Strong duality in cone constrained nonconvex optimization: a general approach with applications to nonconvex variational problems *

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Abstract

In this paper we deepen the analysis of the conditions that ensure strong duality for a cone constrained nonconvex optimization problem. Our conditions can be used where no previous result is applicable, even in a finite dimensional setting or convex situations. An application to Calculus of Variations without the standard convexity assumption yielding zero duality gap and strong duality is provided.

Key words. Quasi relative interior, strong duality, nonconvex optimization, nonconvex variational problems.

1 Introduction

The purpose of this paper is to develop the analysis of conditions ensuring the existence of strong duality for a cone constrained nonconvex optimization problem. Such conditions are based on the notion of quasi relative interior [3], that recently has received great attention in the literature. Our results unify and extend to the nonconvex case some analogous theorems that have been obtained under suitable convexity assumptions on the functions involved [6, 7, 17].

Let X be a real locally convex topological vector space, Y be a Hausdorff locally convex topological vector space, $P \subseteq Y$ be a nonempty closed convex cone with possibly empty topological interior, and C be a nonempty subset of X. Given $f : C \to \mathbb{R}$ and

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$g : C \to Y$, let us consider the cone constrained minimization problem

$$\mu \doteq \inf_{\substack{g(x)\in -P\\ x\in C}} f(x). \tag{1.1}$$

The (Lagrangian) dual problem associated to (P) is

$$\nu \doteq \sup_{\lambda^*\in P^*} \inf_{x\in C} [f(x) + \langle \lambda^*, g(x)\rangle], \tag{1.2}$$

where P^* is the non negative polar cone of P. We say that problem (1.1) has a (Lagrangian) *zero duality gap* if the optimal values of (1.1) and (1.2) coincide, that is, $\mu = \nu$. Problem (1.1) is said to have *strong duality* if it has a zero duality gap and problem (1.2) admits a solution. Our task is to characterize this property without convexity assumptions, and therefore some constraints qualification (CQ) are needed, which may be of Slater-type, or interior-point condition. In some other situation, the validity of strong duality requires a so called closed-cone CQ. Such CQ often restrict some applications.

More precisely, when $X = C = \mathbb{R}^n$ and $P = [0, +\infty[$ with g being a non identically zero quadratic function, the authors in [19] prove that, (1.1) has strong duality for each quadratic function f if, and only if there exists $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) < 0$, which is the classical Slater condition. A similar result is proven in [23, Theorem 3.3] when $X = \mathbb{R}^n$, $C = \{x \in \mathbb{R}^n : \ Hx = d\}$.
Similarly, when g is P-convex ($g(tx_1 + (1-t)x_2) \in tg(x_1) + (1-t)g(x_2) - P$ for every $t \in \ ]0, 1[$ and all $x_1, x_2 \in C$, provided C is convex) and continuous, it is proven in [4] that (1.1) has strong duality for each $f \in X^*$ if, and only if a certain CQ holds. This CQ involves the epigraph of the support function of C and the epigraph of the conjugate of the function $x \mapsto \langle \lambda^*, g(x)\rangle$. This CQ is also equivalent to the fact that (1.1) has strong duality for each continuous and convex function f ([18]).

Stable zero duality gaps in convex programming (g is continuous, P-convex, and f is lower semicontinuous proper convex function), which means that strong duality holds for each linear perturbation of f, were characterized in terms of a similar CQ as above, see [20, 22] for details.
Several sufficient conditions of the zero duality gap have been also established in the literature, see [14, 1, 2, 36, 4, 6, 7].

Unlike some of the above results, which involve conditions on g and C that guarantee (1.1) has strong duality for every f in a certain class of functions, our approach allows us to derive conditions on f, g and C, jointly, that ensure (1.1) has strong duality holds under no convexity assumption. Thus, we provide results where none of those in [14, 4, 20, 18, 5, 6, 7, 21] is applicable.

At the same time, because of many applications, our purpose is also to consider convex cones P possibly with empty topological interior. This happens for instance if $(1 < p < +\infty)$

$$P = L^p_+ \doteq \{u \in L^p(\Omega) : \; u \geq 0 \quad \text{a.e. } x \in \Omega\},$$

or if P is of the form $P = Q \times \{0\}$ with $\text{int}\, Q \neq \emptyset$. The former case appears when dealing with constrained best interporlation problems, see the nice work by Qi, [31] (see also [26]).

A good substitute for the topological interior is the quasi interior and even the quasi-relative interior. Borwein and Lewis in [3] introduced the quasi-relative interior of a convex set $A \subseteq Y$, although the concept of quasi interior was introduced earlier. We use both notions, and since the sets considered are not necessarily convex, the convex hull arises naturally.

The paper is structured as follows. Section 2 provides the basic definitions, notations and preliminaries on quasi (relative) interior of convex sets. In Section 3 we first establish a characterization of strong duality without additional assumption; then we present two main theorems on the validity of strong duality under no convexity assumptions, which extend and unify previous existing results in the literature. Furthermore, we show instances where no previous result is applicable. Consequences and comparison with other previous results are discussed in Section 4. Finally, Section 5 exhibits an application to a nonconvex variational problem showing a characterization of the zero duality gap and the validity of the strong duality property.

2 Basic notations and preliminares

Throughout the paper, Y is a real Hausdorff locally convex topological vector space, its topological dual space is Y^*, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between Y and Y^*.

A set $P \subseteq Y$ is said to be a cone if $tP \subseteq P$, $\forall\, t \geq 0$; given $A \subseteq Y$, $\text{cone}(A)$ stands for the smallest cone containing A, that is,

$$\text{cone}(A) = \bigcup_{t \geq 0} tA,$$

whereas $\overline{\text{cone}}(A)$ denotes the smallest closed cone containing A: obviously $\overline{\text{cone}}(A) = \overline{\text{cone}(A)}$, where $\overline{A}$ denotes the closure of A. Additionally, we set

$$\text{cone}_+(A) \doteq \bigcup_{t > 0} tA.$$

Evidently, $\text{cone}(A) = \text{cone}_+(A) \cup \{0\}$ and therefore $\overline{\text{cone}(A)} = \overline{\text{cone}_+(A)}$.

Some elementary properties of cones are collected in the next lemma, where $\operatorname{co}(A)$, $\operatorname{int} A$, stand for the convex hull of A which is the smallest convex set containing A, and topological interior of A, respectively. We denote $\mathbb{R}_+ \doteq [0, +\infty[$.

Given a convex set $A \subseteq Y$ and $x \in A$, $N_A(x)$ stands for the normal cone to A at x, defined by $N_A(x) = \{\xi \in Y^* : \langle \xi, a - x \rangle \leq 0, \ \ \forall\, a \in A\}$. We say that $x \in A$ is a (see for instance [7]):

(*a*) quasi interior point of A, denoted by $x \in \operatorname{qi} A$, if $\overline{\operatorname{cone}}(A - x) = Y$, or equivalently, $N_A(x) = \{0\}$;

(*b*) quasi relative interior of A, denoted by $x \in \operatorname{qri} A$, if $\overline{\operatorname{cone}}(A - x)$ is a linear subspace of Y, or equivalently, $N_A(x)$ is a linear subspace of Y^*.

For any convex set A, we have that ([26, 7]) $\operatorname{qi} A \subseteq \operatorname{qri} A$ and, $\operatorname{int} A \neq \emptyset$ implies $\operatorname{int} A = \operatorname{qi} A$. Similarly, if $\operatorname{qi} A \neq \emptyset$, then $\operatorname{qi} A = \operatorname{qri} A$. Moreover [3], if Y is a finite dimensional space, then $\operatorname{qi} A = \operatorname{int} A$ and $\operatorname{qri} A = \operatorname{ri} A$, where $\operatorname{ri} A$ means the relative interior of A, which is the interior with respect to the affine hull of A, denoted by $\operatorname{aff} A$.

We recall the definition of pointedness for a cone that is not necessarily convex (see for instance [34]).

Definition 2.1. A cone $P \subseteq Y$ is called "pointed" if $x_1 + \cdots + x_k = 0$ is impossible for $x_1, x_2, \ldots, x_k$ in P unless $x_1 = x_2 = \cdots = x_k = 0$.

It is easy to see that a cone P is pointed if, and only if $\operatorname{co}(P) \cap (-\operatorname{co}(P)) = \{0\}$ if, and only if 0 is a extremal point of $\operatorname{co}(P)$.

The positive polar of the convex cone $P \subseteq Y$ is defined by:

$$P^* \doteq \{y^* \in Y^* : \langle y^*, x \rangle \geq 0, \ \forall x \in P\}.$$

Lemma 2.2. *Let $\emptyset \neq M \subseteq Y$. The following relations hold:*

(*a*) $\operatorname{co}(M \cup \{0\}) \subseteq \operatorname{cone}(\operatorname{co} M)$;

(*b*) $\operatorname{cone}(\operatorname{co}(M \cup \{0\})) = \operatorname{cone}(\operatorname{co} M)$;

(*c*) *If $\emptyset \neq N \subseteq Y$ is a convex set then* $\operatorname{co}(M + N) = \operatorname{co}(M) + N$;

(*d*) $0 \in \operatorname{qri}[\operatorname{co}(M \cup \{0\})] \iff 0 \in \operatorname{qri}[\operatorname{cone}(\operatorname{co} M)]$;

(*e*) $0 \in \operatorname{qi}[\operatorname{co}(M \cup \{0\})] \iff 0 \in \operatorname{qi}[\operatorname{cone}(\operatorname{co} M)]$.

