A posteriori error analysis of twofold saddle point variational formulations for nonlinear boundary value problems*

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Abstract

In this paper we recast the analysis of twofold saddle point variational formulations for several nonlinear boundary value problems arising in continuum mechanics, and derive reliable and efficient residual-based a posteriori error estimators for the associated Galerkin schemes. We illustrate the main results with nonlinear elliptic equations modelling heat conduction and hyperelasticity. The main tools of our analysis include a global inf-sup condition for a linearization of the problem, Helmholtz's decompositions, local approximation properties of the Raviart-Thomas and Clément interpolation operators, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions. Finally, several numerical results confirming the theoretical properties of the estimator and showing the behaviour of the associated adaptive algorithms, are provided.

Key words: twofold saddle point equation, mixed finite element method, a posteriori error estimator

Mathematics Subject Classifications (1991): 65N15, 65N30, 65N50, 74B05

1 Introduction

In this paper we derive new a posteriori error estimators for the Galerkin solutions of a class of nonlinear twofold saddle point operator equations. This kind of saddle point problem, also called dual-dual variational formulations, arised some time ago from the necessity of applying dual-mixed finite element methods to several nonlinear boundary value problems appearing in potential theory and elasticity. Before it, one of the most common ideas for treating nonlinear elliptic equations was based on the inversion, thanks to the implicit function theorem, of the constitutive equations involved. In heat conduction, for instance, the gradient of the temperature

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is expressed as a function of the temperature and the flux variable. This procedure has been studied, including h and p versions and extensions to nonlinear parabolic problems, in several works (see, e.g. [35], [36], and [37] and the references therein). Then, for the case of constitutive equations that are not explicitly invertible, a new methodology was introduced first in [32] and [33], in connection with the coupling of mixed finite element and boundary integral equation methods for solving nonlinear transmission problems. This approach is based on the introduction of the gradient (in potential theory and heat conduction) or the strain tensor (in elasticity and fluid mechanics) as an additional unknown, which yields twofold saddle point operator equations as the resulting weak formulations (see [3], [6], [7], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [30], [31] for details and diverse applications). In particular, the abstract framework that is needed for the solvability analysis of these problems, which constitutes a natural extension of the classical Babuška-Brezzi theory, was developed in [18] and [27]. In addition, suitable numerical methods for solving the systems arising from the associated Galerkin schemes, have been proposed in [23], [24], [25], and [26]. It must be mentioned, however, that the idea of introducing further unknowns to deal with the nonlinearities of the problem was also employed, independently, in [13] and [14], where it was called an expanded mixed finite element method. Actually, the use of an expanded mixed formulation had already been proposed before for some elasticity problems in [17]. Nevertheless, the twofold saddle point structure has only been obtained and studied in the above mentioned works.

On the other hand, in order to obtain good convergence behavior of the Galerkin solution of linear and nonlinear boundary value problems, one normally requires to apply adaptive algorithms that are based on a-posteriori error estimates. This adaptivity is specially necessary when applying finite elements (FEM), mixed finite elements (mixed-FEM), boundary elements (BEM), or any combination of them to nonlinear problems, since in general no a-priori hints on how to build suitable meshes are available in these cases. While the list of references on a-posteriori error analysis for linear and nonlinear problems is nowadays quite extense, which includes some important contributions in recent years, most of the main ideas and associated techniques can be found in [1], [40] and the references therein. Indeed, the first results for mixed formulations of elliptic partial differential equations of second order, which consider a-posteriori error estimators of explicit residual type, the solution of local problems, and the eventual derivation of reliability and efficiency properties, among other issues, go back to [39], [2], [8] and [12]. Furthermore, the classical Bank-Weiser estimator from [5], which involves the solution of equilibrated local Neumann problems, has also been applied to mixed formulations of linear and nonlinear problems. In particular, we refer to [10], where the large-strain elasticity case, including incompressibility, is considered, and also to [11], where implicit residual error estimators for the coupling of finite elements and boundary elements are obtained.

In turn, several of the above mentioned papers concerning twofold saddle point variational formulations for nonlinear problems also include the development of a posteriori error analyses for their associated Galerkin schemes (see, e.g. [3], [6], [7], [19], [21], [31]). Moreover, all the approaches employed in these works are based on the combination of the Bank-Weiser method with the utilization of Ritz projectors and suitable local problems. In particular, a fully explicit and reliable a-posteriori error estimator is derived first in [7] for the dual-mixed formulation of a nonlinear problem in plane elasticity. The related case of nonlinear incompressible elasticity is initially analyzed in [31] and later on in [19]. Similar estimates to those given in [7] are also presented in [3] and [21] for the dual-mixed formulations of nonlinear elliptic equations in divergence form and quasi-Newtonian Stokes flows, respectively. Then the results from [7] and [31] are extended in [6] to the coupling of dual mixed-FEM and BEM as applied to the linear-nonlinear transmission problem in plane hyperelasticity with mixed boundary conditions that is studied in [22]. The main contribution of [6] consists of a reliable a-posteriori error estimator that depends on the solution of local Dirichlet problems and on residual terms on the transmission and Neumann boundaries, which are given in a negative order Sobolev norm. In addition, for certain specific subspaces, two fully local a-posteriori error estimates, in which the residual terms are bounded by weighted local L^2-norms, are provided.

