Exponential and the lack of exponential stability in transmission problems with localized Kelvin-Voigt dissipation

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Abstract

In this paper we consider the transmission problem of a material composed by three components, one of them is a Kelvin-Voigt viscoelastic material, the second is an elastic material (no dissipation) and the third is an elastic material inserted with a frictional damping mechanism. The main result of this paper is that the rate of decay will depend of the position of each component. When the viscoelastic component is not in the middle of the material, then there exists exponential stability of the solution. Instead, when the viscoelastic part is in the middle of the material, then there is not exponential stability. In this case we show that the decay is polynomial as $1/t^2$. Moreover we show that the rate of decay is optimal over the domain of the infinitesimal generator. Finally using a second order scheme that ensures the decay of energy (Newmark-β method), we give some numerical examples which demonstrate these asymptotic behavior.

1 Introduction

The wave equation with localized frictional damping was studied by several authors and by now it is very well known that the semigroup defined by this equation is exponentially stable no matter the size nor the location of the subinterval where the damping mechanism is effective. See for example [6, 7, 8, 11, 14, 15, 16, 20] to quote such a few.

K. Liu and Z. Liu in [10] proved a similar result to the Euler Bernoulli beam equation with localized Kelvin-Voigt damping. That is to say, no matter the size nor the position of the damping mechanism is effective, the semigroup defined by the solution of the model is

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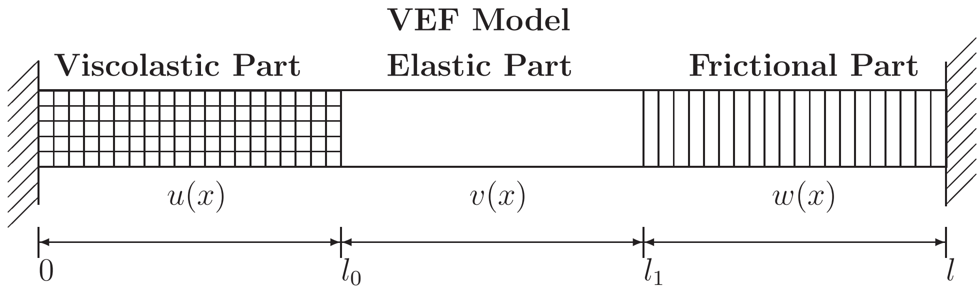
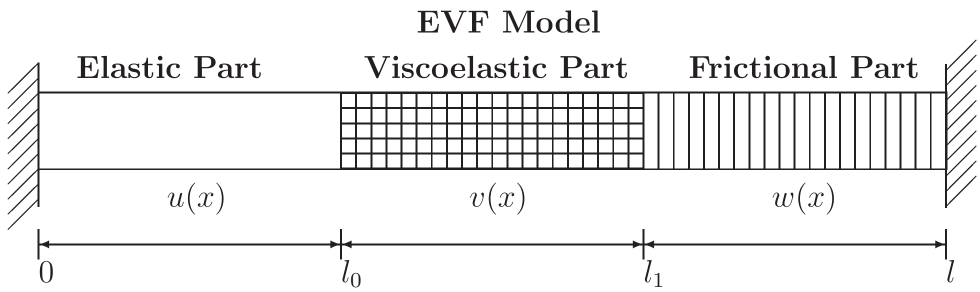
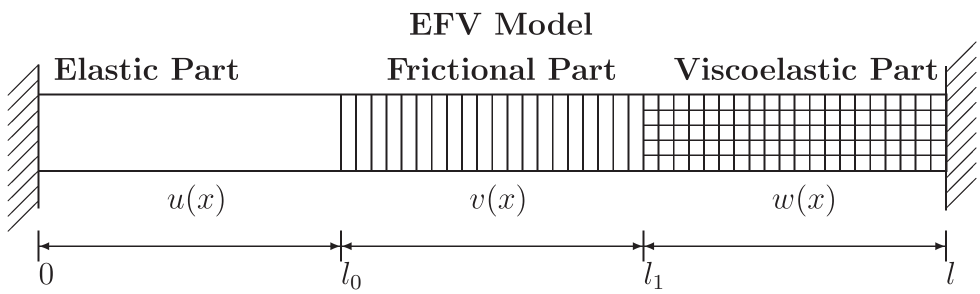
always exponentially stable. Under the light of this result one can arrive to the conclusion that the semigroup defined by the solution of the wave equations with localized Kelvin-Voigt damping is also exponentially stable. This is clearly not true as proved in [10]. That is localized Kelvin-Voigt damping does not produce exponential stability.

In this paper we consider the transmission problem with localized viscoelasticity of Kelvin-Voigt type. Here we consider a beam composed by three different components, one of them is of viscoelastic type, the other is only an elastic part and finally the third component of elastic type with a frictional damping mechanism. The main result of this paper is that the position of this component (optimal design) plays an important role in the study of the stabilization. For example if we consider a beam of the forms given below







The longitudinal displacement ν is divided into two parts

$$\nu = \begin{cases} u(x) & \text{if} \quad x \in]0, l_0[\\ v(x) & \text{if} \quad x \in]l_0, l_1[\\ w(x) & \text{if} \quad x \in]l_1, l[\end{cases}$$

where each component u, v and w, represents the displacement of the first, second and third component of the beam, respectively. There exist six possible combinations of the material. Two possibilities occur when the elastic part is at the center of the material. Other two possibilities when the viscous part is in the middle of the beam, and finally when the elastic part with frictional mechanics is at the center of the beam. Performing the change of varible $s = l - x$ this six posibilitites can be reduced to three. We refer to each model as **VEF**, **EVF** and **EFV**. The **VEF** model is given by

$$\rho_1 u_{tt} - \kappa_1 u_{xx} - \kappa_0 u_{xxt} = 0 \quad \text{in} \quad]0, l_0[\times]0, \infty[, \tag{1.1}$$

$$\rho_2 v_{tt} - \kappa_2 v_{xx} = 0 \quad \text{in} \quad]l_0, l_1[\times]0, \infty[, \tag{1.2}$$

$$\rho_3 w_{tt} - \kappa_3 w_{xx} + \gamma w_t = 0 \quad \text{in} \quad]l_1, l[\times]0, \infty[. \tag{1.3}$$

where κ_0, κ_1, κ_2 and κ_3 are elastic positive constants, and ρ_1, ρ_2 stands for the mass density functions. The transmission conditions are given by

$$u(l_0, t) = v(l_0, t), \quad \kappa_1 u_x(l_0, t) + \kappa_0 u_{xt}(l_0, t) = \kappa_2 v_x(l_0, t), \quad t \geqslant 0, \tag{1.4}$$

$$v(l_1, t) = w(l_1, t), \quad \kappa_2 v_x(l_0, t) = \kappa_3 w_x(l_0, t), \quad t \geqslant 0. \tag{1.5}$$

The boundary conditions

$$u(0, t) = 0, \quad w(l, t) = 0, \quad t \geqslant 0, \tag{1.6}$$

and the initial data

$$\begin{aligned} u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad]0, l_0[, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x) \quad \text{in} \quad]l_0, l_1[, \\ w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x) \quad \text{in} \quad]l_1, l[. \end{aligned} \tag{1.7}$$

Instead, the **EVF** model is given by

$$\rho_1 u_{tt} - \kappa_1 u_{xx} = 0 \quad \text{in} \quad]0, l_0[\times]0, \infty[, \tag{1.8}$$

$$\rho_2 v_{tt} - \kappa_2 v_{xx} - \kappa_0 v_{xxt} = 0 \quad \text{in} \quad]l_0, l_1[\times]0, \infty[, \tag{1.9}$$

$$\rho_3 w_{tt} - \kappa_3 w_{xx} + \gamma w_t = 0 \quad \text{in} \quad]l_1, l[\times]0, \infty[. \tag{1.10}$$

The transmission conditions are given by

$$u(l_0,\,t)=v(l_0,\,t),\quad \kappa_1\,u_x(l_0,\,t)=\kappa_2\,v_x(l_0,\,t)+\kappa_0\,v_{xt}(l_0,\,t),\quad t\geqslant 0, \tag{1.11}$$

$$v(l_1,\,t)=w(l_1,\,t),\quad \kappa_2\,v_x(l_1,\,t)+\kappa_0\,v_{xt}(l_1,\,t)=\kappa_3\,w_x(l_1,\,t),\quad t\geqslant 0, \tag{1.12}$$

with the same boundary condition and initial data (1.6)-(1.7). Finally, we consider the **EFV** model

$$\rho_1\,u_{tt}-\kappa_1\,u_{xx} \;=\; 0 \quad \text{in} \quad]0,\,l_0[\times]0,\,\infty[, \tag{1.13}$$

$$\rho_2\,v_{tt}-\kappa_2\,v_{xx}+\gamma\,v_t \;=\; 0 \quad \text{in} \quad]l_0,\,l_1[\times]0,\,\infty[, \tag{1.14}$$

$$\rho_3\,v_{tt}-\kappa_3\,w_{xx}-\kappa_0\,w_{xxt} \;=\; 0 \quad \text{in} \quad]l_1,\,l[\times]0,\,\infty[. \tag{1.15}$$

The transmission conditions are given by

$$u(l_0,\,t)=v(l_0,\,t),\quad \kappa_1\,u_x(l_0,\,t)=\kappa_2\,v_x(l_0,\,t),\quad t\geqslant 0, \tag{1.16}$$

$$v(l_1,\,t)=w(l_1,\,t),\quad \kappa_2\,v_x(l_1,\,t)=\kappa_3\,w_x(l_1,\,t)+\kappa_0\,w_{xt}(l_1,\,t)\qquad t\geqslant 0, \tag{1.17}$$

with the same boundary condition and initial data (1.6)-(1.7).
The main result of this paper is to show that the solutions of the above models are exponentially stable if and only if the viscous part is not at the center of the beam. Otherwise, the model is not exponentially stable. In this later case we will show that the solution decays to zero polynomially as t^{-2}. Moreover we prove that the rate of decay is optimal.

Our main tool to prove the exponential stability and the lack of exponential stability is a result due to Prüss [19]

Theorem 1.1 *Let $(\mathcal{S}(t))_{t\geqslant 0}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$ generated by $\mathcal{A}$. Then the semigroup is exponentially stable is and only if*

$$i\,\mathbb{R}\subset\varrho(\mathcal{A}),\quad \textit{and}\quad \|(i\,\lambda\,I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}\leqslant C,\qquad \forall\,\lambda\in\mathbb{R}.$$

To show the polynomial decay and the optimality we use a result due to Borichev and Tomilov [5].

Theorem 1.2 *Let $(\mathcal{S}(t))_{t\geqslant 0}$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$ such that $i\,\mathbb{R}\subset\varrho(\mathcal{A})$. Then*

$$\frac{1}{|\lambda|^{\alpha}}\,\|(i\,\lambda\,I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}\leqslant C,\quad \forall\,\lambda\in\mathbb{R}\quad\Leftrightarrow\quad \|\mathcal{S}(t)\mathcal{A}^{-1}\|_{\mathcal{D}(\mathcal{A})}\leqslant\frac{C}{t^{1/\alpha}}.$$

The remaining part of this paper is organized as follows. In Section 2 we show that the corresponding models are well possed. In Section 3 we show that the corresponding semigroup is exponentially stable provided that the viscous component is not in the middle of the beam. In Section 4 we consider the case when the viscous component is in the middle of the beam and we prove that there is a lack of exponential stability. Finally, in Section 5 we prove that when the system is not exponentially stable then the semigroup decays polynomially to zero as t^{-2}. Moreover we show that the rate of decay is optimal for any initial data belonging to $\mathcal{D}(\mathcal{A})$.