Proof. (*a*): Let $\bar{y} \in \operatorname{co}(M \cup \{0\})$. Then

$$\bar{y} = \sum_{i=1}^{p} \alpha_i m_i, \quad \text{for some} \quad \alpha_i \geq 0,\ m_i \in M,\ i = 1, \ldots, p.$$

If $\sum_{i=1}^{p} \alpha_i = 0$, then $\bar{y} = 0 \in \text{cone } M \subseteq \text{cone}(\text{co } M)$.
If $\sum_{i=1}^{p} \alpha_i > 0$, then

$$\bar{y} = (\sum_{i=1}^{p} \alpha_i) \sum_{i=1}^{p} \frac{\alpha_i}{\sum_{i=1}^{p} \alpha_i} m_i \in \text{cone}(\text{co } M).$$

(b): The inclusion $(\supseteq)$ is obvious; the other comes from (a).
(c): $\subseteq)$ Let $m_i \in M, p_i \in N,\ i = 1, \dots, q,\ \alpha_i \geq 0,\ \sum_{i=1}^{q} \alpha_i = 1$. Then

$$\sum_{i=1}^{q} \alpha_i(m_i + p_i) = \sum_{i=1}^{q} \alpha_i m_i + \sum_{i=1}^{q} \alpha_i p_i \in \text{co}(M) + N.$$

$\supseteq)$ Let $m_i \in M,\ i = 1, \dots, q,\ \alpha_i \geq 0,\ \sum_{i=1}^{q} \alpha_i = 1,\ p \in N$.
Then

$$\sum_{i=1}^{q} \alpha_i m_i + p = \sum_{i=1}^{q} \alpha_i(m_i + p) \in \text{co}(M + N).$$

(d) It is a consequence of the following equalities obtained from (a)

$$\overline{\text{cone}}[\text{co}(M \cup \{0\})] = \overline{\text{cone}}(\text{co } M) = \overline{\text{cone}}[\text{cone}(\text{co } M)].$$

Therefore, $\overline{\text{cone}}[\text{co}(M \cup \{0\})]$ is a linear subspace of Y if and only if $\overline{\text{cone}}[\text{cone}(\text{co } M)]$ is a linear subspace of Y, or, equivalently, $0 \in \text{qri}[\text{cone}(\text{co } M)]$.
(e) It is analogous to the proof of part (c).

□

The following separation theorem is a direct consequence of the equivalent characterization of the quasi relative interior and of Lemma 2.2. As mentioned in [3], the quasi relative interior of a set M consists of the points of $x \in M$ for which it is not possible to find a supporting hyperplane to M at x.

Theorem 2.3. *Let $\emptyset \neq M \subseteq Y$. Then, $0 \notin \text{qri}[\text{cone}(\text{co } M)]$ (or, equivalently, $0 \notin \text{qri}[\text{co}(M \cup \{0\})]$) if, and only if there exists $x^* \in Y^* \setminus \{0\}$ such that $\langle x^*, x \rangle \leq 0, \quad \forall\, x \in M$, with strict inequality for some $\bar{x} \in M$.*

Proof. The necessity part is as follows. Since $0 \in \text{co}(M \cup \{0\}))$, then $0 \notin \text{qri}[\text{co}(M \cup \{0\})]$ if and only if $N_{\text{co}(M \cup \{0\})}(0)$ is not a linear subspace of Y^*, i.e., there exists $x^* \in Y^* \setminus \{0\}$ such that

$$\langle x^*, x \rangle \leq 0, \quad \forall x \in \text{co}(M \cup \{0\}),$$

and, furthermore, there is $\hat{x} \in \text{co}(M \cup \{0\})$ such that

$$\langle x^*, \hat{x} \rangle < 0. \tag{2.1}$$

Since $\hat{x} \in \text{co}(M \cup \{0\})$, then, for some integer $p \geq 1$,

$$\hat{x} = \sum_{i=1}^{p} \alpha_i x_i, \quad \alpha_i \geq 0,\ x_i \in M,\ i = 1, \ldots, p.$$

It follows that there exists at least one $i \in 1, \ldots, p$, such that $\langle x^*, x_i \rangle < 0$, otherwise (2.1) would be contradicted, which proves the necessity part.
The sufficiency is straightforward. □

The next result [3] is a useful characterization of the quasi-relative interior.

Theorem 2.4. [*3, Theorem* 3.10] *Let Y be locally convex, partially ordered by a convex cone P with $\overline{P - P} = Y$. Then:*

$$y \in \text{qri } P \quad \Longleftrightarrow \quad y \in P \text{ and } \langle y^*, y \rangle > 0,\ \forall y^* \in P^* \setminus \{0\}.$$

Proposition 2.5. *Let $A \subseteq Y$ be a convex set. Then,*

(a) $\text{cone}(A - A) = \text{cone } A - \text{cone } A$ *provided* $0 \in A$;

(b) $[0 \in \text{qi } A] \Longleftrightarrow [0 \in \text{qi}(A - A) \text{ and } 0 \in \text{qri } A]$.

Proof. (a): It is straightforward.
(b): From (a) it follows that

$$\overline{\text{cone}}(A - A) = \overline{\text{cone}(A) - \text{cone}(A)} = \overline{\overline{\text{cone}}(A) - \overline{\text{cone}}(A)}. \tag{2.2}$$

This along with the equivalence $0 \in \text{qi } A \Longleftrightarrow 0 \in A$ and $\overline{\text{cone}}(A) = Y$, allow us to get $\overline{\text{cone}}(A - A) = Y$ and therefore,

$$[0 \in \text{qi } A] \Longrightarrow [0 \in \text{qi}(A - A) \text{ and } 0 \in \text{qri } A].$$

The converse implication follows from (2.2) as well. □

Notice that (b) can also be found in [17].

Proposition 2.6. *Let $P \subseteq Y$ be a convex cone such that $\overline{P - P} = Y$. Then*

$$\text{qri } P = \text{qi } P.$$

Proof. We only need to prove that $\text{qri } P \subseteq \text{qi } P$. Since $y \in \text{qri } P$ if and only if $0 \in \text{qri}(P - y)$ and $y \in P$; by virtue of the previous proposition, we need to check that $0 \in \text{qi}(P - y - (P - y)) = \text{qi}(P - P)$, which is true by assumption: $\overline{\text{cone}}(P - P) = \overline{P - P} = Y$. □

3 Lagrangian strong duality: main results and regularity conditions

Let Y be as in the preceding section and X be a Hausdorff topological vector space, $f : X \to \mathbb{R}$, $g : X \to Y$, $C \subseteq X$ and P be a closed and convex cone in Y. We consider the problem

$$\mu \doteq \inf_{x \in K} f(x), \tag{3.1}$$

where $K \doteq \{x \in C : g(x) \in -P\}$. We assume that μ is finite and that the feasible region K of (3.1) is nonempty.

Notice that the requirement of f taking finite values is not restrictive since no additional structure like convexity or closedness on C is imposed. Thus for functions $f : X \to \mathbb{R} \cup \{+\infty\}$ the set C may be the effective domain $\mathrm{dom}\, f \doteq \{x \in X : \ f(x) < +\infty\}$. Such a situation is present in the model we deal with in Section 5.

Let $F \doteq (f, g)$ and consider the set:

$$\mathcal{E}_\mu \doteq F(C) - \mu(1, 0) + (\mathbb{R}_+ \times P).$$

This set or its conic hull arises in a natural way when dealing with duality results or in deriving alternative theorems, see [24, 14, 10, 17, 12]. Giannessi [15] used it in a systematic manner for a constrained extremum problem giving rise to the image space analysis.

Proposition 3.1. *The following assertions hold.*

(*a*) *Assume that* $\mu \in \overline{f(K)}$. *Then,* $\mu = \inf_{x \in K} f(x)$ *if and only if*

$$\mathcal{E}_\mu \cap \mathcal{H} = \emptyset, \tag{3.2}$$

where $\mathcal{H} \doteq \{(u, v) \in \mathbb{R} \times Y : u < 0, v \in -P\}$. *Furthermore,*

$$\mathcal{E}_\mu \cap \mathcal{H} = \emptyset \iff \mathrm{cone}(\mathcal{E}_\mu) \cap \mathcal{H} = \emptyset.$$

(*b*) $\inf_{x \in K} f(x) = -\infty$ *if and only if*

$$\mathcal{E}_\rho \cap \mathcal{H} \neq \emptyset, \quad \forall\, \rho \in \mathbb{R}, \tag{3.3}$$

where $\mathcal{E}_\rho$ *is* $\mathcal{E}_\mu$ *with* μ *replaced by* ρ.

Proof. Since (*b*) is obvious, we only prove (*a*). We preliminarily observe that (3.2) is equivalent to the fact that the system

$$f(x) - \mu + t < 0, \quad g(x) + p \in -P, \quad (x, t, p) \in C \times \mathbb{R}_+ \times P \tag{3.4}$$

is impossible.
Let $\mu = \inf_{x \in K} f(x)$ and assume that (3.2) does not hold. Then, there exists a solution $(\tilde{x}, \tilde{t}, \tilde{p}) \in (C \times \mathbb{R}_+ \times P)$ of system (3.4), i.e.,

$$f(\tilde{x}) < \mu - \tilde{t} \leq \mu, \quad g(\tilde{x}) \in -\tilde{p} - P \in -P,$$

which contradicts the definition of μ. Conversely, assume that (3.4) is impossible, then, in particular, setting $t = 0$ and $p = 0$, we have that

$$f(x) \geq \mu, \quad \forall x \in K$$

and, since $\mu \in \overline{f(K)}$, then (3.1) holds.
For the equivalence, one implication is obvious; whereas the other follows from $(0, 0) \notin \mathcal{H}$. □

Strong duality for (3.1) requires the existence of a linear continuous functional that separates the sets $\mathcal{E}_\mu$ and $\mathcal{H}$. Actually, we need more than that as the next theorem asserts. Let us introduce the Lagrangian

$$L(\gamma^*, \lambda^*, x) = \gamma^* f(x) + \langle \lambda^*, g(x) \rangle$$

associated with (3.1).

Theorem 3.2. *The following assertions are equivalent:*

(*a*) *Strong duality holds for* (3.1)*, i.e., there exists* $\lambda_0^* \in P^*$ *such that*

$$\inf_{x \in K} f(x) = \inf_{x \in C} L(1, \lambda_0^*, x); \tag{3.5}$$

(*b*) $\overline{\text{cone}}(\text{co } \mathcal{E}_\mu) \cap (-\mathbb{R}_{++} \times \{0\}) = \emptyset$.