Nevertheless, no one of the a posteriori error estimators developed so far for twofold saddle point variational formulations had been shown to be efficient until [29]. The closest result in this direction had been given by the *quasi-efficiency* property, which, introduced in [3], [31], [21] and [19], refers to the idea of obtaining efficiency up to one or more terms. However, these extra terms usually depend on empirically chosen auxiliary functions, whence there is no theoretical support for them to be of higher order. Only recently, and in connection with a velocity-pseudostress approach for a class of quasi-Newtonian Stokes flows, a reliable and efficient residual-based a posteriori error estimator for the resulting nonlinear twofold saddle point operator equation was provided in [29]. The key aspects of the analysis in [29] are a global inf-sup condition for a linearized version of the original problem, and a conveniently constructed Helmholtz decomposition of the space containing the stresses of the fluid, together with its discrete counterpart. Motivated by the above, the main purpose of the present work is to extend the results from [29] to any nonlinear twofold saddle point variational formulation. According to this, the rest of the paper is organized as follows. In Section 2 we recall from [18] and [27] the main theoretical results needed for the continuous and discrete analyses of nonlinear twofold saddle point operator equations. A linearization technique is then utilized in Section 3 to deduce an abstract a posteriori error estimate for formulations of this kind. Next, this theory is applied in Sections 4 and 5 to derive reliable and efficient residual-based a posteriori error estimators for nonlinear problems from heat conduction and hyperelasticity, respectively. Similarly as for linear problems, the main tools employed include Helmholtz's decompositions, local approximation properties of interpolation operators, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions. Finally, several numerical results confirming the reliability and efficiency of the estimators and illustrating the good performance of the associated adaptive algorithms are reported in Section 6.

2 A class of nonlinear twofold saddle point problems

Let X_1, M_1, and M be Hilbert spaces, and consider a nonlinear operator $\mathbb{A}_1 : X_1 \to X_1'$, and linear bounded operators $\mathbb{B}_1 : X_1 \to M_1'$ and $\mathbb{B} : M_1 \to M'$, with transposes $\mathbb{B}_1' : M_1 \to X_1'$ and $\mathbb{B}' : M \to M_1'$, respectively. Then, given $(\mathbb{H}, \mathbb{G}, \mathbb{F}) \in X_1' \times M_1' \times M'$, we are interested in the following nonlinear variational problem: Find $(\mathbf{t}, \boldsymbol{\sigma}, u) \in \mathbf{X} := X_1 \times M_1 \times M$ such that

$$\begin{array}{lclcl} [\mathbb{A}_1(\mathbf{t}), \mathbf{s}] + [\mathbb{B}_1(\mathbf{s}), \boldsymbol{\sigma}] & & & = [\mathbb{H}, \mathbf{s}] & \forall\, \mathbf{s} \in X_1\,, \\ [\mathbb{B}_1(\mathbf{t}), \boldsymbol{\tau}] & + & [\mathbb{B}(\boldsymbol{\tau}), u] & = [\mathbb{G}, \boldsymbol{\tau}] & \forall\, \boldsymbol{\tau} \in M_1\,, \\ \quad [\mathbb{B}(\boldsymbol{\sigma}), v] & & & = [\mathbb{F}, v] & \forall\, v \in M\,, \end{array} \tag{2.1}$$

where $[\cdot,\cdot]$ stands in each case for the duality pairing induced by the corresponding operators and functionals. Note that (2.1) can also be written, equivalently, as the matrix equation: Find

$(\mathbf{t}, \boldsymbol{\sigma}, u) \in \mathbf{X}$ such that

$$\begin{pmatrix} \mathbb{A}_1 & \mathbb{B}_1' & \mathbb{O} \\ \mathbb{B}_1 & \mathbb{O} & \mathbb{B}' \\ \mathbb{O} & \mathbb{B} & \mathbb{O} \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \boldsymbol{\sigma} \\ u \end{pmatrix} = \begin{pmatrix} \mathbb{H} \\ \mathbb{G} \\ \mathbb{F} \end{pmatrix},$$

which clearly shows a twofold saddle point structure.

Furthermore, the abstract theory for this kind of variational formulation is already available in the literature (see [18], [27]), and their main results are collected in what follows.

THEOREM 2.1 *Let* $V := \mathrm{Ker}(\mathbb{B})$, *define* $V_1 := \{\mathbf{s} \in X_1 : \quad [\mathbb{B}_1(\mathbf{s}), \boldsymbol{\tau}] = 0 \ \forall \boldsymbol{\tau} \in V\}$, *and let* $\Pi_1 : X_1' \to V_1'$ *be the operator defined by* $\Pi_1(\mathbb{H}) = \mathbb{H}|_{V_1}$ *for all* $\mathbb{H} \in X_1'$. *Assume that*

i) *the nonlinear operator* $\mathbb{A}_1 : X_1 \to X_1'$ *is Lipschitz continuous with a Lipschitz constant* $\gamma > 0$, *and for any* $\tilde{\mathbf{t}} \in X_1$, *the nonlinear operator* $\Pi_1 \mathbb{A}_1(\cdot + \tilde{\mathbf{t}}) : V_1 \to V_1'$ *is strongly monotone with a monotonicity constant* $\alpha > 0$ *independent of* $\tilde{\mathbf{t}}$.

ii) *there exists* $\beta > 0$ *such that for all* $v \in M$

$$\sup_{\boldsymbol{\tau} \in M_1 \setminus \{\mathbf{0}\}} \frac{[\mathbb{B}(\boldsymbol{\tau}), v]}{||\boldsymbol{\tau}||_{M_1}} \geq \beta \, ||v||_M \,; \tag{2.2}$$

iii) *there exists* $\beta_1 > 0$ *such that for all* $\boldsymbol{\tau} \in V$

$$\sup_{\mathbf{s} \in X_1 \setminus \{\mathbf{0}\}} \frac{[\mathbb{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{||\mathbf{s}||_{X_1}} \geq \beta_1 \, ||\boldsymbol{\tau}||_{M_1} \,; \tag{2.3}$$

Then, for each $(\mathbb{H}, \mathbb{G}, \mathbb{F}) \in X_1' \times M_1' \times M'$ *there exists a unique* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in \mathbf{X}$ *solution of* (2.1). *Moreover, there exists* $C > 0$, *independent of the solution, such that*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u)\|_{\mathbf{X}} \leq C \Big\{ \|\mathbb{H}\| + \|\mathbb{G}\| + \|\mathbb{F}\| + \|\mathbb{A}_1(\mathbf{0})\| \Big\}.$$