2 The Semigroup approach

The aim of this section is to prove the existence and uniqueness of solutions of the **VEF** problem. Let us denote by

$$\mathbb{H}^m = H^m(0,\, l_0)\times H^m(l_0,\, l_1)\times H^m(l_1,\, l), \qquad \mathbb{L}^2 = L^2(0,\, l_0)\times L^2(l_0,\, l_1)\times L^2(l_1,\, l)$$

$$\mathbb{H}^1_l = \left\{(u,\, v,\, w)\in \mathbb{H}^1 : \quad u(0)=w(l)=0, \quad u(l_0)=v(l_0), \quad v(l_1)=w(l_1)\right\}.$$

Under the above conditions we have that the phase space is given by

$$\mathcal{H} = \mathbb{H}^1_l \times \mathbb{L}^2.$$

Denoting by

$$Z_i = (u_i,\, v_i,\, w_i,\, U_i,\, V_i,\, W_i)$$

where $i = 1,\, 2$. Note that this space equipped with the inner product

$$\begin{aligned}\langle Z_1,\, Z_2\rangle_{\mathcal{H}} &= \int_0^{l_0}\left(\rho_1\, U_1\, \overline{U}_2 + \kappa_1\, u_{1,x}\, \overline{u}_{2,x}\right) dx + \int_{l_0}^{l_1}\left(\rho_2\, V_1\, \overline{V}_2 + \kappa_2\, v_{1,x}\, \overline{v}_{2,x}\right) dx \\ &\quad + \int_{l_1}^{l}\left(\rho_3\, W_1\, \overline{W}_2 + \kappa_3\, w_{1,x}\, \overline{w}_{2,x}\right) dx\end{aligned}$$

is a Hilbert space. We also consider the linear operator $\mathcal{A}_i : \mathcal{D}(\mathcal{A}_i)\subset \mathcal{H}\to\mathcal{H}$ for $i = 1,\, 2,\, 3$. Denoting by $\Phi = (u,\, v,\, w,\, U,\, V,\, W)^t$, we define

$$\mathcal{A}_1\Phi = \begin{pmatrix} U \\ V \\ W \\ \frac{1}{\rho_1}\left(\kappa_1\, u_{xx} + \kappa_0\, U_{xx}\right) \\ \frac{\kappa_2}{\rho_2}\, v_{xx} \\ \frac{\kappa_3}{\rho_3}\, w_{xx} - \frac{\gamma}{\rho_3} W \end{pmatrix}, \qquad \mathcal{A}_2\Phi = \begin{pmatrix} U \\ V \\ W \\ \frac{\kappa_1}{\rho_1}\, u_{xx} \\ \frac{1}{\rho_2}\left(\kappa_2\, v_{xx} + \kappa_0\, V_{xx}\right) \\ \frac{\kappa_3}{\rho_3}\, w_{xx} - \frac{\gamma}{\rho_3} W \end{pmatrix},$$

$$
\mathcal{A}_3\Phi = \begin{pmatrix} U \\ V \\ W \\ \frac{\kappa_1}{\rho_1}\, u_{xx} \\ \frac{\kappa_2}{\rho_2}\, v_{xx} - \frac{\gamma}{\rho_2} V \\ \frac{1}{\rho_3}\,(\kappa_3\, w_{xx} + \kappa_0\, W_{xx}) \end{pmatrix},
$$

whose domain $\mathcal{D}(\mathcal{A}_i)$ is given by

$$
\mathcal{D}(\mathcal{A}_1) = \big\{\Phi \in \mathcal{H} :\ (U,\, V,\, W) \in \mathbb{H}_l^1, \quad (\kappa_1\, u + \kappa_0\, \eta,\, v,\, w) \in \mathbb{H}^2, \tag{2.1}
$$

$$
\kappa_1\, u_x(l_0) + \kappa_0\, \eta_x(l_0) = \kappa_2\, v_x(l_0), \quad \kappa_2\, v_x(l_1) = \kappa_3\, w_x(l_1)\big\} \tag{2.2}
$$

$$
\mathcal{D}(\mathcal{A}_2) = \big\{\Phi \in \mathcal{H} :\ (U,\, V,\, W) \in \mathbb{H}_l^1, \quad (u,\, \kappa_2\, v + \kappa_0\, V,\, w) \in \mathbb{H}^2, \tag{2.3}
$$

$$
\kappa_1\, u_x(l_0) = \kappa_2\, v_x(l_0) + \kappa_0\, V_x(l_0), \quad \kappa_2\, v_x(l_1) + \kappa_0\, V(l_1) = \kappa_3\, w_x(l_1)\big\}\,. \tag{2.4}
$$

$$
\mathcal{D}(\mathcal{A}_3) = \big\{\Phi \in \mathcal{H} :\ (U,\, V,\, W) \in \mathbb{H}_l^1, \quad (u,\, v,\, \kappa_3\, w + \kappa_0\, W) \in \mathbb{H}^2, \tag{2.5}
$$

$$
\kappa_1\, u_x(l_0) = \kappa_2\, v_x(l_0), \quad \kappa_2\, v_x(l_1) = \kappa_3\, w_x(l_1) + \kappa_0\, W_x(l_1)\big\}\,. \tag{2.6}
$$

Using $u_t = U,\ v_t = V,$ and $w_t = W$, the system (1.1)-(1.7), (1.8)-(1.12) and (1.13)-(1.16), can be reduced to the following abstract initial value problem for a first-order evolution equation

$$
\frac{d}{dt}\Phi(t) = \mathcal{A}\,\Phi(t), \qquad \Phi(0) = \Phi_0, \qquad \forall\, t > 0,
$$

with $\Phi(t) = (u,\, v,\, w,\, u_t,\, v_t,\, w_t)^T$ and $\Phi_0 = (u_0,\, v_0,\, w_0,\, u_1,\, v_1,\, w_1)^T$. Next, we show that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions over $\mathcal{H}$.

Proposition 2.1 *The operator $\mathcal{A}$ generates a C_0-semigroup $(\mathcal{S}_{\mathcal{A}}(t))_{t\geqslant 0}$ of contractions on the space $\mathcal{H}$.*

Proof. We will show that $\mathcal{A}$ is a dissipative operator and that $0 \in \varrho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. Then our conclusion will follow using the well known Lumer-Phillips theorem (see [18]). We observe that if $\Phi \in \mathcal{D}(\mathcal{A}_1)$, then

$$
\begin{aligned}
\langle \mathcal{A}_1\Phi,\, \Phi\rangle_{\mathcal{H}} &= \kappa_1 \int_0^{l_0} U_x\, \overline{u}_x\, dx + \kappa_2 \int_{l_0}^{l_1} V_x\, \overline{v}_x\, dx + \kappa_3 \int_{l_1}^{l} W_x\, \overline{w}_x\, dx \\
&\quad + \int_0^{l_0} (\kappa_1\, u + \kappa_0\, U)_{xx}\, \overline{U}\, dx + \kappa_2 \int_{l_0}^{l_1} v_{xx}\, \overline{V}\, dx + \int_{l_1}^{l} (\kappa_3\, w_{xx} - \gamma\, W)\, \overline{W}\, dx.
\end{aligned}
$$

Integrating by parts and performing straightforward calculations we obtain

$$Re\,\langle \mathcal{A}_1\,\Phi,\,\Phi\rangle_{\mathcal{H}} \;=\; -\,\kappa_0\int_0^{l_0}|U_x|^2\,dx-\gamma\int_{l_1}^{l}|W|^2\,dx. \tag{2.7}$$

Similarly we have that

$$Re\,\langle \mathcal{A}_2\,\Phi,\,\Phi\rangle_{\mathcal{H}} \;=\; -\,\kappa_0\int_{l_0}^{l_1}|V_x|^2\,dx-\gamma\int_{l_1}^{l}|W|^2\,dx. \tag{2.8}$$

$$Re\,\langle \mathcal{A}_3\,\Phi,\,\Phi\rangle_{\mathcal{H}} \;=\; -\,\gamma\int_{l_0}^{l_1}|V|^2\,dx-\kappa_0\int_{l_1}^{l}|W_x|^2\,dx. \tag{2.9}$$

Hence, $\mathcal{A}_i$ is a dissipative operator. To show that $0\in\varrho(\mathcal{A}_i)$ let us take $F\in\mathcal{H}$. We will show that there exists a unique Φ in $\mathcal{D}(\mathcal{A}_i)$ such that $\mathcal{A}_i\,\Phi=F$, that is,

$$-\,U \;=\; f_1 \tag{2.10}$$

$$-\,\kappa_1\,u_{xx}-\kappa_2\,U_{xx} \;=\; \rho_1\,f_2 \tag{2.11}$$

$$-\,V \;=\; f_3 \tag{2.12}$$

$$-\,\kappa_2\,v_{xx} \;=\; \rho_2\,f_4 \tag{2.13}$$

$$-\,W \;=\; f_5 \tag{2.14}$$

$$-\,\kappa_3\,w_{xx}+\gamma\,W \;=\; \rho_3\,f_6. \tag{2.15}$$

Substituting (2.10) into (2.11) and (2.14) into (2.15) yields

$$-\,\kappa_1\,u_{xx} \;=\; \rho_1\,f_2+\kappa_2\,f_{1,\,xx} \tag{2.16}$$

$$-\,\kappa_3\,v_{xx} \;=\; \rho_2\,f_4 \tag{2.17}$$

$$-\,\kappa_4\,w_{xx} \;=\; \rho_3\,f_6+\gamma\,f_5, \tag{2.18}$$

verifying

$$u(l_0)=v(l_0),\quad \kappa_1\,u_x(l_0)-\kappa_0\,f_{1,\,x}(l_0)=\kappa_2\,v_x(l_0). \tag{2.19}$$

$$v(l_1)=w(l_1),\quad \kappa_3\,v_x(l_1)=\kappa_4\,w_x(l_1), \tag{2.20}$$

with the following boundary conditions.

$$u(0)=0,\quad w(l)=0,\qquad t\geqslant 0. \tag{2.21}$$

A standard procedure shows that the transmission problem (2.10)-(2.21) is well posed. Therefore, we conclude that $0\in\varrho(\mathcal{A}_i)$.

From Proposition 2.1 we can state the following result ([18])

Theorem 2.2 *For any* $\Phi_0 \in \mathcal{H}$ *there exists a unique solution* $\Phi(t) = (u,\, v,\, w,\, u_t,\, v_t,\, w_t)$ *of the* **VEF**, **EVF** *and* **EFV** *models satisfying*

$$(u,\, v,\, w,\, u_t,\, v_t,\, w_t) \in C([0,\, \infty[:\, \mathbb{H}^1_l \times \mathbb{L}^2).$$

If $\Phi_0 \in \mathcal{D}(\mathcal{A}_i)$, *then*

$$(u,\, v,\, w,\, u_t,\, v_t,\, w_t) \in C^1([0,\, \infty[:\, \mathbb{H}^1_l \times \mathbb{L}^2) \cap C([0,\, \infty[:\, \mathcal{D}(\mathcal{A}_i)).$$

3 The exponential stability

In this section we prove that the exponential stability of the semigroup associated to the transmission problem provided that the viscous part is not in the middle of the beam. This means that the **VEF**, **EFV**, **VFE**, **FEV** models, are exponentially stable. Since the proofs are similar we only consider in this section the **VEF** case. The corresponding resolvent equations are given by

$$i\,\lambda\,\Phi - \mathcal{A}_1\,\Phi = F. \tag{3.1}$$

and in terms of its components are given by

$$\begin{aligned}
i\,\lambda\,u - U &= f_1 \quad \text{in} \quad]0,\, l_0[, && (3.2)\\
i\,\lambda\,\rho_1\,U - \kappa_1\,u_{xx} - \kappa_0\,U_{xx} &= \rho_1\,f_2 \quad \text{in} \quad]0,\, l_0[, && (3.3)\\
i\,\lambda\,v - V &= f_3 \quad \text{in} \quad]l_0,\, l_1[, && (3.4)\\
i\,\lambda\,\rho_2\,V - \kappa_2\,v_{xx} &= \rho_2\,f_4 \quad \text{in} \quad]l_0,\, l_1[, && (3.5)\\
i\,\lambda\,w - W &= f_5 \quad \text{in} \quad]l_1,\, l[, && (3.6)\\
i\,\lambda\,\rho_3\,W - \kappa_3\,w_{xx} + \gamma\,W &= \rho_3\,f_6 \quad \text{in} \quad]l_1,\, l[, && (3.7)
\end{aligned}$$

with the following, transmission condition,

$$u(l_0) = v(l_0), \quad \kappa_1\,u_x(l_0) + \kappa_0\,U_x(l_0) = \kappa_2\,v_x(l_0). \tag{3.8}$$

$$v(l_1) = w(l_1), \quad \kappa_2\,v_x(l_1) = \kappa_3\,w_x(l_1), \tag{3.9}$$

and boundary condition

$$u(0) = 0, \quad w(l) = 0. \tag{3.10}$$

Note that the model is dissipative, multiplying equation (3.1) by Φ and using (2.7) we have that

$$\kappa_0 \int_0^{l_0} |U_x|^2\, dx + \gamma \int_{l_1}^{l} |W|^2\, dx \leqslant \|\Phi\|\,\|F\|. \tag{3.11}$$

The following Lemma will play an important role in what follows.