Proof. Assume that strong duality holds, then,

$$f(x) - \mu + \langle \lambda_0^*, g(x) \rangle \geq 0, \quad \forall\ x \in C, \tag{3.6}$$

which implies

$$f(x) + t - \mu + \langle \lambda_0^*, g(x) + p \rangle \geq 0, \quad \forall\ x \in C,\ \forall\ t \geq 0,\ \forall\ p \in P,$$

i.e.,

$$u + \langle \lambda_0^*, v \rangle \geq 0,\ \forall\ (u, v) \in \mathcal{E}_\mu.$$

It follows that

$$u + \langle \lambda_0^*, v \rangle \geq 0, \quad \forall\ (u, v) \in \overline{\text{cone}}(\text{co } \mathcal{E}_\mu).$$

Moreover, observe that

$$u + \langle \lambda_0^*, v \rangle < 0, \quad \forall\ (u, v) \in -\mathbb{R}_{++} \times \{0\},$$

so that (b) follows.
Vice-versa, assume that (b) holds. Let $A := (-1, 0) + (]-\rho, \rho[\times V(0))$, where $V(0)$ is an open convex neighborhood of 0_Y and $\rho > 0$. Then, $\text{cone}_+(A)$ is an open convex set and by (b) it follows that

$$\text{cone}(\text{co}\ \mathcal{E}_\mu) \cap \text{cone}_+(A) = \emptyset$$

for a suitable choice of $\rho > 0$ and $V(0)$. By the separation theorem for convex sets in a t.v.s., there exist $(\gamma_0^*, \lambda_0^*) \in (\mathbb{R} \times Y^*)$, $(\gamma_0^*, \lambda_0^*) \neq (0, 0)$, such that

$$\gamma_0^* u + \langle \lambda_0^*, v \rangle \geq 0, \quad \forall (u, v) \in \text{cone}\ \mathcal{E}_\mu, \tag{3.7}$$

$$\gamma_0^* u + \langle \lambda_0^*, v \rangle \leq 0, \quad \forall (u, v) \in \text{cone}\ A. \tag{3.8}$$

Let us prove that $\gamma_0^* \neq 0$. By contradiction, suppose that $\gamma_0^* = 0$, then from (3.8) it follows that $\langle \lambda_0^*, v \rangle \leq 0, \quad \forall v \in V(0)$, which implies $\lambda_0^* = 0$, thus contradicting $(\gamma_0^*, \lambda_0^*) \neq (0, 0)$. Therefore, $\gamma_0^* \neq 0$ and, with no loss of generality, we can assume $\gamma_0^* = 1$, since by (3.8) (at the point $(-1, 0) \in A$), we have $-\gamma_0^* \leq 0$. Then, (3.7) implies

$$u + \langle \lambda_0^*, v \rangle \geq 0, \quad \forall (u, v) \in \mathcal{E}_\mu, \tag{3.9}$$

and, in turn,

$$f(x) - \mu + \langle \lambda_0^*, g(x) \rangle \geq 0, \quad \forall x \in C,$$

so that

$$\inf_{x \in C} L(1, \lambda_0^*, x) \geq \mu.$$

Let us prove that $\lambda_0^* \in P^*$. For fixed $x_0 \in C$, we obtain from (3.7)

$$\gamma_0^*(f(x_0) - \mu) + \langle \lambda_0^*, g(x_0) \rangle + \langle \lambda_0^*, p \rangle \geq 0, \qquad \forall p \in P,$$

which implies $\lambda_0^* \in P^*$. Then

$$\inf_{x \in C} L(1, \lambda_0^*, x) \leq \mu,$$

which completes the proof. □

We now establish other theorems of a different nature about strong duality, which involve generalized Slater conditions and quasi (relative) interior. To that purpose, we first prove a necessary and sufficient condition for the existence of Fritz

John multipliers, and afterwards, under Slater-type conditions, the desired strong duality result is established. Example 4.3 below shows the difference between both results.

The next theorem yields the existence of an hyperplane which may separate not in a proper sense $(0,0)$ from $\text{cone}(\text{co }\mathcal{E}_\mu)$. Instead, Theorem 3.4 below provides an hyperplane which separates them properly.

Theorem 3.3. *Let us consider problem* (3.1) *and assume that* μ *is finite. The following assertions are equivalent:*

(a) *there exist* $(\gamma_0^*, \lambda_0^*) \in \mathbb{R}_+ \times P^*$, $(\gamma_0^*, \lambda_0^*) \neq (0,0)$, *such that*

$$\gamma_0^* \inf_{x\in K} f(x) = \inf_{x\in C} L(\gamma_0^*, \lambda_0^*, x);$$

(b) $(0,0) \notin \text{qi}[\text{cone}(\text{co }\mathcal{E}_\mu)]$;

(c) $(0,0) \notin \text{qi}[\text{co}(\mathcal{E}_\mu \cup \{(0,0)\})]$.

In case $\text{qi}(\text{co }\mathcal{E}_\mu) \neq \emptyset$, *any of the previous conditions proves the pointedness of* $\text{cone}[\text{qi}(\text{co }\mathcal{E}_\mu)]$. *Consequently, if* $\text{int}(\text{co }\mathcal{E}_\mu) \neq \emptyset$, *then* ($a$) *is equivalent to the pointedness of* $\text{cone}[\text{int}(\text{co }\mathcal{E}_\mu)]$.

Proof. By Lemma 2.2 (d) it follows that (b) and (c) are equivalent. Assume that (b) is fulfilled. Since $(0,0) \in \text{cone}(\text{co }\mathcal{E}_\mu)$, then ($b$) holds if and only if $N_{\text{cone}(\text{co}\mathcal{E}_\mu)}(0,0) \neq \{(0,0)\}$, i.e., there exists $(0,0) \neq (-\gamma_0^*, -\lambda_0^*) \in N_{\text{cone}(\text{co }\mathcal{E}_\mu)}(0,0)$ such that

$$\langle(\gamma_0^*, \lambda_0^*), (u,v)\rangle \geq 0, \quad \forall\ (u,v) \in \text{cone}(\text{co }\mathcal{E}_\mu),$$

or, equivalently,

$$\langle(\gamma_0^*, \lambda_0^*), (u,v)\rangle \geq 0, \quad \forall\ (u,v) \in \mathcal{E}_\mu. \tag{3.10}$$

Note that (3.10) is equivalent to:

$$\gamma_0^*(f(x) + t - \mu) + \langle\lambda_0^*, g(x) + p\rangle \geq 0, \quad \forall\ t \in \mathbb{R}_+,\ \forall\ x \in C,\ \forall\ p \in P.$$

Since $(\gamma_0^*, \lambda_0^*) \in (\mathbb{R}_+ \times P)^* = \mathbb{R}_+ \times P^*$, the previous inequality yields

$$\gamma_0^* f(x) + \langle\lambda_0^*, g(x) + p\rangle \geq \gamma_0^*\mu,\ \forall\ x \in C,\ \forall\ p \in P.$$

Hence

$$\gamma_0^* \inf_{x\in K} f(x) \leq \inf_{x\in C} L(\gamma_0^*, \lambda_0^*, x).$$

The reverse inequality is obvious, so that we obtain (a).
Vice versa, if (a) holds, then the previous relations show that (3.10) is fulfilled for a suitable $(0,0) \neq (\gamma_0^*, \lambda_0^*) \in \mathbb{R}_+ \times P^*$, so that we obtain ($b$).

For the last part we proceed as follows. Let $x, \ -x \in \text{cone}[\text{qi}(\text{co } \mathcal{E}_\mu)]$, $x \neq 0$. Thus, $x, \ -x \in \text{cone}_+[\text{qi}(\text{co } \mathcal{E}_\mu)]$. Then $0 = x + (-x) \in \text{cone}_+[\text{qi}(\text{co } \mathcal{E}_\mu)]$. Therefore $(0,0) \in \text{qi}(\text{co } \mathcal{E}_\mu)$, and so $Y = \overline{\text{cone}}(\text{co } \mathcal{E}_\mu) = \overline{\text{cone}}[\text{cone}(\text{co } \mathcal{E}_\mu)]$. This contradicts (b), proving the desired implication. For the reverse implication, simply notice that pointedness of $\text{cone}[\text{int}(\text{co } \mathcal{E}_\mu)]$ implies that $(0,0) \notin \text{int}(\text{co } \mathcal{E}_\mu)$. Then, use a standard separation theorem to derive (a). □

A similar theorem using quasi relative interior is obtained next.

Theorem 3.4. *Let us consider problem* (3.1) *and assume that μ is finite. The following assertions are equivalent:*

(a) *there exist $(\gamma_0^*, \lambda_0^*) \in \mathbb{R}_+ \times P^*$, $(\gamma_0^*, \lambda_0^*) \neq (0,0)$, $\tilde{x} \in C$, $\tilde{t} \geq 0$ and $\tilde{p} \in P$ such that*

$$\gamma_0^* \inf_{x \in K} f(x) = \inf_{x \in C} L(\gamma_0^*, \lambda_0^*, x) \quad \textit{and} \quad \gamma_0^*(f(\tilde{x}) + \tilde{t}) + \langle \lambda_0^*, g(\tilde{x}) + \tilde{p} \rangle > \mu \gamma_0^*.$$

(b) $(0,0) \notin \text{qri}[\text{cone}(\text{co } \mathcal{E}_\mu)]$;

(c) $(0,0) \notin \text{qri}[\text{co}(\mathcal{E}_\mu \cup \{(0,0)\})]$.

In case $\text{qri}(\text{co } \mathcal{E}_\mu) \neq \emptyset$, *any of the previous conditions implies the pointedness of* $\text{cone}[\text{qri}(\text{co } \mathcal{E}_\mu)]$.

Proof. The equivalences are consequences of Theorem 2.3. The remaining part follows a similar reasoning as in the preceding theorem. □

Looking at Theorems 3.3 and 3.4, we realize that strong duality is obtained under the non-verticality of the linear functional $(\gamma_0^*, \lambda_0^*)$, that is, we need $\gamma_0^* > 0$. It holds whenever a Slater-type condition is imposed as the following two theorems show.

