LOCKING-FREE FINITE ELEMENT METHOD FOR A BENDING MOMENT FORMULATION OF TIMOSHENKO BEAMS

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Abstract. In this paper we study a finite element formulation for Timoshenko beams. It is known that standard finite elements applied to this model lead to wrong results when the thickness of the beam t is small. Here, we consider a mixed formulation in terms of the transverse displacement, rotation, shear stress and bending moment. By using the classical Babuška-Brezzi theory it is proved that the resulting variational formulation is well posed. We discretize it by continuous piecewise linear finite elements for the shear stress and bending moment, and discontinuous piecewise constant finite elements for the displacement and rotation. We prove an optimal (linear) order of convergence in terms of the mesh size h for the natural norms and a double order (quadratic) in L^2-norms for the shear stress and bending moment, all with constants independent of the beam thickness. Moreover, these constants depend on norms of the solution that can be a priori bounded independently of the beam thickness, which leads to the conclusion that the method is locking-free. Numerical tests are reported in order to support our theoretical results.

1. Introduction

Beams used in practice, like in buildings and bridges as well as in aircrafts, cars, ships, etc., commonly present continuous and discontinuous variations of the geometry and the physical parameters. They may also have appreciable thickness where the shear stress is not negligible. As a result, the thick beam model based on the Timoshenko theory have gained more popularity (see for instance [2, 9, 10, 14, 19]).

In this paper, we study the numerical approximation of the bending of a non-homogeneous beam modeled by Timoshenko equations. Despite its simplicity, the numerical approximation of this problem often presents some difficulties. Indeed, it is very well known that standard finite element methods applied to models of thin structures, like beams, rods and plates, are subject to the so-called locking phenomenon. This means that they produce very unsatisfactory results when the thickness is small with respect to the other dimensions of the structure (see [8]). Indeed, several methods for this model have been rigorously shows to be free from locking and

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optimally convergent by resorting to a mixed formulations, discontinuous Galerkin methods, considering p and $h-p$ versions of the finite element method or based on reduced integration; let us mention, for instance [1, 5, 6, 7, 11, 12, 13, 15].

The purpose of this paper is to propose a mixed finite element method for a bending moment formulation of non-homogeneous Timoshenko beams and to provide an a priori error analysis. This approach follows the strategy used in [3] for Reissner-Mindlin plates, where a finite element method was introduced for the approximation of the bending of a clamped plate. However, the one-dimensional character of the problem allows us to give simpler proofs valid in a more general context. In particular, the results of this paper are valid for non-homogeneous beams, whose physical and geometrical properties may be discontinuous, and the error estimates are fully independent of the beam thickness. To cover such cases, a key point in our analysis is an improved regularity result, which we are able to prove exploiting, once more, the one-dimensional character of the problem.

In the present paper we consider a bending moment formulation for the Timoshenko beam problem. We introduce the bending moment together with the shear stress as new unknowns in the model (we note that the bending moment usually represents a quantity of major interest in engineering applications), which together with the rotation and the transverse displacement lead us to a mixed variational formulation. Using the Babuška-Brezzi theory, we show that the proposed variational formulation is well posed and stable in the natural norms of the considered Sobolev spaces . For the numerical approximation, piecewise linear and continuous finite elements are used for the bending moment and the shear stress and piecewise constants for the transverse displacement and rotation. We prove a uniform inf-sup condition with respect to the discretization parameter h and the thickness t. We note that one advantage of this formulation is that, the bending moment and the shear stress are computed directly instead of by a post-process which may produces loss of accuracy. The method is proved to have an optimal (linear) order of convergence in terms of the mesh size h for the natural norms and a double order (quadratic) in L^2-norms for the shear stress and bending moment. Moreover, the obtained estimates only depend on norms of quantities which are known to be bounded independently of t. Therefore, the method turns out to be thoroughly locking-free.

The outline of this paper is as follows: In Section 2, we recall the differential equations governing the problem, and state a mixed variational formulation. Then, we prove the unique solvability and stability properties of the proposed formulation and some regularity results. In Section 3, we present the finite element discretization of our variational formulation, for which we prove a discrete inf-sup condition uniformly with respect to the beam thickness t and the mesh parameter h. Then, we establish the linear convergence of the method for the natural norms and a quadratic order in L^2-norm for the shear stress and bending moment. In Section 4, we report some numerical tests which confirm the theoretical error estimates and allow us to assess the performance of the proposed method. Finally, we summarize some conclusions in Section 5.

We will use standard notations for Sobolev spaces, norms and seminorms. For $l \geq 0$, $\|\cdot\|_{l,\mathrm{I}}$ stands the norm of the Hilbertian Sobolev space $H^l(\mathrm{I})$, with the convention $H^0(\mathrm{I}) := L^2(\mathrm{I})$. Moreover, we will denote with c and C, with or without subscripts, tildes or hats, a generic positive constant, possibly different at different occurrences, independent of the beam thickness t and the mesh parameter h introduced in the next section.

2. Timoshenko beam model.

Let us consider an elastic beam which satisfies the Timoshenko hypotheses for the admissible displacements. We assume that the geometry and the physical parameters of the beam may change along the axial direction. The deformation of the beam is described in terms of the transverse displacement w and the rotation of the transversal fibers β. Let x be the coordinate in the axial direction.

The equations for the bending of a clamped Timoshenko beam subjected to a distributed load $p(x)$ reads as follows (see [16, 17, 18]):

Find $(\beta(x), w(x)) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$ *such that*

$$\int_{\mathrm{I}} E(x)\mathbb{I}(x)\beta'(x)\eta'(x)\,dx + \int_{\mathrm{I}} G(x)A(x)k_c(x)(\beta(x) - w'(x))(\eta(x) - v'(x))\,dx = \int_{\mathrm{I}} p(x)v(x)\,dx \tag{2.1}$$

for all $(\eta(x), v(x)) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$, where $\mathrm{I} := (0, L)$, L being the length of the beam, $E(x)$ is the Young modulus, $\mathbb{I}(x)$ the moment of inertia of the cross-section, $A(x)$ the area of the cross-section, $G(x) := E(x)/(2(1 + \nu(x)))$ the shear modulus, with $\nu(x)$ the Poisson ratio, and $k_c(x)$ a correction factor. We consider that $E(x)$, $\mathbb{I}(x)$, $A(x)$ and $\nu(x)$ are piecewice smooth in I, the most usual case being when all those coefficients are piecewise constant. Moreover, primes denote derivatives with respect to the x-coordinate.

