Wiener integrals with respect to the Hermite random field and applications to the wave equation

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Abstract

The Hermite random field has been introduced as a limit of some weighted Hermite variations of the fractional Brownian sheet. In this work we define it as a multiple integral with respect to the standard Brownian sheet and introduce Wiener integrals with respect to it. As an application we study the wave equation driven by the Hermite sheet. we prove the existence of the solution and we study the regularity of its sample paths, the existence of the density and of its local times.

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1 Introduction

The random fields or multiparameter stochastic processes have focused a significant amount of attention among scientists due to the wide range of applications that they have.

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Particularly, self-similar random fields find some of their applications in various kind of phenomena, going from hydrology and surface modeling to network traffic analysis and mathematical finance, to name a few. From other side, this type of processes are also quite interesting when they appear as solutions to Stochastic Parial Differential Equations (SPDE's) in several dimensions, such as the wave or heat equations.

A class of processes that lies in the family described above are the Hermite random fields or Hermite sheets (from now on). Inside this class we can find the well-known and studied fractional Brownian sheet and the Rosenblatt processes, among others.

The Hermite processes of order $q \geq 1$ are self-similar with stationary increments and live in the qth Wiener chaos, that is, it can be expressed as a q times iterated integral with respect to the Wiener process. The class of Hermite process includes the fractional Brownian motion which is the only Gaussian process in this family. Their practical aspects are striking: they provide a wide class of processes from which to model long memory, self-similarity and Hölder-regularity, allowing significant deviation from fBm and other Gaussian processes. Since they are non-Gaussian and self-similar with stationary increments, the Hermite processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian.

The Hermite sheet of order q it is only known in his representation as a non-central limit of a particularly normalized Hermite variation of the fractional Brownian sheet, see [21] for the two-parameter case and [8] for the general d-parametric case. In both cases the authors also prove self-similarity, stationary increments and Hölder continuity.

In the present work we deal directly with the multi-parametric case building the Hermite sheet as a natural extension of the expression for the Hermite process studied as a non-central limit in [14] and [25].

Fix $d \in \mathbb{N}\setminus\{0\}$ and let $\mathbf{H} = (H_1, H_2, ..., H_d) \in (\frac{1}{2}, 1)^d$ a multi-Hurst index

$$
\begin{aligned}
Z^q_{\mathbf{H}}(\mathbf{t}) &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{t_1} \cdots \int_0^{t_d} \left(\prod_{j=1}^{q} (s_1 - y_{1,j})_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)} \cdots (s_d - y_{d,j})_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)} \right) \\
&\quad ds_d \dots ds_1 \quad dW(y_{1,1}, ..., y_{d,1}) \dots dW(y_{1,q}, ..., y_{d,q}) \\
&= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \dots dW(\mathbf{y}_q). \qquad (1)
\end{aligned}
$$

The above integrals are Wiener-Itô multiple integrals of order q with respect to the d-parametric standard Brownian sheet $(W(\mathbf{y}))_{\mathbf{y}\in\mathbb{R}^d}$ (see [18] for the definition) and $c(\mathbf{H}, q)$ is a positive normalization constant depending only on $\mathbf{H}$ and q. We designate the process $Z^q_{\mathbf{H}}(\mathbf{t})$ as the *Hermite sheet* or *Hermite random field*.

From expression (1) is possible to note that for $d = 1$ we recover the *Hermite process* which represent a family that has been recently studied in [12], [16] and [20]. As a particular case ($q = 1$) we recover the most known element of this family, the *fractional Brownian motion*, which has been largely studied due to its various applications. Recently, a rich theory of stochastic integration with respect to this process has been introduced and stochastic differential equations driven by the fractional Brownian motion have been considered for several purposes. The process obtained in (1) for $d = 1, q = 2$ is known as the *Rosenblatt process*, it was introduced by Rosenblatt in [22] and it has been called in this way by Taqqu in [24]. Lately, this process have been increasingly studied by his different interesting aspects like wavelet type expansion or extremal properties, parameter estimations, discrete approximations and others potential applications (see [1], [2], [5], [11], [27]).

As far as we know, the only well-known multiparameter process that can be obtained from (1) is the *fractional Brownian sheet* ($d > 1$ and $q = 1$). This processes has been recently studied as a driving noise for stochastic differential equations and stochastic calculus with respect to it have been developed. We refer to [3], [15], [28] for only a few works on various aspects of the fractional Brownian sheet.

In one hand the purpose of this article is to study the basic properties of the multiparamarmeter Hermite process and then to introduce Wiener integrals with respect to the Hermite sheet in order to generalize and continue the line introduced in [16] putting a new brick in the construction of stochastic calculus driven by this class of processes in several dimensions. As in [8] the covariance structure of the Hermite sheet is like the one of the fractional Brownian sheet, enabling the use of the same classes of deterministic integrands as in the fractional Brownian sheet profiting its well-known properties.

Also in the aim of this work lives the idea of making an approach to the study of stochastic partial differential equations in several dimensions driven by non-Gaussian noises, giving a specific expression for the driving noise allowing to use in a better way the properties of the equations by taking advantage of the results already existent in the literature. Is in this sense that, inspired by the works [4], [10] or [13] and exploiting these, we present a stochastic wave equation with respect to the Hermite sheet in spatial dimension $d \geq 1$ and we study the existence, regularity, and other properties of the solution, including the existence of local times and of the joint density.

We organize our paper as follows. Section 2 present the necessary notations and prove several properties of the Hermite sheet. In Section 3, we construct Wiener integrals with respect to this process. Section 4 is devoted to present the wave equation and discuss the existence and regularity of the solution and other properties.

2 Notation and the Hermite sheet

Throughout the work we use the notation introduced in [8]. Fix $d \in \mathbb{N} \setminus \{0\}$ and consider multi-parametric processes indexed in $\mathbb{R}^d$. We shall use bold notation for multi-indexed quantities, i.e., $\mathbf{a} = (a_1, a_2, ..., a_d)$, $\mathbf{ab} = (a_1 b_1, a_2 b_2, ..., a_d b_d)$, $\mathbf{a}/\mathbf{b} = (a_1/b_1, a_2/b_2, ..., a_d/b_d)$, $[\mathbf{a}, \mathbf{b}] = \prod_i^d [a_i, b_i]$, $(\mathbf{a}, \mathbf{b}) = \prod_i^d (a_i, b_i)$, $\sum_{\mathbf{i} \in [\mathbf{0}, \mathbf{N}]} a_{\mathbf{i}} = \sum_{i_1}^{N_1} \sum_{i_2}^{N_2} \cdots \sum_{i_d}^{N_d} a_{i_1, i_2, ..., i_d}$, $\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^d a_i^{b_i}$, and $\mathbf{a} < \mathbf{b}$ iff $a_1 < b_1, a_2 < b_2, ..., a_d < b_d$ (analogously for the other inequalities).

