-stability of the numerical scheme (3.6), without the need for symmetrization or other modifications of the convective terms; see also the discusssion in [8, 9]. These stability properties are established in the next subsection.

3.3 Discrete stability properties

3.3.1 Broken spaces and norms

We introduce the broken space

$$\mathbf{H}^r(\mathcal{T}_h) = \{\,\mathbf{v}\in\mathbf{L}^2(\Omega)\,:\,\mathbf{v}|_K\in\mathbf{H}^r(K),\ K\in\mathcal{T}_h\,\},\qquad r\ge 0. \tag{3.8}$$

We shall mostly work with $r = 1$ and $r = 2$; in these cases we use the broken norms

$$\|\mathbf{v}\|_{1,\mathcal{T}_h}^2 = \sum_{K\in\mathcal{T}_h} \|\nabla_h \mathbf{v}\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h} a_0 h_e^{-1} \|[\![\mathbf{v}]\!]\|_{0,e}^2, \qquad \mathbf{v} \in \mathbf{H}^1(\mathcal{T}_h), \tag{3.9}$$

$$\|\mathbf{v}\|_{2,\mathcal{T}_h}^2 = \|\mathbf{v}\|_{1,\mathcal{T}_h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 |\mathbf{v}|_{2,K}^2, \qquad \mathbf{v} \in \mathbf{H}^2(\mathcal{T}_h). \tag{3.10}$$

By the inverse estimate $|p|_{2,K} \le C h_K^{-1} |p|_{1,K}$ for all $K \in \mathcal{T}_h$, $p \in \mathbb{P}_k(K)$, we see that

$$\|\mathbf{v}\|_{2,\mathcal{T}_h} \le C \|\mathbf{v}\|_{1,\mathcal{T}_h}, \qquad \mathbf{v} \in \mathbf{V}_h. \tag{3.11}$$

We recall the following broken version of the usual Sobolev embeddings: for $d = 2, 3$, and any $p \in I(d) \subset \mathbb{R}$ there exists a constant $C > 0$ such that

$$\|\mathbf{v}\|_{L^p(\Omega)} \le C \|\mathbf{v}\|_{1,\mathcal{T}_h}, \qquad \mathbf{v} \in \mathbf{H}^1(\mathcal{T}_h), \tag{3.12}$$

where $I(2) = [1, \infty)$ and $I^\star(3) = [1, 6]$. For $d = 2$, this has been proved in [14, Lemma 6.2]. In the case $d = 3$, the proof follows along the lines of to [22, Lemma 5.15, Theorem 5.16]. In the following, we shall explicitly write C_{emb} for the embedding constant in the case $p = 3$.

Moreover, we introduce the broken $\mathbf{C}^1$-space given by

$$\mathbf{C}^1(\mathcal{T}_h) = \left\{ \mathbf{u} \in \mathbf{H}^1(\mathcal{T}_h) \,:\, \mathbf{u}|_K \in \mathbf{C}^1(\overline{K}),\ K \in \mathcal{T}_h \right\}, \tag{3.13}$$

equipped with the broken $W^{1,\infty}$-norm

$$\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} = \max_{K\in\mathcal{T}_h} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(K)}. \tag{3.14}$$

We shall also make use of the augmented H^1-norm

$$\|\psi\|_{1,\mathcal{E}_h}^2 = \|\psi\|_{1,\Omega}^2 + \sum_{e\in\mathcal{E}_h(\mathcal{T}_h)} h_e^{-1} \|\psi\|_{0,e}^2, \qquad \psi \in H^1(\Omega). \tag{3.15}$$

3.3.2 Continuity

First, we establish continuity properties of the elliptic forms A_{S}^h and A_{T}, respectively. To that end, we recall that by (2.13), the form A_{T} is a bounded bilinear form over $H^1(\Omega) \times H^1(\Omega)$. To bound DG form A_{S}^h, we proceed in a standard way; see [2], for instance. Indeed, by using the standard trace inequalities

$$\|v\|_{0,\partial K} \le C \left(h_K^{-1/2} \|v\|_{0,K} + h_K^{1/2} |v|_{1,K} \right), \qquad v \in H^1(K), \tag{3.16}$$

$$\|p\|_{0,\partial K} \le C h_K^{-1/2} \|p\|_{0,K}, \qquad p \in \mathbb{P}_k(K), \tag{3.17}$$

and the inverse inequality in (3.11), we obtain the following result.

Lemma 3.2 *There holds*

$$|A_{\mathrm{S}}^h(\cdot; \mathbf{u}, \mathbf{v})| \le C \|\mathbf{u}\|_{2,\mathcal{T}_h} \|\mathbf{v}\|_{1,\mathcal{T}_h}, \qquad \mathbf{u} \in \mathbf{H}^2(\mathcal{T}_h),\ \mathbf{v} \in \mathbf{V}_h, \tag{3.18}$$

$$|A_{\mathrm{S}}^h(\cdot; \mathbf{u}, \mathbf{v})| \le \tilde{C}_A \|\mathbf{u}\|_{1,\mathcal{T}_h} \|\mathbf{v}\|_{1,\mathcal{T}_h}, \qquad \mathbf{u}, \mathbf{v} \in \mathbf{V}_h. \tag{3.19}$$

Moreover, the elliptic forms are Lipschitz continuons with respect to the first argument. For the conforming form A_{T}, this follows from (2.16). The following result holds for the DG form A_{S}^h.

Lemma 3.3 *Let $\psi_1,\ \psi_2 \in H^1(\Omega)$, $\mathbf{u} \in \mathbf{C}^1(\mathcal{T}_h)$, and $\mathbf{v} \in \mathbf{V}_h$. Then there holds*

$$\left| A_{\mathrm{S}}^h(\psi_1; \mathbf{u}, \mathbf{v}) - A_{\mathrm{S}}^h(\psi_2; \mathbf{u}, \mathbf{v}) \right| \le \tilde{C}_{\mathrm{lip}} \nu_{\mathrm{lip}} \, \|\psi_1 - \psi_2\|_{1,\mathcal{E}_h} \, \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} \|\mathbf{v}\|_{1,\mathcal{T}_h}. \tag{3.20}$$

In addition, if $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, then

$$\left| A_{\mathrm{S}}^h(\psi_1; \mathbf{u}, \mathbf{v}) - A_{\mathrm{S}}^h(\psi_2; \mathbf{u}, \mathbf{v}) \right| \le \tilde{C}_{\mathrm{lip}} \nu_{\mathrm{lip}} \, \|\psi_1 - \psi_2\|_{1,\Omega} \, \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} \|\mathbf{v}\|_{1,\mathcal{T}_h}. \tag{3.21}$$

The constant $\tilde{C}_{\mathrm{lip}} > 0$ is independent of the mesh size.

Proof. As before, we note that

$$\left| A_{\mathrm{S}}^h(\psi_1; \mathbf{u}, \mathbf{v}) - A_{\mathrm{S}}^h(\psi_2; \mathbf{u}, \mathbf{v}) \right| \le |T_1| + |T_2| + |T_3| + |T_4|,$$

with

$$T_1 = \int_\Omega (\nu(\psi_1) - \nu(\psi_2)) \nabla_h \mathbf{u} : \nabla_h \mathbf{v}, \qquad T_2 = \sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \int_e (\nu(\psi_1) - \nu(\psi_2)) \{\!\{\nabla \mathbf{u}\}\!\} : [\![\mathbf{v}]\!],$$

$$T_3 = \sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \int_e (\nu(\psi_1) - \nu(\psi_2)) \{\!\{\nabla \mathbf{v}\}\!\} : [\![\mathbf{u}]\!], \qquad T_4 = \sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \frac{a_0}{h_e} \int_e (\nu(\psi_1) - \nu(\psi_2)) [\![\mathbf{u}]\!] : [\![\mathbf{v}]\!].$$

For T_1, the Lipschitz continuity of ν in (2.6) readily yields the bound

$$|T_1| \le \nu_{\mathrm{lip}} \|\psi_1 - \psi_2\|_{0,\Omega} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} \|\nabla_h \mathbf{v}\|_{0,\Omega}.$$

To estimate T_2, we notice that, since $\mathbf{u} \in \mathbf{C}^1(\mathcal{T}_h)$, we have $\|\{\!\{\nabla_h \mathbf{u}\}\!\}_e\|_{L^\infty(e)} \le \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)}$ for all $e \in \mathcal{E}_h(\mathcal{T}_h)$. Hence, from the Lipschitz continuity of ν it follows that

$$|T_2| \le \nu_{\mathrm{lip}} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} \sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \|\psi_1 - \psi_2\|_{0,e} \, \|[\![\mathbf{v}]\!]\|_{0,e}.$$

By applying the discrete Cauchy-Schwarz inequality, the shape-regularity of the meshes, and the trace inequality (3.16), the sum over the edges (faces) can be bounded by

$$\begin{aligned}
\sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \|\psi_1 - \psi_2\|_{0,e} \, \|[\![\mathbf{v}]\!]\|_{0,e} &\le \Big(\sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} h_e \|\psi_1 - \psi_2\|_{0,e}^2 \Big)^{1/2} \Big(\sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} h_e^{-1} \|[\![\mathbf{v}]\!]\|_{0,e}^2 \Big)^{1/2} \\
&\le C \Big(\sum_{K \in \mathcal{T}_h} h_K \|\psi_1 - \psi_2\|_{0,\partial K}^2 \Big)^{1/2} \|\mathbf{v}\|_{1,\mathcal{T}_h} \\
&\le C \|\psi_1 - \psi_2\|_{1,\Omega} \|\mathbf{v}\|_{1,\mathcal{T}_h}.
\end{aligned}$$

This yields

$$|T_2| \le C \nu_{\mathrm{lip}} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} \|\psi_1 - \psi_2\|_{1,\Omega} \|\mathbf{v}\|_{1,\mathcal{T}_h}.$$

For the term T_3, we have $\|[\![\mathbf{u}]\!]\|_{\mathbf{L}^\infty(e)} \leq 2\|\mathbf{u}\|_{\mathbf{L}^\infty(\Omega)} \leq 2\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)}$ for any $e \in \mathcal{E}_h(\mathcal{T}_h)$. Hence, the Lipschitz continuity of ν, the Cauchy-Schwarz inequality the shape-regularity of the meshes, and the polynomial trace inequality (3.17),

$$\begin{aligned}|T_3| &\leq C\nu_{\mathrm{lip}}\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} \sum_{e\in\mathcal{E}_h(\mathcal{T}_h)} \|\psi_1 - \psi_2\|_{0,e}\|\{\!\{\nabla \mathbf{v}\}\!\}\|_{0,e} \\ &\leq C\nu_{\mathrm{lip}}\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)}\Big(\sum_{e\in\mathcal{E}_h(\mathcal{T}_h)} h_e^{-1}\|\psi_1-\psi_2\|_{0,e}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h} h_K\|\nabla\mathbf{v}\|_{0,\partial K}^2\Big)^{1/2} \\ &\leq C\nu_{\mathrm{lip}}\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)}\|\psi_1-\psi_2\|_{1,\mathcal{E}_h}\|\nabla_h\mathbf{v}\|_{0,\Omega}.\end{aligned}$$

Similarly, T_4 can be bounded by:

$$|T_4| \leq C\nu_{\mathrm{lip}}\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)}\|\psi_1-\psi_2\|_{1,\mathcal{E}_h}\|\mathbf{v}\|_{1,\mathcal{T}_h}.$$

Gathering the above bounds for T_1 through T_4 implies the estimate (3.20).