It is well known that standard finite element procedures, when used in formulations such as (2.1) for very thin structures are subject to numerical locking, a phenomenon induced by the difference of magnitude between the coefficients in front of the different terms (see [1]). The appropriate framework for analysing this difficulty is obtained by rescaling formulation (2.1) so as to identify a family of problems with a well-posed limit as the thickness becomes infinitely small. With this aim, we introduce the following nondimensional parameter, characteristic of the thickness of the beam,

$$t^2 := \frac{1}{L}\int_{\mathrm{I}} \frac{\mathbb{I}(x)}{A(x)L^2}\,dx, \tag{2.2}$$

which we assume may take values in the range $(0, t_{\max}]$.

We define

$$f(x) := \frac{p(x)}{t^3}, \quad \hat{\mathbb{I}}(x) := \frac{\mathbb{I}(x)}{t^3}, \quad \hat{A}(x) := \frac{A(x)}{t},$$
$$\mathbb{E}(x) := E(x)\hat{\mathbb{I}}(x) \quad \text{and} \quad \kappa(x) := G(x)\hat{A}(x)k_c(x).$$

Then, problem (2.1) can be equivalently written as follows:

Find $(\beta, w) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$ *such that*

$$\int_{\mathrm{I}} \mathbb{E}(x)\beta'(x)\eta'(x)\,dx + \frac{1}{t^2}\int_{\mathrm{I}} \kappa(x)(\beta(x) - w'(x))(\eta(x) - v'(x))\,dx = \int_{\mathrm{I}} f(x)v(x)\,dx \tag{2.3}$$

for all $(\eta(x), v(x)) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$.

Now, we assume that there exist constants $\underline{\mathbb{E}}, \overline{\mathbb{E}}, \underline{\kappa}, \overline{\kappa} \in \mathbb{R}^+$ such that

$$\begin{aligned} \overline{\mathbb{E}} \geq \mathbb{E}(x) \geq \underline{\mathbb{E}} > 0 \quad \forall x \in \mathrm{I}, \\ \overline{\kappa} \geq \kappa(x) \geq \underline{\kappa} > 0 \quad \forall x \in \mathrm{I}. \end{aligned} \tag{2.4}$$

In such a case, for each $t > 0$, the bilinear form on the left hand side of (2.3) is elliptic and hence this problem has a unique solution.

The aim of this paper is to consider a bending moment formulation of this problem. With this end, by introducing the bending moment $\sigma(x) := \mathbb{E}(x)\beta'(x)$ and the shear stress $\gamma(x) := t^{-2}\kappa(x)(\beta(x) - w'(x))$ as new unknowns in the model, problem (2.3) can be equivalently rewritten as follows (see [5]):

$$\begin{cases} \sigma(x) = \mathbb{E}(x)\beta'(x) & \text{in} \quad \mathrm{I}, \\ -\sigma'(x) + \gamma(x) = 0 & \text{in} \quad \mathrm{I}, \\ \gamma'(x) = f(x) & \text{in} \quad \mathrm{I}, \\ \gamma(x) = t^{-2}\kappa(x)(\beta(x) - w'(x)) & \text{in} \quad \mathrm{I}, \\ w(0) = \beta(0) = w(L) = \beta(L) = 0. \end{cases} \tag{2.5}$$

Finally, testing the equations (2.5) with adequate functions and integrating by parts, we obtain the following variational formulation, where from now on we omit the dependence on the axial variable x:

Find $((\sigma, \gamma), (\beta, w)) \in V \times Q$ *such that*

$$\begin{aligned} \int_{\mathrm{I}} \frac{\sigma\tau}{\mathbb{E}} + t^2 \int_{\mathrm{I}} \frac{\gamma\xi}{\kappa} + \int_{\mathrm{I}} \beta(\tau' - \xi) - \int_{\mathrm{I}} w\xi' = 0 \quad \forall(\tau, \xi) \in V, \\ \int_{\mathrm{I}} \eta(\sigma' - \gamma) - \int_{\mathrm{I}} v\gamma' = -\int_{\mathrm{I}} fv \quad \forall(\eta, v) \in Q, \end{aligned} \tag{2.6}$$

where

$$V := H^1(\mathrm{I}) \times H^1(\mathrm{I}),$$

and

$$Q := L^2(\mathrm{I}) \times L^2(\mathrm{I}),$$

each one endowed with the corresponding product norm.

Finally, we endow $V \times Q$ with the corresponding product norm.

We rewrite the variational problem (2.6) as follows:

Find $((\sigma, \gamma), (\beta, w)) \in V \times Q$ *such that*

$$a((\sigma, \gamma), (\tau, \xi)) + b((\tau, \xi), (\beta, w)) = 0 \qquad \forall(\tau, \xi) \in V, \tag{2.7}$$

$$b((\sigma, \gamma), (\eta, v)) = F(\eta, v) \qquad \forall(\eta, v) \in Q, \tag{2.8}$$

where the bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ and the linear functional $F : Q \to \mathbb{R}$ are defined by

$$a((\sigma, \gamma), (\tau, \xi)) := \int_{\mathrm{I}} \frac{\sigma\tau}{\mathbb{E}} + t^2 \int_{\mathrm{I}} \frac{\gamma\xi}{\kappa}, \tag{2.9}$$

$$b((\tau, \xi), (\eta, v)) := \int_{\mathrm{I}} \eta(\tau' - \xi) - \int_{\mathrm{I}} v\xi', \tag{2.10}$$

and

$$F(\eta, v) := -\int_{\mathrm{I}} fv,$$

for all $(\sigma, \gamma), (\tau, \xi) \in V$ and $(\eta, v) \in Q$.

Next, we will prove that problem (2.7)–(2.8) satisfies the hypotheses of the Babuška-Brezzi theory, which yields the unique solvability and continuous dependence on the data of this variational formulation.

We first observe that the bilinear forms a and b and the linear functional F are bounded with constants independent of the beam thickness t.

Let

$$K := \{(\tau, \xi) \in V : \ b((\tau, \xi), (\eta, v)) = 0 \ \forall (\eta, v) \in Q\}$$

be the so-called continuous kernel; hence (cf. (2.10))

$$K = \{(\tau, \xi) \in V : \ (\tau' - \xi) = 0, \text{ and } \xi' = 0 \text{ in I}\} = \{(\tau, \tau') : \tau \in \mathbb{P}_1(\mathrm{I})\}.$$

The following lemma shows that the bilinear form a is V-elliptic in K uniformly in t.