Lemma 3.1 *Any strong solution of the system*

$$i\,\lambda\,\psi - \Psi \;=\; f_1 \quad \text{in} \quad]a,\, b[, \tag{3.12}$$

$$i\,\lambda\,\rho\,\Psi - \kappa\,\psi_{xx} + \gamma\,\Psi \;=\; f_2 \quad \text{in} \quad]a,\, b[, \tag{3.13}$$

verifies

$$|\psi_x(a)|^2 + |\Psi(a)|^2 + |\psi_x(b)|^2 + |\Psi(b)|^2 \leqslant C \int_{l_0}^{l_1} \left(|\Psi|^2 + |\psi_x|^2\right) dx + C\,\|Z\|\,\|F\|. \tag{3.14}$$

$$\int_{l_0}^{l_1} \left(|\Psi|^2 + |\psi_x|^2\right) dx \leqslant C \left(|\psi_x(a)|^2 + |\Psi(a)|^2\right) + C\,\|Z\|\,\|F\|. \tag{3.15}$$

$$\int_{l_0}^{l_1} \left(|\Psi|^2 + |\psi_x|^2\right) dx \leqslant C\, \left(|\psi_x(b)|^2 + |\Psi(b)|^2\right) + C\,\|Z\|\,\|F\|, \tag{3.16}$$

where $Z = (\psi,\, \Psi)$ *and* $F = (f_1,\, f_2)$.

Proof. Multiplying equation (3.13) by $\left(x - \frac{a+b}{2}\right)\overline{\psi}_x$, taking real part, using integration by parts and using equation (3.12) our conclusion follows. To get inequalities (3.15) and (3.16) we multiply the equation (3.13) by $(x-b)\,\overline{\psi}_x$ and $(x-a)\,\overline{\psi}_x$ respectively.

Theorem 3.2 *The semigroup associated to the transmission problem decays exponentially as time goes to infinity provided that the viscous component is not in the middle of the beam.*

Proof. Note that $\mathcal{A}$ is a closed operator, such that $D(\mathcal{A})$ has compact embedding over the phase space $\mathcal{H}$. Therefore the spectrum set of $\mathcal{A}$ denoted as $\sigma(\mathcal{A})$, consist only of eigenvalues. Thus, to prove that the imaginary axes is contained in the resovent set of $\mathcal{A}$ it is enough to prove that there is not imaginary eigenvalues. To see that let us reasoning by contradiction. Let us suppose that there exists an imaginary eigen value $i\lambda$, with $\lambda \in \mathbb{R}$ such that $i\lambda\Phi - \mathcal{A}\Phi = 0$. Using relation (3.11) for $F = 0$ we get $W = U = 0$ which implies that $u = w = 0$. From (3.4)–(3.5) we have that

$$-\,\lambda^2\,\rho_2\,v - \kappa_2\,v_{xx} = 0$$

Satisfying

$$v(l_0) = v(l_1) = 0, \qquad v_x(l_0) = v_x(l_1) = 0$$

Bacause $u = w = 0$. Considering the above problem as an initial value problem (at $x = l_0$ or $x = l_1$) we conclude that $v = 0$. Therefore we get that $\Phi = 0$. This is contradictory, therefore is not possible that there exists imaginary eigenvalues. Thus, $i\mathbb{R} \subset \varrho(\mathcal{A})$.

Finally, let us prove that the resolvent operator is uniformly bounded over the imaginary axes. Multiplying equation (3.7) by $\overline{w}$ we get

$$i\,\lambda\,\rho_3 \int_{l_1}^{l} W\,\overline{w}\,dx - \kappa_3 \int_{l_1}^{l} w_{xx}\,\overline{w}\,dx + \gamma \int_{l_1}^{l} W\,\overline{w}\,dx = \rho_3 \int_{l_1}^{l} f_6\,\overline{w}\,dx.$$

It follows that

$$\begin{aligned}\kappa_3 \int_{l_1}^{l} |w_x|^2\,dx &\leqslant \operatorname{Re}\kappa_3\,w_x(l_1)\,\overline{w}(l_1) + \rho_3 \int_{l_1}^{l} |W|^2\,dx + \gamma\operatorname{Re}\int_{l_1}^{l} W\,\overline{w}\,dx + \rho_3 \operatorname{Re}\int_{l_1}^{l} f_6\,\overline{w}\,dx \\ &\leqslant \kappa_3 \operatorname{Re} w_x(l_1)\,\overline{w}(l_1) + C \int_{l_1}^{l} |W|^2\,dx + C\,\|\Phi\|\,\|F\|. \end{aligned} \tag{3.17}$$

Note that

$$w_x(l_1)\,\overline{w}(l_1) = -\,\frac{1}{i\,\lambda}\,w_x(l_1)\,\overline{i\,\lambda\,w}(l_1) = -\,\frac{1}{i\,\lambda}\,w_x(l_1)\,[\overline{W(l_1) + f_1(l_1)}].$$

Using Lemma 3.1 we get

$$\begin{aligned}|w_x(l_1)\,\overline{w}(l_1)| &\leqslant \frac{1}{|\lambda|}\,|w_x(l_1)|\,|W(l_1)| + \frac{1}{|\lambda|}\,|w_x(l_1)\,f_1(l_1)| \\ &\leqslant \frac{C}{|\lambda|}\int_{l_1}^{l} \left(|w_x|^2 + |W|^2\right)dx + \frac{C}{|\lambda|}\,\|\Phi\|\,\|F\|.\end{aligned}$$

Substitution of this inequality into (3.17) yields

$$\kappa_3 \int_{l_1}^{l} |w_x|^2\,dx \;\leqslant\; C \int_{l_1}^{l} |W|^2\,dx + C\,\|\Phi\|\,\|F\|,$$

provided λ is large enough. From inequality (3.11) we get

$$\kappa_3 \int_{l_1}^{l} |w_x|^2\,dx \;\leqslant\; C\,\|\Phi\|\,\|F\|,$$

which implies

$$\int_{l_1}^{l} \left(|W|^2 + |w_x|^2\right)dx \;\leqslant\; C\,\|\Phi\|\,\|F\|.$$

Using inequality (3.16) from Lemma 3.1 to v we get

$$\int_{l_0}^{l_1} \left(|V|^2 + |v_x|^2\right) dx \;\; \leqslant \;\; C\left(|V(l_1)|^2 + |v_x(l_1)|^2\right) + C\,\|\Phi\|\,\|F\|.$$

From the transmission conditions we get

$$\int_{l_0}^{l_1} \left(|V|^2 + |v_x|^2\right) dx \;\; \leqslant \;\; C\left(|W(l_1)|^2 + |w_x(l_1)|^2\right) + C\,\|\Phi\|\,\|F\|.$$

Using Lemma 3.1 once more we get

$$\int_{l_0}^{l_1} \left(|V|^2 + |v_x|^2\right) dx \;\; \leqslant \;\; C\,\|\Phi\|\,\|F\|.$$

Multiplying equations (3.3), (3.5), (3.7) by $\overline{u},\ \overline{v},\ \overline{w}$ and summing up the product result and using the transmission conditions we get

$$\begin{aligned} &\kappa_1 \int_0^{l_0} |u_x|^2\, dx + \kappa_2 \int_{l_0}^{l_1} |v_x|^2\, dx + \kappa_3 \int_{l_1}^{l} |w_x|^2\, dx \\ \leqslant\;\; & C \int_0^{l_0} |U_x|^2\, dx + C \int_{l_0}^{l_1} |V|^2\, dx + C \int_{l_1}^{l} |W|^2\, dx + C\,\|\Phi\|\,\|F\| \\ \leqslant\;\; & C\,\|\Phi\|\,\|F\|. \end{aligned}$$

From the above inequalities we get

$$\|\Phi\|^2 \leqslant C\,\|\Phi\|\,\|F\|,$$

which implies the exponential decay.

4 The lack of exponential stability EVF, FVE

In this section we show that the semigroup associated to the **EVF**, **FVE** models are not exponentially stable. Since the **FVE** model, can be obtained from **EVF** by making the change of variable $\sigma = l - x,$ it is enough to show the result to the **EVF** model. In fact, the resolvent system associated to model **EVF** is given by

$$i\,\lambda\,\Phi - \mathcal{A}_2\,\Phi = F, \tag{4.1}$$

which in terms of its components is given by

$$i\,\lambda\,u - U = f_1 \tag{4.2}$$

$$i\,\lambda\,\rho_1\,U - \kappa_1\,u_{xx} = \rho_1\,f_2 \tag{4.3}$$

$$i\,\lambda\,v - V = f_3 \tag{4.4}$$

$$i\,\lambda\,\rho_2\,V - \kappa_2\,v_{xx} - \kappa_0\,V_{xx} = \rho_2\,f_4 \tag{4.5}$$

$$i\,\lambda\,w - W = f_5 \tag{4.6}$$

$$i\,\lambda\,\rho_3\,W - \kappa_3\,w_{xx} + \gamma\,W = \rho_3\,f_6, \tag{4.7}$$

with transmission condition

$$u(l_0) = v(l_0), \quad \kappa_1\,u_x(l_0) = \kappa_2\,v_x(l_0) + \kappa_0\,V_x(l_0), \tag{4.8}$$

$$v(l_1) = w(l_1), \quad \kappa_2\,v_x(l_1) + \kappa_0\,V_x(l_1) = \kappa_3\,w_x(l_1), \tag{4.9}$$

and boundary condition.

$$u(0) = 0, \quad w(l) = 0. \tag{4.10}$$

Here we will show that the **EVF** partial viscoelastic model is not exponentially stable. To do this we will consider the functions

$$f_1 = f_3 = f_4 = f_5 = f_6 = 0, \quad \rho_2\,f_2 = q.$$

Therefore, the system (4.2)-(4.7) can be written as

$$\begin{aligned} -\lambda^2\,\rho_1\,u - \kappa_1\,u_{xx} &= q \\ -\,\lambda^2\,\rho_2\,v - \kappa_2\,v_{xx} - i\,\kappa_0\,\lambda\,v_{xx} &= 0 \\ -\,\lambda^2\,\rho_3\,w - \kappa_3\,w_{xx} + i\,\gamma\,\lambda\,w &= 0. \end{aligned}$$