Before introducing the *Hermite sheet* we briefly recall the *fractional Brownian sheet* and the *standard Brownian sheet.*

The *d-parametric anisotropic fractional Brownian sheet* is the centered Gaussian process
$\{B^{\mathbf{H}}_{\mathbf{t}} : \mathbf{t} = (t_1, ..., t_d) \in \mathbb{R}^d\}$ with Hurst multi-index $\mathbf{H} = (H_1, ..., H_d) \in (0, 1)^d$. It is equal to zero on the hyperplanes $\{\mathbf{t} : t_i = 0\}$, $1 \leq i \leq d$, and its covariance function is given by

$$\begin{aligned} R_{\mathbf{H}}(\mathbf{s}, \mathbf{t}) &= \mathbb{E}[B^{\mathbf{H}}_{\mathbf{s}} B^{\mathbf{H}}_{\mathbf{t}}] \\ &= \prod_{i}^{d} R_{H_i}(s_i, t_i) = \prod_{i}^{d} \frac{s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}}{2}. \end{aligned} \tag{2}$$

The *d-parametric standard Brownian sheet* is the Gaussian process $\{W_{\mathbf{t}} : \mathbf{t} = (t_1, ..., t_d) \in \mathbb{R}^d\}$ equal to zero on the hyperplanes $\{\mathbf{t} : t_i = 0\}$, $1 \leq i \leq d$, and covariance function given by

$$R(\mathbf{s}, \mathbf{t}) = \mathbb{E}[W_{\mathbf{s}}, W_{\mathbf{t}}] = \prod_{i}^{d} R(s_i, t_i) = \prod_{i}^{d} s_i \wedge t_i. \tag{3}$$

Let $q \geq 1, q \in \mathbb{Z}$ and the Hurst multi-index $\mathbf{H} = (H_1, H_2, ..., H_d) \in (\frac{1}{2}, 1)^d$. The *Hermite sheet of order* q is given by

$$\begin{aligned} Z^q_{\mathbf{H}}(\mathbf{t}) &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{t_1} \cdots \int_0^{t_d} \left(\prod_{j=1}^{q} (s_1 - y_{1,j})_+^{-\left(\frac{1}{2} + \frac{1-H_1}{q}\right)} \cdots (s_d - y_{d,j})_+^{-\left(\frac{1}{2} + \frac{1-H_d}{q}\right)} \right) \\ &\quad ds_d \dots ds_1 \quad dW(y_{1,1}, ..., y_{d,1}) \dots dW(y_{1,q}, ..., y_{d,q}) \\ &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}-\mathbf{H}}{\mathbf{q}}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \dots dW(\mathbf{y}_q). \end{aligned} \tag{4}$$

where $x_+ = max(x, 0)$. For a better understanding about multiple stochastic integrals we refer to [18]. As pointed out before, when $q = 1$, (4) is the fractional Brownian sheet with Hurst multi-index $\mathbf{H} = (H_1, H_2, ..., H_d) \in (\frac{1}{2}, 1)^d$. For $q \geq 2$ the process $Z^q_{\mathbf{H}}(\mathbf{t})$ is not Gaussian and for $q = 2$ we denominate it as the *Rosenblatt sheet.*

Now let's calculate the covariance $R^q_{\mathbf{H}}(\mathbf{s}, \mathbf{t})$ of the Hermite sheet. Using the isometry of multiple Wiener-Itô integrals and Fubini one get

$$
\begin{aligned}
R^q_{\mathbf{H}}(\mathbf{s},\mathbf{t}) &= \mathbb{E}[Z^q_{\mathbf{H}}(\mathbf{s})Z^q_{\mathbf{H}}(\mathbf{t})] \\
&= \mathbb{E}\Bigg\{ c(\mathbf{H},q)^2 \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{s}} \prod_{j=1}^q (\mathbf{u}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{u} \quad dW(\mathbf{y}_1)\dots dW(\mathbf{y}_q) \\
&\quad \cdot \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^q (\mathbf{v}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{v} \quad dW(\mathbf{y}_1)\dots dW(\mathbf{y}_q) \Bigg\} \\
&= c(\mathbf{H},q)^2 \int_{\mathbb{R}^{d\cdot q}} \Bigg\{ \int_0^{s_1} \dots \int_0^{s_d} \prod_{j=1}^q \prod_{i=1}^d (u_i - y_{i,j})_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)} du_d \dots du_1 \\
&\quad \cdot \int_0^{t_1} \dots \int_0^{t_d} \prod_{j=1}^q \prod_{i=1}^d (v_i - y_{i,j})_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)} dv_d \dots dv_1 \Bigg\} dy_{1,1}\dots dy_{d,1}\dots dy_{1,q}\dots dy_{d,q} \\
&= c(\mathbf{H},q)^2 \int_0^{t_1} \int_0^{s_1} \int_{\mathbb{R}^q} \prod_{j=1}^q (u_1 - y_{1,j})_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)} (v_1 - y_{1,j})_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)} dy_{1,1}\dots dy_{1,q} du_1 dv_1 \\
&\quad \vdots \\
&\quad \int_0^{t_d} \int_0^{s_d} \int_{\mathbb{R}^q} \prod_{j=1}^q (u_d - y_{d,j})_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)} (v_d - y_{d,j})_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)} dy_{d,1}\dots dy_{d,q} du_d dv_d
\end{aligned}
$$

but

$$
\begin{aligned}
&\int_{\mathbb{R}^q} \prod_{j=1}^q (u-x_j)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} (v-x_j)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} dx_1 \dots dx_q \\
&= \left[\int_{\mathbb{R}} (u-x)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} (v-x)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \right]^q, \qquad (5)
\end{aligned}
$$

so

$$
\begin{aligned}
R^q_{\mathbf{H}}(\mathbf{s},\mathbf{t}) &= c(\mathbf{H},q)^2 \int_0^{t_1} \int_0^{s_1} \left[\int_{\mathbb{R}} (u_1-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)} (v_1-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H_1}{q}\right)} \right]^q du_1 dv_1 \\
&\quad \vdots \\
&\quad \int_0^{t_d} \int_0^{s_d} \left[\int_{\mathbb{R}} (u_d-y_d)_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)} (v_d-y_d)_+^{-\left(\frac{1}{2}+\frac{1-H_d}{q}\right)} \right]^q du_d dv_d.
\end{aligned}
$$