Proof. See [18, Theorem 2.4] (see also [27, Theorem 2.1] or [30, Theorem 4.1]). □

Now, let $X_{1,h}$, $M_{1,h}$ and M_h be finite dimensional subspaces of X_1, M_1 and M, respectively. Then the Galerkin scheme associated with (2.1) reads as follows: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in \mathbf{X}_h := X_{1,h} \times M_{1,h} \times M_h$ such that

$$\begin{aligned} [\mathbb{A}_1(\mathbf{t}_h), \mathbf{s}_h] \ + \ [\mathbb{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] \qquad\qquad &= [\mathbb{H}, \mathbf{s}_h] \quad \forall\, \mathbf{s}_h \in X_{1,h}\,, \\ [\mathbb{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] \qquad\qquad + \ [\mathbb{B}(\boldsymbol{\tau}_h), u_h] \ &= [\mathbb{G}, \boldsymbol{\tau}_h] \quad \forall\, \boldsymbol{\tau}_h \in M_{1,h}\,, \\ [\mathbb{B}(\boldsymbol{\sigma}_h), v_h] \qquad\qquad &= [\mathbb{F}, v_h] \quad \forall\, v_h \in M_h\,. \end{aligned} \tag{2.4}$$

The discrete analogue of Theorem 2.1 is established next.

THEOREM 2.2 *Let* $V_h := \{\boldsymbol{\tau}_h \in M_{1,h} : \quad [\mathbb{B}(\boldsymbol{\tau}_h), v_h] = 0 \quad \forall\, v_h \in M_h\}$, *define the space* $V_{1,h} := \{\mathbf{s}_h \in X_{1,h} : \quad [\mathbb{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h] = 0 \quad \forall\, \boldsymbol{\tau}_h \in V_h\}$ *and let* $\Pi_{1,h} : X_{1,h}' \to V_{1,h}'$ *be the operator defined by* $\Pi_{1,h}(\mathbb{H}_h) = \mathbb{H}_h|_{V_{1,h}}$ *for all* $\mathbb{H}_h \in X_{1,h}'$. *Further, let* $\mathbb{A}_{1,h} := p_h' \mathbb{A}_1 : X_1 \to X_{1,h}'$ *where* $p_h : X_{1,h} \to X_1$ *is the canonical injection with adjoint* $p_h' : X_1' \to X_{1,h}'$. *Assume that*

i) *the nonlinear operator* $\mathbb{A}_{1,h} : X_1 \to X'_{1,h}$ *is Lipschitz-continuous with a Lipschitz constant* $\gamma_h > 0$, *and for any* $\tilde{\mathbf{t}} \in X_{1,h}$, *the nonlinear operator* $\Pi_{1,h}\mathbb{A}_{1,h}(\cdot + \tilde{\mathbf{t}}) : V_{1,h} \to V'_{1,h}$ *is strongly monotone with a monotonicity constant* $\alpha_h > 0$ *independent of* $\tilde{\mathbf{t}}$.

ii) *there exists* $\beta_h > 0$ *such that for all* $v_h \in M_h$

$$\sup_{\boldsymbol{\tau}_h \in M_{1,h}\backslash\{\mathbf{0}\}} \frac{[\mathbb{B}(\boldsymbol{\tau}_h), v_h]}{||\boldsymbol{\tau}_h||_{M_1}} \geq \beta_h \, ||v_h||_M;$$

iii) *there exists* $\beta_{1,h} > 0$ *such that for all* $\boldsymbol{\tau}_h \in V_h$

$$\sup_{\mathbf{s}_h \in X_{1,h}\backslash\{\mathbf{0}\}} \frac{[\mathbb{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{||\mathbf{s}_h||_{X_1}} \geq \beta_{1,h} \, ||\boldsymbol{\tau}_h||_{M_1};$$

Then, for each $(\mathbb{H}, \mathbb{G}, \mathbb{F}) \in X'_1 \times M'_1 \times M'$ *there exists a unique* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in \mathbf{X}_h$ *solution of* (2.4)*. Moreover, there exists* $C_h > 0$, *independent of the solution, but depending on* h, *such that*

$$\|(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\|_{\mathbf{X}} \leq C_h \left\{ \|\mathbb{H}_h\| + \|\mathbb{G}_h\| + \|\mathbb{F}_h\| + \|\mathbb{A}_{1,h}(\mathbf{0})\| \right\},$$

where $\mathbb{H}_h := \mathbb{H}|_{X_{1,h}}$, $\mathbb{G}_h := \mathbb{G}|_{M_{1,h}}$, *and* $\mathbb{F}_h := \mathbb{F}|_{M_h}$.

Proof. See [18, Theorem 3.2] (see also [27, Theorem 3.1] or [30, Theorem 4.2]). □

Finally, concerning the error analysis, we have the following result.