Rewriting the system

$$\begin{aligned} u_{xx} + \alpha^2\,u &= -\,q \\ v_{xx} + \beta^2\,v &= 0 \\ w_{xx} + \sigma^2\,w &= 0, \end{aligned}$$

where

$$\alpha^2 = \frac{\rho_1}{\kappa_1}\,\lambda^2, \qquad \beta^2 = \frac{\rho_2}{\kappa_2 + i\,\lambda\,\kappa_0}\,\lambda^2, \qquad \sigma^2 = \frac{\rho_3\,\lambda^2 - i\,\lambda\,\gamma}{\kappa_3}.$$

Note that

$$u(x) = u(l_0)\,\frac{\sin(\alpha\,x)}{\sin(\alpha\,l_0)} + \frac{\sin(\alpha\,x)}{\alpha\sin(\alpha\,l_0)}\int_0^{l_0} q(s)\,\sin(\alpha\,(l_0 - s))\,ds - \frac{1}{\alpha}\int_0^{x} q(s)\,\sin(\alpha\,(x - s))\,ds.$$

$$v(x) = u(l_0)\,\frac{\sinh(\beta\,x)}{\sinh(\beta\,l_0)} - \left(u(l_0)\,\frac{\sinh(\beta\,l_1)}{\sinh(\beta\,l_0)} - v(l_1)\right)\frac{\sinh(\beta\,(x - l_0))}{\sinh(\beta\,(l_1 - l_0))}.$$

$$w(x) = v(l_1)\,\frac{\sinh(\sigma\,(x - l))}{\sinh(\sigma\,(l_1 - l))}.$$

Using the transmission condition $(\kappa_2 + i\,\lambda\,\kappa_0)\,v_x(l_1) = \kappa_3\,w_x(l_1)$ we get

$$\begin{aligned} & \beta\,u(l_0)\,\frac{\cosh(\beta\,l_1)}{\sinh(\beta\,l_0)} - \beta\,\left(u(l_0)\,\frac{\sinh(\beta\,l_1)}{\sinh(\beta\,l_0)} - v(l_1)\right)\coth(\beta\,(l_1 - l_0)) \\ = \; & v(l_1)\,\frac{\kappa_3\,\sigma}{\kappa_2 + i\,\lambda\,\kappa_0}\,\coth(\sigma\,(l_1 - l_0)). \end{aligned}$$

It follows that

$$\begin{aligned} & \beta\,u(l_0)\,\frac{\cosh(\beta\,l_1) - \sinh(\beta\,l_1)\,\coth(\beta\,(l_1 - l_0))}{\sinh(\beta\,l_0)} \\ = \; & v(l_1)\,\left[\frac{\kappa_3\,\sigma\,\coth(\sigma\,(l_1 - l_0))}{\kappa_2 + i\,\lambda\,\kappa_0} - \beta\,\coth(\beta\,(l_1 - l_0))\right]. \end{aligned}$$

From where it follows that

$$-\frac{\beta\,u(l_0)}{\sinh(\beta\,(l_1 - l_0))} = v(l_1)\,\left[\frac{\kappa_3\,\sigma\,\coth(\sigma\,(l_1 - l_0))}{\kappa_2 + i\,\lambda\,\kappa_0} - \beta\,\coth(\beta\,(l_1 - l_0))\right].$$

$$u(l_0) = h(\lambda)\,v(l_1),$$

where

$$h(\lambda) = -\left[\frac{\kappa_3\,\sigma\,\coth(\sigma\,(l_1 - l_0))}{\beta\,(\kappa_2 + i\,\lambda\,\kappa_0)} - \coth(\beta\,(l_1 - l_0))\right]\sinh(\beta\,(l_1 - l_0)).$$

Note that

$$\frac{1}{|h(\lambda)|} \approx \frac{c_0}{|\sinh(\beta\,(l_1 - l_0))|} \to \infty$$

as $|\beta| \to \infty$, and $c_0 > 0$. Using $\kappa_1\,u_x(l_0) = \kappa_2\,v_x(l_0) + i\,\kappa_0\,\lambda\,v_x(l_0)$ we get

$$\begin{aligned} & \kappa_1\,\alpha\,u(l_0)\,\frac{\cos(\alpha\,l_0)}{\sin(\alpha\,l_0)} + \kappa_1\,\frac{\cos(\alpha\,l_0)}{\sin(\alpha\,l_0)}\int_0^{l_0} q(s)\,\sin(\alpha\,(l_0 - s))\,ds \\ & - \kappa_1\int_0^{l_0} q(s)\,\cos(\alpha\,(l_0 - s))\,ds = \beta\,(\kappa_2 + i\,\kappa_0\,\lambda)\,u(l_0)\,\frac{\cosh(\beta\,l_0)}{\sinh(\beta\,l_0)} \\ & - \left(u(l_0)\,\frac{\sinh(\beta\,l_1)}{\sinh(\beta\,l_0)} - v(l_1)\right)\frac{\beta\,(\kappa_2 + i\,\kappa_0\,\lambda)}{\sinh(\beta\,(l_1 - l_0))}. \end{aligned}$$

It follows that

$$\begin{aligned} & u(l_0)\,[\beta\,(\kappa_2 + i\,\kappa_0\,\lambda)\,\sin(\alpha\,l_0)\,\coth(\beta\,l_0) - \kappa_1\,\alpha\,\cos(\alpha\,l_0) + j\,(\beta)] \\ = \;& \kappa_1\,\cos(\alpha\,l_0)\int_0^{l_0} q(s)\,\sin(\alpha\,(l_0 - s))\,ds - \sin(\alpha\,l_0)\,\kappa_1\int_0^{l_0} q(s)\,\cos(\alpha\,(l_0 - s))\,ds \\ = \;& -\,\kappa_1\int_0^{l_0} q(s)\,\sin(\alpha\,s)\,ds. \end{aligned}$$

Let us take

$$\alpha\,l_0 = 2\,n\,\pi + \frac{1}{\sqrt{n}}, \quad q(s) = \sin(\alpha\,s).$$

So we have that

$$\alpha\,l_0 \approx \frac{2}{l\,\pi}\,n, \quad \sin(\alpha\,l_0) \approx \frac{1}{\sqrt{n}}, \quad \alpha\,\sin(\alpha\,l_0) \approx c_0, \quad \tanh(\alpha\,l) \approx 1$$

as $n \to \infty$ and $0 \neq c_0 \in \mathbb{C}$. This implies that

$$\frac{\kappa_1}{\beta\,(\kappa_2 + i\,\kappa_0\,\lambda)\,\sin(\alpha\,l_0)\,\coth(\beta\,l_0) - \kappa_1\,\alpha\,\cos(\alpha\,l_0) + j\,(\beta)} \approx \frac{c_1}{\lambda}.$$

This implies that

$$u(l_0) = \frac{c_2}{\lambda}.$$

For $0 \neq c_2 \in \mathbb{C}$. Note that the expression

$$\begin{aligned} \beta\,v(x) \;=\; & \beta\,u(0)\,\frac{\sin(\alpha\,(l_0 - x))}{\sin(\alpha\,l_0)} - \frac{\sin(\alpha\,x)}{\sin(\alpha\,l_0)}\int_0^{l} q(s)\,\sin(\alpha\,(l_0 - s))\,ds \\ & + \int_0^{x} q(s)\,\sin(\alpha\,(x - s))\,ds \end{aligned}$$

can be written as

$$\begin{aligned} \beta\,v(x) \;=\; & \left(c_2\,\frac{\sin(\alpha\,(l_0 - x))}{\sin(\alpha\,l_0)} - \frac{\sin(\alpha\,x)}{\sin(\alpha\,l_0)}\right)\int_0^{l} q(s)\,\sin(\alpha\,(l_0 - s))\,ds \\ & + \underbrace{\int_0^{x} q(s)\,\sin(\alpha\,(x - s))\,ds}_{:=Q(x)}. \end{aligned}$$

Then

$$\beta\,v(x) = \left[c_2\,\cos(\alpha\,x) - (c_2\,\cos(\alpha\,l_0) + 1)\,\frac{\sin(\alpha\,x)}{\sin(\alpha\,l)}\right] Q(l) + Q(x).$$

Taking $q(s) = \sin(\beta\, s)$ and squaring and integrating we have

$$
\begin{aligned}
Q(x) &= \int_0^x \big(\sin(\alpha\, s)\, \sin(\alpha\, x)\, \cos(\alpha\, s) - \sin^2(\alpha\, s)\, \cos(\alpha\, x)\big)\, ds \\
&= \sin(\alpha\, x) \int_0^x \sin(\alpha\, s)\, \cos(\alpha\, s)\, ds - \cos(\alpha\, x) \int_0^x \sin^2(\alpha\, s)\, ds \\
&= -\frac{\sin^3(\alpha\, x)}{2\, \alpha\, l_0} - \cos(\alpha\, x) \int_0^x \sin^2(\alpha\, x)\, ds \\
&= -\frac{\sin^3(\alpha\, x)}{2\, \alpha\, l_0} - \frac{x\, \cos(\alpha\, x)}{2} + \frac{\cos(\alpha\, x)\, \sin(2\, \alpha\, x)}{2\, \alpha}. \qquad (4.11)
\end{aligned}
$$

Therefore

$$
Q(l) = -\frac{\pi}{n^{5/2}} - \frac{l\, \cos\alpha}{2} + \frac{\cos(\alpha\, l_0)}{n^{3/2}} \approx -\frac{l}{2}.
$$

Note that

$$
\int_0^l |Q(s)|^2\, ds \geqslant \int_0^l \frac{x^2\, \cos^2(\alpha\, x)}{8}\, dx - \frac{c}{\alpha^2} \geqslant \frac{l^3}{48} - \frac{c}{|\alpha|}. \qquad (4.12)
$$

Finally,

$$
\begin{aligned}
&\int_0^l \left| c_2 \cos(\alpha\, x) - (c_2\, \cos(\alpha\, l_0) + 1)\, \frac{\sin(\alpha\, x)}{\sin(\alpha\, l)} \right|^2\, ds \\
&\geqslant \frac{|c_2\, \cos(\alpha\, l_0) + 1|}{2\, \sin^2(\alpha\, l_0)} \int_0^l \sin^2(\alpha\, x)\, dx - c_0 \\
&\approx c_1\, n - c_0. \qquad (4.13)
\end{aligned}
$$

Inserting inequalities (4.12) and (4.13) into (4.11) we get that there exists a positive constant C such that

$$
\int_0^l |\alpha\, v(x)|^2\, dx \geqslant -\, C + C\, n,
$$

for large n, that is

$$
\frac{1}{n} \int_0^l |\alpha\, v(x)|^2\, dx \geqslant C_0.
$$

In particular, we have that

$$
\|\Phi\|^2 \geqslant \int_0^l |\alpha\, v(x)|^2.
$$

If the rate of decay can be improved then we have that $\frac{1}{n^{1-\epsilon}}\, \|U\|^2$ must be bounded. But

$$
\frac{1}{n^{1-\epsilon}}\, \|\Phi\|^2 \geqslant \int_0^l |\beta\, v(x)|^2 \geqslant C_0\, n^{\epsilon} \qquad (4.14)
$$

from where our conclusion follows

5 Polynomial decay and Optimality

Here we prove that the solutions of the **EVF** model decays polynomially as t^{-2}. Moreover we will show that the rate of decay is optimal.

Theorem 5.1 *The solution of the* **EVF** *model decays polynomially as t^{-2}. Moreover the rate of decay is optimal over $\mathcal{D}(\mathcal{A})$ and*

$$\|\Phi(t)\| \leqslant \frac{c_k}{t^{2\,k}} \|\Phi_0\|_{\mathcal{D}(\mathcal{A}^k)}. \tag{5.1}$$

Proof. Using the same arguments as in the prove of Theorem 3.2 we can show that $i\mathbb{R} \subset \varrho(\mathcal{A})$.