Recalling that the Beta function $\beta(p,q) = \int_0^1 z^{p-1}(1-z)^{p-1}dz, p, q > 0$, satisfies the following identity

$$
\int_{\mathbb{R}} (u-y)_+^{a-1}(v-y)_+^{a-1} dy = \beta(a, 2a-1)|u-v|^{2a-1} \qquad (6)
$$

we see that

$$
\begin{aligned}
R^q_{\mathbf{H}}(\mathbf{s},\mathbf{t}) &= c(\mathbf{H},q)^2 \int_0^{t_1}\int_0^{s_1} \beta\left(\frac{1}{2}-\frac{1-H_1}{q},\frac{2(H_1-1)}{q}\right)^q \cdot |u_1-v_1|^{2(H_1-1)} du_1 dv_1 \\
&\quad \ldots \int_0^{t_d}\int_0^{s_d} \beta\left(\frac{1}{2}-\frac{1-H_d}{q},\frac{2(H_d-1)}{q}\right)^q \cdot |u_d-v_d|^{2(H_d-1)} du_d dv_d \\
&= c(\mathbf{H},q)^2 \beta\left(\frac{1}{2}-\frac{1-H_1}{q},\frac{2(H_1-1)}{q}\right)^q \frac{1}{2H_1(2H_1-1)}\left(s_1^{2H_1}+t_1^{2H_1}-|t_1-s_1|^{2H_1}\right) \\
&\quad \ldots \beta\left(\frac{1}{2}-\frac{1-H_d}{q},\frac{2(H_d-1)}{q}\right)^q \frac{1}{2H_d(2H_d-1)}\left(s_d^{2H_d}+t_d^{2H_d}-|t_d-s_d|^{2H_d}\right)
\end{aligned}
$$

So now we choose

$$
c(\mathbf{H},q)^2 = \left(\frac{\beta\left(\frac{1}{2}-\frac{1-H_1}{q},\frac{2(H_1-1)}{q}\right)^q}{H_1(2H_1-1)}\right)^{-1} \ldots \left(\frac{\beta\left(\frac{1}{2}-\frac{1-H_d}{q},\frac{2(H_d-1)}{q}\right)^q}{H_d(2H_d-1)}\right)^{-1} \tag{7}
$$

in this way we get $\mathbb{E}\left(Z^q_{\mathbf{H}}(\mathbf{t})^2\right) = \mathbf{t}^{2\mathbf{H}} = t_1^{2H_1}\ldots t_d^{2H_d}$, and finally

$$
\begin{aligned}
R^q_{\mathbf{H}}(\mathbf{s},\mathbf{t}) &= \frac{1}{2}\left(s_1^{2H_1}+t_1^{2H_1}-|t_1-s_1|^{2H_1}\right)\ldots\left(s_d^{2H_d}+t_d^{2H_d}-|t_d-s_d|^{2H_d}\right) \\
&= \prod_i^d \frac{s_i^{2H_i}+t_i^{2H_i}-|t_i-s_i|^{2H_i}}{2} \\
&= \prod_i^d R_{H_i}(s_i,t_i) = R_{\mathbf{H}}(\mathbf{s},\mathbf{t})
\end{aligned} \tag{8}
$$

Remark 1 *As mentioned at the beginning, from the previous development we see that the covariance structure is the same for all $q \geq 1$, so it coincides with the covariance of the fractional Brownian sheet.*

We will next prove the basic properties of the Hermite sheet: self-similarity, stationarity of the increments and Hölder continuity.

Let us first recall the concept of self-similarity for multiparameter stochastic processes.

Definition 1 *A stochastic process $(X_{\mathbf{t}})_{\mathbf{t}\in T}$, where $T \subset \mathbb{R}^d$ is called self-similar with self-similarity order $\alpha = (\alpha_1,...,\alpha_d) > 0$ if for any $\mathbf{h} = (h_1,..,h_d) > 0$ the stochastic process $(\hat{X}_{\mathbf{t}})_{\mathbf{t}\in T}$ given by*

$$
\hat{X}_{\mathbf{t}} = \mathbf{h}^{\alpha} X_{\frac{\mathbf{t}}{\mathbf{h}}} = h_1^{\alpha_1}...h_d^{\alpha_d} X_{\frac{t_1}{h_1},...,\frac{t_d}{h_d}}
$$

has the same law as the process X.

Proposition 1 *The Hermite sheet is self-similar of order $\mathbf{H} = (H_1,...,H_d)$.*

Proof: The scaling property of the Wiener sheet implies that for every $0 < \mathbf{c} = (c_1, .., c_d) \in \mathbb{R}^d$ the processes $(W(\mathbf{ct})_{\mathbf{t}\geq 0})$ and $\left(\sqrt{\mathbf{c}}W(\mathbf{t})\right)_{\mathbf{t}\geq 0}$ have the same finite dimensional distributions. Therefore, if $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^d$, using obvious changes of variables in the integrals $d\mathbf{s}$ and dW,

$$\begin{aligned}
\hat{Z}^q_{\mathbf{H}}(t) &= \mathbf{h}^{\mathbf{H}} Z^q_{\frac{\mathbf{t}}{\mathbf{h}}} \\
&= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\frac{\mathbf{t}}{\mathbf{h}}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1)\ldots dW(\mathbf{y}_q) \\
&= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}-1} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\frac{\mathbf{s}}{\mathbf{h}} - \mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1)\ldots dW(\mathbf{y}_q) \\
&= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}-1} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\frac{\mathbf{s}}{\mathbf{h}} - \frac{\mathbf{y}_j}{\mathbf{h}})_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{s} \quad dW(\mathbf{h}^{-1}\mathbf{y}_1)\ldots dW(\mathbf{h}^{-1}\mathbf{y}_q) \\
&= c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}-1}\mathbf{h}^{q\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{s} \quad dW(\mathbf{h}^{-1}\mathbf{y}_1)\ldots dW(\mathbf{h}^{-1}\mathbf{y}_q) \\
&=^{(d)} c(\mathbf{H}, q)\mathbf{h}^{\mathbf{H}-1}\mathbf{h}^{q\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)}\mathbf{h}^{-\frac{q}{2}} \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1)\ldots dW(\mathbf{y}_q) \\
&= Z^q_{\mathbf{H}}(t)
\end{aligned}$$

where $=^{(d)}$ means equivalence of finite dimensional distributions.