If $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, then $T_3 = T_4 = 0$, and the second bound (3.21) follows from the estimates for T_1 and T_2. □

Second, we notice that the forms B and D are bounded by

$$|B(\mathbf{v},q)| \leq \tilde{C}_B\|\mathbf{v}\|_{1,\mathcal{T}_h}\|q\|_{0,\Omega}, \qquad \mathbf{v}\in\mathbf{H}^1(\mathcal{T}_h),\ q\in L_0^2(\Omega), \tag{3.22}$$

$$|D(\psi,\mathbf{v})| \leq \tilde{C}_D\|\mathbf{g}\|_{0,\Omega}\|\psi\|_{1,\Omega}\|\mathbf{v}\|_{1,\mathcal{T}_h}, \qquad \mathbf{v}\in\mathbf{H}^1(\mathcal{T}_h),\ \psi\in H^1(\Omega). \tag{3.23}$$

The estimate for B is straightforward, and the one for D follows from the embedding (3.12) with $p = 4$ and Hölder's inequality.

Third, we discuss the convective forms O_{S}^h and O_{T}, respectively. In contrast to O_{S} and due to the upwind terms, the discrete form O_{S}^h is not linear in the first argument. However, as established in the following lemma, it is Lipschitz continuous.

Lemma 3.4 *There exists a constant $\tilde{C}_{\mathrm{S}} > 0$, independent of the mesh size, such that*

$$|O_{\mathrm{S}}^h(\mathbf{w}_1;\mathbf{u},\mathbf{v}) - O_{\mathrm{S}}^h(\mathbf{w}_2;\mathbf{u},\mathbf{v})| \leq \tilde{C}_{\mathrm{S}}\|\mathbf{w}_1-\mathbf{w}_2\|_{1,\mathcal{T}_h}\|\mathbf{u}\|_{1,\mathcal{T}_h}\|\mathbf{v}\|_{1,\mathcal{T}_h}, \tag{3.24}$$

for any $\mathbf{w}_1, \mathbf{w}_2, \mathbf{u} \in \mathbf{H}^2(\mathcal{T}_h)$ and $\mathbf{v} \in \mathbf{V}_h$.

Proof. The proof of this property in the case $d = 2$ can be found in [8], and makes use of the embedding (3.12) with $p = 4$. In the case $d = 3$, we proceed similarly: to conclude, we use the shape-regularity of the meshes, Hölder's inequality, the embedding (3.12) with $p = 4$, and the trace estimate $h_K^{1/4}\|z\|_{L^4(\partial K)} \leq C\big(\|z\|_{L^4(K)} + \|\nabla z\|_{L^2(K)}\big)$, $z \in W^{1,4}(K)$, from [18, Section 7]. We omit further details. □

The conforming temperature form O_{T} is still trilinear, and there holds

$$|O_{\mathrm{T}}(\mathbf{w};\varphi,\psi)| \leq \tilde{C}_T\|\mathbf{w}\|_{1,\mathcal{T}_h}\|\varphi\|_{1,\Omega}\|\psi\|_{1,\Omega}, \qquad \mathbf{w}\in\mathbf{H}^1(\mathcal{T}_h),\ \psi,\varphi\in H^1(\Omega). \tag{3.25}$$

This follows similarly from Hölder's inequality and the embedding (3.12). We use the following variant of (3.25).

Lemma 3.5 *There is a constant $\tilde{C}_{\mathrm{T},2} > 0$ such that*

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \leq \tilde{C}_{\mathrm{T},2}\|\theta\|_{L^3(\Omega)}\|\mathbf{w}\|_{1,\mathcal{T}_h}\|\psi\|_{1,\Omega}, \qquad \mathbf{w} \in \mathbf{H}_0(\mathrm{div}\,^0;\Omega),\ \theta,\psi \in H^1(\Omega). \tag{3.26}$$

Proof. Integration by parts yields and using that $\mathrm{div}\,\mathbf{w} \equiv 0$ in Ω, $\mathbf{w}\cdot\mathbf{n} = 0$ on Γ yield

$$O_{\mathrm{T}}(\mathbf{w};\theta,\psi) = \int_\Omega (\mathbf{w}\cdot\nabla\theta)\psi = -\int_\Omega \theta(\mathbf{w}\cdot\nabla\psi).$$

From Hölder's inequality we obtain

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \leq C\|\theta\|_{L^3(\Omega)}\|\nabla\psi\|_{0,\Omega}\|\mathbf{w}\|_{L^6(\Omega)}.$$

Hence, the embeddings in (3.12) with $p = 3$, $p = 6$ yield the assertion. □

3.3.3 Coercivity and inf-sup condition

First, we point out that coercivity of A_{T} over the discrete spaces is implied by (2.21). Due to the bounds of ν in (2.7) the DG form A^h_{S} is also elliptic, and we have

$$A^h_{\mathrm{S}}(\cdot,\mathbf{v},\mathbf{v}) \geq \tilde{\alpha}_{\mathrm{S}}\|\mathbf{v}\|^2_{1,\mathcal{T}_h}, \qquad \mathbf{v} \in \mathbf{V}_h, \tag{3.27}$$

provided that $a_0 > 0$ is sufficiently large independently of the mesh size; cf. [2].

To state the positivity of O^h_S and O_T, let $\mathbf{w} \in \mathbf{H}_0(\mathrm{div}\,^0;\Omega)$. Then we have

$$O^h_{\mathrm{S}}(\mathbf{w};\mathbf{u},\mathbf{u}) = \frac{1}{2}\sum_{e\in\mathcal{E}_I(\mathcal{T}_h)}\int_e |\mathbf{w}\cdot\mathbf{n}||[\![\mathbf{u}\otimes\mathbf{n}]\!]|^2\,ds \geq 0, \qquad \mathbf{u} \in \mathbf{V}_h. \tag{3.28}$$

Here, in the integrals over edges (faces) e, the vector $\mathbf{n}$ denotes any unit vector normal to e. This is a standard property of the upwind form O_{S}, see, e.g., [16, 8]. Moreover, integration by parts readily implies that

$$O_{\mathrm{T}}(\mathbf{w};\theta,\theta) = 0, \qquad \theta \in H^1(\Omega). \tag{3.29}$$

Finally, we recall the discrete inf-sup condition for B:

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v}_h,q_h)}{\|\mathbf{v}_h\|_{1,\mathcal{T}_h}} \geq \tilde{\beta}\|q_h\|_{0,\Omega} \qquad \forall\, q_h \in Q_h, \tag{3.30}$$

with $\tilde{\beta} > 0$ independent of the mesh size. The proof of (3.30) follows along the lines of [15] from the surjectivity of $\mathrm{div}\,: \mathbf{H}^1_0(\Omega) \to L^2_0(\Omega)$ and the properties of the BDM projection. We omit further details.

3.4 The reduced problem

The reduced version of (3.6) consists in finding $(\mathbf{u}_h,\theta_h) \in \mathbf{X}_h \times \Psi_h$ such that $\theta_h|_\Gamma = \theta_{D,h}$ and

$$\begin{aligned} A^h_{\mathrm{S}}(\theta_h;\mathbf{u}_h,\mathbf{v}) + O_{\mathrm{S}}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}) - D(\theta_h,\mathbf{v}) &= 0,\\ A_{\mathrm{T}}(\theta_h;\theta_h,\psi) + O_{\mathrm{T}}(\mathbf{u}_h;\theta_h,\psi) &= 0, \end{aligned} \tag{3.31}$$

for all $(\mathbf{v},\psi) \in \mathbf{X}_h \times \Psi_{h,0}$.

Due to the discrete stability properties of Section 3.3, the discrete analog of Lemma 2.1 hold.

Lemma 3.6 *If $(\mathbf{u}_h, p_h, \theta_h) \in \mathbf{V}_h \times Q_h \times \Psi_h$ is a solution of* (3.6)*, then $\mathbf{u}_h \in \mathbf{X}_h$ and $(\mathbf{u}_h, \theta_h)$ is also a solution of* (3.31)*. Conversely, if $(\mathbf{u}_h, \theta_h) \in \mathbf{X}_h \times \Psi_h$ is a solution of* (3.31)*, then there exists a unique pressure $p_h \in Q_h$ such that $(\mathbf{u}_h, p_h, \theta_h)$ is a solution of* (3.6)*.*

In what follows, we shall discuss the existence for the reduced problem (3.31). We notice that the uniqueness of discrete solutions remains an open problem. The main difficulty for adapting Theorem 2.3 to the discrete case is to control the augmented norm (3.15) appearing in the discrete counterpart of (2.37) .

4 Existence of discrete solutions

In this section, we establish the existence of discrete solutions of (3.31) following the continuous arguments proposed in [17] and based on Brouwer's fixed point theorem. Then we adjust this result to a particular choice of the discrete boundary datum $\theta_{D,h}$.

4.1 Stability and existence

We start by proving the following stability property of the discrete solutions under a small data assumption. As in the continuous case, we write the discrete temperature θ_h as $\theta_h = \theta_{h,0} + \theta_{h,1}$, with $\theta_{h,0} \in \Psi_{h,0}$ and

$$\theta_{h,1} \in \Psi_h, \qquad \theta_{h,1}|_\Gamma = \theta_{D,h}. \tag{4.1}$$

(A specific choice of the discrete lifting $\theta_{1,h}$ will be made in Section 4.2 below.)