Let us prove that the resolvent operator is uniformly bounded by $C|\lambda|^{1/2}$ over the imaginary axes. Multiplying equation (4.1) by Φ and using (2.8) we get

$$\kappa_0 \int_{l_0}^{l_1} |V_x|^2\,dx + \gamma \int_{l_1}^{l} |W|^2\,dx = \mathrm{Re}(F,\,\Phi)_{\mathcal{H}}. \tag{5.2}$$

From (4.2) we have

$$|\lambda|\,\|V\|_{-1} \leqslant C\,\|v_x\| + C\,\|V_x\| + C\,\|F\|_{\mathcal{H}} \leqslant C\,\|\Phi\|_{\mathcal{H}}^{1/2}\,\|F\|_{\mathcal{H}}^{1/2} + C\,\|F\|_{\mathcal{H}}.$$

Using interpolation and inequality (5.2) we get

$$\begin{aligned} \|V\|_{L^2}^2 &\leqslant C\,\|V\|_{-1}\,\|V\|_1 \leqslant \frac{C}{|\lambda|}\left[\|\Phi\|_{\mathcal{H}}^{1/2}\,\|F\|_{\mathcal{H}}^{1/2} + \|F\|_{\mathcal{H}}\right]\|V\|_1 \\ &\leqslant \frac{C}{|\lambda|}\left[\|\Phi\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} + \|\Phi\|_{\mathcal{H}}^{1/2}\,\|F\|_{\mathcal{H}}^{3/2}\right]. \end{aligned} \tag{5.3}$$

Multiplying equation (4.5) by $(x-l_0)\,\left(\overline{\kappa_2\,v_x + \kappa_3\,V_x}\right)$ and taking real part we have

$$\begin{aligned} &Re\,i\,\lambda \int_{l_0}^{l_1} \eta\,(x-l_0)\,\left(\overline{\kappa_2\,v_x + \kappa_3\,V_x}\right)dx - \frac{1}{2}\int_{l_0}^{l_1} (x-l_0)\,\frac{d}{dx}\left|\kappa_2\,v_x + \kappa_3\,V_x\right|^2\,dx \\ = \;&\rho_1\,Re \int_{l_0}^{l_1} f_3\,(x-l_0)\,\left(\overline{\kappa_2\,v_x + \kappa_3\,V_x}\right). \end{aligned}$$

Using (4.4), we note that

$$\begin{aligned} \kappa_1\,Re\,i\,\lambda \int_{l_0}^{l_1} V\,(x-l_0)\,\overline{v}_x\,dx &= -\frac{(l_1-l_0)}{2}\,\kappa_2\,|V(0)|^2 + \frac{1}{2}\,\kappa_2 \int_{l_0}^{l_1} |V|^2\,dx \\ &\quad - \kappa_2 \int_{l_0}^{l_1} (x-l_0)\,V\,\overline{f}\,dx. \end{aligned}$$

We denote the functional

$$I_u = \frac{1}{2}\left[\rho_2\,|V(0)|^2 + |\kappa_2\,v_x(0) + \kappa_3\,V_x(0)|^2\right].$$

It follows that

$$\begin{aligned} I_u &= \rho_2\,Re\,i\,\lambda\int_{l_0}^{l_1}(x-l_0)\,V\,\overline{V}_x\,dx + \frac{1}{2}\rho_1\int_{l_0}^{l_1}|V|^2\,dx + \frac{1}{2}\int_{l_0}^{l_1}|\kappa_2\,v_x+\kappa_3\,V_x|^2\,dx \\ &\quad -\rho_2\,Re\int_{l_0}^{l_1} f_3\,(x-l_0)\,\left(\overline{\kappa_2\,v_x+\kappa_2\,V_x}\right)dx - \rho_2\int_{l_0}^{l_1}(x-l_0)\,V\,\overline{f}\,dx \\ &\leqslant C\int_{l_0}^{l_1}\left(|\lambda|\,|V_x|\,|V| + V_x^2 + v_x^2\right)dx + C\,\|\Phi\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} \\ &\leqslant C\,|\lambda|^{1/2}\int_{l_0}^{l_1}|V_x|\left(|\lambda|^{1/2}\,|V|\right)dx + C\,\|\Phi\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}. \end{aligned}$$

Using (5.3) we get

$$I_u \leqslant C\,|\lambda|^{1/2}\left(\|\Phi\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} + \|\Phi\|_{\mathcal{H}}^{3/4}\,\|F\|_{\mathcal{H}}^{5/4}\right), \tag{5.4}$$

for λ large enough. On the other hand, multiplying equation (4.3) by $(x-l_0)\,\overline{u}_x$ we get

$$i\,\lambda\,\rho_1\int_0^{l_0} U\,(x-l_0)\,\overline{u}_x\,dx - \kappa_3\int_0^{l_0} u_{xx}\,(x-l_0)\,\overline{u}_x\,dx = \rho_2\int_0^{l_0}(x-l_0)\,f_2\,\overline{u}_x\,dx.$$

Taking real part and using (4.2) we obtain

$$\begin{aligned} \frac{1}{2}\int_0^{l_0}\left(\rho_1\,|U|^2 + \kappa_1\,|u_x|^2\right)dx &= \frac{1}{2}\rho_1\,l_0\left(|U(l_0)|^2 + \frac{\kappa_1}{\rho_1}\,|u_x(l_0)|^2\right) + \rho_1\,Re\int_0^{l_0}(x-l_0)\,f_2\,\overline{u}_x\,dx \\ &\quad + \rho_1\,Re\int_0^{l_0}(x-l_0)\,U\,\overline{f_1}_x\,dx \end{aligned}$$

Using (1.3), and performing straightforward estimates it follows that

$$\begin{aligned} \frac{1}{2}\int_0^{l_0}\left(\rho_1\,|U|^2 + \kappa_1\,|u_x|^2\right)dx &\leqslant \frac{1}{2}\rho_2\,L\left(|U(l_0)|^2 + \frac{\kappa_1}{\rho_2}\,|u_x(l_0)|^2\right) + C\,\|\Phi\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} \\ &\leqslant C\left[V(l_0)|^2 + |\kappa_1\,u_x(l_0) + \kappa_2\,V_x(l_0)|^2\right] + C\,\|\Phi\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}. \end{aligned}$$

Using inequality (5.4) we get

$$\int_0^{l_0}\left(|U|^2 + |u_x|^2\right)dx \;\leqslant\; C\,|\lambda|^{1/2}\left(\|\Phi\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} + \|\Phi\|_{\mathcal{H}}^{3/4}\,\|F\|_{\mathcal{H}}^{5/4}\right).$$

Multiplying (4.3), (4.5) and (4.7) by $\overline{u}$, $\overline{v}$ and $\overline{w}$ respectively, using (4.2), (4.4) and (4.6) we get that

$$\begin{aligned} &\kappa_1\int_0^{l_0}|u_x|^2\,dx + \kappa_2\int_{l_0}^{l_1}|v_x|^2\,dx + \kappa_4\int_{l_1}^{l}|w_x|^2\,dx \\ &\quad\leqslant C\int_0^{l_0}|U|^2\,dx + C\int_{l_0}^{l_1}|V|^2\,dx + C\int_{l_1}^{l}|W|^2\,dx + C\,\|\Phi\|\,\|F\| \end{aligned}$$

From (5.3)-(5.4) we conclude that

$$\|\Phi\|_{\mathcal{H}}^2 \;\leqslant\; C\,|\lambda|^{1/2}\left(\|\Phi\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}+\|\Phi\|_{\mathcal{H}}^{3/4}\,\|F\|_{\mathcal{H}}^{5/4}\right).$$

Thus

$$\|\Phi\|_{\mathcal{H}} \;\leqslant\; C\,|\lambda|^{1/2}\,\|F\|_{\mathcal{H}}$$

for λ large enough. Therefore, from Theorem 1.2 we get

$$\|\Phi(t)\| \leqslant \frac{c_k}{t^{2\,k}}\,\|\Phi_0\|_{\mathcal{D}(\mathcal{A}^k)}.$$

Inequality (5.1) follows by using a standard semigroup procedure. Finally, to prove the optimality we will use inequality (4.14). In fact, if the rate of decay can be improved for example as

$$\|\Phi(t)\| \leqslant \frac{c_k}{t^{2/(1-2\,\epsilon)}}\,\|\Phi_0\|_{\mathcal{D}(\mathcal{A})}$$

for $\epsilon > 0$ small enough, then we have that the expression

$$\frac{1}{|\lambda|^{1-\epsilon}}\,\|\Phi\|^2,$$

must be bounded, but from (4.14) and from the fact that $|\lambda| \approx c_1\,n$ we get

$$\frac{1}{|\lambda|^{1-\epsilon}}\,\|\Phi\|^2 \geqslant \int_{l_0}^{l_1} |\alpha\,v(x)|^2\,dx \geqslant C_0\,n^{\epsilon} \to \infty$$

Bust this is a contradiction, so we have that the rate of decay can not be improved.

6 Numerical approximations

Here we will verify numerically the polynomial and exponential rate of decay obtained in the previous sections. It is important to note that any numerical approximation is a finite-dimensional simplification of the original problem. Thus, any numerical method used, decay exponentially for large enough times, and this because of its restrictive nature of the finite dimensional space approach.

Denoting by $\mathcal{E}$ the energy

$$\mathcal{E}(t) = \frac{1}{2}\left[\rho_1\int_0^{l_0} u_t^2\,dx + \rho_2\int_{l_0}^{l_1} v_t^2\,dx + \rho_3\int_{l_1}^{l} w_t^2\,dx \right.$$
$$\left. +\kappa_1\int_0^{l_0} u_x^2\,dx + \kappa_2\int_{l_0}^{l_1} v_x^2\,dx + \kappa_3\int_{l_1}^{l} w_x^2\,dx\right],$$

and denoting by $\mathcal{E}_{\mathbf{VEF}}$, $\mathcal{E}_{\mathbf{EVF}}$, $\mathcal{E}_{\mathbf{EFV}}$ the energy for the three respective cases **VEF**, **EVF** and **EFV**, it is not difficult to see that the energy decays for all the cases. More precisely,

$$\frac{d}{dt}\mathcal{E}_{\mathbf{VEF}}(t) = -\,\kappa_0 \int_0^{l_0} u_{xt}^2\,dx - \gamma\int_{l_1}^{l} w_t^2\,dx \tag{6.1}$$

$$\frac{d}{dt}\mathcal{E}_{\mathbf{EVF}}(t) = -\,\kappa_0 \int_{l_0}^{l_1} v_{xt}^2\,dx - \gamma\int_{l_1}^{l} w_t^2\,dx \tag{6.2}$$

$$\frac{d}{dt}\mathcal{E}_{\mathbf{EFV}}(t) = -\,\kappa_0 \int_{l_1}^{l} w_{xt}^2\,dx - \gamma\int_{l_0}^{l_1} v_t^2\,dx \tag{6.3}$$

In this regard, we have a robust numerical method of high order which in turn ensures a natural way (without additional artificial viscosity for example) the decay of energy with the same terms prescribed in identity (6.1).