■

Let us recall the notion of the increment of a d-parameter process X on a rectangle $[\mathbf{s}, \mathbf{t}] \subset \mathbb{R}^d$, $\mathbf{s} = (s_1, ..., s_d), \mathbf{t} = (t_1, ..., t_d)$, with $\mathbf{s} \leq \mathbf{t}$. This increment is denoted by $\Delta X_{[\mathbf{s},\mathbf{t}]}$ and it is given by

$$\Delta X_{[\mathbf{s},\mathbf{t}]} = \sum_{r\in\{0,1\}^d} (-1)^{d-\sum_i r_i} X_{\mathbf{s}+\mathbf{r}\cdot(\mathbf{t}-\mathbf{s})}. \tag{9}$$

When $d = 1$ one obtains $\Delta X_{[\mathbf{s},\mathbf{t}]} = X_t - X_s$ while for $d = 2$ one gets $\Delta X_{[\mathbf{s},\mathbf{t}]} = X_{t_1,t_2} - X_{t_1,s_2} - X_{s_1,t_2} + X_{s_1,s_2}$.

Definition 2 *A process $(X_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}^d)$ has stationary increments if for every $\mathbf{h} > 0, \mathbf{h} \in \mathbb{R}^d$ the stochastic processes $(\Delta X_{[0,\mathbf{t}]}, \mathbf{t} \in \mathbb{R}^d)$ and $(\Delta X_{[\mathbf{h},\mathbf{h}+\mathbf{t}]}, \mathbf{t} \in \mathbb{R}^d)$ have the same finite dimensional distributions.*

Proposition 2 *The Hermite sheet $(Z^q(\mathbf{t}))_{\mathbf{t}\geq 0}$ has stationary increments.*

Proof: Developing the increments of the process using the definition of the Hermite sheet and proceding as in the proof of Proposition 1 using the change of variables $\mathbf{s}' = \mathbf{s} - \mathbf{h}$, it is immediate to see that for every $\mathbf{h} > 0, \mathbf{h} \in \mathbb{R}^d$,

$$\Delta Z^q_{[\mathbf{h},\mathbf{h}+\mathbf{t}]} =^d \Delta Z^q_{[0,\mathbf{t}]}$$

for every $\mathbf{t}$. ■

Proposition 3 *The trajectories of the Hermite sheet* $(Z^q(\mathbf{t}), \mathbf{t} \geq 0)$ *are Hölder continuous of any order* $\delta = (\delta_1, .., \delta_d) \in [0, \mathbf{H}]$ *in the following sense: for every* $\omega \in \Omega$*, there exists a constant* $C_\omega > 0$ *such that for every* $\mathbf{s}, \mathbf{t} \in \mathbb{R}^d, \mathbf{s}, \mathbf{t} \geq 0$*,*

$$|\Delta Z^q_{[\mathbf{s},\mathbf{t}]}| \leq C_\omega |t_1 - s_1|^{\delta_1} ....|t_d - s_d|^{\delta_d} = C_\omega |\mathbf{t} - \mathbf{s}|^{\delta}.$$

Proof: Using the Cencov's criteria (see [9]) and the fact that the process Z^q is almost sure equal to 0 when $t_i = 0$, it suffices to check that

$$\mathbb{E}\left|\Delta Z^q_{[\mathbf{s},\mathbf{t}]}\right|^p \leq C\left(|t_1 - s_1|....|t_d - s_d|\right)^{1+\gamma} \tag{10}$$

for some $p \geq 2$ and $\gamma > 0$. From the self-similarity and the stationarity of the increments of the process Z^q, we have for every $p \geq 2$

$$\mathbb{E}\left|\Delta Z^q_{[\mathbf{s},\mathbf{t}]}\right|^p = \mathbb{E}\left|Z_{\mathbf{1}}\right|^p \left(|t_1 - s_1|....|t_d - s_d|\right)^{p\mathbf{H}}$$

and this obviously implies (10). ■

3 Wiener integrals with respect to the Hermite sheet

Now we are well positioned to present Wiener integrals with respect to the d-parametric Hermite sheet. Let us consider a Hermite sheet $\left(Z^q_{\mathbf{H}}(\mathbf{t})\right)_{t\in\mathbb{R}^d}$. Denote $\mathscr{E}$ the family of elementary functions on $\mathbb{R}^d$ of the form

$$\begin{aligned} f(\mathbf{u}) &= \sum_{l=1}^{n} a_l 1_{(\mathbf{t}_l, \mathbf{t}_{l+1}]}(\mathbf{u}) \qquad (11)\\ &= \sum_{l=1}^{n} a_l 1_{(t_{1,l}, t_{1,l+1}]\times\ldots\times(t_{d,l}, t_{d,l+1}]}(u_1, \ldots, u_d), \quad \mathbf{t}_l < \mathbf{t}_{l+1}, \;\; a_l \in \mathbb{R}, \;\; l = 1, \ldots, n. \end{aligned}$$

For functions like f above we can naturally define it's Wiener integral with respect to the Hermite sheet $Z^q_{\mathbf{H}}$ as

$$\int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u}) = \sum_{l=1}^{n} a_l \Delta(Z^q_{\mathbf{H}})_{[\mathbf{t}_l, \mathbf{t}_{l+1}]} \tag{12}$$

where $(\Delta Z^q_{\mathbf{H}})_{[\mathbf{t}_l,\mathbf{t}_{l+1}]}$ (see (9)) stands for the generalized increments of $Z^q_{\mathbf{H}}$ on the rectangle

$$\Delta_{\mathbf{t}_l} := [\mathbf{t}_l, \mathbf{t}_{l+1}] = \prod_{i=1}^{d} [t_{i,l}, t_{i,l+1}]$$

given by

$$(\Delta Z^q_{\mathbf{H}})_{[\mathbf{t}_l,\mathbf{t}_{l+1}]} = \sum_{\xi\in\{0,1\}^d} (-1)^{d-\sum_{i=1}^d \xi_i} Z^q_{\mathbf{H}}(t_{1,l+\xi_d}, \ldots, t_{d,l+\xi_1}). \tag{13}$$

In the case $d = 1$, we simply have

$$(\Delta Z^q_{\mathbf{H}})_{[\mathbf{t}_l, \mathbf{t}_{l+1}]} = Z^q_{\mathbf{H}}(t_{1,l+1} - t_{1,l})$$

while for $d = 2$

$$(\Delta Z^q_{\mathbf{H}})_{[\mathbf{t}_l, \mathbf{t}_{l+1}]} = Z^q_{\mathbf{H}}(t_{1,l+1}, t_{2,l+1}) - Z^q_{\mathbf{H}}(t_{1,l}, t_{2,l+1}) - Z^q_{\mathbf{H}}(t_{1,l+1}, t_{2,l}) + Z^q_{\mathbf{H}}(t_{1,l}, t_{2,l}).$$