Lemma 2.1. *There exists $\alpha > 0$, independent of t, such that*

$$a((\tau, \xi), (\tau, \xi)) \geq \alpha \|(\tau, \xi)\|_V^2 \qquad \forall (\tau, \xi) \in K.$$

Proof. Given $(\tau, \xi) \in K$, from (2.9) and (2.4) we obtain

$$a((\tau, \xi), (\tau, \xi)) \geq \frac{1}{\overline{\mathbb{E}}} \|\tau\|_{0,\mathrm{I}}^2 + \frac{t^2}{\overline{\kappa}} \|\xi\|_{0,\mathrm{I}}^2 \geq \frac{1}{\overline{\mathbb{E}}} \|\tau\|_{0,\mathrm{I}}^2 \geq C \|\tau\|_{1,\mathrm{I}}^2,$$

where the last inequality because of the equivalence between $\|\cdot\|_{0,\mathrm{I}}$ and $\|\cdot\|_{1,\mathrm{I}}$, with a constant independent of t in $K \cong \mathbb{P}_1(\mathrm{I})$. Thus, the result follows from the fact that

$$\|(\tau, \xi)\|_V^2 = \|\tau\|_{1,\mathrm{I}}^2 + \|\tau'\|_{0,\mathrm{I}}^2 \quad \forall (\tau, \xi) \in K.$$

Therefore, we end the proof. □

Now, we are in a position to prove an inf-sup condition for the bilinear form b uniformly in t.

Lemma 2.2. *There exists $C > 0$, independent of t, such that*

$$\sup_{\substack{(\tau,\xi) \in V \\ (\tau,\xi) \neq 0}} \frac{b((\tau, \xi), (\eta, v))}{\|(\tau, \xi)\|_V} \geq C \|(\eta, v)\|_Q \qquad \forall (\eta, v) \in Q.$$

Proof. Let $(\eta, v) \in Q$. Let $\widetilde{\tau}(r) := \int_0^r \eta(s) ds$, $0 \leq r \leq L$. We have that $\widetilde{\tau}' = \eta \in L^2(\mathrm{I})$. Hence $\widetilde{\tau} \in H^1(\mathrm{I})$, and

$$\|\widetilde{\tau}\|_{1,\mathrm{I}} \leq \left(\frac{L^2 + 2}{2} \right)^{1/2} \|\eta\|_{0,\mathrm{I}}.$$

Therefore,

$$\sup_{\substack{(\tau,\xi) \in V \\ (\tau,\xi) \neq 0}} \frac{b((\tau, \xi), (\eta, v))}{\|(\tau, \xi)\|_V} \geq \frac{b((\widetilde{\tau}, 0), (\eta, v))}{\|\widetilde{\tau}\|_{1,\mathrm{I}}} = \frac{\|\eta\|_{0,\mathrm{I}}^2}{\|\widetilde{\tau}\|_{1,\mathrm{I}}} \geq \left(\frac{2}{L^2 + 2} \right)^{1/2} \|\eta\|_{0,\mathrm{I}}. \tag{2.11}$$

Finally, let $\widetilde{\xi}(r) := -\int_0^r v(s) ds$, $0 \leq r \leq L$. The same arguments as above allow us to prove that

$$\|\widetilde{\xi}\|_{1,\mathrm{I}} \leq \left(\frac{L^2 + 2}{2} \right)^{1/2} \|v\|_{0,\mathrm{I}}.$$

Hence, it follows that

$$\sup_{\substack{(\tau,\xi)\in V\\(\tau,\xi)\neq 0}} \frac{b((\tau,\xi),(\eta,v))}{\|(\tau,\xi)\|_V} \geq \frac{b((0,\widetilde{\xi}),(\eta,v))}{\|\widetilde{\xi}\|_{1,\mathrm{I}}}$$

$$= \frac{\|v\|_{0,\mathrm{I}}^2 - \int_{\mathrm{I}} \eta\widetilde{\xi}}{\|\widetilde{\xi}\|_{1,\mathrm{I}}} \geq \left(\frac{2}{L^2+2}\right)^{1/2} \|v\|_{0,\mathrm{I}} - \|\eta\|_{0,\mathrm{I}}.$$

From this inequality and (2.11), it is immediate to show that

$$\sup_{\substack{(\tau,\xi)\in V\\(\tau,\xi)\neq 0}} \frac{b((\tau,\xi),(\eta,v))}{\|(\tau,\xi)\|_V} \geq \frac{2}{\sqrt{L^2+2}(\sqrt{L^2+2}+\sqrt{2})} \|v\|_{0,\mathrm{I}}.$$

Thus, the proof follows from this estimate, and (2.11). □

We are now in a position to state the main result of this section which yields the solvability of the continuous problem (2.7)–(2.8).

Theorem 2.1. *There exists a unique solution* $((\sigma,\gamma),(\beta,w)) \in V \times Q$ *to problem* (2.7)–(2.8) *and the following continuous dependence result holds:*

$$\|((\sigma,\gamma),(\beta,w))\|_{V\times Q} \leq C\|f\|_{0,\mathrm{I}},$$

where C *is independent of* t.

Proof. By virtue of Lemmas 2.1 and 2.2, the proof follows from a straightforward application of [4, Theorem II.1.1]. □

The following result establish an additional regularity result for the solution of problem (2.7)–(2.8). This result will be the key point to prove the convergence of the propose method.

Proposition 2.1. *Suppose that* $f \in H^l(\mathrm{I})$, $l = 0,1$. *Let* $((\sigma,\gamma),(\beta,w)) \in V \times Q$ *be the solution to problem* (2.7)–(2.8). *Then, there exists a constant* C*, independent of* t *and* f*, such that*

$$\|w\|_{1,\mathrm{I}} + \|\beta\|_{1,\mathrm{I}} + \|\gamma\|_{l+1,\mathrm{I}} + \|\sigma\|_{l+2,\mathrm{I}} \leq C\|f\|_{l,\mathrm{I}}.$$

Proof. Because of the equivalence between problems (2.6) and (2.3), we will consider the latter to prove the result.