Lemma 4.1 *Let $(\mathbf{u}_h, \theta_h)$ be a solution of* (3.31) *with $\theta_h = \theta_{h,0} + \theta_{h,1}$ as in* (4.1)*. Assume that*

$$\tilde{C}_{\mathrm{dep}} \|\mathbf{g}\|_{0,\Omega} \|\theta_{h,1}\|_{L^3(\Omega)} \leq \frac{1}{2}, \tag{4.2}$$

with

$$\tilde{C}_{\mathrm{dep}} = \frac{\tilde{C}_D \tilde{C}_{T,2}}{\tilde{\alpha}_{\mathrm{S}} \alpha_{\mathrm{T}}}, \tag{4.3}$$

then there exist constants $\tilde{C}_u$ and $\tilde{C}_\theta$ only depending on $\|\mathbf{g}\|_{0,\Omega}$ and the stability constants in Section 3.3, such that

$$\|\mathbf{u}_h\|_{1,\mathcal{T}_h} \leq \tilde{C}_u \|\theta_{h,1}\|_{1,\Omega}, \qquad \|\theta_h\|_{1,\Omega} \leq \tilde{C}_\theta \|\theta_{h,1}\|_{1,\Omega}. \tag{4.4}$$

(Explicit expressions for $\tilde{C}_u$ and $\tilde{C}_\theta$ can be found in (4.8) *and* (4.9)*, respectively.)*

Proof. We choose the test function $(\mathbf{v}, \psi) = (\mathbf{u}_h, \theta_{h,0})$ in (3.31), and use (3.29) to obtain the two equations

$$\begin{aligned} A_{\mathrm{S}}^h(\theta_h; \mathbf{u}_h, \mathbf{u}_h) + O_{\mathrm{S}}^h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{u}_h) \;&=\; D(\theta_{h,0}, \mathbf{u}_h) + D(\theta_{h,1}, \mathbf{u}_h), \\ A_{\mathrm{T}}(\theta_h; \theta_{h,0}, \theta_{h,0}) \;&=\; -\, A_{\mathrm{T}}(\theta_h; \theta_{h,1}, \theta_{h,0}) \;-\; O_{\mathrm{T}}(\mathbf{u}_h; \theta_{h,1}, \theta_{h,0}). \end{aligned} \tag{4.5}$$

In the first identity of (4.5), the coercivity of $A_{\rm S}^h$ in (3.27), the positivity of $O_{\rm S}^h$ in (3.28), and the boundedness of D in (3.23) imply

$$\|\mathbf{u}\|_{1,\mathcal{T}_h} \leq \tilde{\alpha}_{\rm S}^{-1}\tilde{C}_D\|\mathbf{g}\|_{0,\Omega}\|\theta_{h,0}\|_{1,\Omega} + \tilde{\alpha}_{\rm S}^{-1}\tilde{C}_D\|\mathbf{g}\|_{0,\Omega}\|\theta_{h,1}\|_{1,\Omega}. \tag{4.6}$$

In the second equation of (4.5), we employ the coercivity and boundedness of $A_{\rm T}$ in (2.21) and (2.13), respectively, along with the bound for $O_{\rm T}$ in Lemma 3.5. We conclude that

$$\|\theta_{h,0}\|_{1,\Omega} \leq \alpha_T^{-1}\kappa_2\|\theta_{h,1}\|_{1,\Omega} + \alpha_T^{-1}\tilde{C}_{T,2}\|\theta_{h,1}\|_{L^3(\Omega)}\|\mathbf{u}_h\|_{1,\mathcal{T}_h}. \tag{4.7}$$

Then, using the bound (4.7) in (4.6) yields

$$\|\mathbf{u}_h\|_{1,\mathcal{T}_h} \leq \tilde{\alpha}_{\rm S}^{-1}\alpha_{\rm T}^{-1}\tilde{C}_D\tilde{C}_{T,2}\|\mathbf{g}\|_{0,\Omega}\|\theta_{h,1}\|_{L^3(\Omega)}\|\mathbf{u}_h\|_{1,\mathcal{T}_h} + \tilde{\alpha}_{\rm S}^{-1}\alpha_{\rm T}^{-1}\tilde{C}_D\|\mathbf{g}\|_{0,\Omega}\big(\alpha_{\rm T}+\kappa_2\big)\|\theta_{h,1}\|_{1,\Omega}.$$

Hence, referring to assumption (4.2), we obtain

$$\|\mathbf{u}_h\|_{1,\mathcal{T}_h} \leq \tilde{C}_u\|\theta_{h,1}\|_{1,\Omega} \qquad \text{with} \qquad \tilde{C}_u = 2\tilde{\alpha}_{\rm S}^{-1}\alpha_{\rm T}^{-1}\tilde{C}_D\|\mathbf{g}\|_{0,\Omega}\big(\alpha_{\rm T}+\kappa_2\big). \tag{4.8}$$

Moreover, by using the triangle inequality, estimate (4.8), the definition of $\tilde{C}_{\rm dep}$ and assumption (4.2) we find that

$$\begin{aligned}
\|\theta_h\|_{1,\Omega} &\leq \|\theta_{h,0}\|_{1,\Omega} + \|\theta_{h,1}\|_{1,\Omega}\\
&\leq (\alpha_T^{-1}\kappa_2+1)\|\theta_{h,1}\|_{1,\Omega} + \alpha_T^{-1}\tilde{C}_{T,2}\|\theta_{h,1}\|_{L^3(\Omega)}\|\mathbf{u}_h\|_{1,\mathcal{T}_h}\\
&\leq (\alpha_{\rm T}^{-1}\kappa_2+1)\|\theta_{h,1}\|_{1,\Omega} + 2\alpha_{\rm T}^{-1}\tilde{C}_{\rm dep}\|\mathbf{g}\|_{0,\Omega}\|\theta_{h,1}\|_{L^3(\Omega)}(\alpha_{\rm T}+\kappa_2)\|\theta_{h,1}\|_{1,\Omega}\\
&\leq (\alpha_{\rm T}^{-1}\kappa_2+1)\|\theta_{h,1}\|_{1,\Omega} + \alpha_{\rm T}^{-1}(\alpha_{\rm T}+\kappa_2)\|\theta_{h,1}\|_{1,\Omega}.
\end{aligned}$$

Hence,

$$\|\theta_h\|_{1,\Omega} \leq \tilde{C}_\theta\|\theta_{h,1}\|_{1,\Omega} \qquad \text{with} \qquad \tilde{C}_\theta = 2(1+\alpha_{\rm T}^{-1}\kappa_2). \tag{4.9}$$

This completes the proof. □

We are now ready to state our main existence result.

Theorem 4.2 *Let $\theta_{h,1}$ be a discrete lifting satisfying* (4.2). *Then there exists a discrete solution $(\mathbf{u}_h, \theta_h) \in \mathbf{X}_h \times \Psi_h$ to the reduced problem* (3.31) *satisfying the stability bound* (4.4).

The proof of Theorem 4.2 is carried out in detail in Section 4.3.

4.2 Discrete boundary datum

A natural choice for $\theta_{D,h} \in \Lambda_h$ is the nodal interpolant of θ_D. In this section, we adapt the existence result in Theorem 4.2 to this particular choice.

In what follows, we shall thus always assume that

$$\theta_D \in C(\overline{\Gamma}). \tag{4.10}$$

We denote by $\mathcal{I} : C(\overline{\Omega}) \to \Psi_h$ the classical nodal interpolation operator with respect to an unisolvent set of Lagrange interpolation nodes for Ψ. Its restriction to the boundary nodes is denoted by $\mathcal{I}_\Gamma : C(\overline{\Gamma}) \to \Lambda_h$. A natural choice of the discrete boundary datum $\theta_{D,h} \in \Lambda_h$ is the nodal interpoland of θ_D. That is, we take

$$\theta_{D,h} = \mathcal{I}_\Gamma \theta_D. \tag{4.11}$$

The following result holds.

Lemma 4.3 *Assume* (4.10) *and* (4.11). *Then there is a lifting* $\theta_{h,1} \in \Psi_h$ *such that* $\theta_{h,1}|_\Gamma = \theta_{D,h}$ *and*

$$\|\theta_{h,1}\|_{1,\Omega} \;\leq\; C_{\text{lift}}\, \|\theta_{D,h}\|_{H^{1/2}(\Gamma)},$$

with a constant $C_{\text{lift}} > 0$ *independent of the mesh size.*

Proof. From the definition of the $H^{1/2}(\Gamma)$-norm, we infer the existence of a constant $K > 1$ and a function $\theta \in H^1(\Omega)$ such that $\theta|_\Gamma = \theta_{D,h}$ and

$$\|\theta\|_{1,\Omega} \leq K\|\theta_{D,h}\|_{H^{1/2}(\Gamma)}.$$

Set $\theta_{h,1} = \mathcal{SZ}_h\theta \in \Psi_h$, where $\mathcal{SZ}_h$ is the Scott-Zhang interpolation constructed in [11, Section 1.6.2]. By its definition and due to assumption (4.10), it coincides with the Lagrange interpolation operator on $\overline{\Gamma}$, i.e., we have

$$(\mathcal{SZ}_h\theta)|_\Gamma = \mathcal{I}_\Gamma(\theta|_\Gamma) = \theta_{D,h}.$$

Moreover, the stability result in [11, Lemma 1.130] yields

$$\|\theta_{h,1}\|_{1,\Omega} \leq C\|\theta\|_{1,\Omega},$$

with a constant $C > 0$ independent of the mesh size. This implies the desired result with $C_{\text{lift}} = CK$. □

We also have the following approximation result.

Lemma 4.4 *Assume* (4.10) *and* (4.11). *Let* $\theta \in C(\overline{\Omega})$ *be such that* $\theta|_\Gamma = \theta_D$. *Then we have*

$$\|\theta_D - \theta_{D,h}\|_{H^{1/2}(\Gamma)} \leq \|\theta - \mathcal{I}\theta\|_{1,\Omega}.$$

Proof. By construction, $(\theta - \mathcal{I}\theta)|_\Gamma = \theta_D - \theta_{D,h}$. Hence, the definition of the $H^{1/2}(\Gamma)$-norm implies the assertion. □

We shall now establish the following alternate existence and stability results for the specific choice of $\theta_{D,h}$ in (4.11). They hold under a natural small data assumption involving the $H^{1/2}(\Gamma)$-norm of the discrete datum $\theta_{D,h}$, which is slightly more restrictive than the one in (4.2).