With the purpose of extend the definition (12) to a larger family of integrands, we will point out some observations before. Let's consider the mapping J on the set of functions $f : \mathbf{R}^d \to \mathbf{R}$ to the set of functions $f : \mathbf{R}^{d\cdot q} \to \mathbf{R}$ such that

$$\begin{aligned} J(f)(\mathbf{y}_1, \ldots, \mathbf{y}_q) &= c(\mathbf{H}, q) \int_{\mathbb{R}^d} f(\mathbf{u}) \prod_{j=1}^{q} (\mathbf{u} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{q}\right)} d\mathbf{u} & (14) \\ &= c(\mathbf{H}, q) \int_{\mathbb{R}^d} f(u_1, \ldots, u_d) \prod_{j=1}^{q} \prod_{i=1}^{d} (u_i - u_{i,j})_+^{-\left(\frac{1}{2} + \frac{1-H_i}{q}\right)} du_1, \ldots du_d. \end{aligned}$$

Using the mapping J we see that definition (4) can be re-expressed as follows

$$\begin{aligned} Z^q_{\mathbf{H}}(\mathbf{t}) &= c(\mathbf{H}, q) \int_{\mathbb{R}^{d\cdot q}} \int_0^{\mathbf{t}} \prod_{j=1}^{q} (\mathbf{s} - \mathbf{y}_j)_+^{-\left(\frac{1}{2} + \frac{1-\mathbf{H}}{q}\right)} d\mathbf{s} \quad dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q) \\ &= \int_{\mathbb{R}^{d\cdot q}} J(1_{[0,t_1]\times \ldots \times [0,t_d]})(\mathbf{y}_1, \ldots, \mathbf{y}_q) dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q). & (15) \end{aligned}$$

As J is clearly linear, definition (12) can be tailored to

$$\begin{aligned} \int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u}) &= \sum_{l=1}^{n} a_l \Delta_{\mathbf{t}_l} \left(Z^q_{\mathbf{H}}(\mathbf{t}_l) \right) \\ &= \sum_{l=1}^{n} a_l \left(\sum_{\xi \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d \xi_i} Z^q_{\mathbf{H}}(t_{1,l+\xi_1}, \ldots, t_{d,l+\xi_d}) \right) \\ &= \sum_{l=1}^{n} a_l \sum_{\xi \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d \xi_i} \int_{\mathbb{R}^{d\cdot q}} J(1_{[0,t_{1,l+\xi_1}]\times \ldots \times [0,t_{d,l+\xi_d}]})(\mathbf{y}_1, \ldots, \mathbf{y}_q) dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q) \\ &= \sum_{l=1}^{n} a_l \int_{\mathbb{R}^{d\cdot q}} J(1_{[t_{1,l}, t_{1,l+1}]\times \ldots \times [t_{d,l}, t_{d,l+1}]})(\mathbf{y}_1, \ldots, \mathbf{y}_q) dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q) \\ &= \int_{\mathbb{R}^{d\cdot q}} J(f)(\mathbf{y}_1, \ldots, \mathbf{y}_q) dW(\mathbf{y}_1) \ldots dW(\mathbf{y}_q). & (16) \end{aligned}$$

In this way we introduce the space

$$\mathcal{H} = \left\{ f : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^{d\cdot q}} \left(J(f)(\mathbf{y}_1, \ldots, \mathbf{y}_q) \right)^2 d\mathbf{y}_1, \ldots, d\mathbf{y}_q < \infty \right\} \qquad (17)$$

equipped with the norm

$$\|f\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^{d\cdot q}} (J(f)(\mathbf{y}_1,\dots,\mathbf{y}_q))^2 \, d\mathbf{y}_1,\dots,d\mathbf{y}_q. \tag{18}$$

Working the expression for the norm we see that

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 &= c(\mathbf{H},q)^2 \int_{\mathbb{R}^{d\cdot q}} \left\{ \left(\int_{\mathbb{R}^d} f(\mathbf{u}) \prod_{j=1}^{q} (\mathbf{u}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{u} \right) \right.\\
&\quad \left. \cdot \left(\int_{\mathbb{R}^d} f(\mathbf{v}) \prod_{j=1}^{q} (\mathbf{v}-\mathbf{y}_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathbf{H}}{q}\right)} d\mathbf{v} \right) \right\} d\mathbf{y}_1,\dots,d\mathbf{y}_q \\
&= c(\mathbf{H},q)^2 \int_{\mathbb{R}^{d\cdot q}} \left\{ \left(\int_{\mathbb{R}^d} f(u_1,\dots,u_d) \prod_{j=1}^{q} \prod_{i=1}^{d} (u_i - y_{i,j})_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)} du_1,\dots du_d \right) \right.\\
&\quad \left. \cdot \left(\int_{\mathbb{R}^d} f(v_1,\dots,v_d) \prod_{j=1}^{q} \prod_{i=1}^{d} (v_i - y_{i,j})_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)} dv_1,\dots dv_d \right) \right\} d\mathbf{y}_1,\dots,d\mathbf{y}_q
\end{aligned}$$

Using (5), (6) and (7) we get that

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 &= c(\mathbf{H},q)^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u_1,\dots,u_d) f(v_1,\dots,v_d) \\
&\quad \left\{ \prod_{i=1}^{d} \int_{\mathbb{R}^q} \prod_{j=1}^{q} (u_i - y_{i,j})_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)} (v_i - y_{i,j})_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)} dy_{i,1},\dots dy_{i,q} \right\} du_1,\dots du_d dv_1,\dots dv_d \\
&= c(\mathbf{H},q)^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u_1,\dots,u_d) f(v_1,\dots,v_d) \\
&\quad \cdot \prod_{i=1}^{d} \left(\int_{\mathbb{R}} (u_i - y)_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)} (v_i - y)_+^{-\left(\frac{1}{2}+\frac{1-H_i}{q}\right)} dy \right)^q du_1,\dots du_d dv_1,\dots dv_d \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u_1,\dots,u_d) f(v_1,\dots,v_d) \prod_{i=1}^{d} H_i(2H_i-1)|u-v|^{2H_i-2} du_1,\dots du_d dv_1,\dots dv_d \\
&= \mathbf{H}(\mathbf{2H}-\mathbf{1}) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u}-\mathbf{v}|^{\mathbf{2H-2}} d\mathbf{u} d\mathbf{v}, \qquad (19)
\end{aligned}$$

hence

$$\mathcal{H} = \left\{ f : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u}-\mathbf{v}|^{\mathbf{2H-2}} d\mathbf{u} d\mathbf{v} < +\infty \right\} \tag{20}$$

and

$$\|f\|_{\mathcal{H}}^2 = \mathbf{H}(\mathbf{2H}-\mathbf{1}) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}) f(\mathbf{v}) |\mathbf{u}-\mathbf{v}|^{\mathbf{2H-2}} d\mathbf{u} d\mathbf{v}.$$