Consider the following decomposition for the scaled shear stress:

$$\gamma = \psi' + k, \tag{2.12}$$

with $\psi \in H_0^1(\mathrm{I})$ and $k := (\frac{1}{L}\int_{\mathrm{I}}\gamma) \in \mathbb{R}$. We have that problem (2.3) and the following are equivalent:

Given $f \in H^l(\mathrm{I})$, $l = 0, 1$, *find* $(\psi, \beta, k, w) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I}) \times \mathbb{R} \times H_0^1(\mathrm{I})$ *such that*

$$\left\{\begin{array}{l} \displaystyle\int_{\mathrm{I}} \psi' v' = -\int_{\mathrm{I}} f v \quad \forall v \in H_0^1(\mathrm{I}), \\ \displaystyle\int_{\mathrm{I}} \mathbb{E}\beta'\eta' + \int_{\mathrm{I}} k\eta = -\int_{\mathrm{I}} \psi'\eta \quad \forall \eta \in H_0^1(\mathrm{I}), \\ \displaystyle\int_{\mathrm{I}} \beta q - t^2 \int_{\mathrm{I}} \frac{kq}{\kappa} = -t^2 \int_{\mathrm{I}} \frac{\psi' q}{\kappa} \quad \forall q \in \mathbb{R}, \\ \displaystyle\int_{\mathrm{I}} w'\delta' = \int_{\mathrm{I}} \beta\delta' + t^2 \int_{\mathrm{I}} \frac{\psi'\delta'}{\kappa} \quad \forall \delta \in H_0^1(\mathrm{I}). \end{array}\right. \tag{2.13}$$

For this problem, we have that for any $t \in (0, t_{\max}]$ and $f \in H^l(\mathrm{I})$, $l = 0, 1$, there exists a unique solution $(\psi, \beta, k, w) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I}) \times \mathbb{R} \times H_0^1(\mathrm{I})$. Moreover, there exists a constant C independent of t and f, such that

$$\|\psi\|_{l+2,\mathrm{I}} + \|\beta\|_{1,\mathrm{I}} + |k| + \|w\|_{1,\mathrm{I}} \leq C\|f\|_{l,\mathrm{I}}.$$

In fact, given $f \in H^l(\mathrm{I})$, $l = 0, 1$, from problem $(2.13)_1$ and the Lax-Milgram's Theorem, we have that there exists a unique $\psi \in H_0^1(\mathrm{I}) \cap H^{l+2}(\mathrm{I})$, solution and $\|\psi\|_{l+2,\mathrm{I}} \leq C\|f\|_{l,\mathrm{I}}$, $l = 0, 1$. Now, for all $t \in (0, t_{\max}]$ we can apply Theorem 5.1 of [1] to obtain that there exists a unique solution $(\beta, k) \in H_0^1(\mathrm{I}) \times \mathbb{R}$ of problem $(2.13)_{2-3}$, moreover,

$$\|\beta\|_{1,\mathrm{I}} + |k| \leq C\|\psi\|_{1,\mathrm{I}} \leq C\|f\|_{0,\mathrm{I}},$$

where the constant C is independent of t.

Finally, we obtain from the Lax-Milgram's lemma, that there exists a unique solution $w \in H_0^1(\mathrm{I})$ of problem $(2.13)_4$, and taking $\xi = w$, we obtain

$$\|w\|_{1,\mathrm{I}} \leq C(\|\beta\|_{0,\mathrm{I}} + \|\psi'\|_{0,\mathrm{I}}) \leq C\|f\|_{0,\mathrm{I}}.$$

Consequently, by virtue of (2.12), (2.4), the first and the second equation of (2.5) and the equivalence between problems (2.3) and (2.13), we have that there exists C independent of t and f such that for $l = 0, 1$

$$\|\beta\|_{1,\mathrm{I}} + \|w\|_{1,\mathrm{I}} + \|\gamma\|_{l+1,\mathrm{I}} + \|\sigma\|_{l+2,\mathrm{I}} \leq C\|f\|_{l,\mathrm{I}}.$$

The proof is complete. □

Remark 2.1. *It is possible to refine the proposition above by considering piecewise smooth loads. Suppose that there exists a partition* $0 = s_0 < \cdots < s_n = L$, *of the interval* I. *If we denote* $S_i := (s_{i-1}, s_i)$, $i = 1, \ldots, n$, *then, we assume that* $f \in H^1(S_i)$. *Then, repeating the arguments used in the proof, we have the following result for the solution of problem* (2.7)–(2.8)*: There exist a constant* C*, independent of* t *and* f*, such that*

$$\|w\|_{1,\mathrm{I}} + \|\beta\|_{1,\mathrm{I}} + \|\gamma\|_{1,\mathrm{I}} + \left(\sum_{i=1}^n \|\gamma''\|_{0,S_i}^2\right)^{1/2} + \|\sigma\|_{2,\mathrm{I}} + \left(\sum_{i=1}^n \|\sigma'''\|_{0,S_i}^2\right)^{1/2} \leq C\left(\|f\|_{0,\mathrm{I}}^2 + \sum_{i=1}^n \|f'\|_{0,S_i}^2\right)^{1/2}.$$

3. The mixed finite element scheme

In this section, we present our discrete methods for the Timoshenko beam problem. With this aim, first we consider a family of partitions of I

$$\mathcal{T}_h := 0 = x_0 < \cdots < x_N = L.$$

We denote $\mathrm{I}_j = (x_{j-1}, x_j)$, with length $h_j = x_j - x_{j-1}$, $j = 1, \ldots, N$, and the maximun subinterval length is denoted $h := \max_{1\leq j\leq N} h_j$.

To approximate the shear stress and the bending moment, we consider the space of piecewise linear continuous finite elements:

$$W_h := \{\xi_h \in H^1(\mathrm{I}) \ : \ \xi_h|_{\mathrm{I}_j} \in \mathbb{P}_1(\mathrm{I}_j),\ j = 1, \ldots, N\}.$$

Let $\mathcal{L}(\xi) \in W_h$ be the Lagrange interpolant of $\xi \in H^1(\mathrm{I})$, we recall that

$$\|\xi - \mathcal{L}(\xi)\|_{1,\mathrm{I}} \leq Ch|\xi|_{2,\mathrm{I}} \qquad \forall \xi \in H^2(\mathrm{I}), \tag{3.1}$$

$$\|\xi - \mathcal{L}(\xi)\|_{0,\mathrm{I}} \leq Ch^2|\xi|_{2,\mathrm{I}} \qquad \forall \xi \in H^2(\mathrm{I}). \tag{3.2}$$

To approximate the transverse displacement and the rotation, we will use the space of piecewise constant functions:

$$Z_h := \{v_h \in L^2(\mathrm{I}) \ : \ v_h|_{\mathrm{I}_j} \in \mathbb{P}_0(\mathrm{I}_j),\ j = 1, \ldots, N\}.$$

We also consider the L^2-proyector onto Z_h:

$$\mathcal{P} : L^2(\mathrm{I}) \to Z_h,$$
$$v \mapsto \mathcal{P}(v) := \bar{v} \in Z_h \ : \ \int_{\mathrm{I}} (v - \bar{v})q_h = 0 \quad \forall q_h \in Z_h.$$

Then, we have

$$\|v - \mathcal{P}(v)\|_{0,\mathrm{I}} \leq Ch|v|_{1,\mathrm{I}} \qquad \forall v \in H^1(\mathrm{I}). \tag{3.3}$$