THEOREM 2.3 *Assume that the hypotheses of Theorems* 2.1 *and* 2.2 *are satisfied, and let* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in \mathbf{X}$ *and* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in \mathbf{X}_h$ *be the unique solutions of* (2.1) *and* (2.4)*, respectively. In addition, suppose that there exist positive constants* $\tilde{\gamma}$, $\tilde{\alpha}$, $\tilde{\beta}$, *and* $\tilde{\beta}_1$ *such that* $\gamma_h \leq \tilde{\gamma}$, $\alpha_h \geq \tilde{\alpha}$, $\beta_h \geq \tilde{\beta}$, *and* $\beta_{1,h} \geq \tilde{\beta}_1$ *for all* h. *Then, there exists* $C > 0$, *independent of* h, *such that the following Céa error estimate holds:*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\|_{\mathbf{X}} \leq C \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \\ \in \mathbf{X}_h}} \|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{s}_h, \boldsymbol{\tau}_h, v_h)\|_{\mathbf{X}}.$$

Proof. See [18, Section 4] (see also [27, Theorem 3.3]). □

3 An abstract a posteriori error estimate

We begin by assuming that the nonlinear operator $\mathbb{A}_1 : X_1 \to X'_1$ is Gâteaux differentiable. This means that for each $\mathbf{x} \in X_1$ there exists a bounded and linear operator $\mathcal{D}\mathbb{A}_1(\mathbf{x}) : X_1 \to X'_1$ such that

$$\mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r}) = \lim_{\epsilon \to 0} \frac{\mathbb{A}_1(\mathbf{x} + \epsilon\, \mathbf{r}) - \mathbb{A}_1(\mathbf{x})}{\epsilon} \qquad \forall\, \mathbf{r} \in X_1.$$

It follows that for each $\mathbf{x} \in X_1$, $\mathcal{D}\mathbb{A}_1(x)$ can be considered as a bilinear form acting from $X_1 \times X_1$ into $\mathbb{R}$ as follows:

$$\mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r}, \mathbf{s}) := \mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r})(\mathbf{s}) \quad \forall\, \mathbf{x}, \mathbf{r}, \mathbf{s} \in X_1.$$

Hence, we also suppose that the family $\{\mathcal{D}\mathbb{A}_1(\mathbf{x})\}_{\mathbf{x}\in X_1}$ is uniformly bounded and uniformly elliptic on $X_1 \times X_1$ and $V_1 \times V_1$, respectively, that is that there exist $M, \alpha > 0$ such that for each $\mathbf{x} \in X_1$ there hold

$$|\mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r},\mathbf{s})| \leq M\|\mathbf{r}\|\,\|\mathbf{s}\| \quad \forall (\mathbf{r},\mathbf{s}) \in X_1 \times X_1,$$

and

$$\mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r},\mathbf{r}) \geq \alpha\,\|\mathbf{r}\|^2 \quad \forall\, \mathbf{r} \in V_1\,,$$

where, as set in Theorem 2.1, $V := \mathrm{Ker}(\mathbb{B})$ and $V_1 := \{\mathbf{s} \in X_1 : \quad [\mathbb{B}_1(\mathbf{s}), \boldsymbol{\tau}] = 0\ \forall\, \boldsymbol{\tau} \in V\}$. In addition, we assume that the linear operators $\mathbb{B}$ and $\mathbb{B}_1$ satisfy the continuous inf-sup conditions (2.2) and (2.3) (cf. Theorem 2.1).

Therefore, as a consequence of the continuous dependence result provided by the linear version of Theorem 2.1 (cf. (2.1) with $\mathbb{A}_1$ linear), we conclude that the linear operator $\mathcal{L}$ obtained by adding the three equations of the left hand side of (2.1), after replacing $\mathbb{A}_1$ by the Gâteaux derivative $\mathcal{D}\mathbb{A}_1(\mathbf{x})$ at any $\mathbf{x} \in X_1$, satisfies a global inf-sup condition. More precisely, there exists a constant $C > 0$, independent of $\mathbf{x} \in X_1$, such that

$$c\,\|(\mathbf{r},\boldsymbol{\zeta},w)\|_{\mathbf{X}} \leq \sup_{(\mathbf{s},\boldsymbol{\tau},v)\in \mathbf{X}\backslash\{0\}} \frac{[\mathcal{L}(\mathbf{s},\boldsymbol{\tau},v),(\mathbf{r},\boldsymbol{\zeta},w)]}{\|(\mathbf{s},\boldsymbol{\tau},v)\|_{\mathbf{X}}} \tag{3.1}$$

for all $(\mathbf{x},(\mathbf{r},\boldsymbol{\tau},w)) \in X_1 \times \mathbf{X}$, where

$$[\mathcal{L}(\mathbf{s},\boldsymbol{\tau},v),(\mathbf{r},\boldsymbol{\zeta},w)] := \mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r},\mathbf{s}) + [\mathbb{B}_1(\mathbf{s}),\boldsymbol{\zeta}] + [\mathbb{B}_1(\mathbf{r}),\boldsymbol{\tau}] + [\mathbb{B}(\boldsymbol{\tau}),w] + [\mathbb{B}(\boldsymbol{\zeta}),v]. \tag{3.2}$$

Then, we have the following abstract a posteriori error estimate.

THEOREM 3.1 *Let* $(\mathbf{t},\boldsymbol{\sigma},u) \in \mathbf{X}$ *and* $(\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h) \in \mathbf{X}_h$ *be the unique solutions of the continuous and discrete formulations (*2.1) and (2.4), respectively. Then, there exists $C > 0$, independent of h, such that

$$\|(\mathbf{t},\boldsymbol{\sigma},u) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h)\|_{\mathbf{X}} \leq C\Big\{\|R_1\|_{X_1'} + \|R_2\|_{M_1'} + \|R_3\|_{M'}\Big\}, \tag{3.3}$$

where

$$R_1(\mathbf{s}) := [\mathbb{H},\mathbf{s}] - [\mathbb{A}_1(\mathbf{t}_h),\mathbf{s}] - [\mathbb{B}_1(\mathbf{s}),\boldsymbol{\sigma}_h] \quad \forall\, \mathbf{s} \in X_1, \tag{3.4}$$

$$R_2(\boldsymbol{\tau}) := [\mathbb{G},\boldsymbol{\tau}] - [\mathbb{B}_1(\mathbf{t}_h),\boldsymbol{\tau}] - [\mathbb{B}(\boldsymbol{\tau}),u_h] \quad \forall\, \boldsymbol{\tau} \in M_1, \tag{3.5}$$

$$R_3(v) := [\mathbb{F},v] - [\mathbb{B}(\boldsymbol{\sigma}_h),v] \quad \forall\, v \in M. \tag{3.6}$$