Corollary 4.5 *Assume* (4.10), (4.11), *and let* $\theta_{h,1} \in \Psi_h$ *be the lifting constructed in Lemma 4.3. Assume further that*

$$\tilde{C}_{\text{dep}} C_{\text{emb}} C_{\text{lift}} ||\mathbf{g}||_{0,\Omega} ||\theta_{D,h}||_{1/2,\Gamma} \leq 1/2, \tag{4.12}$$

with $\tilde{C}_{\rm dep}$ defined in (4.3)*, $C_{\rm emb} > 0$ the embedding constant in* (3.12) *for $p = 3$, and $C_{\rm lift}$ the constant in Lemma 4.3, Then, there is a solution $(\mathbf{u}_h, \theta_h)$ to* (3.31) *with $\theta_h = \theta_{h,0} + \theta_{h,1}$ as in* (4.1) *and satisfying the stability bounds*

$$\|\mathbf{u}_h\|_{1,\mathcal{T}_h} \le \tilde{C}_u C_{\rm lift} \|\theta_{D,h}\|_{1/2,\Gamma}, \qquad \|\theta_h\|_{1,\Omega} \le \tilde{C}_\theta C_{\rm lift} \|\theta_{D,h}\|_{1/2,\Gamma}, \tag{4.13}$$

where $\tilde{C}_u$ and $\tilde{C}_\theta$ are the constants in (4.4)*.*

Proof. We apply Theorem 4.2. To that end, let us verify that (4.12) implies (4.2). Indeed, there holds

$$\begin{aligned}\tilde{C}_{\rm dep}\|\mathbf{g}\|_{0,\Omega}\|\theta_{h,1}\|_{L^3(\Omega)} &\le \tilde{C}_{\rm dep}C_{\rm emb}\|\mathbf{g}\|_{0,\Omega}\|\|\theta_{h,1}\|_{1,\Omega} \\ &\le \tilde{C}_{\rm dep}C_{\rm emb}C_{\rm lift}\|\mathbf{g}\|_{0,\Omega}\|\|\theta_{D,h}\|_{H^{1/2}(\Gamma)} \le \frac{1}{2},\end{aligned}$$

where we have used the embedding (3.12) with $p = 3$, and the bound in Lemma 4.3.

Hence, Theorem 4.2 implies the existence of a solution $(\mathbf{u}_h, \theta_h)$ satisfying the bound (4.4). Employing Lemma 4.3 once more shows that (4.13) holds true. $\square$

Remark 4.6 *If we assume the exact temperature θ of* (2.1)*–* (2.5) *to belong to $H^2(\Omega)$. Then, by the triangle inequality, Lemma 4.4 and standard approximation results for the nodal interpolant $\mathcal{I}$, we have*

$$\|\theta_{D,h}\|_{H^{1/2}(\Gamma)} \le \|\theta_D - \theta_{D,h}\|_{H^{1/2}(\Gamma)} + \|\theta_D\|_{H^{1/2}(\Gamma)} \le Ch\|\theta\|_{2,\Omega} + \|\theta_D\|_{H^{1/2}(\Gamma)}.$$

Hence, $\|\theta_{D,h}\|_{H^{1/2}(\Gamma)}$ is bounded as $h \to 0$. This property is crucial in the stability bounds (4.13) *and will be used in the subsequent error analysis.*

4.3 Proof of Theorem 4.2

To prove Theorem 4.2, we shall now make use of Brouwer's fixed point theorem in the following form [6]: Let $\mathcal{K}$ be a non-empty compact convex subset of a finite dimensional normed space, and let $\mathcal{L}$ be a continuous mapping of $\mathcal{K}$ into itself. Then $\mathcal{L}$ has a fixed point in $\mathcal{K}$. We proceed in several steps.

Step 1: We introduce the finite dimensional set

$$\mathcal{K} = \left\{ \begin{array}{ll} (\mathbf{u}_h, \theta_h) \in \mathbf{X}_h \times \Psi_h & : \|\mathbf{u}_h\|_{1,\mathcal{T}_h} \le \tilde{C}_u \|\theta_{h,1}\|_{1,\Omega}, \quad \|\theta_h\|_{1,\Omega} \le \tilde{C}_\theta \|\theta_{h,1}\|_{1,\Omega} \\ & \quad \text{and} \quad \theta_h = \theta_{h,0} + \theta_{h,1} \end{array} \right\}, \tag{4.14}$$

with $\tilde{C}_{\mathbf{u}}$ and $\tilde{C}_\theta$ the constants defined in (4.8) and (4.9), respectively. It is convex and compact. We then define the mapping

$$\mathcal{L} : (\mathbf{z}_h, \varphi_h) \in \mathbf{X}_h \times \Psi_h \mapsto (\mathbf{u}_h, \theta_h := \theta_{h,0} + \theta_{h,1}) \in \mathbf{X}_h \times \Psi_h$$

as the solution to the following linearized version of problem (3.31): find $(\mathbf{u}_h, \theta_h) \in \mathbf{X}_h \times \Psi_h$ such that

$$\begin{aligned} A^h_{\rm S}(\varphi_h; \mathbf{u}_h, \mathbf{v}) + O^h_{\rm S}(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}) - D(\varphi_h, \mathbf{v}) &= 0, \\ A_{\rm T}(\varphi_h; \theta_{h,0}, \psi) + O_{\rm T}(\mathbf{z}_h; \theta_{h,0}, \psi) &= -A_{\rm T}(\varphi_h; \theta_{h,1}, \psi) - O_{\rm T}(\mathbf{z}_h; \theta_{h,1}, \psi) \end{aligned} \tag{4.15}$$

for all $\mathbf{v} \in \mathbf{X}_h$ and $\psi \in \Psi_{h,0}$. With the stability properties in Section 3.3, it is not difficult to see that problem (4.15) is uniquely solvable, and hence the operator $\mathcal{L}$ is well defined.

Step 2: Let us prove that $\mathcal{L}$ maps from $\mathcal{K}$ into $\mathcal{K}$. To that end, let $(\mathbf{z}_h, \varphi_h) \in \mathcal{K}$ be given, and denote by $(\mathbf{u}_h, \theta_h) \in \mathbf{V}_h \times \Psi_h$ the solution to the problem (4.15). Then, as in the proof of Lemma 4.1, we take the test function $(\mathbf{v}, \psi) = (\mathbf{u}_h, \theta_{h,0})$. In the first of the two resulting equations, we use the coercivity of A_{S}^h in (3.27), the positivity of O_{T}^h in (3.28), and the boundedness of D in (3.23). This results in

$$\|\mathbf{u}_h\|_{1,\mathcal{T}_h}^2 \le \tilde{\alpha}_{\mathrm{S}}^{-1} |D(\varphi_h, \mathbf{u}_h)| \le \tilde{\alpha}_{\mathrm{S}}^{-1} \tilde{C}_D \|\mathbf{g}\|_{0,\Omega} \|\varphi_h\|_{1,\Omega} \|\mathbf{u}_h\|_{1,\mathcal{T}_h}.$$

Division by $\|\mathbf{u}\|_{1,\mathcal{T}_h}$ and the bound $\|\varphi_h\|_{1,\Omega} \le \tilde{C}_\theta \|\theta_{h,1}\|_{1,\Omega}$ then give

$$\|\mathbf{u}_h\|_{1,\mathcal{T}_h} \le \tilde{\alpha}_{\mathrm{S}}^{-1} \tilde{C}_D \tilde{C}_\theta \|\mathbf{g}\|_{0,\Omega} \|\theta_{h,1}\|_{1,\Omega} \; = \; \tilde{C}_u \|\theta_{h,1}\|_{1,\Omega},$$

where we have also used the identity

$$\tilde{C}_u = \tilde{\alpha}_{\mathrm{S}}^{-1} \tilde{C}_D \|\mathbf{g}\|_{0,\Omega} \tilde{C}_\theta. \tag{4.16}$$

In the second of the two resulting equations, we use the coercivity of A_{T} in (2.21), property (3.29), the boundedness of A_{T} and O_{T} in (2.13) and Lemma 3.5, respectively, the bound $\|\mathbf{z}_h\|_{1,\mathcal{T}_h} \le \tilde{C}_u \|\theta_{h,1}\|_{1,\Omega}$, and division by $\|\theta_{h,0}\|_{1,\Omega}$, to find that

$$\|\theta_{h,0}\|_{1,\Omega} \le \alpha_{\mathrm{T}}^{-1} \kappa_2 \|\theta_{h,1}\|_{1,\Omega} + \alpha_{\mathrm{T}}^{-1} \tilde{C}_{T,2} \tilde{C}_u \|\theta_{h,1}\|_{1,\Omega} \|\theta_{h,1}\|_{L^3(\Omega)}.$$

Then, from the identity (4.16) and assumption (4.2),

$$\begin{aligned} \|\theta_{h,0}\|_{1,\Omega} &\le \alpha_{\mathrm{T}}^{-1} \kappa_2 \|\theta_{h,1}\|_{1,\Omega} + \tilde{\alpha}_{\mathrm{S}}^{-1} \alpha_{\mathrm{T}}^{-1} \tilde{C}_D \tilde{C}_{T,2} \|\mathbf{g}\|_{0,\Omega} \tilde{C}_\theta \|\theta_{h,1}\|_{1,\Omega} \|\theta_{h,1}\|_{L^3(\Omega)} \\ &\le \alpha_{\mathrm{T}}^{-1} \kappa_2 \|\theta_{h,1}\|_{1,\Omega} + \frac{\tilde{C}_\theta}{2} \|\theta_{h,1}\|_{1,\Omega}. \end{aligned}$$

Then, the triangle inequality and the definition $C_\theta = 2(1 + \alpha_{\mathrm{T}}^{-1} \kappa_2)$ in (4.9) imply

$$\begin{aligned} \|\theta_h\|_{1,\Omega} &\le \|\theta_{h,0}\|_{1,\Omega} + \|\theta_{h,1}\|_{1,\Omega} \\ &\le (1 + \alpha_{\mathrm{T}}^{-1} \kappa_2) \|\theta_{h,1}\|_{1,\Omega} + \frac{\tilde{C}_\theta}{2} \|\theta_{h,1}\|_{1,\Omega} \le C_\theta \|\theta_{h,1}\|_{1,\Omega}. \end{aligned}$$

Hence, we have $(\mathbf{u}_h, \theta_h) \in \mathcal{K}$. It is now clear that the existence of a fixed point of $\mathcal{L} : \mathcal{K} \to \mathcal{K}$ is equivalent to the solvability of (3.31) as stated in the assertion.