Theorem 3.5. *Assume that μ is finite and* $\overline{\text{cone}}(\text{co}(g(C)) + P) = Y$, *i.e.,* $0 \in \text{qi}(\text{co}(g(C) + P))$. *Then, any of the assumptions* (b) *or* (c) *of Theorem 3.3 is equivalent to* (3.5) *for some* $\lambda_0^* \in P^*$. *In such a situation,*

$$\inf_{x \in K} f(x) = \inf_{\substack{\langle \lambda_0^*, g(x) \rangle \leq 0 \\ x \in C}} f(x). \tag{3.11}$$

Hence, if $\bar{x}$ is a solution to (3.1) *then* $\langle \lambda_0^*, g(\bar{x}) \rangle = 0$.

Proof. As regards (3.5), from Theorem 3.3, we have only to prove that $\gamma_0^* > 0$. If, on the contrary, $\gamma_0^* = 0$, then

$$0 = \inf_{x \in C} L(0, \lambda_0^*, x) \leq \langle \lambda_0^*, g(x) \rangle, \quad \forall \ x \in C.$$

It implies that $\langle \lambda_0^*, v\rangle \geq 0, \ \forall \ v \in g(C) + P$, which yields

$$\langle \lambda_0^*, v\rangle \geq 0, \ \forall \ v \in \overline{\text{cone}}(\text{co}(g(C)) + P).$$

Therefore, by assumption, we obtain $\lambda_0^* = 0$, which cannot happen as $(\gamma_0^*, \lambda_0^*) \neq (0,0)$. Hence $\gamma_0^* > 0$, and the conclusion follows.

For the equality in (3.11), we observe that the inequality "$\geq$" is obvious. The reverse inequality is a consequence of (3.5):

$$\inf_{\substack{\langle \lambda_0^*, g(x)\rangle \leq 0 \\ x\in C}} f(x) \geq \inf_{\substack{\langle \lambda_0^*, g(x)\rangle \leq 0 \\ x\in C}} L(1, \lambda_0^*, x) \geq \inf_{x\in C} L(1,\lambda_0^*, x) = \inf_{x\in K} f(x).$$

□

Now, we consider a Slater-type condition involving the quasi relative interior of the set $\text{co}(g(C) + P)$.

Theorem 3.6. *Assume that* μ *is finite and* $0 \in \text{qri}(\text{co}(g(C) + P))$. *Then, any of the assumptions* (b) *or* (c) *of Theorem 3.4 is equivalent to the following:*

$$\inf_{x\in K} f(x) = \inf_{x\in C} L(1, \lambda_0^*, x). \tag{3.12}$$

for some $\lambda_0^* \in P^*$. *In such a case, if* $\bar{x}$ *is a solution to* (3.1) *then* $\langle \lambda_0^*, g(\bar{x})\rangle = 0$.

Proof. By Theorem 3.4, we have only to prove that $\gamma_0^* > 0$, taking into account that, in such a case, the second assertion in Theorem 3.4(a),

$$\gamma_0^*(f(\tilde{x}) + \tilde{t}) + \langle \lambda_0^*, g(\tilde{x}) + \tilde{p}\rangle > \gamma_0^* \mu,$$

for some $(\gamma_0^*, \lambda_0^*) \in \mathbb{R}_+ \times P^*$, $\tilde{x} \in C$, $\tilde{t} > 0$, $\tilde{p} \in P$, is automatically satisfied if the feasible set K is nonempty.

Lemma 2.2 (a) proves that (b) and (c) of Theorem 3.4 are equivalent. By Theorem 2.3 where we have set $M = \mathcal{E}_\mu$, (b) or (c) holds if and only if there exists $(0,0) \neq (\gamma_0^*, \lambda_0^*) \in \mathbb{R} \times Y^*$ such that

$$\langle (\gamma_0^*, \lambda_0^*), (u, v)\rangle \leq 0, \quad \forall \ (u,v) \in \mathcal{E}_\mu,$$

with strict inequality for some $(\bar{u}, \bar{v}) \in \mathcal{E}_\mu$.

Suppose on the contrary, that $\gamma_0^* = 0$, then $\lambda_0^* \neq 0$ and

$$\langle \lambda_0^*, v\rangle \leq 0, \quad \forall \ v \in \text{co}(g(C) + P), \tag{3.13}$$

i.e., $\lambda_0^* \in N_{\text{co}(g(C)+P)}(0)$, recalling that $0 \in \text{co}(g(C) + P)$ because the feasible set is nonempty. Since $0 \in \text{qri}(\text{co}(g(C) + P))$ is equivalent to say that $N_{\text{co}(g(C)+P)}(0)$ is a linear subspace, then $-\lambda_0^* \in N_{\text{co}(g(C)+P)}(0)$, and it follows that

$$\langle \lambda_0^*, v\rangle = 0, \quad \forall \ v \in \text{co}(g(C) + P),$$

which contradicts that strict inequality holds in (3.13) for $v = \bar{v}$. □

The next two theorems provides certain regularity conditions on cone(co $\mathcal{E}_\mu$) under the Slater-type assumptions It is really an important and interesting fact since, for instance in finite dimension, such a Slater assumption guarantees that int[cone(co $\mathcal{E}_\mu$)] $\neq \emptyset$, under the assumptions of Theorem 3.5.

Proposition 3.7.

(*a*) *If* $0 \in \text{qi}[\text{co}(g(C)+P)]$ *then* $\text{qi}[\text{co}(\mathcal{E}_\mu \cup \{(0,0)\})] \neq \emptyset$ *and so* $\text{qi}[\text{cone}(\text{co } \mathcal{E}_\mu)] \neq \emptyset$.

(*b*) *If* $0 \in \text{qri}[\text{co}(g(C)+P)]$ *then* $\text{qri}[\text{co}(\mathcal{E}_\mu \cup \{(0,0)\})] \neq \emptyset$ *and* $\text{qri}[\text{cone}(\text{co } \mathcal{E}_\mu)] \neq \emptyset$.

Proof. (*a*) Since we assume that the feasible region of (3.1) is nonempty, then there exists $\tilde{x} \in C$ such that $0 \in g(\tilde{x}) + P$ (therefore, our assumption is equivalent to say that $0 \in \text{qi}[\text{co}(g(C)+P)]$). Let $g(\tilde{x}) = -\tilde{p}$, where $\tilde{p} \in P$. We will prove that

$$(f(\tilde{x}) + t - \mu, g(\tilde{x}) + \tilde{p}) \in \text{qi}(\text{co}(\mathcal{E}_\mu \cup \{(0,0)\})), \quad \forall\, t > 0. \tag{3.14}$$

Let $t > 0$, be fixed and consider

$$(u^*, v^*) \in N_{\text{co}(\mathcal{E}_\mu \cup \{(0,0)\})}((f(\tilde{x}) + t - \mu, g(\tilde{x}) + \tilde{p})).$$

Therefore $u^* \in \mathbb{R}$, $v^* \in Y^*$ and

$$u^*(u - ((f(\tilde{x}) + t - \mu)) + \langle v^*, v - (g(\tilde{x}) + \tilde{p})\rangle \leq 0, \quad \forall (u,v) \in \text{co}(\mathcal{E}_\mu \cup \{(0,0)\}). \tag{3.15}$$

Setting $u \doteq f(\tilde{x}) + \frac{t}{2} - \mu$, $v = g(\tilde{x}) + \tilde{p}$, we obtain $-u^*\frac{t}{2} \leq 0$ and setting $u \doteq f(\tilde{x}) + \frac{3}{2}t - \mu$, $v = g(\tilde{x}) + \tilde{p}$, we obtain $u^*\frac{t}{2} \leq 0$. Since $t > 0$, it must be $u^* = 0$. Therefore, (3.15) becomes:

$$\langle v^*, v - (g(\tilde{x}) + \tilde{p})\rangle \leq 0, \quad \forall v \in \text{co}(g(C) + P). \tag{3.16}$$

Since $g(\tilde{x}) + \tilde{p} = 0$, then (3.16) implies that

$$\langle v^*, v - (g(\tilde{x}) + \tilde{p})\rangle \leq 0, \quad \forall v \in \overline{\text{cone}}[\text{co}(g(C)+P)] = Y,$$

so that it must be $v^* = 0$.

Hence,

$$N_{\text{co}(\mathcal{E}_\mu \cup \{(0,0)\})}((f(\tilde{x}) + t - \mu, g(\tilde{x}) + \tilde{p})) = \{(0,0)\}, \tag{3.17}$$

which proves (3.14).

In order to complete the proof, we simply observe that $\text{co}(\mathcal{E}_\mu \cup \{(0,0)\}) \subseteq \text{cone}(\text{co } \mathcal{E}_\mu)$ implies that $\text{qi}[\text{co}(\mathcal{E}_\mu \cup \{(0,0)\})] \subseteq \text{qi}[\text{cone}(\text{co } \mathcal{E}_\mu)] \neq \emptyset$, see Lemma 2.2(*a*).

(*b*) With the same $\tilde{x} \in C$ and $\tilde{p} \in P$ as in (*a*), we will prove that

$$(f(\tilde{x}) + t - \mu, g(\tilde{x}) + \tilde{p}) \in \text{qri}(\text{co}(\mathcal{E}_\mu \cup \{(0,0)\})), \quad \forall\, t > 0. \tag{3.18}$$

Indeed, similar to (a), we can check that

$$\langle v^*, v - (g(\tilde{x}) + \tilde{p})\rangle \leq 0, \quad \forall\ v \in \mathrm{co}(g(C) + P), \tag{3.19}$$

which says $v^* \in N_{\mathrm{co}(g(C))+P}(g(\tilde{x}) + \tilde{p})$. By assumption, $0 = g(\tilde{x}) + \tilde{p} \in \mathrm{qri}(\mathrm{co}(g(C)) + P)$, so that, by the equivalent characterization of the quasi relative interior, $N_{\mathrm{co}(g(C)+P)}(g(\tilde{x}) + \tilde{p})$ is a linear subspace of Y^*. Then we have also $-v^* \in N_{\mathrm{co}(g(C)+P)}(g(\tilde{x}) + \tilde{p})$, i.e.,

$$\langle -v^*, v - (g(\tilde{x}) + \tilde{p})\rangle \leq 0, \quad \forall v \in \mathrm{co}(g(C)) + P. \tag{3.20}$$

¿From (3.19) and (3.20), it follows that

$$(0, \pm v^*) \in N_{\mathrm{co}(\mathcal{E}_\mu \cup \{(0,0)\})}((f(\tilde{x}) + t - \mu, g(\tilde{x}) + \tilde{p})). \tag{3.21}$$

We have already proved that, if $(u^*, v^*) \in N_{\mathrm{co}(\mathcal{E}_\mu)}((f(\tilde{x}) + t - \mu, g(\tilde{x}) + \tilde{p}))$, then $u^* = 0$. Thus, (3.21) implies that $N_{\mathrm{co}(\mathcal{E}_\mu \cup \{(0,0)\})}((f(\tilde{x}) + t - \mu, g(\tilde{x}) + \tilde{p}))$ is a linear subspace and (3.18) holds.