Proof. We first observe, thanks to the mean value theorem, that there exists a convex combination of $\mathbf{t}$ and $\mathbf{t}_h$, say $\widetilde{\mathbf{t}}_h \in X_1$, such that

$$\mathcal{D}\mathbb{A}_1(\widetilde{\mathbf{t}}_h)(\mathbf{t}-\mathbf{t}_h,\mathbf{s}) = [\mathbb{A}_1(\mathbf{t}),\mathbf{s}] - [\mathbb{A}_1(\mathbf{t}_h,\mathbf{s})] \quad \forall\, \mathbf{s} \in X_1. \tag{3.7}$$

Then, applying (3.1) and (3.2) to the error $(\mathbf{r},\boldsymbol{\zeta},w) := (\mathbf{t},\boldsymbol{\sigma},u) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h)$, and using the identity (3.7) and the fact that

$$\|(\mathbf{s},\boldsymbol{\tau},v)\|_{\mathbf{X}} \geq \max\{\|\mathrm{s}\|_{\mathrm{X}_1}, \|\boldsymbol{\tau}\|_{\mathrm{M}_1}, \|\mathrm{v}\|_{\mathrm{M}}\} \quad \forall\, (\mathbf{s},\boldsymbol{\tau},\mathrm{v}) \in \mathbf{X},$$

we find that

$$c\,\|(\mathbf{t},\boldsymbol{\sigma},u)-(\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h)\|_{\mathbf{X}} \,\le\, \sup_{(\mathbf{s},\boldsymbol{\tau},v)\in\mathbf{X}\backslash\{0\}} \left\{\frac{R_1(\mathbf{s})+R_2(\boldsymbol{\tau})+R_3(v)}{\|(\mathbf{s},\boldsymbol{\tau},v)\|_{\mathbf{X}}}\right\}$$
$$\le\, \|R_1\|_{X_1'}+\|R_2\|_{M_1'}+\|R_3\|_{M'}\,,$$

where $R_1\in X_1', R_2\in M_1'$, and $R_3\in M'$, are given by

$$R_1(\mathbf{s}) := [\mathbb{A},(\mathbf{t}),\mathbf{s}]-[\mathbb{A}_1(\mathbf{t}_h),\mathbf{s}]+[\mathbb{B}_1(\mathbf{s}),\boldsymbol{\sigma}-\boldsymbol{\sigma}_h] \quad \forall\,\mathbf{s}\in\,X_1\,,$$

$$R_2(\boldsymbol{\tau}) := [\mathbb{B}_1(\mathbf{t}-\mathbf{t}_h),\boldsymbol{\tau}]+[\mathbb{B}(\boldsymbol{\tau}),u-u_h] \quad \forall\,\boldsymbol{\tau}\in M_1\,,$$

$$R_3(v) := [\mathbb{B}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),v] \quad \forall\,v\in M\,,$$

Then, according to the three equations of the continuous formulation (2.1), the above functionals become as given in (3.4), (3.5), and (3.6). $\square$

We remark that this theorem provides the key estimate for the reliability of a residual-based a posteriori error estimator for our Galerkin scheme (2.1). Moreover, in most of the applications that we have in mind, particularly in the ones to be developed in the following sections, the norms of the functionals R_1 and R_3 are simply estimated from the identities:

$$\|R_1\|_{X_1'} \,=\, \|\mathbb{H}-\mathbb{A}_1(\mathbf{t}_h)-\mathbb{B}_1'(\boldsymbol{\sigma}_h)\|_{X_1'} \,=\, \sup_{\mathbf{s}\in X_1\backslash\{\mathbf{0}\}} \frac{[\mathbb{H}-\mathbb{A}_1(\mathbf{t}_h)-\mathbb{B}_1'(\boldsymbol{\sigma}_h),\mathbf{s}]}{\|\mathbf{s}\|_{X_1}}\,,$$

and

$$\|R_3\|_{M'} \,=\, \|\mathbb{F}-\mathbb{B}(\boldsymbol{\sigma}_h)\|_{M'} \,=\, \sup_{v\in M\backslash\{\mathbf{0}\}} \frac{[\mathbb{F}-\mathbb{B}(\boldsymbol{\sigma}_h),v]}{\|v\|_M}\,,$$

which follow straightforwardly from (3.4) and (3.6). In turn, the estimate for $\|R_2\|_{M_1'}$ could also be obtained analogously from the expression

$$\|R_2\|_{M_1'} \,=\, \|\mathbb{G}-\mathbb{B}_1(\mathbf{t}_h)-\mathbb{B}'(u_h)\|_{M_1'} \,=\, \sup_{\boldsymbol{\tau}\in M_1\backslash\{\mathbf{0}\}} \frac{[\mathbb{G}-\mathbb{B}_1(\mathbf{t}_h)-\mathbb{B}'(u_h),\boldsymbol{\tau}]}{\|\boldsymbol{\tau}\|_{M_1}}\,. \tag{3.8}$$