Step 3: To apply Brouwer's fixed point theorem, it remains to show that $\mathcal{L}$ is a continuous operator. To do so, assume we are given $(\mathbf{z}, \varphi) \in \mathcal{K}$ and a sequence $\{(\mathbf{z}_m, \varphi_m)\}_{m \in \mathbb{N}} \subset \mathcal{K}$, such that

$$\|\mathbf{z}_m - \mathbf{z}\|_{1,\mathcal{T}_h} \overset{m\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \|\varphi_m - \varphi\|_{1,\Omega} \overset{m\to\infty}{\longrightarrow} 0.$$

We note that by the trace inequality (3.16) and for a fixed mesh size, there also holds $\lim\limits_{m\to\infty} \|\varphi_m - \varphi\|_{1,\mathcal{E}_h} = 0$. Thus, setting $(\mathbf{u}, \theta) = \mathcal{L}(\mathbf{z}, \varphi)$ and $(\mathbf{u}_m, \theta_m) = \mathcal{L}(\mathbf{z}_m, \varphi_m)$, $m \in \mathbb{N}$, we need to prove that

$$\|\mathbf{u}_m - \mathbf{u}\|_{1,\mathcal{T}_h} \overset{m\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \|\theta_m - \theta\|_{1,\Omega} \overset{m\to\infty}{\longrightarrow} 0. \tag{4.17}$$

From the definition of $\mathcal{L}$ in (4.15) we see that there hold

$$\begin{aligned} A_{\mathrm{S}}^h(\varphi_m;\mathbf{u}_m,\mathbf{v}) + O_{\mathrm{S}}^h(\mathbf{z}_m;\mathbf{u}_m,\mathbf{v}) - D(\varphi_m,\mathbf{v}) &= 0,\\ A_{\mathrm{T}}(\varphi_m;\theta_m,\psi) + O_{\mathrm{T}}(\mathbf{z}_m;\theta_m,\psi) &= 0, \end{aligned}$$

and

$$\begin{aligned} A_{\mathrm{S}}^h(\varphi;\mathbf{u},\mathbf{v}) + O_{\mathrm{S}}^h(\mathbf{z};\mathbf{u},\mathbf{v}) - D(\varphi,\mathbf{v}) &= 0,\\ A_{\mathrm{T}}(\varphi;\theta,\psi) + O_{\mathrm{T}}(\mathbf{z};\theta,\psi) &= 0, \end{aligned}$$

for all $\mathbf{v} \in \mathbf{X}_h$, $\psi \in \Psi_{h,0}$ and $m \in \mathbb{N}$. Subtracting the two systems from each other yields the equations

$$A_{\mathrm{S}}^h(\varphi_m;\mathbf{u}_m,\mathbf{v}) - A_{\mathrm{S}}^h(\varphi;\mathbf{u},\mathbf{v}) + O_{\mathrm{S}}^h(\mathbf{z}_m;\mathbf{u}_m,\mathbf{v}) - O_{\mathrm{S}}^h(\mathbf{z};\mathbf{u},\mathbf{v}) - D(\varphi_m-\varphi,\mathbf{v}) = 0, \tag{4.18}$$

for all $\mathbf{v} \in \mathbf{X}_h$, and

$$A_{\mathrm{T}}(\varphi_m;\theta_m,\psi) - A_{\mathrm{T}}(\varphi;\theta,\psi) + O_{\mathrm{T}}(\mathbf{z}_m;\theta_m,\psi) - O_{\mathrm{T}}(\mathbf{z};\theta,\psi) = 0, \tag{4.19}$$

for all $\psi \in \Psi_{h,0}$.

We first consider (4.18). Elementary manipulations then yield

$$\begin{aligned} A_{\mathrm{S}}^h(\varphi_m;\mathbf{u}-\mathbf{u}_m,\mathbf{v}) + O_{\mathrm{S}}^h(\mathbf{z}_m;\mathbf{u}-\mathbf{u}_m,\mathbf{v}) = &- [A_{\mathrm{S}}^h(\varphi;\mathbf{u},\mathbf{v}) - A_{\mathrm{S}}^h(\varphi_m;\mathbf{u},\mathbf{v})]\\ &- [O_{\mathrm{S}}^h(\mathbf{z};\mathbf{u},\mathbf{v}) - O_{\mathrm{S}}^h(\mathbf{z}_m;\mathbf{u},\mathbf{v})] + D(\varphi_m-\varphi,\mathbf{v}). \end{aligned}$$

We take $\mathbf{v} = \mathbf{u} - \mathbf{u}_m$, use the ellipticity property of A_{S}^h and O_{S}^h in (3.27) and (3.28), respectively, as well as the continuity of O_{S}^h and C_D, to get

$$\begin{aligned} \tilde{\alpha}_{\mathrm{S}}\|\mathbf{u}-\mathbf{u}_m\|_{1,\mathcal{T}_h}^2 \le{}& \left|A_{\mathrm{S}}^h(\varphi;\mathbf{u},\mathbf{u}-\mathbf{u}_m) - A_{\mathrm{S}}^h(\varphi_m;\mathbf{u},\mathbf{u}-\mathbf{u}_m)\right|\\ &+ \tilde{C}_{\mathrm{S}}\,\|\mathbf{z}-\mathbf{z}_m\|_{1,\mathcal{T}_h}\|\mathbf{u}\|_{1,\mathcal{T}_h}\|\mathbf{u}-\mathbf{u}_m\|_{1,\mathcal{T}_h} + \tilde{C}_D\|\mathbf{g}\|_{0,\Omega}\|\varphi-\varphi_m\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}_m\|_{1,\mathcal{T}_h}. \end{aligned}$$

With the continuity property (3.20) for A_{S}^h and division by $\|\mathbf{u}-\mathbf{u}_m\|_{1,\mathcal{T}_h}$, it follows that

$$\|\mathbf{u}-\mathbf{u}_m\|_{1,\mathcal{T}_h} \le C\Big(\|\varphi-\varphi_m\|_{1,\mathcal{E}_h}\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} + \|\mathbf{z}-\mathbf{z}_m\|_{1,\mathcal{T}_h}\|\mathbf{u}\|_{1,\mathcal{T}_h} + \|\varphi-\varphi_m\|_{1,\Omega}\Big).$$

Hence, we find that

$$\lim_{m\to\infty}\|\mathbf{u}-\mathbf{u}_m\|_{1,\mathcal{T}_h} = 0. \tag{4.20}$$

Next, we consider equation (4.19). By proceeding as before, we rewrite it as

$$\begin{aligned} A_{\mathrm{T}}(\varphi_m;\theta-\theta_m,\psi) + O_{\mathrm{T}}(\mathbf{z}_m;\theta-\theta_m,\psi) = &- [A_{\mathrm{T}}(\varphi;\theta,\psi) - A_{\mathrm{T}}(\varphi_m;\theta,\psi)]\\ &- [O_{\mathrm{T}}(\mathbf{z};\theta,\psi) - O_{\mathrm{T}}(\mathbf{z}_m;\theta,\psi)]. \end{aligned}$$

Then, we take $\psi = \theta - \theta_m \in \Psi_{h,0}$, note that $O_{\mathrm{T}}(\mathbf{z}_m;\theta-\theta_m,\theta-\theta_m) = 0$, by (3.29), and apply the continuity property (2.16), the ellipticity (2.21), and the bound (3.25) for O_T. Dividing the resulting inequality by $\|\theta-\theta_m\|_{1,\Omega}$ results in

$$\|\theta-\theta_m\|_{1,\Omega} \le C\Big(\|\varphi-\varphi_m\|_{1,\Omega}\|\theta\|_{W^{1,\infty}(\Omega)} + \|\mathbf{z}-\mathbf{z}_m\|_{1,\mathcal{T}_h}\|\theta\|_{1,\Omega}\Big).$$

Therefore,

$$\lim_{m\to\infty}\|\theta-\theta_m\|_{1,\Omega} = 0. \tag{4.21}$$

Referring to (4.20) and (4.21) shows the claim in (4.17), which completes the proof.

5 Error analysis

In this section, we carry out the error analysis of the finite element approximation in (3.6). We start by stating our error bounds. Then, we present the details of the proofs in several steps.

5.1 Error estimates

We shall prove the following error estimates.

Theorem 5.1 *Assume* (4.10), (4.11), *and the small data assumption* (4.12). *Let* $(\mathbf{u}, p, \theta)$ *be a solution of* (2.8), *and let* $(\mathbf{u}_h, p_h, \theta_h)$ *be an approximate solution to* (3.6) *satisfying the stability bounds* (4.13) *in Corollary 4.5. Assume further that*

$$\max\left\{ \|\mathbf{g}\|_{0,\Omega}, \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)}, \|\theta\|_{W^{1,\infty}(\Omega)} \right\} \leq \min\{M, \tilde{M}\}, \tag{5.1}$$

with M *and* $\tilde{M}$ *sufficiently small, as specified in* (2.42) *and* (5.18) *below. We further suppose that, for* $k = 1$,

$$\mathbf{u} \in \mathbf{C}^1(\overline{\Omega}) \cap \mathbf{H}^2(\Omega) \cap \mathbf{X}, \qquad p \in H^1(\Omega), \qquad \theta \in W^{1,\infty}(\Omega) \cap H^2(\Omega), \tag{5.2}$$

and, for $k \geq 2$,

$$\mathbf{u} \in \mathbf{H}^{k+1}(\Omega) \cap \mathbf{X}, \qquad p \in H^k(\Omega), \qquad \theta \in H^{k+1}(\Omega). \tag{5.3}$$

Then there exist two constants $C > 0$ *independent of the mesh size such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{2,\mathcal{T}_h} + \|\theta - \theta_h\|_{1,\Omega} \leq Ch^k(\|\mathbf{u}\|_{k+1,\Omega} + \|\theta\|_{k+1,\Omega}), \tag{5.4}$$

and

$$\|p - p_h\|_{0,\Omega} \leq Ch^k(\|p\|_{k,\Omega} + \|\mathbf{u}\|_{k+1,\Omega} + \|\theta\|_{k+1,\Omega}). \tag{5.5}$$

The proof of Theorem 5.1 is presented in Section 5.2.

Remark 5.2 *In our analysis, we shall need the base regularity* $(\mathbf{u}, \theta) \in \mathbf{C}^1(\overline{\Omega}) \times W^{1,\infty}(\Omega)$ *as assumed in the lowest-order case* $k = 1$ *in* (5.2)*; cf. Lemma 3.3 and* (2.16). *Notice that for* $k \geq 2$, *the regularity assumption* $(\mathbf{u}, \theta) \in \mathbf{H}^{k+1}(\Omega) \times H^{k+1}(\Omega)$ *in* (5.3) *implies* $(\mathbf{u}, \theta) \in \mathbf{C}^1(\overline{\Omega}) \times C^1(\overline{\Omega})$.

Remark 5.3 *Observe that under the small solution assumption* (5.1), *the exact solution to* (2.8) *is unique, in agreement to Theorem 2.2. On the other hand and as mentioned above, an analogous uniqueness result for the discrete solution remains an open question.*

5.2 Proof of Theorem 5.1

We present the proof of Theorem 5.1 in several steps.