The mapping

$$f \to \int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u}) \tag{21}$$

provides an isometry from $\mathscr{E}$ to $L^2(\Omega)$. Indeed, for f like (11) it holds that

$$\begin{aligned}
& \mathbb{E}\left\{\left(\int_{\mathbb{R}^d} f(\mathbf{u}) dZ^q_{\mathbf{H}}(\mathbf{u})\right)^2\right\} \\
= & \sum_{k,l=0}^{n-1} a_k a_l \mathbb{E}\left(\Delta_{\mathbf{t}_k}\left(Z^q_{\mathbf{H}}(\mathbf{t}_k)\right) \cdot \Delta_{\mathbf{t}_l}\left(Z^q_{\mathbf{H}}(\mathbf{t}_l)\right)\right) \qquad (22) \\
= & \sum_{k,l=0}^{n-1} a_k a_l \sum_{\xi \in \{0,1\}^d} (-1)^{d-\sum_{i=1}^d \xi_i} \sum_{\rho \in \{0,1\}^d} (-1)^{d-\sum_{j=1}^d \rho_j} \mathbb{E}\left\{Z^q_{\mathbf{H}}(\mathbf{t}_{k+\xi}) Z^q_{\mathbf{H}}(\mathbf{t}_{l+\rho})\right\} \\
= & \sum_{k,l=0}^{n-1} a_k a_l \sum_{\xi \in \{0,1\}^d} (-1)^{d-\sum_{i=1}^d \xi_i} \sum_{\rho \in \{0,1\}^d} (-1)^{d-\sum_{j=1}^d \rho_j} R_{\mathbf{H}}(\mathbf{t}_{k+\xi}, \mathbf{t}_{l+\rho}) \\
= & \sum_{k,l=0}^{n-1} a_k a_l H_1(2H_1-1) \dots H_d(2H_d-1) \int_{t_{1,k}}^{t_{1,k+1}} \dots \int_{t_{d,k}}^{t_{d,k+1}} \cdot \int_{t_{1,l}}^{t_{1,l+1}} \dots \int_{t_{d,l}}^{t_{d,l+1}} \\
& |u_1 - v_1|^{2H_1-2} \dots |u_d - v_d|^{2H_d-2} du_1 \dots du_d dv_1 \dots dv_d \\
= & \sum_{k,l=0}^{n-1} a_k a_l < 1_{[t_{1,k}, t_{1,k+1}] \times [t_{d,k}, t_{d,k+1}]}, 1_{[t_{1,l}, t_{1,l+1}] \times [t_{d,l}, t_{d,l+1}]} >_{\mathcal{H}} \\
= & < f, f >_{\mathcal{H}}, \qquad (23)
\end{aligned}$$

where we have made a slight abuse of notation, $\mathbf{t}_{k+\xi} = (t_{1,k+\xi_1}, \dots t_{d,k+\xi_d})$.

On the other hand, from what shown in [19] it follows that the set of elementary functions $\mathcal{E}$ is dense in $\mathcal{H}$. As a consequence the mapping (14) can be extended to an isometry from $\mathcal{H}$ to $L^2(\Omega)$ and relation (15) still holds.

Remark 2 *The elements of $\mathcal{H}$ may be not functions but distributions; it is therefore more practical to work with subspaces of $\mathcal{H}$ that are sets of functions. Such a subspace is*

$$|\mathcal{H}| = \left\{ f : \mathbb{R}^d \to \mathbb{R} \;\Big|\; \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(\mathbf{u})||f(\mathbf{v})||\mathbf{u} - \mathbf{v}|^{\mathbf{2H-2}} d\mathbf{v} d\mathbf{u} < \infty \right\}.$$

Then $|\mathcal{H}|$ is a strict subspace of $\mathcal{H}$ and we actually have the inclusions

$$L^{\mathbf{2}}(\mathbb{R}^d) \cap L^{\mathbf{1}}(\mathbb{R}^d) \subset L^{\frac{1}{\mathbf{H}}}(\mathbb{R}^d) \subset |\mathcal{H}| \subset \mathcal{H}, \tag{24}$$

where $L^{\mathbf{p}}$ denotes $L^{p_1} \otimes \dots \otimes L^{p_d}$.
The space $|\mathcal{H}|$ is not complete with respect to the norm $\|\cdot\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm

$$\|f\|^2_{|\mathcal{H}|} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(\mathbf{u})||f(\mathbf{v})||\mathbf{u} - \mathbf{v}|^{\mathbf{2H-2}} d\mathbf{v} d\mathbf{u}$$

Remark 3 *Expression (16) present a useful interpretation for the Wiener integrals with respect to the Hermite sheet; as elements in the q-th Wiener chaos generated by the d-parametric standard Brownian field.*

4 Application: The Hermite stochastic wave equation

In this section we present the linear stochastic wave equation as an example of equations driven by a Hermite sheet. We show the existence of the solution and study some properties of it thanks to the definition of the Wiener integrals with respect to the Hermite sheet.

Consider the linear stochastic wave equation driven by an infinite-dimensional Hermite sheet $Z^q_{\mathbf{H}}$ with Hurst multi-index $\mathbf{H} \in (1/2, 1)^{(d+1)}$. That is

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{Z}^q_{\mathbf{H}}(t,\mathbf{x}), \quad t>0, \mathbf{x} \in \mathbb{R}^d \\ u(0,x) &= 0, \quad \mathbf{x} \in \mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{25}$$

Here Δ is the Laplacian on $\mathbb{R}^d$ and $Z^q_{\mathbf{H}} = \{Z^q_{\mathbf{H}}(t,\mathbf{x}); t \geq 0, \mathbf{x} \in \mathbb{R}^d\}$ is the $(d+1)$-parametric Hermite sheet whose covariance is given by

$$\mathbb{E}\left\{Z^q_{\mathbf{H}}(s,\mathbf{x}) Z^q_{\mathbf{H}}(t,\mathbf{y})\right\} = R_H(t,s) R_{\mathbf{H}_0}(\mathbf{x},\mathbf{y})$$

if $\mathbf{H} = (H, H_1, .., H_d)$ and we denoted by $\mathbf{H}_0 = (H_1, ..., H_d)$. Equivalently we can write

$$\mathbb{E}\left\{\dot{Z}^q_{\mathbf{H}}(s,\mathbf{x}) \dot{Z}^q_{\mathbf{H}}(t,\mathbf{y})\right\} = H(2H-1)|t-s|^{2H-2} \prod_{i=1}^{d} (H_i(2H_i-1) \cdot |x_i - y_i|^{2H_i-2}) \tag{26}$$