Defining $V_h := W_h \times W_h$ and $Q_h := Z_h \times Z_h$, the discretization of problem (2.7)-(2.8) reads as follows:

Find $((\sigma_h, \gamma_h), (\beta_h, w_h)) \in V_h \times Q_h$ *such that*

$$a((\sigma_h, \gamma_h), (\tau_h, \xi_h)) + b((\tau_h, \xi_h), (\beta_h, w_h)) = 0 \qquad \forall (\tau_h, \xi_h) \in V_h, \tag{3.4}$$

$$b((\sigma_h, \gamma_h), (\eta_h, v_h)) = F(\eta_h, v_h) \qquad \forall (\eta_h, v_h) \in Q_h. \tag{3.5}$$

Our next goal is to show the corresponding discrete versions of Lemmas 2.1 and 2.2 to conclude the solvability and stability of problem (3.4)-(3.5). With this aim, we note that the discrete null space of the bilinear form b reduces to:

$$K_h := \{(\tau_h, \xi_h) \in V_h : b((\tau_h, \xi_h), (\eta_h, v_h)) = 0 \quad \forall (\eta_h, v_h) \in Q_h\}.$$

Let $(\tau_h, \xi_h) \in K_h$, taking $(0, v_h) \in Q_h$ and using that $\xi_h'|_{\mathrm{I}_j}$ is a constant, since $v_h|_{\mathrm{I}_j}$ is also a constant, we conclude that $\xi_h' = 0$ in I.

Now, taking $(\eta_h, 0) \in Q_h$, since $\tau_h'|_{\mathrm{I}_j}$ is a constant and $\xi_h|_{\mathrm{I}_j}$ is also a constant, we conclude that $(\tau_h' - \xi_h) = 0$ in I_j and hence $\tau_h' = \xi_h$ in I. Thus, we obtain

$$K_h = \{(\tau_h, \xi_h) \in V_h \ : \ (\tau_h' - \xi_h) = 0 \ \text{ and } \ \xi_h' = 0 \ \text{ in } \ \mathrm{I}\} = \{(\tau_h, \tau_h') : \tau_h \in \mathbb{P}_1(\mathrm{I})\}.$$

Hence, we have that K_h coincides with K, thus, we have

Lemma 3.1. *There exists* $\alpha > 0$ *independent of* h *and* t *such that*

$$a((\tau_h, \xi_h), (\tau_h, \xi_h)) \geq \alpha \|(\tau_h, \xi_h)\|_V^2 \quad \forall (\tau_h, \xi_h) \in K_h.$$

We continue with the following lemma establishing the discrete analogue to Lemma 2.2.

Lemma 3.2. *There exists $C > 0$, independent of h and t, such that*

$$\sup_{\substack{(\tau_h,\xi_h)\in V_h\\(\tau_h,\xi_h)\neq 0}} \frac{b((\tau_h,\xi_h),(\eta_h,v_h))}{\|(\tau_h,\xi_h)\|_V} \geq C\|(\eta_h,v_h)\|_Q \quad \forall(\eta_h,v_h)\in Q_h.$$

Proof. Let $(\eta_h, v_h) \in Q_h$. The arguments used in the proof of Lemma 2.2 can be applied. In fact, $\widetilde{\tau}(r) := \int_0^r \eta_h(s)ds$ lies in W_h, and the same happens with $\widetilde{\xi}(r) := \int_0^r v_h(s)ds$. □

We are now in a position to establish the unique solvability, the stability, and the convergence properties of the discrete problem (3.4)-(3.5).

Theorem 3.1. *There exists a unique $((\sigma_h,\gamma_h),(\beta_h,w_h)) \in V_h \times Q_h$ solution of the discrete problem* (3.4)-(3.5)*. Moreover, there exist $\widetilde{C}, C > 0$, independent of h and t, such that*

$$\|((\sigma_h,\gamma_h),(\beta_h,w_h))\|_{V\times Q} \leq \widetilde{C}\|f\|_{0,\mathrm{I}},$$

and

$$\begin{aligned}\|((\sigma,\gamma),(\beta,w)) - ((\sigma_h,\gamma_h),(\beta_h,w_h))\|_{V\times Q}\\ \leq C \inf_{((\tau_h,\xi_h),(\eta_h,v_h))\in V_h\times Q_h} \|((\sigma,\gamma),(\beta,w)) - ((\tau_h,\xi_h),(\eta_h,v_h))\|_{V\times Q},\end{aligned} \tag{3.6}$$

where $((\sigma,\gamma),(\beta,w)) \in V \times Q$ is the unique solution of the mixed variational formulation (2.7)-(2.8)*.*

Proof. Existence and uniqueness of problem (3.4)-(3.5) and the error bound (3.6) follow from the abstract theory for the saddle point problem [4, Theorem II.2.1] and Lemmas 3.1 and 3.2. □

The following theorem provides the rate of convergence of our mixed finite element scheme (3.4)-(3.5).

Theorem 3.2. *Let $((\sigma,\gamma),(\beta,w)) \in V \times Q$ and $((\sigma_h,\gamma_h),(\beta_h,w_h)) \in V_h \times Q_h$ be the unique solutions of the continuous and discrete problems* (2.7)-(2.8) *and* (3.4)-(3.5)*, respectively. If $f \in H^1(\mathrm{I})$, then,*

$$\|((\sigma,\gamma),(\beta,w)) - ((\sigma_h,\gamma_h),(\beta_h,w_h))\|_{V\times Q} \leq Ch\|f\|_{1,\mathrm{I}},$$

where the constant $C > 0$ is independent of h and t.

Proof. From Theorem 3.1 we have

$$\begin{aligned}\|((\sigma,\gamma),(\beta,w)) - ((\sigma_h,\gamma_h),(\beta_h,w_h))\|_{V\times Q}\\ \leq C\|((\sigma,\gamma),(\beta,w)) - ((\mathcal{L}(\sigma),\mathcal{L}(\gamma)),(\mathcal{P}(\beta),\mathcal{P}(w)))\|_{V\times Q}.\end{aligned} \tag{3.7}$$

Then, the proof follows from the term above, and using error estimates for $\mathcal{P}$ (see (3.3)) and $\mathcal{L}$ (see (3.1)-(3.2)), and Proposition 2.1. □

The theorem above, implies that $\|\sigma - \sigma_h\|_{0,\mathrm{I}} + \|\gamma - \gamma_h\|_{0,\mathrm{I}} = O(h)$, but it is possible to refine this estimate as show the following result.