5.2.1 Preliminaries

Let $(\mathbf{u}, p, \theta)$ be a solution of problem (2.8), and $(\mathbf{u}_h, p_h \theta_h)$ a finite element approximation obtained by its discrete counterpart (3.6). To simplify the subsequent analysis, we write $\mathbf{e}_\mathbf{u} = \mathbf{u} - \mathbf{u}_h$, $\mathrm{e}_\theta = \theta - \theta_h$ and $\mathrm{e}_p = p - p_h$. As usual, we shall then decompose these errors into

$$
\begin{aligned}
\mathbf{e}_\mathbf{u} &= \boldsymbol{\xi}_\mathbf{u} + \boldsymbol{\chi}_\mathbf{u} = (\mathbf{u} - \tilde{\mathbf{v}}_h) + (\tilde{\mathbf{v}}_h - \mathbf{u}_h),\\
\mathrm{e}_\theta &= \xi_\theta + \chi_\theta = (\theta - \tilde{\psi}_h) + (\tilde{\psi}_h - \theta_h),\\
\mathrm{e}_p &= \xi_p + \chi_p = (p - \tilde{q}_h) + (\tilde{q}_h - p_h),
\end{aligned}
\tag{5.6}
$$

where we take $\tilde{\mathbf{v}}_h$ as the BDM projection of $\mathbf{u}$, $\tilde{\psi}_h = \mathcal{I}\theta \in \Psi_h$ is the nodal projection of θ, as introduced in Section 4.2, and $\tilde{q}_h$ is the L^2-projection of p into Q_h.

We recall that for $\mathbf{u} \in \mathbf{X}$, we have $\tilde{\mathbf{v}}_h \in \mathbf{X}_h$; see, e.g., [7]. Then, we also have $\boldsymbol{\chi}_\mathbf{u} \in \mathbf{X}_h$. The following approximation properties are standard:

$$
\|\boldsymbol{\xi}_\mathbf{u}\|_{2,\mathcal{T}_h} \le C h^k \|\mathbf{u}\|_{k+1,\Omega}, \qquad \|\xi_\theta\|_{1,\Omega} \le C h^k \|\theta\|_{k+1,\Omega}, \qquad \|\xi_p\|_{0,\Omega} \le C h^k \|p\|_{k,\Omega}. \tag{5.7}
$$

Then, according to the triangle inequality and the inverse inequality (3.11), we see that

$$
\begin{aligned}
\|\mathbf{e}_\mathbf{u}\|_{2,\mathcal{T}_h} &\le \|\boldsymbol{\xi}_\mathbf{u}\|_{2,\mathcal{T}_h} + \|\boldsymbol{\chi}_\mathbf{u}\|_{2,\mathcal{T}_h} \le C\, h^k \|\mathbf{u}\|_{k+1,\Omega} + C\|\boldsymbol{\chi}_\mathbf{u}\|_{1,\mathcal{T}_h},\\
\|\mathrm{e}_\theta\|_{1,\Omega} &\le \|\xi_\theta\|1,\Omega + \|\chi_\theta\|_{1,\Omega} \le C\, h^k \|\theta\|_{k+1,\Omega} + \|\chi_\theta\|_{1,\Omega},\\
\|\mathrm{e}_p\|_{0,\Omega} &\le \|\xi_p\|_{0,\Omega} + \|\chi_p\|_{0,\Omega} \le C\, h^k \|p\|_{k,\Omega} + \|\chi_p\|_{0,\Omega}.
\end{aligned}
\tag{5.8}
$$

Hence, to prove the error estimate (5.1), we need to show the optimal convergence of $\|\boldsymbol{\chi}_\mathbf{u}\|_{1,\mathcal{T}_h}$, $\|\chi_\theta\|_{1,\Omega}$, and $\|\chi_p\|_{0,\Omega}$.

To do so, we shall employ the following Galerkin orthogonality property.

Lemma 5.4 *Assume that* $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{X}$. *Then we have*

$$
\begin{aligned}
\left[A_\mathrm{S}^h(\theta; \mathbf{u}, \mathbf{v}) - A_\mathrm{S}^h(\theta_h; \mathbf{u}_h, \mathbf{v})\right] + \left[O_\mathrm{S}^h(\mathbf{u}; \mathbf{u}, \mathbf{v}) - O_\mathrm{S}^h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v})\right] - B(\mathbf{v}, \mathrm{e}_p) - D(\mathrm{e}_\theta, \mathbf{v}) &= 0,\\
B(\mathrm{e}_\mathbf{u}, q) &= 0,\\
\left[A_\mathrm{T}(\theta; \theta, \psi) - A_\mathrm{T}(\theta_h; \theta_h, \psi)\right] + \left[O_\mathrm{T}(\mathbf{u}; \theta, \psi) - O_\mathrm{T}(\mathbf{u}_h; \theta_h, \psi)\right] &= 0,
\end{aligned}
$$

for all $(\mathbf{v}, q, \psi) \in \mathbf{V}_h \times Q_h \times \Psi_{h,0}$.

Proof. As we assume $\mathbf{H}^2(\Omega)$-regularity for the velocity field $\mathbf{u}$, it can be readily seen by integration by parts that the exact solution $(\mathbf{u}, p, \theta)$ satisfies

$$
A_\mathrm{S}^h(\theta; \mathbf{u}, \mathbf{v}) + O_\mathrm{S}^h(\mathbf{u}; \mathbf{u}, \mathbf{v}) - B(\mathbf{v}, p) - D(\theta, \mathbf{v}) = 0,
$$

for all $\mathbf{v} \in \mathbf{V}_h$; see also [2]. This implies the first equation. The second and third equations are readily verified. □

5.2.2 Error estimates in the velocity and temperature

We now start by analyzing the convergence of $\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}$ and $\|\chi_\theta\|_{1,\Omega}$.

Lemma 5.5 *There exists a constant $C_1 > 0$ independent of the mesh size such that*

$$\begin{aligned}(\tilde{\alpha}_{\mathrm{S}} - \tilde{C}_{\mathrm{S}} C_\infty \tilde{M}) \|\boldsymbol{\chi}_{\mathbf{u}}\|^2_{1,\mathcal{T}_h} \leq & C_1 \Big(\|\boldsymbol{\xi}_{\mathbf{u}}\|_{2,\mathcal{T}_h} + \|\xi_\theta\|_{1,\Omega} \Big) \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \\ & + \tilde{M}(\tilde{C}_{\mathrm{lip}} \nu_{\mathrm{lip}} + \tilde{C}_D) \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \|\chi_\theta\|_{1,\Omega}.\end{aligned}$$

Proof. First, note that $\boldsymbol{\chi}_{\mathbf{u}} \in \mathbf{X}_h$. From the ellipticity of A^h_{S} in (3.27) and elementary calculations, it is not difficult to see that

$$\tilde{\alpha}_{\mathrm{S}} \|\boldsymbol{\chi}_{\mathbf{u}}\|^2_{1,\mathcal{T}_h} \leq A^h_{\mathrm{S}}(\theta_h; \boldsymbol{\chi}_{\mathbf{u}}, \boldsymbol{\chi}_{\mathbf{u}}) = A^1_{\mathrm{S}} + A^2_{\mathrm{S}} + A^3_{\mathrm{S}} + A^4_{\mathrm{S}}, \tag{5.9}$$

with the terms A^1_{S} through A^4_{S} given by

$$\begin{aligned} A^1_{\mathrm{S}} &= A^h_{\mathrm{S}}(\theta_h; \mathbf{u}, \boldsymbol{\chi}_{\mathbf{u}}) - A^h_{\mathrm{S}}(\tilde{\psi}_h; \mathbf{u}, \boldsymbol{\chi}_{\mathbf{u}}), \\ A^2_{\mathrm{S}} &= A^h_{\mathrm{S}}(\tilde{\psi}_h; \mathbf{u}, \boldsymbol{\chi}_{\mathbf{u}}) - A^h_{\mathrm{S}}(\theta; \mathbf{u}, \boldsymbol{\chi}_{\mathbf{u}}), \\ A^3_{\mathrm{S}} &= A^h_{\mathrm{S}}(\theta; \mathbf{u}, \boldsymbol{\chi}_{\mathbf{u}}) - A^h_{\mathrm{S}}(\theta_h; \mathbf{u}_h, \boldsymbol{\chi}_{\mathbf{u}}), \\ A^4_{\mathrm{S}} &= -A^h_{\mathrm{S}}(\theta_h; \boldsymbol{\xi}_{\mathbf{u}}, \boldsymbol{\chi}_{\mathbf{u}}). \end{aligned}$$

Similarly, thanks to the positivity of O^h_{S} in (3.28), we obtain

$$0 \leq O^h_{\mathrm{S}}(\mathbf{u}_h; \boldsymbol{\chi}_{\mathbf{u}}, \boldsymbol{\chi}_{\mathbf{u}}) = O^1_{\mathrm{S}} + O^2_{\mathrm{S}} + O^3_{\mathrm{S}} + O^4_{\mathrm{S}}, \tag{5.10}$$

with O^1_{S} through O^4_{S} given by

$$\begin{aligned} O^1_{\mathrm{S}} &= O^h_{\mathrm{S}}(\mathbf{u}_h; \mathbf{u}, \boldsymbol{\chi}_{\mathbf{u}}) - O^h_{\mathrm{S}}(\tilde{\mathbf{v}}_h; \mathbf{u}, \boldsymbol{\chi}_{\mathbf{u}}), \\ O^2_{\mathrm{S}} &= O^h_{\mathrm{S}}(\tilde{\mathbf{v}}_h; \mathbf{u}, \boldsymbol{\chi}_{\mathbf{u}}) - O^h_{\mathrm{S}}(\mathbf{u}; \mathbf{u}, \boldsymbol{\chi}_{\mathbf{u}}), \\ O^3_{\mathrm{S}} &= O^h_{\mathrm{S}}(\mathbf{u}; \mathbf{u}, \boldsymbol{\chi}_{\mathbf{u}}) - O^h_{\mathrm{S}}(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\chi}_{\mathbf{u}}), \\ O^4_{\mathrm{S}} &= -O^h_{\mathrm{S}}(\mathbf{u}_h; \boldsymbol{\xi}_{\mathbf{u}}, \boldsymbol{\chi}_{\mathbf{u}}). \end{aligned}$$

From the first error equation in Lemma 5.4, it further follows that

$$A^3_{\mathrm{S}} + O^3_{\mathrm{S}} = D(\mathrm{e}_\theta, \boldsymbol{\chi}_{\mathbf{u}}) = D(\xi_\theta, \boldsymbol{\chi}_{\mathbf{u}}) + D(\chi_\theta, \boldsymbol{\chi}_{\mathbf{u}}), \tag{5.11}$$

where we have used the fact that $B(\boldsymbol{\chi}_{\mathbf{u}}, \mathrm{e}_p) = 0$ since $\boldsymbol{\chi}_{\mathbf{u}} \in \mathbf{X}_h$ is exactly divergence-free.