The second part of the proof follows a similar reasoning as above. More precisely, we observe that

$$(u^*, v^*) \in N_{\mathrm{cone}(\mathrm{co}\ \mathcal{E}_\mu)}((f(\tilde{x}) + t - \mu, g(\tilde{x}) + \tilde{p})), \quad \text{for some } t > 0,$$

implies that $u^* = 0$, so that $v^* \in N_{\mathrm{cone}(\mathrm{co}(g(C)+P))}(g(\tilde{x}) + \tilde{p})$. The assumption $0 \in \mathrm{qri}(\mathrm{co}(g(C) + P))$ is equivalent to the fact that $N_{\mathrm{cone}(\mathrm{co}(g(C)+P))}(g(\tilde{x}) + \tilde{p})$ is a linear subspace (see Remark 3.9) which implies that $N_{\mathrm{cone}(\mathrm{co}\ \mathcal{E}_\mu)}((f(\tilde{x}) + t - \mu, g(\tilde{x}) + \tilde{p}))$ is a linear subspace, i.e., $(f(\tilde{x}) + t - \mu, g(\tilde{x}) + \tilde{p}) \in \mathrm{qri}[\mathrm{cone}(\mathrm{co}\ \mathcal{E}_\mu)]$, which completes the proof. □

Remark 3.8. Denoted by $\mathrm{aff}(A)$ the affine hull of A, one of the referees proposes the following alternate proof of the second part of (b). It follows observing that: if $A \subseteq B$ and $\mathrm{aff}(A) = \mathrm{aff}(B)$, then one has $\mathrm{qri}(A) \subseteq \mathrm{qri}(B)$. We apply it to the sets $A = \mathrm{co}(\mathcal{E}_\mu \cup \{(0,0)\})$, $B = \mathrm{cone}(\mathrm{co}\ \mathcal{E}_\mu)$. However, our proof is selfcontained.

Remark 3.9. By Lemma 2.2 (a) and since $K \neq \emptyset$, the hypothesis $0 \in \mathrm{qri}(\mathrm{co}(g(C)+P))$ is equivalent to $0 \in \mathrm{qri}(\mathrm{cone}(\mathrm{co}(g(C) + P)))$; likewise the hypothesis $0 \in \mathrm{qi}(\mathrm{co}(g(C) + P))$ is equivalent to $0 \in \mathrm{qi}(\mathrm{cone}(\mathrm{co}(g(C) + P)))$.

We end this section by noting that our results are closely related with saddle point conditions for the Lagrangian $L(1, \lambda^*, x)$ in case the infimum of (3.1) is attained. The saddle point characterization of strong duality clarifies the importance of such a property for nonconvex optimization problems both from the theoretical and algorithmic point of view.

Theorem 3.10. *Let Y be locally convex and assume that $0 \in \text{qi}(\text{co}(g(C)+P))$. Then, μ is attained at $\bar{x} \in K$ and any of the assumptions (b) or (c) of Theorem 3.3 holds if and only if there exists λ_0^* such that $(\lambda_0^*, \bar{x}) \in P^* \times C$ is a saddle point for $L(1, \lambda^*, x)$ on $P^* \times C$, i.e.,*

$$L(1, \lambda^*, \bar{x}) \leq L(1, \lambda_0^*, \bar{x}) \leq L(1, \lambda_0^*, x), \quad \forall (\lambda^*, x) \in P^* \times C. \tag{3.22}$$

Proof. It is enough to recall that $(\lambda_0^*, \bar{x})$ is a saddle point for $L(1, \lambda^*, x)$ on $(P^* \times C)$, if and only if $\bar{x} \in K$, $f(\bar{x}) = \inf_{x \in C} L(1, \lambda_0^*, x)$ and $\langle \lambda_0^*, g(\bar{x}) \rangle = 0$, so that the thesis follows from Theorem 3.5. □

Similarly, from Theorem 3.6 we obtain the following result.

Theorem 3.11. *Let Y be locally convex and assume that $0 \in \text{qri}(\text{co}(g(C)+P))$. Then, μ is attained at $\bar{x} \in K$ and any of the assumptions (b) or (c) of Theorem 3.4 holds if and only if there exists λ_0^* such that $(\lambda_0^*, \bar{x}) \in P^* \times C$ is a saddle point for $L(1, \lambda^*, x)$ on $P^* \times C$.*

4 Some consequences and comparison with other existing results

We observe that the convex hull appearing in the results of the previous section can be deleted everywhere simply by requiring the convexity of the sets $\overline{\text{cone}}\ \mathcal{E}_\mu$ and $\overline{\text{cone}}(g(C)+P)$, since in this situation,

$$\overline{\text{cone}}\ \mathcal{E}_\mu = \overline{\text{cone}}(\text{co}\ \mathcal{E}_\mu), \quad \overline{\text{cone}}(\text{co}(g(C)) + P) = \overline{\text{cone}}(g(C)+P).$$

An important class of vector functions implying the convexity of the sets $\mathcal{E}_\mu$ and $g(C)+P$ which satisfy more verifiable conditions, is that introduced in [24]: given a convex set $C \subseteq X$ with X as above, a real locally convex topological vector space Z along with a convex cone $Q \subseteq Z$, a mapping $G : C \to Z$ is called $*$-quasiconvex if $\langle q^*, G(\cdot) \rangle$ is quasiconvex for all $q^* \in Q^*$. Independently, the author in [35] says that G is naturally Q-quasiconvex if for all $x, y \in C$, $G([x,y]) \subseteq [G(x), G(y)] - Q$. Both classes coincide as shown in [10, Proposition 3.9] when int $Q \neq \emptyset$, and [13, Theorem 2.3] for general Q. See also [25].
It is known from Corollary 3.11 in [10], that every $*$-quasiconvex function $G : C \to Z$ satisfying (4.1):

$$\forall q^* \in Q^*, \text{ the restriction of } \langle q^*, G(\cdot) \rangle \text{ on any line segment of } C \text{ is lower semicontinuous,} \tag{4.1}$$

is such that $G(C) + P$ is convex, so that $G(C') + P$ is also convex for every convex set $C' \subseteq C$.
Therefore, by setting $F = (f, g)$, and assuming the convexity of C, the lower semicontinuity on any line segment of C of $\langle q^*, F(\cdot)\rangle$ for all $q^* \in \mathbb{R}_+ \times P^*$, and the $*$-quasiconvexity of $F : C \to \mathbb{R} \times Y$, we get: the convexity of $F(C) + (\mathbb{R}_+ \times P)$, (and so $\mathcal{E}_\mu$ is convex as well) and the quasiconvexity of the functions f and $\langle p^*, g(\cdot)\rangle$ on C for all $p^* \in P^*$. Hence, $f(C) + \mathbb{R}_+$ and $g(C) + P$ are convex sets as well.

Obviously there are vector functions F such that $F(C)+(\mathbb{R}_+\times P)$ is convex without being $*$-quasiconvex. The convexity of $F(C) + (\mathbb{R}_+ \times P)$ was imposed in [7, 17]. Hence, our Theorem 3.6 is more general, even in the convex case, than Theorem 4.4 in [7] and Theorem 10 in [17], since the last two theorems require the stronger condition $0 \in \mathrm{qi}(g(C) + P)$. This is shown by Example 4.3 below. To be more precise, Theorem 4.4 in [7] reads as follows

Theorem 4.1. [7, *Theorem* 4.4] *Suppose that* $F(C)+(\mathbb{R}_+\times P)$ *is convex,* $0 \in \mathrm{qi}(g(C)+P)$ *and* $(0, 0) \notin \mathrm{qri}[\mathrm{co}(\mathcal{E}_\mu \cup \{(0, 0)\})]$*. Then, there exists* $\lambda_0^* \in P^*$ *such that* (3.12) *holds.*

In order to prove the previous theorem, the authors show first that "Fenchel and Lagrange duality" (so, some convexity assumptions are imposed) are equivalent, generalizing an earlier result due to Magnanti [27]. Then, from such an equivalence the strong duality is obtained.

On the other hand, from Proposition 2.5(b), it follows

$$0 \in \mathrm{qi}(\mathrm{co}(g(C))+P) \iff 0 \in \mathrm{qi}[\mathrm{co}(g(C))+P-(\mathrm{co}(g(C))+P)] \text{ and } 0 \in \mathrm{qri}(\mathrm{co}(g(C))+P). \tag{4.2}$$

This implies that Theorems 4.2 and 4.4 in [7] are identical provided $g(C)+P$ is convex.

Furthermore, we point out that Theorems 3.5 and 3.6 apply to more general situations, even to non quasiconvex functions with equality and inequality constraints and possibly where $\operatorname*{argmin}_K f$ is empty as the next example shows.

Example 4.2. Notice this example shows our approach applies even if $\mathrm{int}\, P = \emptyset$.
Take $C = \mathbb{R}^2$, $P = \{0\} \times \mathbb{R}_+$, $x = (x_1, x_2)$, $f(x) = x_1^2 + 2e^{-x_2^2}$,

$$g_1(x) = x_1^4 - e^{-x_2^2}, \quad g_2(x) = x_1^2 - x_2^2,$$

and consider the problem

$$\mu \doteq \inf\{f(x) : \ g_1(x) = 0, \ g_2(x) \le 0, \ x \in C\}.$$

Thus, $P^* = \mathbb{R} \times \mathbb{R}_+$ and $\mu = 0$, although the set of minimizers is empty. Setting $F(x) = (f(x), g_1(x), g_2(x))$, $x \in C$, we obtain $F(x) = (x_1^2 + 2e^{-x_2^2}, x_1^4 - e^{-x_2^2}, x_1^2 - x_2^2)$.