Theorem 3.3. *Under the assumptions of Theorem 3.2*

$$\|(\sigma-\sigma_h,\gamma-\gamma_h)\|_Q := \left(\|\sigma-\sigma_h\|_{0,\mathrm{I}}^2 + \|\gamma-\gamma_h\|_{0,\mathrm{I}}^2\right)^{1/2} \leq Ch^2\|f\|_{1,\mathrm{I}},$$

where $C > 0$ is independent of h and t.

Proof. We resort to a duality arguments. First, we consider the following well posed problem: Given $g = (g_1, g_2) \in L^2(\mathrm{I})^2$, find $((\phi, \rho), (\chi, u)) \in V \times Q$ such that

$$a((\tau, \xi), (\phi, \rho)) + b((\tau, \xi), (\chi, u)) = (g, (\tau, \xi))_{0,\mathrm{I}} \qquad \forall (\tau, \xi) \in V, \tag{3.8}$$

$$b((\phi, \rho), (\eta, v)) = 0 \qquad \forall (\eta, v) \in Q. \tag{3.9}$$

The unique solution of the problem above satisfies the following additional regularity result: There exists a constant $C > 0$, independent of t and g such that

$$\|u\|_{1,\mathrm{I}} + \|\chi\|_{1,\mathrm{I}} + \|\rho\|_{2,\mathrm{I}} + \|\phi\|_{2,\mathrm{I}} \leq C\|g\|_{0,\mathrm{I}}. \tag{3.10}$$

In fact, considering a decomposition similar to (2.12) for the variable ρ (namely, writing $\rho = \lambda' + r$, with $\lambda \in H_0^1(\mathrm{I})$ and $r := (\frac{1}{L}\int_{\mathrm{I}} \rho) \in \mathbb{R}$), we have that problem (3.8)–(3.9) and the following are equivalent: *Given* $g = (g_1, g_2) \in L^2(\mathrm{I})^2$, *find* $(\lambda, \chi, r, u) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I}) \times \mathbb{R} \times H_0^1(\mathrm{I})$ *such that*

$$\left\{\begin{aligned} &\int_{\mathrm{I}} \lambda' v' = 0 \quad \forall v \in H_0^1(\mathrm{I}),\\ &\int_{\mathrm{I}} \mathbb{E}\chi'\eta' + \int_{\mathrm{I}} r\eta = -\int_{\mathrm{I}} \mathbb{E} g_1 \eta' - \int_{\mathrm{I}} \lambda' \eta \quad \forall \eta \in H_0^1(\mathrm{I}),\\ &\int_{\mathrm{I}} \chi q - t^2 \int_{\mathrm{I}} \frac{rq}{\kappa} = -\int_{\mathrm{I}} g_2 q - t^2 \int_{\mathrm{I}} \frac{\lambda' q}{\kappa} \quad \forall q \in \mathbb{R},\\ &\int_{\mathrm{I}} u'\delta' = \int_{\mathrm{I}} \chi\delta' + \int_{\mathrm{I}} g_2 \delta' + t^2 \int_{\mathrm{I}} \frac{\lambda'\delta'}{\kappa} \quad \forall \delta \in H_0^1(\mathrm{I}). \end{aligned}\right. \tag{3.11}$$

From the first of these equation we have that $\lambda = 0$. Then, reapeating the arguments used to prove Proposition 2.1, we conclude the additional regularity (3.10).

On the other hand, by choosing $(\tau, \xi) = (\sigma - \sigma_h, \gamma - \gamma_h)$ in problem (3.8)–(3.9), we obtain

$$(g, (\sigma - \sigma_h, \gamma - \gamma_h))_{0,\mathrm{I}} = a((\sigma - \sigma_h, \gamma - \gamma_h), (\phi, \rho)) + b((\sigma - \sigma_h, \gamma - \gamma_h), (\chi, u)). \tag{3.12}$$

Substracting (2.7) and (3.4) and using (3.9), we have

$$\begin{aligned} a((\sigma - \sigma_h, \gamma - \gamma_h), (\phi_h, \rho_h)) &= -b((\phi_h, \rho_h), (\beta - \beta_h, w - w_h))\\ &= b((\phi - \phi_h, \rho - \rho_h), (\beta - \beta_h, w - w_h)) \quad \forall (\phi_h, \rho_h) \in V_h. \end{aligned}$$

Moreover, from (2.8) and (3.5), we also have

$$b((\sigma - \sigma_h, \gamma - \gamma_h), (\chi_h, u_h)) = 0 \quad \forall (\chi_h, u_h) \in Q_h.$$

Substituting the last two terms into (3.12) we obtain:

$$\begin{aligned} &(g, (\sigma - \sigma_h, \gamma - \gamma_h))_{0,\mathrm{I}} = a((\sigma - \sigma_h, \gamma - \gamma_h), (\phi - \phi_h, \rho - \rho_h))\\ &\quad + b((\sigma - \sigma_h, \gamma - \gamma_h), (\chi - \chi_h, u - u_h)) + b((\phi - \phi_h, \rho - \rho_h), (\beta - \beta_h, w - w_h)) \end{aligned}$$

for all $(\phi_h, \rho_h) \in V_h$, and for all $(\chi_h, u_h) \in Q_h$. Hence,

$$\begin{aligned} |(g, (\sigma - \sigma_h, \gamma - \gamma_h))_{0,\mathrm{I}}| &\leq C(\|(\sigma - \sigma_h, \gamma - \gamma_h)\|_V \|(\phi - \phi_h, \rho - \rho_h)\|_V\\ &\quad + \|(\sigma - \sigma_h, \gamma - \gamma_h)\|_V \|(\chi - \chi_h, u - u_h)\|_Q\\ &\quad + \|(\phi - \phi_h, \rho - \rho_h)\|_V \|(\beta - \beta_h, w - w_h)\|_Q)\\ &\leq Ch\|f\|_{1,\mathrm{I}}(\|(\phi - \phi_h, \rho - \rho_h)\|_V + \|(\chi - \chi_h, u - u_h)\|_Q) \end{aligned}$$

for all $(\phi_h, \rho_h) \in V_h$ and $(\chi_h, u_h) \in Q_h$, where in the last inequality we have utilized Theorem 3.2. Taking in particular $(\phi_h, \rho_h) := (\mathcal{L}(\phi), \mathcal{L}(\rho))$ and $(\chi_h, u_h) := (\mathcal{P}(\chi), \mathcal{P}(u))$ in the above estimate we obtain:

$$
\begin{aligned}
|(g, (\sigma - \sigma_h, \gamma - \gamma_h))_{0,\mathrm{I}}| \leq & Ch\|f\|_{1,\mathrm{I}}(\|(\phi - \mathcal{L}(\phi), \rho - \mathcal{L}(\rho))\|_V + \|(\chi - \mathcal{P}(\chi), u - \mathcal{P}(u))\|_Q) \\
\leq & Ch^2\|f\|_{1,\mathrm{I}}(\|\phi\|_{2,\mathrm{I}} + \|\rho\|_{2,\mathrm{I}} + \|\chi\|_{1,\mathrm{I}} + \|u\|_{1,\mathrm{I}}) \\
\leq & Ch^2\|f\|_{1,\mathrm{I}}\|g\|_{0,\mathrm{I}},
\end{aligned}
$$

where in the last inequality, first we consider the error estimates for $\mathcal{P}$ (see (3.3)) and $\mathcal{L}$ (see (3.1)-(3.2)), and then additional regularity result (3.10).