6.1 Linear equation of Motion

First, we approximate the displacement vector $[u, v, w]^\top$ in space using finite elements $P2$. For that, we consider the variational problem

$$\begin{aligned} \rho_1 \int_0^{l_0} u_{tt}\,\varphi_1\,dx + \rho_2\int_{l_0}^{l_1} v_{tt}\,\varphi_2\,dx + \rho_3\int_{l_1}^{l} w_{tt}\,\varphi_3\,dx \\ +\kappa_1\int_0^{l_0} u_x\,\varphi_{1,x}\,dx + \kappa_2\int_{l_0}^{l_1} v_x\,\varphi_{2,x}\,dx + \kappa_3\int_{l_1}^{l} w_x\,\varphi_{2,x}\,dx, \\ = \mathcal{R}(u, v, w; \varphi_1, \varphi_2, \varphi_3) \end{aligned} \tag{6.4}$$

for all $(\varphi_1, \varphi_2, \varphi_3) \in V = \{(\varphi_1, \varphi_2, \varphi_3, \varphi_{1,x}, \varphi_{2,x}, \varphi_{3,x}) \in \mathcal{D}(\mathcal{A}_i)\}$ for $i = 1, 2$ and 3, defined in (2.2)-(2.6), and where $\mathcal{R}$ take the values $\mathcal{R}_{\mathbf{VEF}}$, $\mathcal{R}_{\mathbf{EVF}}$, $\mathcal{R}_{\mathbf{EFV}}$, for the different cases, respectively, given by

$$\mathcal{R}_{\mathbf{VEF}}(u, v, w; \varphi_1, \varphi_2, \varphi_3) = -\,\kappa_0 \int_0^{l_0} u_{xt}^2\,\varphi_{1,x}\,dx - \gamma\int_{l_1}^{l} w_t^2\,\varphi_3\,dx \tag{6.5}$$

$$\mathcal{R}_{\mathbf{EVF}}(u, v, w; \varphi_1, \varphi_2, \varphi_3) = -\,\kappa_0 \int_{l_0}^{l_1} v_{xt}^2\,\varphi_{2,x}\,dx - \gamma\int_{l_1}^{l} w_t^2\,\varphi_3\,dx \tag{6.6}$$

$$\mathcal{R}_{\mathbf{EFV}}(u, v, w; \varphi_1, \varphi_2, \varphi_3) = -\,\kappa_0 \int_{l_1}^{l} w_{xt}^2\,\varphi_{3,x}\,dx - \gamma\int_{l_0}^{l_1} v_t^2\,\varphi_2\,dx. \tag{6.7}$$

The variational problem (6.4) have a unique solution in the same sense of Theorem 2.2, which we approach by two-degree piecewise polynomial basis functions (see [1, 2, 3, 4]). Then, we choose J_1 values of x in the interval $(0, l_0)$, J_2 values of x in the interval (l_0, l_1), and J_3 values of x in the interval (l_1, l), with a total of $J = J_1 + J_2 + J_3 - 1$ nodes for the unknowns. That is,

$$0 = x_0 < x_1 < \ldots < x_{n_1} = l_0 < x_{n_1+1} < \ldots < x_{n_2} = l_1 < x_{n_2+1} < \ldots < x_{n_3} = l.$$

We obtain a vector $[\mathbf{u}_\delta(t), \mathbf{v}_\delta(t), \mathbf{w}_\delta(t)]^\top$ approximation of $[u, v, w]^\top$ in $\mathbb{R}^J \times \mathbb{R}^J \times \mathbb{R}^J$. Additionally, let us define $[\mathbf{U}_\delta(t), \mathbf{V}_\delta(t), \mathbf{W}_\delta(t)]^\top$ the approximation of the velocity $[U, V, W]^\top$, where $\mathbf{U}_\delta(t) = \dot{\mathbf{u}}_\delta(t)$, $\mathbf{V}_\delta(t) = \dot{\mathbf{v}}_\delta(t)$ and $\mathbf{W}_\delta(t) = \dot{\mathbf{w}}_\delta(t)$. Using the boundary and transmission condition, we easily obtain the linear equation of motion

$$\mathbf{M}\begin{bmatrix} \dot{\mathbf{U}}_h \\ \dot{\mathbf{V}}_h \\ \dot{\mathbf{W}}_h \end{bmatrix} + \frac{1}{\delta x^2}\mathbf{C}_{visc}\begin{bmatrix} \mathbf{U}_h \\ \mathbf{V}_h \\ \mathbf{W}_h \end{bmatrix} + \mathbf{C}_{frict}\begin{bmatrix} \mathbf{U}_h \\ \mathbf{V}_h \\ \mathbf{W}_h \end{bmatrix} + \frac{1}{\delta x^2}\mathbf{K}\begin{bmatrix} \mathbf{u}_h \\ \mathbf{v}_h \\ \mathbf{w}_h \end{bmatrix} = \mathbf{0}, \quad (6.8)$$

where $\mathbf{M}$, $\mathbf{C}_{visc}$, $\mathbf{C}_{frict}$ and $\mathbf{K}$ are the mass, viscoelastic damping, frictional damping and stiffness matrices of the system in $\mathcal{M}_{3J}(\mathbb{R})$. We remark, that the matrices $\mathbf{C}_{visc}$ and $\mathbf{C}_{frict}$, have several null rows depending of each one of the tree cases. For instance, for the **VEF** case, the matrix $\mathbf{C}_{visc}$ have only the first J rows nonzero, and the matrix $\mathbf{C}_{frict}$ have only the last J rows nonzero.

6.2 Time discretization

Regarding now to the time discretization, it is desirable that the algorithm has at least second-order accuracy too, and because the spatial discretization used in structural dynamics often leads to inclusion of high-frequency modes in the model, it is also desirable to have unconditional stability. The method consists of updating the displacement, velocity and acceleration vectors at current time $t^n = n\delta t$ to the time $t^{n+1} = (n+1)\delta t$, a small time interval δt later. The Newmark algorithm [17] is based on a set of two relations expressing the forward displacement $[\mathbf{u}_\delta^{n+1}, \mathbf{v}_\delta^{n+1}, \mathbf{w}_\delta^{n+1}]^\top$ and velocity $[\mathbf{U}_\delta^{n+1}, \mathbf{V}_\delta^{n+1}, \mathbf{W}_\delta^{n+1}]^\top$ in terms of their current values and the forward and current values of the acceleration,

$$\begin{aligned}
\mathbf{U}_\delta^{n+1} &= \mathbf{U}_\delta^n + (1-\gamma)\delta t\,\dot{\mathbf{U}}_\delta^n + \gamma\delta t\,\dot{\mathbf{U}}_\delta^{n+1} && (6.9)\\
\mathbf{u}_\delta^{n+1} &= \mathbf{u}_\delta^n + \left(\frac{1}{2}-\beta\right)\delta t^2\,\dot{\mathbf{U}}_\delta^n + \beta\delta t^2\,\dot{\mathbf{U}}_\delta^{n+1} && (6.10)\\
\mathbf{V}_\delta^{n+1} &= \mathbf{V}_\delta^n + (1-\gamma)\delta t\,\dot{\mathbf{V}}_\delta^n + \gamma\delta t\,\dot{\mathbf{V}}_\delta^{n+1} && (6.11)\\
\mathbf{v}_\delta^{n+1} &= \mathbf{v}_\delta^n + \left(\frac{1}{2}-\beta\right)\delta t^2\,\dot{\mathbf{V}}_\delta^n + \beta\delta t^2\,\dot{\mathbf{V}}_\delta^{n+1}, && (6.12)\\
\mathbf{W}_\delta^{n+1} &= \mathbf{W}_\delta^n + (1-\gamma)\delta t\,\dot{\mathbf{W}}_\delta^n + \gamma\delta t\,\dot{\mathbf{W}}_\delta^{n+1} && (6.13)\\
\mathbf{w}_\delta^{n+1} &= \mathbf{w}_\delta^n + \left(\frac{1}{2}-\beta\right)\delta t^2\,\dot{\mathbf{W}}_\delta^n + \beta\delta t^2\,\dot{\mathbf{W}}_\delta^{n+1}, && (6.14)
\end{aligned}$$

where β and γ are parameters of the methods that will be fixed later. Replacing (6.9)-(6.14) in the equation of motion (6.8), we obtain

$$\begin{aligned}
\left(\delta x^2\mathbf{M} + \gamma\delta t\,\mathbf{C} + \beta\delta t^2\mathbf{K}\right)\begin{bmatrix}\dot{\mathbf{U}}_\delta^{n+1}\\ \dot{\mathbf{V}}_\delta^{n+1}\\ \dot{\mathbf{W}}_\delta^{n+1}\end{bmatrix} &= -\mathbf{C}_{visc}\left(\begin{bmatrix}\mathbf{U}_\delta^n\\ \mathbf{V}_\delta^n\\ \mathbf{W}_\delta^n\end{bmatrix} + (1-\gamma)\delta t\begin{bmatrix}\dot{\mathbf{U}}_\delta^n\\ \dot{\mathbf{V}}_\delta^n\\ \dot{\mathbf{W}}_\delta^n\end{bmatrix}\right)\\
&\quad -\delta x^2\,\mathbf{C}_{frict}\left(\begin{bmatrix}\mathbf{U}_\delta^n\\ \mathbf{V}_\delta^n\\ \mathbf{W}_\delta^n\end{bmatrix} + (1-\gamma)\delta t\begin{bmatrix}\dot{\mathbf{U}}_\delta^n\\ \dot{\mathbf{V}}_\delta^n\\ \dot{\mathbf{W}}_\delta^n\end{bmatrix}\right)\\
&\quad -\mathbf{K}\left(\begin{bmatrix}\mathbf{u}_\delta^n\\ \mathbf{v}_\delta^n\\ \mathbf{w}_\delta^n\end{bmatrix} + \delta t\begin{bmatrix}\mathbf{U}_\delta^n\\ \mathbf{V}_\delta^n\\ \mathbf{W}_\delta^n\end{bmatrix} + \left(\frac{1}{2}-\beta\right)\delta t^2\begin{bmatrix}\dot{\mathbf{U}}_\delta^n\\ \dot{\mathbf{V}}_\delta^n\\ \dot{\mathbf{W}}_\delta^n\end{bmatrix}\right) \quad (6.15)
\end{aligned}$$

The acceleration $[\dot{\mathbf{U}}_\delta^{n+1}, \dot{\mathbf{V}}_\delta^{n+1}, \dot{\mathbf{W}}_\delta^{n+1}]^\top$ is found from (6.15). On the other hand, the velocity $[\mathbf{U}_\delta^{n+1}, \mathbf{V}_\delta^{n+1}, \mathbf{W}_\delta^{n+1}]^\top$ follow from (6.9), (6.11) and (6.13), respectively. Finally, the displacement $[\mathbf{u}_\delta^{n+1}, \mathbf{v}_\delta^{n+1}, \mathbf{w}_\delta^{n+1}]^\top$ follow from (6.10), eqref405 and (6.14), respectively by simple vector operations.