However, this procedure will usually yield reliability but not efficiency of our estimate. Hence, in order to overcome this difficulty, one needs to introduce a suitable bounded linear operator $\Pi_h : M_1 \to M_{1,h}$ so that $(I-\Pi_h)$ gives rise to the additional terms that are needed for efficiency. Indeed, noting from the second equation of (2.4) that $R_2(\boldsymbol{\tau}_h)=0 \quad \forall\,\boldsymbol{\tau}_h\in M_{1,h}$, we can write $R_2(\boldsymbol{\tau}) = R_2(\boldsymbol{\tau}-\Pi_h(\boldsymbol{\tau})) \quad \forall\,\boldsymbol{\tau}\in M_1$, and then replace (3.8) by

$$\|R_2\|_{M_1'} \,=\, \sup_{\substack{\boldsymbol{\tau}\in M_1\\ \boldsymbol{\tau}\neq 0}} \frac{R_2(\boldsymbol{\tau}-\Pi_h(\boldsymbol{\tau}))}{\|\boldsymbol{\tau}\|_{M_1}} \,=\, \sup_{\substack{\boldsymbol{\tau}\in M_1\\ \boldsymbol{\tau}\neq 0}} \frac{[\mathbb{G}-\mathbb{B}_1(\mathbf{t}_h)-\mathbb{B}'(u_h),\boldsymbol{\tau}-\Pi_h(\boldsymbol{\tau})]}{\|\boldsymbol{\tau}\|_{M_1}}. \tag{3.9}$$

In this way, the extra terms arising from $\boldsymbol{\tau}-\Pi_h(\boldsymbol{\tau})$, mostly given by powers of the meshsizes multiplied by local norms of $\boldsymbol{\tau}$, will be crucial for the efficiency of the a posteriori error estimator. This comment will be better understood throughout the examples shown next.

4 A problem from heat conduction

We first illustrate the application of the abstract estimate provided in the previous section with the nonlinear elliptic problem in divergence form analyzed in [3] and [27]. In order to describe the boundary value problem of interest, we let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary $\Gamma := \partial\Omega$. Then, given $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$ and a scalar function $\boldsymbol{\kappa} : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, we look for $u \in H^1(\Omega)$ such that

$$-\mathrm{div}(\boldsymbol{\kappa}(\cdot, |\nabla u|)\nabla u) = f \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \Gamma, \tag{4.1}$$

where $|\cdot|$ stands for the euclidean norm in $\mathbb{R}^2$ and div denotes the usual divergence operator. This kind of nonlinear elliptic problem in divergence form appears in several applications, such as steady heat conduction and the computation of the magnetic field of electromagnetic devices. In what follows, we assume that $\boldsymbol{\kappa} \in C^1(\Omega \times \mathbb{R}^+)$ and that there exist constants $\kappa_0, \kappa_1 > 0$ such that for all $(x, \rho) \in \Omega \times \mathbb{R}^+$:

$$\begin{aligned} &\kappa_0 \le \boldsymbol{\kappa}(x,\rho) \le \kappa_1, \\ &\kappa_0 \le \boldsymbol{\kappa}(x,\rho) + \rho \frac{\partial}{\partial \rho}\boldsymbol{\kappa}(x,\rho) \le \kappa_1, \\ &|\nabla_x \boldsymbol{\kappa}(x,\rho)| \le \kappa_1. \end{aligned} \tag{4.2}$$

4.1 The continuous twofold saddle point formulation

We now establish the dual-mixed variational formulation of (4.1). For this purpose we introduce the further unknowns $\mathbf{t} := \nabla u$ and $\boldsymbol{\sigma} = \boldsymbol{\kappa}(\cdot, |\nabla u|)\, \nabla u$ in Ω so that (4.1) is rewritten as the nonlinear first order system:

$$\mathbf{t} = \nabla u \quad \text{in} \quad \Omega\,, \qquad \boldsymbol{\sigma} = \boldsymbol{\kappa}(\cdot, |\mathbf{t}|)\,\mathbf{t} \quad \text{in} \quad \Omega\,,$$

$$\mathrm{div}\,\boldsymbol{\sigma} = -f \quad \text{in} \quad \Omega\,, \qquad u = g \quad \text{on} \quad \Gamma\,.$$

In this way, proceeding in the usual form (see e.g. [3] or [27] for details), we arrive at the following problem: Find $(\mathbf{t}, \boldsymbol{\sigma}, u) \in [L^2(\Omega)]^2 \times H(\mathrm{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{array}{rrrcl} \displaystyle\int_\Omega \boldsymbol{\kappa}(\cdot,|\mathbf{t}|)\mathbf{t}\cdot\mathbf{s} & \displaystyle -\int_\Omega \boldsymbol{\sigma}\cdot\mathbf{s} & & = & 0 \\ \displaystyle -\int_\Omega \boldsymbol{\tau}\cdot\mathbf{t} & & \displaystyle -\int_\Omega u\,\mathrm{div}\,\boldsymbol{\tau} & = & -\langle \boldsymbol{\tau}\cdot\nu,\, g\rangle \\ & \displaystyle -\int_\Omega v\,\mathrm{div}\boldsymbol{\sigma} & & = & \displaystyle\int_\Omega f v \end{array} \tag{4.3}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, v) \in [L^2(\Omega)]^2 \times H(\mathrm{div};\Omega) \times L^2(\Omega)$, where $\langle\cdot,\cdot\rangle$ stands for the duality pairing of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$ - inner product, and ν denotes the unit outward normal to Γ. We also recall here that $H(\mathrm{div};\Omega)$ is the space of functions $\boldsymbol{\tau} \in [L^2(\Omega)]^2$ such that $\mathrm{div}\,\boldsymbol{\tau} \in L^2(\Omega)$. It is well known that $H(\mathrm{div};\Omega)$ is a Hilbert space with the norm $\|\cdot\|_{\mathrm{div},\Omega}$ induced by the scalar product

$$\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{\mathrm{div},\Omega} := \int_\Omega \Big(\boldsymbol{\zeta}\cdot\boldsymbol{\tau} + \mathrm{div}\boldsymbol{\zeta}\,\mathrm{div}\boldsymbol{\tau}\Big), \tag{4.4}$$

and that for all $\boldsymbol{\tau} \in H(\mathrm{div};\Omega), \boldsymbol{\tau}\cdot\nu \in H^{-1/2}(\Gamma)$ with $\|\boldsymbol{\tau}\cdot\nu\|_{-1/2,\Gamma} \le \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}$.