Finally, from the estimate above and the definition by duality of $\|\cdot\|_Q$ we have that:

$$
\|(\sigma - \sigma_h, \gamma - \gamma_h)\|_Q = \sup_{\substack{g \in L^2(\mathrm{I})^2 \\ g \neq 0}} \frac{(g, (\sigma - \sigma_h, \gamma - \gamma_h))_{0,\mathrm{I}}}{\|g\|_{0,\mathrm{I}}} \leq Ch^2\|f\|_{1,\mathrm{I}},
$$

where the constant C is independent of h and t.

The proof is complete. □

Remark 3.1. *It is possible to refine Theorems 3.2 and 3.3, considering piecewise smooth loads. Considering a family of partitions of* I

$$
\widetilde{\mathcal{T}}_h := 0 = x_0 < \cdots < x_N = L,
$$

which are refinements of the initial partition $0 = s_0 < \cdots < s_n = L$ *(see Remark 2.1). We denote* $\mathrm{I}_j = (x_{j-1}, x_j)$*,* $j = 1, \ldots, N$*, and note that for any mesh* $\widetilde{\mathcal{T}}_h$*, each* I_j *is contained in some subinterval* S_i*,* $i = 1, \ldots, n$*. Hence, if* $f \in H^1(S_i)$*, then,*

$$
\|((\sigma, \gamma), (\beta, w)) - ((\sigma_h, \gamma_h), (\beta_h, w_h))\|_{V \times Q} \leq Ch \left(\|f\|_{0,\mathrm{I}}^2 + \sum_{i=1}^{n} \|f'\|_{0,S_i}^2 \right)^{1/2},
$$

and

$$
\|(\sigma - \sigma_h, \gamma - \gamma_h)\|_Q \leq \hat{C}h^2 \left(\|f\|_{0,\mathrm{I}}^2 + \sum_{i=1}^{n} \|f'\|_{0,S_i}^2 \right)^{1/2},
$$

where the constants $C, \hat{C} > 0$ *are independent of* h *and* t*. In fact, the first estimate follows from inequality* (3.7)*, the local character of the Lagrange interpolant of* γ *and the additional regularity result given in Remark 2.1. In its turn, the second one follows from the first one and Theorem 3.3.*

4. Numerical results.

We report in this section some numerical experiments which confirm the theoretical results proved above. The numerical method analyzed has been implemented in a MATLAB code.

In what follows, N denotes the number of degrees of freedom, namely, $N := \dim(V_h \times Q_h)$. Moreover, we define the individual errors by:

$$
\mathbf{e}_0(\sigma) := \|\sigma - \sigma_h\|_{0,\mathrm{I}}, \quad \mathbf{e}_1(\sigma) := \|\sigma' - \sigma_h'\|_{0,\mathrm{I}}, \quad \mathbf{e}_0(\gamma) := \|\gamma - \gamma_h\|_{0,\mathrm{I}},
$$
$$
\mathbf{e}_1(\gamma) := \|\gamma' - \gamma_h'\|_{0,\mathrm{I}}, \quad \mathbf{e}_0(\beta) := \|\beta - \beta_h\|_{0,\mathrm{I}}, \qquad \mathbf{e}_0(w) := \|w - w_h\|_{0,\mathrm{I}},
$$

where $((\sigma, \gamma), (\beta, w)) \in V \times Q$ and $((\sigma_h, \gamma_h), (\beta_h, w_h)) \in V_h \times Q_h$ are the solutions of problems (2.7)-(2.8) and (3.4)-(3.5), respectively.

We define the experimental rates of convergence (rc_i) for the errors $\mathbf{e}_i(\cdot)$ by

$$\mathrm{rc}_i(\cdot) := \frac{\log(\mathbf{e}_i(\cdot)/\mathbf{e}'_i(\cdot))}{\log(h/h')} \qquad i = 0, 1,$$

where h and h' denote two consecutive meshsizes and $\mathbf{e}$ and $\mathbf{e}'$, denote the corresponding errors.

4.1. **Test 1.** As a first test, we take $\mathrm{I} := (0, 1)$ and solve the equations (2.5) with $f(x) = e^x$, $\mathbb{E}(x) = e^x$, and $\kappa(x) = e^{-x}$. Thus, we obtain the following exact solution:

$$\gamma(x) = e^x + c_1,$$

$$\sigma(x) = e^x + c_1 x + c_2,$$

$$\beta(x) = x - c_1((x+1)e^x - 1) - c_2(e^{-x} - 1),$$

$$w(x) = \frac{x^2}{2} + c_1((x+2)e^{-x} + x + t^2(1 - e^x) - 2) + c_2(e^{-x} + x - 1) + \frac{t^2}{2}(1 - e^{2x}),$$

where

$$c_1 = \frac{t^2(e^2 - 1) - \frac{2}{1-e} - 1}{6e^{-1} + 2t^2(1-e) - \frac{2(e^{-1}-1)}{1-e}}, \quad \text{and} \qquad c_2 = \frac{1 - c_1(2e^{-1} - 1)}{e^{-1} - 1}.$$

First, we analyze the convergence properties of the elements proposed here, and keep the thickness fixed $t = 0.01$. Then, we also show an analysis for various thickness in order to assess the locking free nature of the proposed method.

Tables 1 and 2 show the convergence history of the mixed finite element scheme (3.4) applied to our test problem.

Table 1. Convergence analysis for $t = 0.01$. Errors and experimental rates of convergence for variables σ and γ.