Let G_1 be the fundamental solution of $u_{tt} - \Delta u = 0$. It is known that $G_1(t,\cdot)$ is a distribution in $\mathcal{S}'(\mathbb{R}^d)$ with rapid decrease, and

$$\mathcal{F} G_1(t,\cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \tag{27}$$

for any $\xi \in \mathbb{R}^d, t > 0, d \geq 1$ (see e.g. [26]). In particular,

$$\begin{aligned} G_1(t,\mathbf{x}) &= \frac{1}{2} 1_{\{|x|<t\}}, \quad \text{if } d=1 \\ G_1(t,\mathbf{x}) &= \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x|<t\}}, \quad \text{if } d=2 \\ G_1(t,\mathbf{x}) &= c_d \frac{1}{t} \sigma_t, \quad \text{if } d=3, \end{aligned}$$

where σ_t denotes the surface measure on the 3-dimensional sphere of radius t.

The *mild* solution of (25) is a square-integrable process $u = \{u(t,\mathbf{x}); t \geq 0, \mathbf{x} \in \mathbb{R}^d\}$ defined by:

$$u(t,\mathbf{x}) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s,\mathbf{x}-\mathbf{y}) Z^q_{\mathbf{H}}(ds, d\mathbf{y}). \tag{28}$$

The above integral is a Wiener integral with respect to the Hermite sheet, as introduced in Section 2.

4.1 Existence and regularity of the solution

By definition, $u(t,\mathbf{x})$ exists if and only if the stochastic integral above is well-defined, i.e. $g_{t,\mathbf{x}} := G_1(t-\cdot, \mathbf{x}-\cdot) \in \mathcal{H}$. In this case, $\mathbb{E}|u(t,x)|^2 = \|g_{t,\mathbf{x}}\|^2_{\mathcal{H}}$.

We state the result on the existence and the regularity of the solution to (25).

Proposition 4 *Let $Z^q_{\mathbf{H}}(t,\mathbf{x})$ be the $(d+1)$-parametric Hermite sheet of order q. Denote by*

$$\beta = d - \sum_{i=1}^d (2H_i - 1). \tag{29}$$

Then the following statements are true

a.- *The stochastic wave equation (25) admits an unique mild solution $(u(t,\mathbf{x}))_{t\in[0,1],\mathbf{x}\in\mathbb{R}^d}$ if and only if*

$$\sum_{i=1}^d (2H_i - 1) > d - 2H - 1. \tag{30}$$

b.- *Assume $\beta > 2H-1$ and let t_0 and $\mathbf{x} \in \mathbb{R}^d$ fixed. Then there exists positive constants c_1, c_2 such that for every $s, t \in [t_0, 1]$*

$$c_1|t-s|^{2H+1-\beta} \leq \mathbb{E}\,|u(t,\mathbf{x}) - u(s,\mathbf{x})|^2 \leq c_2|t-s|^{2H+1-\beta}.$$

Also for every fixed $\mathbf{x} \in \mathbb{R}^d$ the application

$$t \to u(t,\mathbf{x})$$

is almost surely Hölder continuous of order $\delta \in \left(0, \frac{2H+1-\beta}{2}\right)$.

c.- *Fix $t \in [t_0, T]$. Then there exist positive constants c_3, c_4 such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$*

$$c_3|\mathbf{x}-\mathbf{y}|^{2H+1-\beta} \leq \mathbb{E}\,|u(t,\mathbf{x}) - u(t,\mathbf{y})|^2 \leq c_4|\mathbf{x}-\mathbf{y}|^{2H+1-\beta}.$$

Also, for any $t \in [t_0, 1]$ the application

$$\mathbf{x} \to u(t,\mathbf{x})$$

is almost surely Hölder continuous of order $\delta \in \left(0, \left(\frac{2H+1-\beta}{2}\right) \wedge 1\right)$.

d.- *Denote by Δ the following metric on $[0,T]\times\mathbb{R}^d$*

$$\Delta\left((t,\mathbf{x});(s,\mathbf{y})\right)=|t-s|^{2H+1-\beta}+|\mathbf{x}-\mathbf{y}|^{2H+1-\beta}. \tag{31}$$

Fix $M>0$ and assume (30). For every $t,s\in[t_0,1]$ and $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$ there exist positive constants C_1,C_2 such that

$$C_1\Delta\left((t,\mathbf{x});(s,\mathbf{y})\right)\leq\mathbb{E}\left|u(t,\mathbf{x})-u(s,\mathbf{y})\right|^2\leq C_2\Delta\left((t,\mathbf{x});(s,\mathbf{y})\right). \tag{32}$$

Proof: By the isometry of the Wiener integral with respect to the Hermite sheet, the L^2 norm will be

$$\begin{aligned}\mathbb{E}u(t,\mathbf{x})^2 &= \alpha_H\int_0^t du\int_0^t dv|u-v|^{2H-2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}d\mathbf{y}d\mathbf{z}G_1(t-u,\mathbf{x}-\mathbf{y})G_1(t-v,\mathbf{x}-\mathbf{z})\\ &\quad\times\prod_{i=1}^d(H_i(2H_i-1))|x_i-y_i|^{2H_i-2}\\ &= \alpha_H\int_0^t du\int_0^t dv|u-v|^{2H-2}\int_{\mathbb{R}^d}\frac{\sin(u|\xi|)\sin(v|\xi|)}{|\xi|^2}\mu(d\xi)\end{aligned}$$

where

$$\mu(d\xi)=c_{\mathbf{H}}\prod_{i=1}^d|\xi_i|^{-(2H_i-1)} \tag{33}$$

with $\xi=(\xi_1,...,\xi_d)$. This is, $u(t,\mathbf{x})$ has the same L^2 norm as in the case $q=1$, that means, when the noise of the equation is a fractional Brownian sheet. It therefore follows from [4], Theorem 3.1 that the above integral is finite if and only if

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^{H+\frac{1}{2}}\mu(d\xi)<\infty$$

with μ given by (33). The above condition is equivalent to $\sum_{i=1}^d(2H_i-1)>d-2H-1$, see Example 3.4 in [4].