6.3 Energy balance of the Newmark algorithm

We define the discrete energy as

$$\mathcal{E}_\delta^n := \frac{1}{2}\left[\mathbf{U}_\delta^\top, \mathbf{V}_\delta^\top, \mathbf{W}_\delta^\top\right]\mathbf{M}\begin{bmatrix}\mathbf{U}_\delta\\ \mathbf{V}_\delta\\ \mathbf{W}_\delta\end{bmatrix} + \frac{1}{2\delta x^2}\left[\mathbf{u}_\delta^\top, \mathbf{v}_\delta^\top, \mathbf{w}_\delta^\top\right]\mathbf{K}\begin{bmatrix}\mathbf{u}_\delta\\ \mathbf{v}_\delta\\ \mathbf{w}_\delta\end{bmatrix}$$

which is an approximation of that defined in (6.1) for the continuous case. The increment of this energy can be expressed in terms of mean values and increments of the displacement and velocity by the following identity:

$$\begin{aligned}
\mathcal{E}_\delta^{n+1} - \mathcal{E}_\delta^n &= \left[\frac{1}{2}\left[\mathbf{U}_\delta^\top, \mathbf{V}_\delta^\top, \mathbf{W}_\delta^\top\right]\mathbf{M}\begin{bmatrix}\mathbf{U}_\delta\\ \mathbf{V}_\delta\\ \mathbf{W}_\delta\end{bmatrix} + \frac{1}{2\delta x^2}\left[\mathbf{u}_\delta^\top, \mathbf{v}_\delta^\top, \mathbf{w}_\delta^\top\right]\mathbf{K}\begin{bmatrix}\mathbf{u}_\delta\\ \mathbf{v}_\delta\\ \mathbf{w}_\delta\end{bmatrix}\right]_n^{n+1}\\
&= \begin{bmatrix}\mathbf{U}_\delta^{n+\frac{1}{2}}\\ \mathbf{V}_\delta^{n+\frac{1}{2}}\\ \mathbf{W}_\delta^{n+\frac{1}{2}}\end{bmatrix}^\top \mathbf{M}\begin{bmatrix}\Delta\mathbf{U}_\delta\\ \Delta\mathbf{V}_\delta\\ \Delta\mathbf{W}_\delta\end{bmatrix} + \frac{1}{\delta x^2}\begin{bmatrix}\mathbf{u}_\delta^{n+\frac{1}{2}}\\ \mathbf{v}_\delta^{n+\frac{1}{2}}\\ \mathbf{w}_\delta^{n+\frac{1}{2}}\end{bmatrix}^\top \mathbf{K}\begin{bmatrix}\Delta\mathbf{u}_\delta\\ \Delta\mathbf{v}_\delta\\ \Delta\mathbf{w}_\delta\end{bmatrix}
\end{aligned}$$

where $\mathbf{u}^{n+\frac{1}{2}} = \frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2}$ and $\Delta\mathbf{u} = \mathbf{u}^{n+1} - \mathbf{u}^n$. Now, in order to derive the required energy estimates, we rely on calculations and notations similar to S. Krenk [9] to finally obtain

$$\begin{aligned}
&\left[\frac{1}{2}\begin{bmatrix}\mathbf{U}_h\\ \mathbf{V}_h\\ \mathbf{W}_h\end{bmatrix}^\top \mathbf{M}_*\begin{bmatrix}\mathbf{U}_h\\ \mathbf{V}_h\\ \mathbf{W}_h\end{bmatrix}\right.\\
&\quad\left. +\frac{1}{2\delta x^2}\begin{bmatrix}\mathbf{u}_h\\ \mathbf{v}_h\\ \mathbf{w}_h\end{bmatrix}^\top \mathbf{K}\begin{bmatrix}\mathbf{u}_h\\ \mathbf{v}_h\\ \mathbf{w}_h\end{bmatrix} + \left(\beta - \frac{1}{2}\gamma\right)\frac{\delta t^2}{2}\begin{bmatrix}\dot{\mathbf{U}}_h\\ \dot{\mathbf{V}}_h\\ \dot{\mathbf{W}}_h\end{bmatrix}^\top \mathbf{M}_*\begin{bmatrix}\dot{\mathbf{U}}_h\\ \dot{\mathbf{V}}_h\\ \dot{\mathbf{W}}_h\end{bmatrix}\right]_n^{n+1}\\
&= \left(\gamma - \frac{1}{2}\right)\left\{\frac{1}{\delta x^2}\begin{bmatrix}\Delta\mathbf{u}_h\\ \Delta\mathbf{v}_h\\ \Delta\mathbf{w}_h\end{bmatrix}^\top \mathbf{K}\begin{bmatrix}\Delta\mathbf{u}_h\\ \Delta\mathbf{v}_h\\ \Delta\mathbf{w}_h\end{bmatrix} + \left(\beta - \frac{1}{2}\gamma\right)\delta t^2\begin{bmatrix}\Delta\dot{\mathbf{U}}_h\\ \Delta\dot{\mathbf{V}}_h\\ \Delta\dot{\mathbf{W}}_h\end{bmatrix}^\top \mathbf{M}_*\begin{bmatrix}\Delta\dot{\mathbf{U}}_h\\ \Delta\dot{\mathbf{V}}_h\\ \Delta\dot{\mathbf{W}}_h\end{bmatrix}\right\}\\
&\quad -\frac{\delta t}{2\delta x^2}\left\{\delta t^{-2}\begin{bmatrix}\Delta\mathbf{u}_h\\ \Delta\mathbf{v}_h\\ \Delta\mathbf{w}_h\end{bmatrix}^\top \mathbf{C}_{visc}\begin{bmatrix}\Delta\mathbf{u}_h\\ \Delta\mathbf{v}_h\\ \Delta\mathbf{w}_h\end{bmatrix} + \begin{bmatrix}\mathbf{U}_h^{n+\frac{1}{2}}\\ \mathbf{V}_h^{n+\frac{1}{2}}\\ \mathbf{W}_h^{n+\frac{1}{2}}\end{bmatrix}^\top \mathbf{C}_{visc}\begin{bmatrix}\mathbf{U}_h^{n+\frac{1}{2}}\\ \mathbf{V}_h^{n+\frac{1}{2}}\\ \mathbf{W}_h^{n+\frac{1}{2}}\end{bmatrix}\right\}\\
&\quad +\frac{\delta t^3}{2\delta x^2}\left(\beta - \frac{1}{2}\gamma\right)^2\begin{bmatrix}\Delta\dot{\mathbf{U}}_h\\ \Delta\dot{\mathbf{V}}_h\\ \Delta\dot{\mathbf{W}}_h\end{bmatrix}^\top \mathbf{C}_{visc}\begin{bmatrix}\Delta\dot{\mathbf{U}}_h\\ \Delta\dot{\mathbf{V}}_h\\ \Delta\dot{\mathbf{W}}_h\end{bmatrix}\\
&\quad -\frac{\delta t}{2}\left\{\delta t^{-2}\begin{bmatrix}\Delta\mathbf{u}_h\\ \Delta\mathbf{v}_h\\ \Delta\mathbf{w}_h\end{bmatrix}^\top \mathbf{C}_{frict}\begin{bmatrix}\Delta\mathbf{u}_h\\ \Delta\mathbf{v}_h\\ \Delta\mathbf{w}_h\end{bmatrix} + \begin{bmatrix}\mathbf{U}_h^{n+\frac{1}{2}}\\ \mathbf{V}_h^{n+\frac{1}{2}}\\ \mathbf{W}_h^{n+\frac{1}{2}}\end{bmatrix}^\top \mathbf{C}_{frict}\begin{bmatrix}\mathbf{U}_h^{n+\frac{1}{2}}\\ \mathbf{V}_h^{n+\frac{1}{2}}\\ \mathbf{W}_h^{n+\frac{1}{2}}\end{bmatrix}\right\}\\
&\quad +\frac{\delta t^3}{2}\left(\beta - \frac{1}{2}\gamma\right)^2\begin{bmatrix}\Delta\dot{\mathbf{U}}_h\\ \Delta\dot{\mathbf{V}}_h\\ \Delta\dot{\mathbf{W}}_h\end{bmatrix}^\top \mathbf{C}_{frict}\begin{bmatrix}\Delta\dot{\mathbf{U}}_h\\ \Delta\dot{\mathbf{V}}_h\\ \Delta\dot{\mathbf{W}}_h\end{bmatrix}
\end{aligned}$$

where $\mathbf{M}_* = \mathbf{M} + \left(\gamma - \frac{1}{2}\right)\delta t \left(\frac{1}{\delta x^2}\mathbf{C}_{visc} + \mathbf{C}_{frict}\right)$. Then, we choose $\gamma = \frac{1}{2}$ and $\beta = \frac{\gamma}{2}$, reducing the above expression to

$$\begin{aligned}
&\left[\frac{1}{2}\begin{bmatrix}\mathbf{U}_h\\ \mathbf{V}_h\\ \mathbf{W}_h\end{bmatrix}^\top \mathbf{M}\begin{bmatrix}\mathbf{U}_h\\ \mathbf{V}_h\\ \mathbf{W}_h\end{bmatrix} + \frac{1}{2\delta x^2}\begin{bmatrix}\mathbf{u}_h\\ \mathbf{v}_h\\ \mathbf{w}_h\end{bmatrix}^\top \mathbf{K}\begin{bmatrix}\mathbf{u}_h\\ \mathbf{v}_h\\ \mathbf{w}_h\end{bmatrix}\right]_n^{n+1}\\
&= -\frac{\delta t}{2\delta x^2}\left\{\begin{bmatrix}\frac{\Delta \mathbf{u}_h}{\delta t}\\ \frac{\Delta \mathbf{v}_h}{\delta t}\\ \frac{\Delta \mathbf{w}_h}{\delta t}\end{bmatrix}^\top \mathbf{C}_{visc}\begin{bmatrix}\frac{\Delta \mathbf{u}_h}{\delta t}\\ \frac{\Delta \mathbf{v}_h}{\delta t}\\ \frac{\Delta \mathbf{w}_h}{\delta t}\end{bmatrix} + \begin{bmatrix}\mathbf{U}_h^{n+\frac{1}{2}}\\ \mathbf{V}_h^{n+\frac{1}{2}}\\ \mathbf{W}_h^{n+\frac{1}{2}}\end{bmatrix}^\top \mathbf{C}_{visc}\begin{bmatrix}\mathbf{U}_h^{n+\frac{1}{2}}\\ \mathbf{V}_h^{n+\frac{1}{2}}\\ \mathbf{W}_h^{n+\frac{1}{2}}\end{bmatrix}\right\}\\
&\quad -\frac{\delta t}{2}\left\{\begin{bmatrix}\frac{\Delta \mathbf{u}_h}{\delta t}\\ \frac{\Delta \mathbf{v}_h}{\delta t}\\ \frac{\Delta \mathbf{w}_h}{\delta t}\end{bmatrix}^\top \mathbf{C}_{frict}\begin{bmatrix}\frac{\Delta \mathbf{u}_h}{\delta t}\\ \frac{\Delta \mathbf{v}_h}{\delta t}\\ \frac{\Delta \mathbf{w}_h}{\delta t}\end{bmatrix} + \begin{bmatrix}\mathbf{U}_h^{n+\frac{1}{2}}\\ \mathbf{V}_h^{n+\frac{1}{2}}\\ \mathbf{W}_h^{n+\frac{1}{2}}\end{bmatrix}^\top \mathbf{C}_{frict}\begin{bmatrix}\mathbf{U}_h^{n+\frac{1}{2}}\\ \mathbf{V}_h^{n+\frac{1}{2}}\\ \mathbf{W}_h^{n+\frac{1}{2}}\end{bmatrix}\right\}\\
&\leqslant 0 \qquad (6.16)
\end{aligned}$$

Figure 1: Initial conditions u_0, v_0 and w_0.