It follows that

$$\mathcal{E}_0 = \{(u, v_1, v_2) \in \mathbb{R}^3 : u \geq x_1^2 + 2e^{-x_2^2},\ v_1 = x_1^4 - e^{-x_2^2},\ v_2 \geq x_1^2 - x_2^2,\ (x_1, x_2) \in \mathbb{R}^2\},$$

which is nonconvex (see afterwards). Then, because of the condition $u \geq x_1^2 + 2e^{-x_2^2},\ \forall (x_1, x_2) \in \mathbb{R}^2$, we have co $\mathcal{E}_0 \subseteq \{(u, v_1, v_2) \in \mathbb{R}^3 : u > 0\}$ and, therefore,

$$(0, 0, 0) \notin \operatorname{int}(\operatorname{cone}(\operatorname{co}(F(C)) - \mu(1, 0, 0) + \mathbb{R}_+ \times P)).$$

Observe that $\mathcal{E}_0 \cap \mathcal{H} = \emptyset$, where $\mathcal{H} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u < 0, v_1 = 0, v_2 \leq 0\}$. Moreover,

$$(g_1, g_2)(C) + P = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = x_1^4 - e^{-x_2^2}, v_2 \geq x_1^2 - x_2^2,\ (x_1, x_2) \in \mathbb{R}^2\}$$

is nonconvex. In fact, taking $(x_1, x_2) = (0, 0), (0, 2\sqrt{2})$, we have that $(v_1, v_2) = (-1, 0), (-e^{-8}, -8)$ belongs to $(g_1, g_2)(C) + P$ but the convex combination $(\frac{-1-e^{-8}}{2}, -4) \notin (g_1, g_2)(C) + P$. To prove this, observe that the system

$$\begin{cases} \frac{-1-e^{-8}}{2} = x_1^4 - e^{-x_2^2},\ (x_1, x_2) \in \mathbb{R}^2; \\ -4 \geq x_1^2 - x_2^2, \end{cases}$$

is not possible. Otherwise, for a suitable $(x_1, x_2) \in \mathbb{R}^2$ it should be

$$\begin{cases} e^{-x_2^2} = x_1^4 + \frac{1+e^{-8}}{2} \geq \dfrac{1}{2}; \\ x_2^2 \geq x_1^2 + 4 \geq 4, \end{cases}$$

that is clearly impossible. This also proves that $\mathcal{E}_0$ is nonconvex.
It is easy to see that

$$(0, 0) \in \operatorname{int}(\operatorname{co}((g_1, g_2)(C) + P))$$

that is, the generalized Slater condition is satisfied. (Actually, taking $(x_1, x_2) = (0, 0), (1, 0), (1, \sqrt{2})$, we have that $(v_1, v_2) = (-1, 0), (0, 1), (1 - e^{-2}, -1)$ belongs to $(g_1, g_2)(C) + P$, so that the previous relation follows). On the other hand, given $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}_+$, we obtain

$$L(\lambda, x) = x_1^2 + 2e^{-x_2^2} + \lambda_1(x_1^4 - e^{-x_2^2}) + \lambda_2(x_1^2 - x_2^2) =$$

$$\lambda_1 x_1^4 + (1 + \lambda_2)x_1^2 + (2 - \lambda_1)e^{-x_2^2} - \lambda_2 x_2^2.$$

Hence, for $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \geq 0$, we get

$$\inf_{x \in \mathbb{R}^2} L(\lambda, x) = \begin{cases} 0 & \text{if} \quad \lambda_2 = 0,\ 0 \leq \lambda_1 \leq 2, \\ 2 - \lambda_1 & \text{if} \quad \lambda_2 = 0,\ \lambda_1 > 2, \\ -\infty & \text{if} \quad \lambda_2 = 0, \lambda_1 < 0 \text{ or } \lambda_2 > 0. \end{cases}$$

and therefore,

$$\max_{(\lambda_1,\lambda_2)\in P^*}\ \inf_{x\in\mathbb{R}^2} L(\lambda,x)=0=\mu,$$

$$\inf_{x\in\mathbb{R}^2} L(\lambda^*,x)=0,\quad \lambda^*=(\lambda_1^*,0),\quad 0\le\lambda_1^*\le 2.$$

The following example shows that even in the convex case, our Theorem 3.6 is applicable but Theorem 4.4 in [7] or Theorem 10 in [17] are not.

Example 4.3. This example shows an application to a convex problem with int $P=\emptyset$. Take $C=\{(x_1,x_2,x_3)\in\mathbb{R}^3: x_3=0\}$, $P=\{0\}\times\{0\}\times\mathbb{R}_+$, $x=(x_1,x_2,x_3)$, $f(x)=x_1^2$,

$$g_1(x)=x_1+x_2+x_3,\quad g_2(x)=x_1+x_2-x_3,\quad g_3(x)=x_1^2+x_2^2-1,$$

and consider the problem

$$\mu\doteq\inf\{f(x):\ g_1(x)=0,\ g_2(x)=0,\ g_3(x)\le 0,\ x\in C\}.$$

Thus, $\mu=0$ and $\bar{x}=(0,0,0)$ is the optimal solution. Setting $F(x)=(x_1^2,x_1+x_2+x_3,x_1+x_2-x_3,x_1^2+x_2^2-1)$, it follows that

$$\mathcal{E}_0=\{(u,v_1,v_2,v_3)\in\mathbb{R}^4: u\ge x_1^2, v_1=x_1+x_2+x_3, v_2=x_1+x_2-x_3,\ v_3\ge x_1^2+x_2^2-1,\ x\in\mathbb{R}^3\},$$

is convex, and it is not difficult to check that $\overline{\text{cone}}(\mathcal{E}_0)\cap\mathcal{H}=\emptyset$, which implies that $\overline{\text{cone}}(\mathcal{E}_0)\cap(-\mathbb{R}_{++}\cap\{0\})=\emptyset$, where

$$\mathcal{H}=\{(u,v_1,v_2,v_3)\in\mathbb{R}^3: u<0, v_1=0, v_2=0, v_3\le 0\}.$$

The former equality allows us to apply Theorem 3.2. It is easy to see that

$$(g_1,g_2,g_3)(C)+P=\{(v_1,v_2,v_3)\in\mathbb{R}^3: v_1=v_2=x_1+x_2,\ v_3\ge x_1^2+x_2^2-1,\ (x_1,x_2)\in\mathbb{R}^2\}$$

is a convex set with empty interior so that $\text{qi}((g_1,g_2,g_3)(C)+P)=\emptyset$. However, since

$$(0,0,0)\in\text{qri}((g_1,g_2,g_3)(C)+P)$$

Theorem 3.6 can be applied.

The next example shows a problem where the Slater condition does not hold while (b) of Theorem 3.2 is fulfilled.

Example 4.4. Take $f(x)=x^2$, $C=\mathbb{R}$, $g(x)=x^2+x^4$, $P=\mathbb{R}_+$ and consider the problem

$$\mu\doteq\min_{g(x)\le 0} f(x)$$

It is easy to see that $\mu=0$ and $\bar{x}=0$ is the optimal solution. On the other hand, $F(C)=\{(x^2,x^2+x^4):\ \ x\in\mathbb{R}\}=\{(u,v)\in\mathbb{R}\times\mathbb{R}:\ v=u+u^2,\ u\in\mathbb{R}_+\}$. Then $F(C)+(\mathbb{R}_+\times P)=\mathbb{R}^2_+$ is a closed convex cone and (b) of Theorem 3.2 is fulfilled. However, $g(C)+P=\mathbb{R}_+$ and therefore $0\notin\text{qi}(g(C)+P)=\text{qri}(g(C)+P)$.

A simple consequence of Theorem 3.5 is the following.

Corollary 4.5. *Assume that* μ *is finite and* $0 \in \mathrm{co}(g(C)) + \mathrm{qi}\ P$*. Then, any of the assumptions* (*b*) *or* (*c*) *of Theorem 3.3 is equivalent to the existence of* $\lambda_0^* \in P^*$ *such that* (3.5) *is fulfilled.*

Proof. The result follows from Theorem 3.5 once we observe that $\mathrm{co}(g(C)) + \mathrm{qi}\ P \subseteq \mathrm{qi}(\mathrm{co}(g(C))+P)$. Such an inclusion easily follows since any $x = q+p$ with $q \in \mathrm{co}(g(C))$ and $p \in \mathrm{qi}\ P$ satisfies $\overline{\mathrm{cone}}(P - (x-q)) = Y$, which yields

$$Y = \overline{\mathrm{cone}}(P + q - x) \subseteq \overline{\mathrm{cone}}(P + \mathrm{co}(g(C)) - x) \subseteq Y,$$

proving the desired result. □

The next corollary is a generalization to the nonconvex case of Theorem 4.1 of [6].

Corollary 4.6. *Let* P *be a convex cone in* Y *such that* $\overline{P-P} = Y$*. Assume that* μ *is finite and* $0 \in \mathrm{co}(g(C)) + \mathrm{qri}\ P$*. Then, any of the assumptions* (*b*) *or* (*c*) *of Theorem 3.3 is equivalent to the existence of* $\lambda_0^* \in P^*$ *such that* (3.5) *is fulfilled.*

Proof. Since $\overline{P-P} = Y$, by Proposition 2.6 we obtain that $\mathrm{qri}\ P = \mathrm{qi}\ P$ so that our assumptions imply that $0 \in \mathrm{co}(g(C)) + \mathrm{qi}\ P$ and from Corollary 4.5 the conclusion follows. □

When the topological interior is employed instead of the quasi relative interior, we obtain the following theorem which is a consequence of Theorems 3.3 and 3.5, already appeared in [11] and applies to situations when int P may be empty. In what follows, $\mathbb{R}_{++} = \ ]0, +\infty[$.