N	h	$\mathbf{e}_0(\sigma)$	$\mathrm{rc}_0(\sigma)$	$\mathbf{e}_1(\sigma)$	$\mathrm{rc}_1(\sigma)$	$\mathbf{e}_0(\gamma)$	$\mathrm{rc}_0(\gamma)$
34	0.125	1.6539e-3	–	6.4508e-2	–	2.6697e-3	–
66	0.0625	4.1355e-4	1.98	3.2249e-2	0.99	6.6905e-4	2.00
130	0.03125	1.0339e-4	1.99	1.6124e-2	1.00	1.6736e-4	2.00
258	0.015625	2.5848e-5	2.00	8.0617e-3	1.00	4.1847e-5	2.00
514	0.0078125	6.4621e-6	2.00	4.0309e-3	1.00	1.0462e-5	2.00

Table 2. Convergence analysis for $t = 0.01$. Errors and experimental rates of convergence for variables γ, β and w.

N	h	$\mathbf{e}_1(\gamma)$	$\mathrm{rc}(\gamma)$	$\mathbf{e}_0(\beta)$	$\mathrm{rc}_0(\beta)$	$\mathbf{e}_0(w)$	$\mathrm{rc}_0(w)$
34	0.125	6.4474e-2	–	1.3298e-3	–	2.1197e-4	–
66	0.0625	3.2245e-2	0.99	6.7574e-4	0.98	1.0523e-4	1.01
130	0.03125	1.6123e-2	1.00	3.3923e-4	1.00	5.2507e-5	1.00
258	0.015625	8.0618e-3	1.00	1.6978e-4	1.00	2.6240e-5	1.00
514	0.0078125	4.0310e-3	1.00	8.4912e-5	1.00	1.3118e-5	1.00

We observe from Tables 1 and 2 that the rates of convergence $O(h)$ and $O(h^2)$ predicted by Theorems 3.2 and 3.3 are attained for all the variables.

Secondly, we solve the same problem with a varying thickness to asses the locking free character of the method. We report in Table 3 the errors and the rates of convergence for the transverse displacement.

TABLE 3. Locking free analysis for variable w ($\mathbf{e}_0(w)$).

		t=1.0e-3		t=1.0e-4		t=1.0e-5	
N	h	$\mathbf{e}_0(w)$	$\mathrm{rc}_0(w)$	$\mathbf{e}_0(w)$	$\mathrm{rc}_0(w)$	$\mathbf{e}_0(w)$	$\mathrm{rc}_0(w)$
34	0.125	2.0928e-4	–	2.0929e-4	–	2.0925e-4	–
66	0.0625	1.0388e-4	1.01	1.0387e-4	1.01	1.0386e-4	1.01
130	0.03125	5.1831e-5	1.00	5.1824e-5	1.00	5.1824e-5	1.00
258	0.015625	2.5902e-5	1.00	2.5898e-5	1.00	2.5898e-5	1.00
514	0.0078125	1.2949e-5	1.00	1.2947e-5	1.00	1.2947e-5	1.00

We observe from Table 3 that our method does not deteriorate when the thickness parameter becomes small. The same happens with all the other variables, so that we can assert that the method is locking free.

4.2. **Test 2.** As a second test, we take $\mathrm{I} := (0,1)$ and solve the equations (2.5) with

$$f(x) = \begin{cases} x, & 0 \le x \le 0.5, \\ e^{-x}, & 0.5 < x \le 1, \end{cases} \qquad \mathbb{E}(x) = \begin{cases} 1, & 0 \le x \le 0.5, \\ e^{-x}, & 0.5 < x \le 1, \end{cases}$$

and

$$\kappa(x) = \begin{cases} e^{x}, & 0 \le x \le 0.5, \\ 1, & 0.5 < x \le 1. \end{cases}$$

In this case, we have consided a piecewise smooth load $f(x)$. As required by the theory (see Remark 3.1), we analyze the convergence properties of the elements proposed here, keep the thickness fixed $t = 0.01$ and taking meshes where the point $x = 0.5$ is always a node.

For this particular test, the analytical solution can be obtained by solving the corresponding problems in $(0, 0.5)$ and $(0.5, 1)$ in terms of the unknowns values at $x = 0.5$ and matching the solutions at this point.

Tables 4 and 5 show the convergence history of the mixed finite element scheme (3.4)-(3.5) applied to our test problem.

TABLE 4. Convergence analysis for $t = 0.01$. Errors and experimental rates of convergence for variables σ and γ.

N	h	$\mathbf{e}_0(\sigma)$	$\mathrm{rc}_0(\sigma)$	$\mathbf{e}_1(\sigma)$	$\mathrm{rc}_1(\sigma)$	$\mathbf{e}_0(\gamma)$	$\mathrm{rc}_0(\gamma)$
34	0.125	4.6083e-4	–	1.4357e-2	–	1.2335e-3	–
66	0.0625	1.1605e-4	1.99	7.1723e-3	0.99	3.0852e-4	1.99
130	0.03125	2.9063e-5	2.00	3.5854e-3	1.00	7.7138e-5	2.00
258	0.015625	7.2691e-6	2.00	1.7926e-3	1.00	1.9285e-5	2.00
514	0.0078125	1.8175e-6	2.00	8.9628e-4	1.00	4.8213e-6	2.00

TABLE 5. Convergence analysis for $t = 0.01$. Errors and experimental rates of convergence for variables γ, β and w.

N	h	$\mathbf{e}_1(\gamma)$	$\mathrm{rc}(\gamma)$	$\mathbf{e}_0(\beta)$	$\mathrm{rc}_0(\beta)$	$\mathbf{e}_0(w)$	$\mathrm{rc}_0(w)$
34	0.125	2.8322e-2	–	9.6598e-4	–	1.5040e-4	–
66	0.0625	1.4163e-2	0.99	4.9102e-4	0.98	7.3356e-5	1.02
130	0.03125	7.0817e-3	1.00	2.4652e-4	0.99	3.6417e-5	1.00
258	0.015625	3.5409e-3	1.00	1.2338e-4	1.00	1.8175e-5	1.00
514	0.0078125	1.7705e-3	1.00	6.1707e-5	1.00	9.0830e-6	1.00

We observe from Tables 4 and 5 that the rates of convergence $O(h)$ and $O(h^2)$ predicted by Remark 3.1 are attained for all the variables.

5. CONCLUSIONS.

In the present paper, we analyzed a mixed finite element method to approximate the bending of a non-homogeneous Timoshenko beam. We proposed a mixed variational formulation in terms of the bending moment, shear stress, rotation and transverse displacement, which has been shown to be well posed by the Babuška-Brezzi theory. The proofs cover the cases of non-homogeneous beams with varying geometry and physical parameters. The formulation was discretized by continuous $P1$ and discontinuous $P0$ finite elements and we proved linear convergence with respect to the mesh size in the natural norm and a quadratic order for the bending moment and the shear stress in L^2-norm, all the estimates being uniform in the beam thickness. Finally, we reported numerical results that confirm the numerical analysis of the proposed method.

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