The proof of the other two items items is strongly held in the covariance structure of the Hermite sheet, which is the same as for the fractional Brownian sheet. By a carefully revision of the proofs of Theorem 3.1 in [4], Propositions 1, 2, 3 and Corollary 1 in [10], is possible to appreciate that the computations are also valid for any process with a covariance structure like the one presented in these articles, in particular our case.

- The bounds for the increments are consequence of Proposition 1 in [10], and the Hölder regularity comes from Corollary 1 in [10].
- The bounds are deduced from Proposition 2 in [10], and the space Hölder regularity is direct from Proposition 3 in [10].
- Point **d** follows from **b** and **c** by following the lines of the proof of Theorem 2 in [10].

■

4.2 Existence of local times

We will show that the solution to (25), viewed as a process in (t,x), admits a square integrable local time.

Let us define the local time of a stochastic process $(X_t)_{t\in T}$. Here T denotes a subset of $\mathbb{R}^d$. For any Borel set $I \subset T$ the occupation measure of X on I is defined as

$$\mu_I(A) = \lambda\left(t \in I, X_t \in A\right), \qquad A \in \mathcal{B}(\mathbb{R})$$

where λ denotes the Lebesque measure. If μ_I is absolutely continuous with respect to the Lebesque measure, we say that X has local time on I. The local time is defined as the Radon-Nykodim derivative of μ_I

$$L(I,x) = \frac{d\mu_I}{d\lambda}(x), \qquad x \in \mathbb{R}.$$

We will use the notation

$$L(\mathbf{t},x) := L([0,\mathbf{t}],x), \qquad \mathbf{t} \in \mathbb{R}^d_+, x \in \mathbb{R}.$$

The local time satisfies the occupation time formula

$$\int_I f(X_{\mathbf{t}})d\mathbf{t} = \int_{\mathbb{R}} f(y)L(I,y)dy \tag{34}$$

for any Borel set I in T and for any measurable function $f : \mathbb{R}^d \to \mathbb{R}$.

Proposition 5 *Let $u(t,\mathbf{x}), t \geq 0, \mathbf{x} \in \mathbb{R}^d$ be the solution to (25) and assume $\beta > 2H - 1$ where β is given by (29). Then on each set $[a,b] \times [A,B] \subset [0,\infty) \times \mathbb{R}^d$ the process $\left(u(t,x), t \geq 0, \mathbf{x} \in \mathbb{R}^d\right)$ admits a local time $(L([a,b] \times [A,B], y), y \in \mathbb{R})$ which is square integrable with respect to y*

$$\mathbb{E}\int_{\mathbb{R}} L([a,b] \times [A,B], y)^2 dy < \infty \ \textit{a.s.}\ .$$

Proof: It is well known from [7] (see also Lemma 8.1 in [29]) that, for a jointly measurable zero-mean stochastic process $X = (X(\mathbf{t}), t \in [0,\mathbf{T}])$ ($\mathbf{T}$ belongs to $\mathbb{R}^d$) with bounded variance, the condition

$$\int_{[0,\mathbf{T}]}\int_{[0,\mathbf{T}]} (\mathbb{E}[X(\mathbf{t}) - X(\mathbf{s})]^2)^{-1/2} d\mathbf{s}d\mathbf{t} < \infty$$

is sufficient for the local time of X to exist on $[0,\mathbf{T}]$ almost surely and to be square integrable as a function of u.

According to the inequality (32), for all $I = [a,b] \times [A,B]$ interval included in $[0,\infty) \times \mathbb{R}^d$ we have,

$$\int_I\int_I (\mathbb{E}(u(t,\mathbf{x}) - u(s,\mathbf{y}))^{-1/2} dtd\mathbf{x}dsd\mathbf{y} < C\int_I\int_I \left(|t-s|^{2H+1-\beta} + |\mathbf{x}-\mathbf{y}|^{2H+1-\beta}\right)^{-\frac{1}{2}} dtd\mathbf{x}dsd\mathbf{y}$$

and this is finite for $\beta > 2H - 1$. Thus almost surely the local time of u exists and is square integrable. ■

Remark 4 *It follows as a consequence of Lemma 8.1 in [29] that the local time of the solution u admits the following L^2 representation*

$$L([a,b]\times[A,B],x) = \frac{1}{2\pi}\int_{\mathbb{R}} dz e^{-izx}\int_{[a,b]\times[A,B]} dsd\mathbf{y}e^{iu(s,\mathbf{y})z}$$

for every $x\in\mathbb{R}$.

4.3 Existence of the joint density for the solution in the Rosenblatt case

It is possible to obtain the existence of the joint density of the random vector $(u(t,x),u(s,y))$ with $s\neq t$ or $\mathbf{x}\neq\mathbf{y}$ in the case when the wave equation (25) is driven by a Hermite sheet of order $q=2$ (the Rosenblatt sheet). The result is based on a criterium for the existence of densities for vectors of multiple integrals which has recently been proven in [17].

Let us state our result.

Proposition 6 *Let $u(t,x), t\geq 0, \mathbf{x}\in\mathbb{R}^d$ be the mild solution to (25). Then for every $(t,\mathbf{x})\neq(s,\mathbf{y})$, $(t,\mathbf{x}),(s,\mathbf{y})\in(0,\infty)\times\mathbb{R}^d$, the random vector*

$$(u(t,\mathbf{x}),u(s,\mathbf{y}))$$

admits a density.

Proof: Note that for every $t\geq 0$ and $x\in\mathbb{R}^d$, the random variable $u(t,\mathbf{x})$ is a multiple integral of order 2 with respect to the d-parametric Brownian sheet. A result present in [17] states that a two-dimensional vector of multiple integrals of order 2 admits a density if and only if the determinant of the covariance matrix is strictly positive. Denote by $C(t,s,\mathbf{x},\mathbf{y})$ the covariance matrix of $(u(t,\mathbf{x}),u(s,\mathbf{y}))$. The determinant of this matrix is the same for every $q\geq 1$, from the covariance structure of the Hermite sheet. It is clear that for $q=1$ obviously $\det C(t,s,\mathbf{x},\mathbf{y})$ is strictly positive, since the vector $(u(t,\mathbf{x}),u(s,\mathbf{y}))$ is a Gaussian vector and hence admits a density when $(t,\mathbf{x})\neq(s,\mathbf{y})$. This implies that $\det C(t,s,\mathbf{x},\mathbf{y})$ is also strictly positive for $q=2$ and so the vector $(u(t,\mathbf{x}),u(s,\mathbf{y}))$ admits a density also for $q=2$. ■

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