Theorem 4.7. *Let us consider problem* (3.1) *and assume that* μ *is finite,*

$$\mathrm{int}[\mathrm{co}(F(C)) + (\mathbb{R}_+ \times P)] \neq \emptyset, \quad (\text{or } \mathrm{int}[\mathrm{co}\ \mathcal{E}_\mu] \neq \emptyset)$$

and $\overline{\mathrm{cone}}(\mathrm{co}(g(C) + P)) = Y$*. The following assertions are equivalent:*

(*a*) *there exists* $\lambda_0^* \in P^*$ *such that* (3.12) *hold;*

(*b*) $\mathrm{cone}[\mathrm{int}[\mathrm{co}(F(C)) - \mu(1,0) + (\mathbb{R}_+ \times P)]]$ *is pointed.*

(*b'*) $\mathrm{cone}[\mathrm{co}(F(C)) - \mu(1,0) + (\mathbb{R}_{++} \times \mathrm{int}\ P)]$ *is pointed, provided* $\mathrm{int}\ P \neq \emptyset$*.*

Remark 4.8. (**The case of finite dimensional spaces**)
Theorem 3.4, when specialized to finite dimension, reduces to Theorem 3.2 in [16]; whereas the finite dimensional version of Theorem 3.6 strengthens Theorem 3.6 in [16].

Remark 4.9. (**Connection with the S-lemma**)
We now provide a connection with the well-known S-lemma. This expresses the following: given $f, g : \mathbb{R}^n \to \mathbb{R}$ and $C \subseteq \mathbb{R}^n$, the question is under which conditions the implication

$$g(x) \le 0, \; x \in C \implies f(x) \ge 0$$

is satisfied, or equivalently when the system $g(x) \le 0, x \in C, \; f(x) < 0$ has no solution. The important case, when f and g are quadratic, with $C = \mathbb{R}^n$, was studied by Yakubovich, see the survey by Pólik and Terlaky in [30]. Its proof uses the Dines theorem which asserts the convexity of the set $\{(f(x), g(x)) \in \mathbb{R}^2 : \; x \in \mathbb{R}^n\}$ when f and g are homogeneous quadratic functions. The S-lemma due to Yakubovich says the following:
assume f and g as above and that there is $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) < 0$. Then, (a) and (b) are equivalent:

(a) There is no $x \in \mathbb{R}^n$ such that

$$f(x) < 0, \; g(x) \le 0.$$

(b) There is $\lambda \ge 0$ such that

$$f(x) + \lambda g(x) \ge 0, \quad \forall \; x \in \mathbb{R}^n.$$

Let us sketch a proof. Obviously $(b) \implies (a)$ always holds. Assume therefore that (a) holds. This means that $g(x) \le 0$ implies $f(x) \ge 0$, that is, $0 \le \mu \doteq \inf_{g(x) \le 0} f(x)$. By Proposition 3.1 we have that $\text{cone}(\mathcal{E}_\mu) \cap \mathcal{H} = \emptyset$, where (set $F = (f, g)$)

$$\mathcal{E}_\mu \doteq F(\mathbb{R}^n) - \mu(1, 0) + \mathbb{R}^2_+ \text{ and } \mathcal{H} \doteq \{(u, v) \in \mathbb{R}^2 \; : u < 0, v \le 0\}.$$

By Dines theorem [30, Proposition 2.3], $F(\mathbb{R}^n)$ is convex, and therefore $\mathcal{E}_\mu$ is convex. It follows that

$$\text{ri}(\text{cone } \mathcal{E}_\mu) \cap \text{ri } \mathcal{H} = \emptyset \quad \iff \quad \text{ri}(\text{cone } \mathcal{E}_\mu) \cap \text{ri}(\overline{\mathcal{H}}) = \emptyset$$

or, equivalently, (recalling that for any nonempty convex sets $C_1, \; C_2 \subseteq \mathbb{R}^n$, $\text{ri}(C_1 + C_2) = \text{ri}C_1 + \text{ri}C_2$, see [33, Corollary 6.6.2])

$$(0, 0) \notin \text{ri}[\text{cone } \mathcal{E}_\mu - \overline{\mathcal{H}}] = \text{ri}[\text{cone}(\mathcal{E}_\mu - \overline{\mathcal{H}})] = \text{ri}(\text{cone } \mathcal{E}_\mu).$$

Therefore, $(0, 0) \notin \text{ri}(\text{cone } \mathcal{E}_\mu)$. Moreover, we observe that the set $g(\mathbb{R}^n) + \mathbb{R}_+$ is convex since g takes values in $\mathbb{R}$. We can apply Theorem 3.6 to obtain the existence of $\lambda \ge 0$ such that $f(x) + \lambda g(x) \ge \mu \ge 0$ for all $x \in \mathbb{R}^n$.

We end this remark by pointing out that the Dines theorem was extended to the case when $\mathbb{R}^n$ is substituted by a cone K such that $K \cup (-K)$ is a subspace of $\mathbb{R}^n$ in [21, Theorem 3.2].

5 Zero duality gap and strong duality for a nonconvex variational problem

In this section we shall deal with the problem $P(a)$ defined by

$$\inf \int_0^1 f_0(t, z(t))dt \quad s.t. \quad z \in K(a) \doteq \Big\{ z \in L^1([0,1], \mathbb{R}^n) : \int_0^1 g_0(t, z(t))dt \in -P + a \Big\}, \tag{5.1}$$

where $f_0 : [0,1] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $g_0 : [0,1] \times \mathbb{R}^n \to (\mathbb{R} \cup \{+\infty\})^m$ and $f_0(t, \cdot)$ is lower semicontinuous and $g_0(t, \cdot)$ is continuous for a.e. $t \in [0,1]$; f_0 is a Borel function and $g_0(\cdot, u)$ is measurable (with respect to the Lebesgue measure) for all $u \in \mathbb{R}^n$ such that $g_0(\cdot, z(\cdot)) \in L^1([0,1], \mathbb{R}^m)$ for all $z \in L^1([0,1], \mathbb{R}^n)$; P is a closed convex cone in $\mathbb{R}^m$ and $a \in \mathbb{R}^m$. We consider the functions $f : L^1([0,1], \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ and $g : L^1([0,1], \mathbb{R}^n) \to \mathbb{R}^m$ defined by

$$f(z) = \int_0^1 f_0(t, z(t))dt, \quad g(z) = \int_0^1 g_0(t, z(t))dt,$$

Here, dom $g = L^1([0,1], \mathbb{R}^m)$ by assumption. Furthermore, in order to avoid technicalities we impose the following linear growth condition of f_0: there exist $\alpha \in \mathbb{R}^n$, $\beta \in L^1([0,1], \mathbb{R})$ such that

$$f_0(t, u) \geq \langle \alpha, u \rangle + \beta(t), \quad \text{for a.e. } t \in [0,1], \text{ all } u \in \mathbb{R}^n.$$

Under this assumption, $f(z) > -\infty$ for all $z \in L^1([0,1], \mathbb{R}^n)$.

We associate with our problem the optimal value function ψ defined as follows

$$\psi(a) = \begin{cases} \inf \Big\{ \int_0^1 f_0(t, z(t))dt : g(z) \in -P + a \Big\} & \text{if } K(a) \neq \emptyset; \\ +\infty & \text{otherwise.} \end{cases}$$

Consider the Lagrangian dual (D) associated with $P(0)$ and defined by

$$v_D \doteq \sup_{\lambda \in P^*} \inf_{z \in C} L(1, \lambda, z).$$

Here $C \doteq L^1([0,1], \mathbb{R}^n)$ and

$$L(1, \lambda, z) = f(z) + \langle \lambda, g(z) \rangle, \quad \lambda \in P^*, \quad z \in L^1([0,1], \mathbb{R}^n). \tag{5.2}$$

We remark that the analysis can be equivalently carried out with slight modifications if in (5.1) we set $z \in C$ where C is a closed subset of $L^1([0,1],\mathbb{R}^n)$.

We recall an extension of the Lyapunov theorem proved in [28]. Given a set $K \subseteq L^1([0,1],\mathbb{R}^k)$, define the set

$$I(K) \doteq \Big\{ \int_0^1 \phi(t)dt : \phi \in K \Big\}.$$

K is said to be decomposable if, for every measurable set $B \subseteq [0,1]$ and all $u, v \in K$:

$$u \cdot \chi_B + v \cdot \chi_{[0,1]\setminus B} \in K,$$

where χ_B is the characteristic function of the set B.

Theorem 5.1. *If $K \subseteq L^1([0,1],\mathbb{R}^k)$ is decomposable, then $I(K)$ is convex and $I(K) = I(\text{co } K)$. If, in addition, K is (strongly) closed and $\overline{I(K)}$ contains neither a line nor an extremal halfline, then $I(K)$ is closed.*

In what follows, given $h : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, $\overline{h}$, $\overline{\text{co}}\ h$ stand for the greatest lower semicontinuous function bounded above by h and for the greatest convex and lower semicontinuous function bounded above by h, respectively. To be coherent with our previous notation we need the following definition of epigraph of a function: epi $h \doteq \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : \ h(x) \le t\}$.

It is known that

$$\text{epi } \overline{h} = \overline{\text{epi } h}; \quad \overline{\text{co}}(\text{epi } h) = \text{epi } \overline{\text{co}}\ h.$$

Moreover, if $\overline{\text{co}}\ h(x) > -\infty$ for all $x \in \mathbb{R}^n$ (which is satisfied when h is bounded below by a linear function for instance) then $\overline{\text{co}}\ h(x) = h^{**}(x)$ for all $x \in \mathbb{R}^n$, where h^{**} is the bipolar or biconjugate of h, that is, the conjugate of h^*. There are examples showing the assumption $\overline{\text{co}}\ h(x) > -\infty$ for all $x \in \mathbb{R}^n$ is necessary to get the equality. In general we have $h^{**} \le \overline{\text{co}}\ h \le h$.

Set $C_0 \doteq \text{dom } f = \{z \in C : \ f(z) < +\infty\}$ and

$$K_0 \doteq \Big\{(u,v) \in L^1([0,1],\mathbb{R}^{1+m}) : \ \exists\ z \in C, u(t) \ge f_0(t,z(t)), \\ v(t) \ge_P g_0(t,z(t)), \text{for a.e. } t \in [0,1]\Big\}. \quad (5.3)$$

The following result holds.