Now, it is easy to see that (4.3) has the twofold saddle point structure considered in Section 2. In fact, let $X_1 := [L^2(\Omega)]^2$, $\quad M_1 := H(\mathrm{div};\Omega)$, $\quad M := L^2(\Omega)$, $\quad \mathbf{X} := X_1 \times M_1 \times M$, and define the nonlinear operator $\mathbb{A}_1 : X_1 \to X_1'$, the bounded linear operators $\mathbb{B}_1 : X_1 \to M_1'$ and $\mathbb{B} : M_1 \to M'$ and the functionals $\mathbb{H} \in X_1'$, $\quad \mathbb{G} \in M_1'$ and $\mathbb{F} \in M'$, as follows:

$$\begin{aligned}
[\mathbb{A}_1(\mathbf{r}), \mathbf{s}] &:= \int_\Omega \boldsymbol{\kappa}(\cdot, |\mathbf{r}|)\mathbf{r}\cdot\mathbf{s}, \\
[\mathbb{B}_1(\mathbf{r}), \boldsymbol{\tau}] &:= -\int_\Omega \mathbf{r}\cdot\boldsymbol{\tau}, \\
[\mathbb{B}(\boldsymbol{\zeta}), v] &:= -\int_\Omega v \,\mathrm{div}\,\boldsymbol{\zeta},
\end{aligned} \tag{4.5}$$

$$[\mathbb{H}, \mathbf{s}] := 0, \quad [\mathbb{G}, \boldsymbol{\tau}] = -\langle \boldsymbol{\tau}\cdot\nu, g\rangle, \quad \text{and} \quad [\mathbb{F}, v] := \int_\Omega f\, v, \tag{4.6}$$

for all $(\mathbf{r}, \boldsymbol{\zeta}), (\mathbf{s}, \boldsymbol{\tau}) \in X_1 \times M_1$ and for all $v \in M$. We remark here that the first condition in (4.2) ensures that $\mathbb{A}_1(\mathbf{r}) \in X_1'$ for all $\mathbf{r} \in X_1$, which confirms that $\mathbb{A}_1$ is well defined.

Then, it is clear that, with the above definitions, our variational formulation (4.3) can be written in the twofold saddle point structure given by (2.1).

The solvability of (4.3) was proved in [27]. It reduces to show that $\mathbb{A}_1$, $\mathbb{B}_1$ and $\mathbb{B}$ satisfy the hypotheses of Theorem 2.1. The corresponding result is stated as follows.

THEOREM 4.1 *There exists a unique* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X$ *solution of the nonlinear twofold saddle point problem* (4.3)*. Moreover, there exists* $C > 0$*, independent of the solution, such that*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u)\|_{\mathbf{X}} \le C \left\{ \|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma} \right\}.$$

Proof. See [27, Theorem 4.1]. □

4.2 The associated Galerkin scheme

In order to define the Galerkin scheme associated with (4.3), we assume from now on that Γ is a polygonal curve and let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of Ω, made up of triangles T of diameter h_T, such that $h := \sup\{h_T : T \in \mathcal{T}_h\}$ and $\overline{\Omega} := \bigcup\{T : T \in \mathcal{T}_h\}$. Given an integer $\ell \ge 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathbb{P}_\ell(S)$ the space of polynomials of total degree at most ℓ defined on S. In addition, for each $T \in \mathcal{T}_h$ and for each integer $k \ge 0$ we define the local Raviart-Thomas space of order k (see, e.g. [9], [38])

$$\mathbb{RT}_k(T) := [\mathbb{P}_k(T)]^2 \oplus \mathbb{P}_k(T)\,\mathbf{x},$$

where $\mathbf{x}$ is a generic vector of $\mathbb{R}^2$. Then, we define the following finite element subspaces

$$X_{1,h} := \{\mathbf{s}_h \in [L^2(\Omega)]^2 : \mathbf{s}_h|_T \in [\mathbb{P}_k(T)]^2 \quad \forall T \in \mathcal{T}_h\}, \tag{4.7}$$

$$M_{1,h} := \{\boldsymbol{\tau}_h \in H(\mathrm{div};\Omega) : \boldsymbol{\tau}_h|_T \in \mathbb{RT}_k(T) \quad \forall T \in \mathcal{T}_h\}, \tag{4.8}$$

$$M_h := \{v_h \in L^2(\Omega) : v_h|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h\}. \tag{4.9}$$

The well-posedness of the Galerkin scheme associated with (4.3) is established as follows.

THEOREM 4.2 *Let k be a non-negative integer and let $X_{1,h}$, $M_{1,h}$, and M_h be given by* (4.7), (4.8), *and* (4.9). *Then there exists a unique solution* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in \mathbf{X}_h := X_{1,h} \times M_{1,h} \times M_h$ *of the discrete scheme* (2.4) *with the operators and functionals defined by* (4.5) *and* (4.6). *Moreover, there exist positive constants c and C, independent of h such that*

$$\|(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\|_{\mathbf{X}} \leq c \Big\{ \|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma} \Big\}, \tag{4.10}$$

and

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\|_{\mathbf{X}} \leq C \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \\ \in \mathbf{X}_h}} \|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{s}_h, \boldsymbol{\tau}_h, v_h)\|_{\mathbf{X}}. \tag{4.11}$$

Proof. Similarly as for Theorem 4.1, it reduces to show that $\mathbb{A}_1$, $\mathbb{B}_1$, and $\mathbb{B}$ satisfy the hypotheses of Theorem 2.2. We omit further details here and refer to [29, Section 2.4] for a very close analysis or to [27, Section 4.2] for a fully discrete version with $k = 0$. □

In turn, the following theorem provides the rate of convergence of the Galerkin scheme associated with (4.3).