Next, we bound each of the terms on the right hand sides of (5.9), (5.10) and (5.11), respectively. We start by estimating those in (5.9). To that end, we use bound (3.21), the continuity of A^h_{S} in (3.18), and the fact that $\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} = \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)} \leq \tilde{M}$ (since $\mathbf{u} \in \mathbf{C}^1(\overline{\Omega})$). We find that

$$\begin{aligned} |A^1_{\mathrm{S}}| &\leq \tilde{C}_{\mathrm{lip}} \nu_{\mathrm{lip}} \|\theta_h - \tilde{\psi}_h\|_{1,\Omega} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \leq \tilde{M} \tilde{C}_{\mathrm{lip}} \nu_{\mathrm{lip}} \|\chi_\theta\|_{1,\Omega} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}, \\ |A^2_{\mathrm{S}}| &\leq \tilde{C}_{\mathrm{lip}} \nu_{\mathrm{lip}} \|\theta - \tilde{\psi}_h\|_{1,\Omega} \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \leq \tilde{M} \tilde{C}_{\mathrm{lip}} \nu_{\mathrm{lip}} \|\xi_\theta\|_{1,\Omega} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}, \\ |A^4_{\mathrm{S}}| &\leq C \|\boldsymbol{\xi}_{\mathbf{u}}\|_{2,\mathcal{T}_h} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}. \end{aligned} \tag{5.12}$$

We proceed similarly for the terms in (5.10). We use the continuity of $O_{\rm S}^h$, cf. (3.24), the continuous dependence of $\mathbf{u}_h$ in (4.13), and note that $\|\mathbf{u}\|_{1,\Omega} \le C_\infty \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)} \le C_\infty \tilde{M}$ by (1.2). This results in

$$\begin{aligned}
|O_{\rm S}^1| &\le \tilde{C}_{\rm S} \|\mathbf{u}\|_{1,\Omega} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}^2 \le \tilde{C}_{\rm S} C_\infty \tilde{M} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}^2, \\
|O_{\rm S}^2| &\le \tilde{C}_{\rm S} \|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \|\mathbf{u}\|_{1,\Omega} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \le \tilde{C}_{\rm S} C_\infty \tilde{M} \|\boldsymbol{\xi}_{\mathbf{u}}\|_{2,\mathcal{T}_h} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}, \\
|O_{\rm S}^4| &\le \tilde{C}_{\rm S} \|\mathbf{u}_h\|_{1,\mathcal{T}_h} \|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \le \tilde{C}_{\rm S} \tilde{C}_u C_{\rm lift} \|\theta_{D,h}\|_{H^{1/2}(\Gamma)} \|\boldsymbol{\xi}_{\mathbf{u}}\|_{2,\mathcal{T}_h} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}.
\end{aligned} \tag{5.13}$$

In the bound for $|O_{\rm S}^4|$, we emphasize that $\|\theta_{D,h}\|_{H^{1/2}(\Gamma)}$ is bounded independently of the mesh size, in agreement to Remark 4.6.

Finally, to estimate the terms in (5.11) we employ the continuity of D and the hypothesis that $\|\mathbf{g}\|_{0,\Omega} \le \tilde{M}$. We conclude that

$$\begin{aligned}
|D(\xi_\theta, \boldsymbol{\chi}_{\mathbf{u}})| &\le \tilde{M} \tilde{C}_D \|\xi_\theta\|_{1,\Omega} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}, \\
|D(\chi_\theta, \boldsymbol{\chi}_{\mathbf{u}})| &\le \tilde{M} \tilde{C}_D \|\chi_\theta\|_{1,\Omega} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}.
\end{aligned} \tag{5.14}$$

Hence, from (5.9), (5.10) and (5.11), and the upper bounds (5.12), (5.13) and (5.14) the assertion follows. □

A corresponding upper bound for $\|\chi_\theta\|_{1,\Omega}$ is established in a similar fashion.

Lemma 5.6 *There exists a constant $C_2 > 0$ independent of the mesh size such that*

$$(\alpha_{\rm T} - \kappa_{\rm lip} \tilde{M}) \|\chi_\theta\|_{1,\Omega}^2 \le C_2 \Big(\|\boldsymbol{\xi}_{\mathbf{u}}\|_{2,\mathcal{T}_h} + \|\xi_\theta\|_{1,\Omega} \Big) \|\chi_\theta\|_{1,\Omega} + \tilde{C}_{\rm T} C_\infty \tilde{M} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \|\chi_\theta\|_{1,\Omega}.$$

Proof. We proceed similarly to the proof of Lemma 5.5. Indeed, by adding and subtracting suitable terms and noting that $\chi_\theta \in H_0^1(\Omega)$, the ellipticity (2.21) of $A_{\rm T}$ and property (3.29) for O_T imply that

$$\begin{aligned}
\alpha_{\rm T} \|\chi_\theta\|_{1,\Omega}^2 &\le A_{\rm T}(\theta_h; \chi_\theta, \chi_\theta) + O_{\rm T}(\mathbf{u}_h; \chi_\theta, \chi_\theta) \\
&= A_{\rm T}^1 + A_{\rm T}^2 + A_{\rm T}^3 + A_{\rm T}^4 + O_{\rm T}^1 + O_{\rm T}^2 + O_{\rm T}^3 + O_{\rm T}^4,
\end{aligned} \tag{5.15}$$

with

$$\begin{aligned}
A_{\rm T}^1 &= A_{\rm T}(\theta_h; \theta, \chi_\theta) - A_{\rm T}(\tilde{\psi}_h; \theta, \chi_\theta), \qquad & A_{\rm T}^2 &= A_{\rm T}(\tilde{\psi}_h; \theta, \chi_\theta) - A_{\rm T}(\theta; \theta, \chi_\theta), \\
A_{\rm T}^3 &= A_{\rm T}(\theta; \theta, \chi_\theta) - A_{\rm T}(\theta_h; \theta_h, \chi_\theta), \qquad & A_{\rm T}^4 &= -A_{\rm T}(\theta_h; \xi_\theta, \chi_\theta),
\end{aligned}$$

and

$$\begin{aligned}
O_{\rm T}^1 &= -O_{\rm T}(\boldsymbol{\chi}_{\mathbf{u}}; \theta, \chi_\theta), \qquad & O_{\rm T}^2 &= -O_{\rm T}(\boldsymbol{\xi}_{\mathbf{u}}; \theta, \chi_\theta), \\
O_{\rm T}^3 &= O_{\rm T}(\mathbf{u}; \theta, \chi_\theta) - O_{\rm T}(\mathbf{u}_h; \theta_h, \chi_\theta), \qquad & O_{\rm T}^4 &= -O_{\rm T}(\mathbf{u}_h; \xi_\theta, \chi_\theta).
\end{aligned}$$

As before, the third error equation in Lemma 5.4 yields

$$A_{\rm T}^3 + O_{\rm T}^3 = 0.$$

Now, by the continuity properties of A_{T} in (2.13), (2.16), and since $\|\theta\|_{W^{1,\infty}(\Omega)} \leq \tilde{M}$, we see that

$$
\begin{aligned}
&|A_{\mathrm{T}}^1| \leq \kappa_{\mathrm{lip}}\|\theta\|_{W^{1,\infty}(\Omega)}\|\theta_h - \tilde{\psi}_h\|_{1,\Omega}\|\chi_\theta\|_{1,\Omega} \leq \kappa_{\mathrm{lip}}\tilde{M}\|\chi_\theta\|_{1,\Omega}^2,\\
&|A_{\mathrm{T}}^2| \leq \kappa_{\mathrm{lip}}\|\theta - \tilde{\psi}_h\|_{1,\Omega}\|\theta\|_{W^{1,\infty}(\Omega)}\|\chi_\theta\|_{1,\Omega} \leq \kappa_{\mathrm{lip}}\tilde{M}\|\xi_\theta\|_{1,\Omega}\|\chi_\theta\|_{1,\Omega},\\
&|A_{\mathrm{T}}^4| \leq \kappa_2\|\xi_\theta\|_{1,\Omega}\|\chi_\theta\|_{1,\Omega}.
\end{aligned}
\tag{5.16}
$$

On the other hand, by employing the bound $\|\theta\|_{1,\Omega} \leq C_\infty\|\theta\|_{W^{1,\infty}(\Omega)} \leq C_\infty\tilde{M}$, the inequality (2.13), and the continuous dependence in (4.13), we obtain

$$
\begin{aligned}
&|O_{\mathrm{T}}^1| \leq \tilde{C}_{\mathrm{T}}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}\|\theta\|_{1,\Omega}\|\chi_\theta\|_{1,\Omega} \leq \tilde{C}_{\mathrm{T}}C_\infty\tilde{M}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}\|\chi_\theta\|_{1,\Omega},\\
&|O_{\mathrm{T}}^2| \leq \tilde{C}_{\mathrm{T}}\|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}\|\theta\|_{1,\Omega}\|\chi_\theta\|_{1,\Omega} \leq \tilde{C}_{\mathrm{T}}C_\infty C_{\mathrm{T}}\tilde{M}\|\boldsymbol{\xi}_{\mathbf{u}}\|_{2,\mathcal{T}_h}\|\chi_\theta\|_{1,\Omega},\\
&|O_{\mathrm{T}}^4| \leq \tilde{C}_{\mathrm{T}}\|\mathbf{u}_h\|_{1,\mathcal{T}_h}\|\xi_\theta\|_{1,\Omega}\|\chi_\theta\|_{1,\Omega} \leq \tilde{C}_{\mathrm{T}}\tilde{C}_u C_{\mathrm{lift}}\|\theta_{D,h}\|_{H^{1/2}(\Gamma)}\|\xi_\theta\|_{1,\Omega}\|_{1,\Omega}\|\chi_\theta\|_{1,\Omega}.
\end{aligned}
\tag{5.17}
$$

The desired result follows from (5.15) and the estimates in (5.16) and (5.17), noting again that $\|\theta_{D,h}\|_{H^{1/2}(\Gamma)}$ is bounded independently of the mesh size; cf. Remark 4.6. ☐

We are now ready to prove the error bound (5.4) of Theorem 5.1.