Remark 6.1 *The identity* (6.16) *corresponds to the discrete version of* (6.1)-(6.3)*. More precisely, the matrices of R^{3J}, $\mathbf{C}_{frict}$ and $\mathbf{C}_{visc}$ have only J nonzero rows, and depending on the distribution of these rows, is that the right term of* (6.16) *coincide with each one of the three cases* **VEF**, **EVF** *and* **EFV** *which correspond to* (6.1), (6.2) *and* (6.3), *respectively. Thus, for example, in the case of the Viscoelastic-Elastic-Frictional model* **VEF**, *it follows that*

$$\mathbf{C}_{visc} = \begin{pmatrix}\kappa_0\widetilde{\mathbf{C}}_1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}, \qquad \mathbf{C}_{frict} = \begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \gamma\widetilde{\mathbf{C}}_2\end{pmatrix},$$

and then, the identity (6.16) *can be rewrite as*

$$\begin{aligned}
\mathcal{E}_\delta^{n+1} - \mathcal{E}_\delta^n = &-\frac{\delta t}{2\delta x^2}\kappa_0\left\{\frac{\Delta \mathbf{u}_h^\top}{\delta t}\widetilde{\mathbf{C}}_1\frac{\Delta \mathbf{u}_h}{\delta t} + \mathbf{U}_h^{n+\frac{1}{2},\top}\widetilde{\mathbf{C}}_1\mathbf{U}_h^{n+\frac{1}{2}}\right\}\\
&-\frac{\delta t}{2}\gamma\left\{\frac{\Delta \mathbf{w}_h^\top}{\delta t}\widetilde{\mathbf{C}}_2\frac{\Delta \mathbf{w}_h}{\delta t} + \mathbf{W}_h^{n+\frac{1}{2},\top}\widetilde{\mathbf{C}}_2\mathbf{W}_h^{n+\frac{1}{2}}\right\},
\end{aligned}$$

which corresponds well to a discretization of (6.1) *consistent with the definition of energy. With this, we expect the rate of decay of energy in the discrete case is an accurate reflection of what happens in the continuous case.*

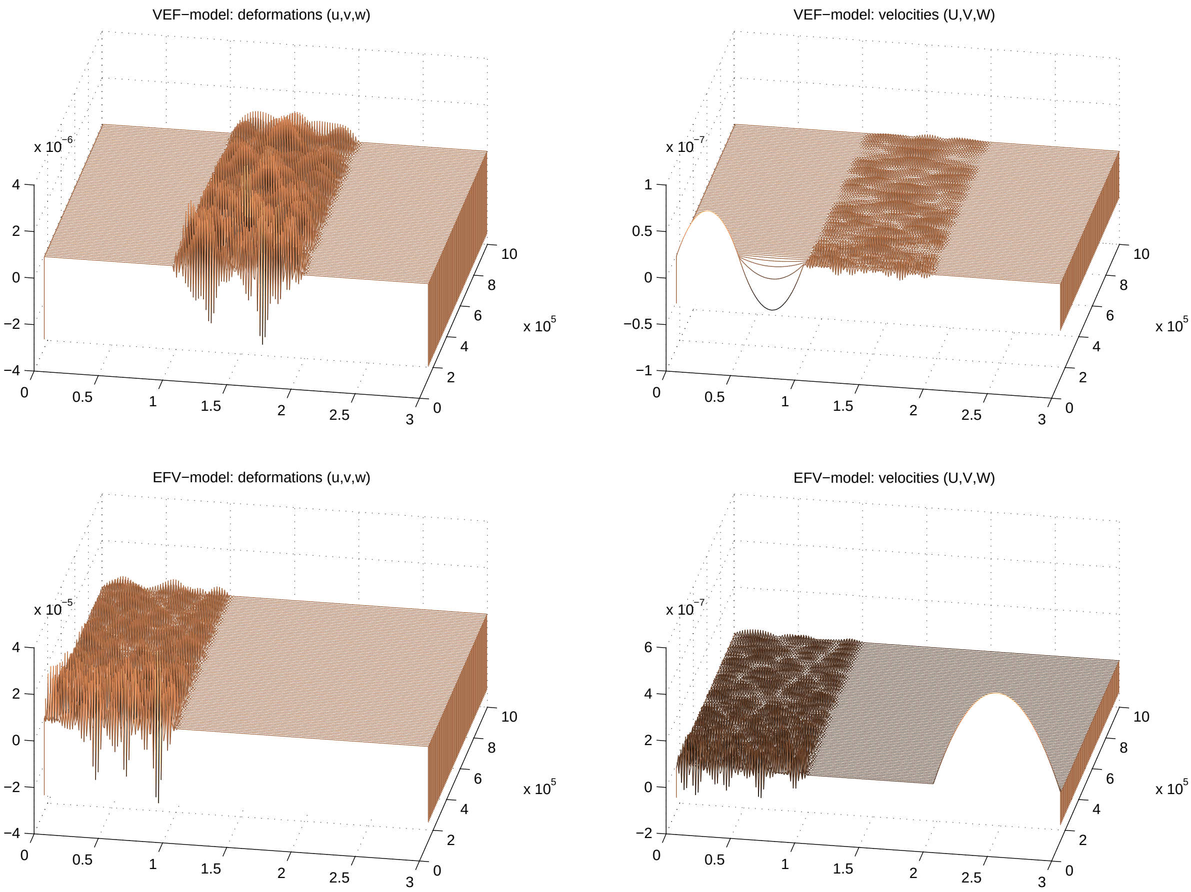


Figure 2: Viscoelastic-Elastic-Frictional (**VEF**) model (top), and Elastic-Frictional-Viscoelastic (**EFV**) model (bottom), simulation for $t \in (15,\, 1000000)$. Exponential Decay, when the frictional part is isolated of the viscoelastic part.

6.4 Numerical Example

Now we present an example with one initial condition in $\mathcal{D}(A)$ for the three cases to illustrate graphically the polynomial and exponential energy decay.

6.4.1 Example 1. Initial conditions with different smoothness

Let us suppose here that $l = 3$ and $T = 1000000$. We will study the asymptotic behavior for a family of initial conditions of the form

$$[\, u_0 \; v_0 \; w_0 \,] \;=\; \begin{cases} \left(x-\frac{1}{2}\right)\left|x-\frac{1}{2}\right|+\frac{1}{4} & \text{if } x \in (0,1) \\ \left(x-\frac{3}{2}\right)\left|x-\frac{3}{2}\right|+\frac{3}{4} & \text{if } x \in (1,2) \\ 2\,x^2 \,+\, 9\,x \,-\, 9 & \text{if } x \in (2,3) \\ 0 & \text{otherwise} \end{cases} \tag{6.17}$$

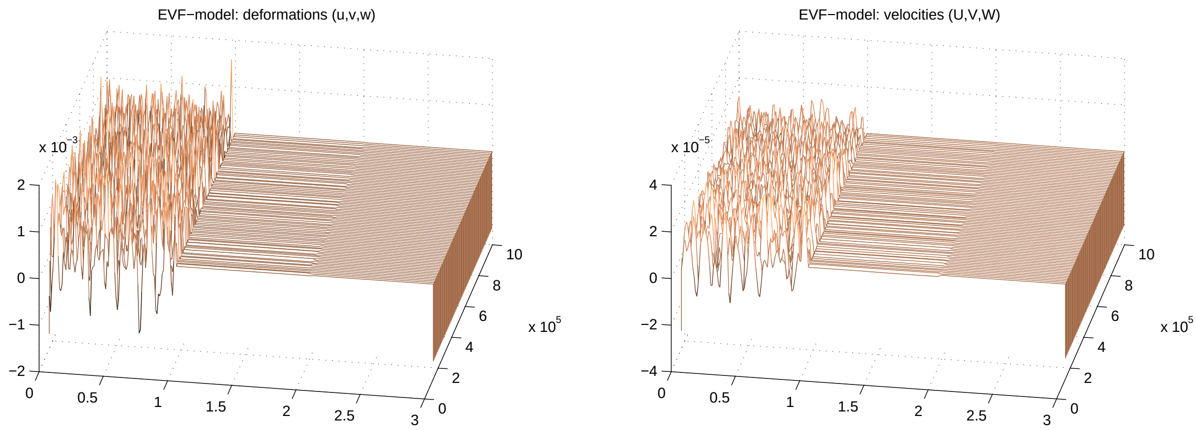


Figure 3: Elastic-Viscoelastic-Frictional (**EVF**) model, simulation for $t \in (15, 1000000)$. Polinomial Decay, when the viscoelastic part is in the middle and in contact with the fractional part.

at rest, that is $U_0 = V_0 = W_0 = 0$ (see Figure 1). We suppose additionally that $\kappa_1 = \kappa_2 = \kappa_3 = 1$, $\kappa_0 = 10000$, $\gamma = 100$. Note that the initial condition verifies be on $\mathcal{D}(\mathcal{A})$, and meets the minimum requirements of regularity for it. The discretization is given by $J = 300$ and $N = 10^6$, that is $\delta x = L/J = 0.01$ and $\delta t = T/N = 1$. Figure 2 shows the evolutionary behavior of cases, Viscoelastic-Elastic-Frictional (**VEF**) model (top), and Elastic-Frictional-Viscoelastic (**EFV**) model (bottom). In both cases, the energy decays exponentially, and correspond to the cases where the viscoelastic part is isolated from the frictional part. Both for the **VEF** model, as well as for the **EFV** model, both for the deformations u, v, w (on the left), as well as for the deformation velocities U, V, W, the viscoelastic and frictional part of these four cases, practically immediately fell to zero. On the other hand, the purely elastic, decays more slowly. Still, the total energy decays exponentially (see Figure 4). In both cases, we plot from the time $t = 15$, in order to improve the visual on the asymptotic behavior (which is the interest in short), and removing the initial behavior while important because it determines the rest, on the other hand, it changes the scale of the global behavior. In Figure 3, the viscoelastic part is on the middle, isolating the elastic part of the frictional part. That is, it correspnds to the Elastic-Viscoelastic-Frictional (**EVF**) model. While both the frictional, as the viscoelastic have a behavior called dissipative, the mere fact that the frictional part is isolated from the elastic, makes the latter not stabilize quickly enough in the case of **VEF** model or **EFV** model, and it decays only polynomial, which is what was shown in theory in the

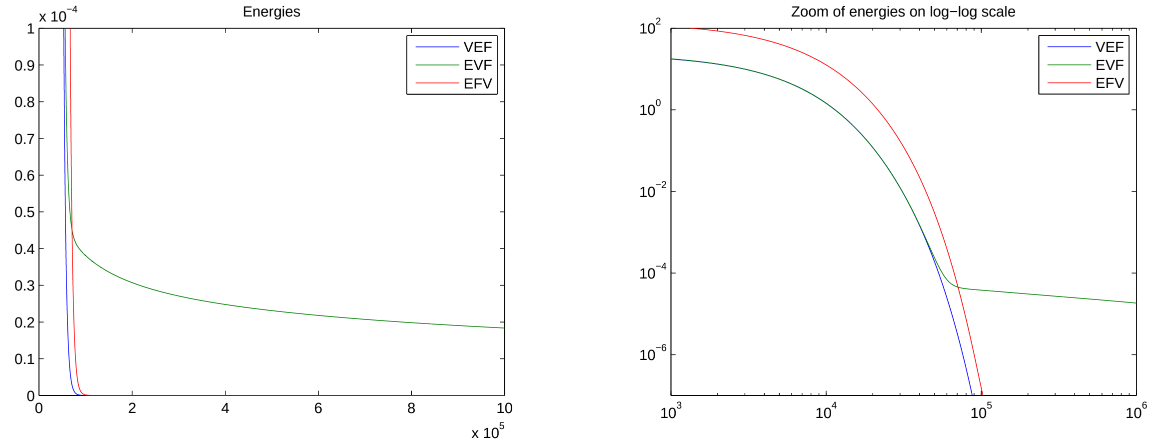


Figure 4: Energy decays for the three cases **VEF**, **EVF** and **EFV**. Left: plot of the energies; Right: zoom of the plot on log-log scale.

previous section, and currently checks in this figure.

Finally, in Figure 4, we plot the decay of the energies. Here, we see clearly the difference between **EFV** and **VEF** cases (whose decay is exponential) v/s **EVF** case (whose decay is polynomial). On the graph on the left, the energy is plotted directly, and clearly the **EVF** case seen well above the other two. The graph on the right is a zoom of the same graph but in log-log scale. In this zoom, shows an asymptotic behavior of the **EVF** case near a straight a line, which is interpreted in a log-log scale, as polynomial behaviour, instead the **VEF** and **VEF** cases decay much faster than just straight (exponential)

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