Proposition 5.2. $I(K_0) = F(C_0) + (\mathbb{R}_+ \times P)$; *it is convex and*

$$\overline{F(C_0) + (\mathbb{R}_+ \times P)} = \overline{\text{epi } \psi} = \text{epi } \overline{\psi} = \text{epi } \overline{\text{co}}\ \psi,$$

where $F(z) \doteq (f(z), g(z))$.

Proof. We observe that K_0 is a decomposable set and $I(K_0) = F(C_0) + (\mathbb{R}_+ \times P)$. Indeed, let $(u_i, v_i) \in K_0$, $i = 1, 2$ and $B \subseteq [0, 1]$ a measurable set. Then,

$$u_i(t) \geq f_0(t, z_i(t)), \quad v_i(t) \geq_P g_0(t, z_i(t)), \text{for a.e. } t \in [0, 1].$$

Setting $\tilde{z} \doteq z_1 \cdot \chi_B + z_2 \cdot \chi_{[0,1]\setminus B} \in C$, we have for a.e. $t \in [0, 1]$:

$$u_1(t)\cdot\chi_B(t)+u_2(t)\cdot\chi_{[0,1]\setminus B}(t) \geq f_0(t, \tilde{z}(t)), \;\; v_1(t)\cdot\chi_B(t)+v_2(t)\cdot\chi_{[0,1]\setminus B}(t) \geq_P g_0(t, \tilde{z}(t)),$$

i.e. $(u_1, v_1) \cdot \chi_B + (u_2, v_2) \cdot \chi_{[0,1]\setminus B} \in K_0$.
It is straightforward that $I(K_0) \subseteq F(C_0) + (\mathbb{R}_+ \times P)$. To prove the reverse inclusion it is enough to observe that if $(u, v) \in F(C_0) + (\mathbb{R}_+ \times P)$, then, for some $z \in C_0$ and $(h, p) \in (\mathbb{R}_+ \times P)$,

$$(u, v) = \Big(\int_0^1 [f_0(t, z(t)) + h]dt, \int_0^1 [g_0(t, z(t)) + p]dt \Big) \in I(K_0).$$

This proves the first equality.
By Theorem 5.1 with K_0 instead of K, the set $F(C_0) + (\mathbb{R}_+ \times P)$ is convex.
We observe that

$$F(C_0) + (\mathbb{R}_+ \times P) \subseteq \text{epi } \psi \subseteq \overline{F(C_0) + (\mathbb{R}_+ \times P)}. \tag{5.4}$$

Taking the convex hulls in (5.4) we obtain

$$F(C_0) + (\mathbb{R}_+ \times P) \subseteq \text{co}(\text{epi } \psi) \subseteq \overline{F(C_0) + (\mathbb{R}_+ \times P)}. \tag{5.5}$$

Taking the closures in (5.4) and (5.5) we complete the proof. □

Let us define the function $G : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ by

$$G(p) = \int_0^1 f^g(t, p)dt, \quad \text{where} \quad f^g(t, p) \doteq \sup_{\xi \in \mathbb{R}^n} \Big\{ \langle p, g_0(t, \xi) \rangle - f_0(t, \xi) \Big\}.$$

It follows that G is lsc and convex.
Let $\mu = \psi(0)$ be the value of $P(0)$. Next result provides a new characterization of the zero duality gap between $P(0)$ and (D) in terms of the set $\mathcal{E}_\mu \doteq F(C_0) + (\mathbb{R}_+ \times P) - \mu(1, 0)$ considered in Section 3.

Theorem 5.3. *Assume that* $K(0) \neq \emptyset$ *and*

$$A \doteq \{p \in -P^* : f^g(\cdot, p) \in L^1[0, 1]\} \neq \emptyset. \tag{5.6}$$

The following statements hold.

(*a*) $\psi(a) \geq \langle p^*, a \rangle - G(p^*) > -\infty, \forall\, a \in \mathbb{R}^n, \;\forall\, p^* \in A$; *consequently* $\overline{\psi} = \overline{\text{co}}\, \psi = \psi^{**}$;

(*b*) *The duality gap between* $P(0)$ *and (D) is zero, i.e.,* $v_D = \psi(0)$*, if and only if*

$$\overline{\mathcal{E}_\mu} \cap -(\mathbb{R}_{++} \times \{0\}) = \emptyset. \tag{5.7}$$

Proof. (a): Let $p^* \in A$. We have

$$G(p^*) \doteq \int_0^1 f^g(t,p^*)dt \geq \int_0^1 \langle p^*, g_0(t,z(t))\rangle dt - \int_0^1 f_0(t,z(t))dt, \quad \forall z \in L^1([0,1],\mathbb{R}^n).$$

Then, for every $a \in \mathbb{R}^m$ and $z \in K(a)$ there exists $p \in P$ such that:

$$\int_0^1 f_0(t_0,z(t))dt \geq \int_0^1 \langle p^*, g_0(t,z(t))\rangle dt - \int_0^1 f^g(t,p^*)dt \geq \int_0^1 \langle p^*, a-p\rangle dt - G(p^*).$$

Since $p^* \in -P^*$ we have:

$$\psi(a) \geq \int_0^1 \langle p^*, a\rangle dt - G(p^*) = \langle p^*, a\rangle - G(p^*).$$

(b) Recalling that $v_D = \psi^{**}(0)$, (see e.g., [32, Theorem 7]), by (a), it follows that $v_D = \overline{\psi}(0)$. Therefore, we only need to prove that $\overline{\psi}(0) = \psi(0)$.

Since $\mu = \psi(0)$ then (5.7) is equivalent to

$$\overline{F(C_0) + (\mathbb{R}_+ \times P)} \cap \{(u,v) \in \mathbb{R} \times \mathbb{R}^n : u < \psi(0), v = 0\} = \emptyset. \tag{5.8}$$

Taking into account Proposition 5.2, we have that (5.8) implies that $\overline{\psi}(0) \geq \psi(0)$. Recalling that

$$\overline{\psi}(a) \leq \psi(a), \quad \forall\ a \in \mathbb{R}^n, \tag{5.9}$$

we obtain $\overline{\psi}(0) = \psi(0)$.

Vice versa, from (5.9) and Proposition 5.2, we immediately obtain that $\overline{\psi}(0) = \psi(0)$ implies (5.8).

□

Remark 5.4. We notice that the set $I(K_0)$ is closed in the simplest case when K_0 is an affine set, i.e., $\forall x,y \in K_0$, $\forall \alpha \in \mathbb{R}$, $\alpha x + (1-\alpha)y \in K_0$. Then, recalling that $I : K_0 \to \mathbb{R}^{n+1}$ is linear, $I(K_0)$ is an affine set in $\mathbb{R}^{n+1}$ and therefore it is closed. Clearly K_0 is affine if $f_0(t,\cdot)$ and $g_0(t,\cdot)$ are affine, for a.e. $t \in [0,1]$ and C is an affine set in $L^1([0,1],\mathbb{R}^n)$. In such a case, from Proposition 5.2 it follows that $\mathcal{E}_\mu$ is closed and by Proposition 3.1, we have that (5.7) is fulfilled.

Recalling that $L(1,\lambda,z)$ is given by (5.2), from Theorem 3.6, the following result on strong duality is obtained, without assuming any coercivity or convexity assumption. Here quasi relative interior, "qri", coincides with relative interior, "ri".

Corollary 5.5. *Assume that* $\mu \in \mathbb{R}$*,* $K(0) \neq \emptyset$ *and* $0 \in \mathrm{ri}(g(C_0) + P)$*. Then, there exists* $\lambda_0 \in P^*$ *such that*

$$\inf_{z \in K(0)} \int_0^1 f_0(t, z(t))dt = \inf_{z \in L^1([0,1],\mathbb{R}^n)} \int_0^1 [f_0(t, z(t)) + \langle \lambda_0, g_0(t, z(t)) \rangle]dt.$$

Moreover, if $\bar{z} \in L^1([0,1],\mathbb{R}^n)$ *is a solution to* $P(0)$*, then* $(\lambda_0, \bar{z})$ *is a saddle point for* $L(1, \lambda, z)$ *on* $(P^* \times C)$*.*

Proof. By Proposition 3.1 we have that $\mathrm{cone}(\mathcal{E}_\mu) \cap \mathcal{H} = \emptyset$, where

$$\mathcal{E}_\mu \doteq F(C_0) - \mu(1,0) + (\mathbb{R}_+ \times P) \text{ and } \mathcal{H} \doteq \{(u,v) \in \mathbb{R} \times \mathbb{R}^m : u < 0, v \in -P\}.$$

By Proposition 5.2, $\mathcal{E}_\mu$ is convex. It follows that

$$\mathrm{ri}(\mathrm{cone}\ \mathcal{E}_\mu) \cap \mathrm{ri}\ \mathcal{H} = \emptyset \quad \Leftrightarrow \quad \mathrm{ri}(\mathrm{cone}\ \mathcal{E}_\mu) \cap \mathrm{ri}(\overline{\mathcal{H}}) = \emptyset$$

or, equivalently,

$$(0,0) \notin \mathrm{ri}[\mathrm{cone}\ \mathcal{E}_\mu - \overline{\mathcal{H}}] = \mathrm{ri}[\mathrm{cone}(\mathcal{E}_\mu - \overline{\mathcal{H}})] = \mathrm{ri}[\mathrm{cone}\ \mathcal{E}_\mu].$$

Therefore, $(0,0) \notin \mathrm{ri}[\mathrm{cone}(F(C_0) - \mu(1,0) + (\mathbb{R}_+ \times P))]$. Moreover, we observe that the set $g(C_0) + P$ is convex since $\mathcal{E}_\mu$ is convex, so that the thesis follows from Theorem 3.6.

The last statement follows from Theorem 3.11. □

In case $g_0(t,z) = z$ and $C_0 = L^1([0,1],\mathbb{R}^n)$, we obtain $g(L^1([0,1],\mathbb{R}^n) = \mathbb{R}^n$. Thus, $0 \in \mathrm{ri}(g(C_0) + P)$ trivially holds whatever P is.

A similar strong duality result with a different proof was established in [8], see also [29].

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