Lemma 5.7 *There is a constant $C > 0$ independent of the mesh size such that*

$$
\|\mathbf{u} - \mathbf{u}_h\|_{2,\mathcal{T}_h} + \|\theta - \theta_h\|_{1,\Omega} \leq Ch^k(\|\mathbf{u}\|_{k+1,\Omega} + \|\theta\|_{k+1,\Omega}).
$$

Proof. Starting from (5.8), it is enough to bound $\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}$ and $\|\chi_\theta\|_{1,\Omega}$. To this end, we set

$$
L(\mathbf{u},\theta) = \|\boldsymbol{\xi}_{\mathbf{u}}\|_{2,\mathcal{T}_h} + \|\xi_\theta\|_{1,\Omega}.
$$

Adding the two bounds in Lemma 5.5 and Lemma 5.6 results in

$$
\begin{aligned}
(\tilde{\alpha}_{\mathrm{S}} - \tilde{C}_{\mathrm{S}}C_\infty\tilde{M})\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}^2 + (\alpha_{\mathrm{T}} - \kappa_{\mathrm{lip}}\tilde{M})\|\chi_\theta\|_{1,\Omega}^2 &\leq CL(\mathbf{u},\theta)\big[\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} + \|\chi_\theta\|_{1,\Omega}\big]\\
&\quad + \tilde{M}(\tilde{C}_{\mathrm{lip}}\nu_{\mathrm{lip}} + \tilde{C}_D)\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}\|\chi_\theta\|_{1,\Omega}\\
&\quad + \tilde{C}_T C_\infty\tilde{M}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}\|\chi_\theta\|_{1,\Omega}.
\end{aligned}
$$

An application of the inequality $|ab| \leq \frac{a^2}{2} + \frac{b^2}{2}$ allows us to bring the last two terms above to the right-hand side. By setting $\tilde{K} = \big((\tilde{C}_{\mathrm{lip}}\nu_{\mathrm{lip}}) + \tilde{C}_D + \tilde{C}_T C_\infty\big)/2$, we obtain

$$
\begin{aligned}
(\tilde{\alpha}_{\mathrm{S}} - (\tilde{C}_{\mathrm{S}}C_\infty + \tilde{K})\tilde{M})\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}^2 + &(\alpha_{\mathrm{T}} - (\kappa_{\mathrm{lip}} + \tilde{K})\tilde{M})\|\chi_\theta\|_{1,\Omega}^2\\
&\leq CL(\mathbf{u},\theta)\big[\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} + \|\chi_\theta\|_{1,\Omega}\big].
\end{aligned}
$$

Hence, if we choose $\tilde{M}$ such that

$$
\tilde{M} < \inf\left\{\frac{\tilde{\alpha}_{\mathrm{S}}}{\tilde{C}_{\mathrm{S}}C_\infty + \tilde{K}}, \frac{\alpha_{\mathrm{T}}}{\kappa_{\mathrm{lip}} + \tilde{K}}\right\},
\tag{5.18}
$$

we readily obtain

$$
\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} + \|\chi_\theta\|_{1,\Omega} \leq C\,L(\mathbf{u},\theta).
\tag{5.19}
$$

From the approximation properties in (5.7), we conclude that

$$
L(\mathbf{u},\theta) \leq C\,h^k\big(\,\|\mathbf{u}\|_{k+1,\Omega} + \|\theta\|_{k+1,\Omega}\,\big),
$$

which implies the desired estimate (5.4). ☐

5.2.3 Error in the pressure

Next, we bound the error in the pressure.

Lemma 5.8 *There is a constant $C > 0$ independent of the mesh size such that*

$$\|\mathrm{e}_p\|_{0,\Omega} \le Ch^k \Big(\|\mathbf{u}\|_{k+1,\Omega} + \|\theta\|_{k+1,\Omega} + \|p\|_{k,\Omega} \Big).$$

Proof. From (5.8), it remains to bound $\|\chi_p\|_{0,\Omega}$. To that end, we invoke the discrete inf-sup condition (3.30) and the boundedness of B in (3.22) to find that

$$\begin{aligned}
\|\chi_p\|_{0,\Omega} \le& \tilde{\beta}^{-1} \sup_{\mathbf{v}\in\mathbf{V}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},\chi_p)}{\|\mathbf{v}\|_{1,\mathcal{T}_h}} \\
\le& \tilde{\beta}^{-1} \sup_{\mathbf{v}\in\mathbf{V}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},\mathrm{e}_p)}{\|\mathbf{v}\|_{1,\mathcal{T}_h}} + \tilde{\beta}^{-1} \sup_{\mathbf{v}\in\mathbf{V}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},-\xi_p)}{\|\mathbf{v}\|_{1,\mathcal{T}_h}} \\
\le& \tilde{\beta}^{-1} \sup_{\mathbf{v}\in\mathbf{V}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},\mathrm{e}_p)}{\|\mathbf{v}\|_{1,\mathcal{T}_h}} + \tilde{\beta}^{-1}\tilde{C}_B\|\xi_p\|_{0,\Omega}.
\end{aligned} \tag{5.20}$$

Then, from the first error equation in Lemma 5.4, we find that, for any $\mathbf{v}\in\mathbf{V}_h$,

$$B(\mathbf{v},\mathrm{e}_p) \le |D(\mathrm{e}_\theta,\mathbf{v})| + |T_1| + |T_2| + |T_3| + |T_4|, \tag{5.21}$$

with

$$\begin{aligned}
T_1 &= [A^h_{\mathrm{S}}(\theta;\mathbf{u},\mathbf{v}) - A^h_{\mathrm{S}}(\theta_h;\mathbf{u},\mathbf{v})], \qquad & T_2 &= A^h_{\mathrm{S}}(\theta_h;\mathbf{e_u},\mathbf{v}),\\
T_3 &= [O^h_{\mathrm{S}}(\mathbf{u};\mathbf{u},\mathbf{v}) - O^h_{\mathrm{S}}(\mathbf{u}_h;\mathbf{u},\mathbf{v})], \qquad & T_4 &= O^h_{\mathrm{S}}(\mathbf{u}_h;\mathbf{e_u},\mathbf{v}).
\end{aligned}$$

Next, we bound the terms T_1 through T_4 appearing on the right hand side of (5.21). For T_1, we use the triangle inequality, the continuity bound in Lemma 3.3, and the assumption $\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)} = \|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\Omega)} \le \tilde{M}$. We obtain

$$|T_1| \le \tilde{C}_{\mathrm{lip}}\nu_{\mathrm{lip}}\|\mathrm{e}_\theta\|_{1,\Omega}\|\mathbf{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)}\|\mathbf{v}\|_{1,\mathcal{T}_h} \le \tilde{C}_{\mathrm{lip}}\nu_{\mathrm{lip}}\tilde{M}\|\mathrm{e}_\theta\|_{1,\Omega}\|\mathbf{v}\|_{1,\mathcal{T}_h}.$$

Furthermore, from the bound (3.18),

$$|T_2| \le C\|\mathbf{e_u}\|_{2,\mathcal{T}_h}\|\mathbf{v}\|_{1,\mathcal{T}_h}.$$

From the Lipschitz continuity of O^h_{S} in (3.24), the stability bound (4.13), and the inequality $\|\mathbf{u}\|_{1,\Omega} \le C_\infty\tilde{M}$, we have the estimates

$$\begin{aligned}
|T_3| \le& \tilde{C}_{\mathrm{S}}\|\mathbf{e_u}\|_{1,\mathcal{T}_h}\|\mathbf{u}\|_{1,\mathcal{T}_h}\|\mathbf{v}\|_{1,\mathcal{T}_h} \le \tilde{C}_{\mathrm{S}}C_\infty\tilde{M}\|\mathbf{e_u}\|_{2,\mathcal{T}_h}\|\mathbf{v}\|_{1,\mathcal{T}_h},\\
|T_4| \le& \tilde{C}_{\mathrm{S}}\|\mathbf{u}_h\|_{1,\mathcal{T}_h}\|\mathbf{e_u}\|_{1,\mathcal{T}_h}\|\mathbf{v}\|_{1,\mathcal{T}_h} \le \tilde{C}_{\mathrm{S}}\tilde{C}_uC_{\mathrm{lift}}\|\theta_{D,h}\|_{H^{1/2}(\Gamma)}\|\mathbf{e_u}\|_{2,\mathcal{T}_h}\|\mathbf{v}\|_{1,\mathcal{T}_h}.
\end{aligned}$$

Finally, note that, by (3.23) and assumption (5.1),

$$|D(\mathrm{e}_\theta,\mathbf{v})| \le \tilde{C}_D\|\mathbf{g}\|_{0,\Omega}\|\mathrm{e}_\theta\|_{1,\Omega}\|\mathbf{v}\|_{1,\mathcal{T}_h} \le \tilde{C}_D\tilde{M}\|e_\theta\|_{1,\Omega}\|\mathbf{v}\|_{1,\mathcal{T}_h}.$$

The above estimates imply

$$|B(\mathbf{v},\mathrm{e}_p)| \le C_3\Big(\|\mathrm{e}_\theta\|_{1,\Omega} + \|\mathbf{e_u}\|_{2,\mathcal{T}_h} \Big)\|\mathbf{v}\|_{1,\mathcal{T}_h}. \tag{5.22}$$

Hence, the desired estimate (5.5) follows from the inequalities in (5.20), (5.21), and (5.22). □

This completes the proof of Theorem 5.1.

6 Conclusions

We have introduced a new mixed finite element method for the numerical simulation of a generalized Boussinesq problem with exactly divergence-free BDM elements of order k for the velocities, discontinuous elements of order $k-1$ for the pressure, and standard continuous elements of order k for the discretization of the temperature. The resulting method yields exactly divergence-free velocity approximations, and thus it is energy-stable without additional modifications of the convection terms. Under suitable hypotheses on the data, we have shown the existence and stability of discrete solutions. Moreover, we have shown optimal a-priori error estimates with respect to the mesh size h for problems with smooth and sufficiently small solutions. More precisely, the broken H^1-norm errors in the velocity, the H^1-norm errors in the temperature, and the L^2-norm errors in the pressure are proved to converge with order $\mathcal{O}(h^k)$.

The uniqueness of (small) discrete solutions remains an open issue: one of the the difficulties in adapting Theorem 2.3 to the discrete level is the appearance of the augmented norm (3.15) in the continuity estimate (3.20). Ongoing research is concerned with finding ways to overcome this problem.

Finally, we emphasize that using conforming elements for the temperature unknown makes the analysis simpler, but may not yield robust approximations in highly convection-dominated problems. In this regime, discontinuous discretizations may be more appropriate for the temperature equation as well. This is also the subject of ongoing work.

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