THEOREM 4.3 *Let k be a non-negative integer and let $X_{1,h}$, $M_{1,h}$, and M_h be given by* (4.7), (4.8), *and* (4.9). *Let* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in \mathbf{X}$ *and* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in \mathbf{X}_h$ *be the unique solutions of* (4.3) *and its associated Galerkin scheme, respectively. Assume that* $\mathbf{t} \in [H^\delta(\Omega)]^2$, $\boldsymbol{\sigma} \in [H^\delta(\Omega)]^2$, $\operatorname{div} \boldsymbol{\sigma} \in H^\delta(\Omega)$, *and* $u \in H^\delta(\Omega)$, *for some* $\delta \in (0, k+1]$. *Then, there exists $C > 0$, independent of h, such that*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\|_{\mathbf{X}} \leq C\, h^\delta \Big\{ \|\mathbf{t}\|_{\delta,\Omega} + \|\boldsymbol{\sigma}\|_{\delta,\Omega} + \|\operatorname{div} \boldsymbol{\sigma}\|_{\delta,\Omega} + \|u\|_{\delta,\Omega} \Big\}. \tag{4.12}$$

Proof. It follows from the Céa estimate (4.11) and the approximation properties of the subspaces $X_{1,h}$, $M_{1,h}$, and M_h (see, e.g. [29, Section 2.4]). □

4.3 The a posteriori error analysis

4.3.1 Preliminaries

We begin by introducing further notations. We let $\mathcal{E}_h$ be the set of all edges of the triangulation $\mathcal{T}_h$, and given $T \in \mathcal{T}_h$, we let $\mathcal{E}(T)$ be the set of its edges. Then we write $\mathcal{E}_h = \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$, where $\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h : e \subseteq \Omega\}$ and $\mathcal{E}_h(\Gamma) := \{e \in \mathcal{E}_h : e \subseteq \Gamma\}$. In what follows, h_e stands for the length of the edge e. Also, for each $e \in \mathcal{E}_h$ we fix a unit normal vector $\nu_e := (\nu_1\,, \nu_2)^{\mathtt{t}}$, and let $s_e := (-\nu_2\,, \nu_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Then, given $e \in \mathcal{E}_h(\Omega)$ and $\boldsymbol{\tau} \in [L^2(\Omega)]^2$ such that $\boldsymbol{\tau}|_T \in [C(T)]^2$ on each $T \in \mathcal{T}_h$, we let $[\boldsymbol{\tau} \cdot s_e]$ be the corresponding jump across e, that is $[\boldsymbol{\tau} \cdot s_e] := \Big(\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'}\Big) \cdot s_e$, where T and T' are the triangles of $\mathcal{T}_h$ having e as a common edge. Abusing notation, when $e \in \mathcal{E}_h(\Gamma)$, we also write $[\boldsymbol{\tau} \cdot s_e] := \boldsymbol{\tau}|_e \cdot s_e$. Similar definitions hold for the tangential jumps of scalar fields $v \in L^2(\Omega)$ such that $v|_T \in C(T)$ on each $T \in \mathcal{T}_h$. From now on, when no confusion arises, we simply write s and ν instead of s_e and ν_e, respectively. Finally, given scalar and vector fields v and $\boldsymbol{\tau} := (\tau_1, \tau_2)^{\mathtt{t}}$, respectively, we let

$$\mathbf{curl}(v) := \Big(\frac{\partial v}{\partial x_2} \quad - \frac{\partial v}{\partial x_1} \Big)^{\mathtt{t}} \tag{4.13}$$

and

$$\operatorname{curl}(\boldsymbol{\tau}) := \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}. \tag{4.14}$$

On the other hand, we let $E_h^k : [H^1(\Omega)]^2 \to M_{1,h}$ be the usual Raviart-Thomas interpolation operator (see e.g. [38], [9]), which, given $\boldsymbol{\tau} \in [H^1(\Omega)]^2$, is characterized by the following identities:

$$\int_e E_h^k(\boldsymbol{\tau}) \cdot \nu \, \psi = \int_e \boldsymbol{\tau} \cdot \nu \, \psi \quad \forall \, \text{edge } e \in \mathcal{T}_h, \quad \forall \, \psi \in \mathbb{P}_k(e), \quad \text{when } k \geq 0, \tag{4.15}$$

and

$$\int_T E_h^k(\boldsymbol{\tau}) \cdot \boldsymbol{\psi} = \int_T \boldsymbol{\tau} \cdot \boldsymbol{\psi} \quad \forall \, T \in \mathcal{T}_h, \quad \forall \, \boldsymbol{\psi} \in [\mathbb{P}_{k-1}(T)]^2, \quad \text{when } k \geq 1. \tag{4.16}$$

It is easy to show, using (4.15) and (4.16), that

$$\operatorname{div}\big(E_h^k(\boldsymbol{\tau})\big) = \mathcal{P}_h^k\big(\operatorname{div}(\boldsymbol{\tau})\big) \quad \forall \boldsymbol{\tau} \in [H^1(\Omega)]^2, \tag{4.17}$$

where $\mathcal{P}_h^k$ is the orthogonal projector from $L^2(\Omega)$ into M_h. It is well known (see, e.g. [15]) that for each $v \in H^m(\Omega)$, with $0 \leq m \leq k+1$, there holds

$$\|v - \mathcal{P}_h^k(v)\|_{0,T} \leq C \, h_T^m \, |v|_{m,T} \quad \forall \, T \in \mathcal{T}_h. \tag{4.18}$$

Furthermore, the operator E_h^k satisfies the following approximation properties (see, e.g. [9], [38]), which, by the way, yielded the terms $h^\delta \|\boldsymbol{\sigma}\|_{\delta,\Omega}$ and $h^\delta \|\operatorname{div} \boldsymbol{\sigma}\|_{\delta,\Omega}$ for $\delta = m$ in the estimate (4.12), that is

$$\|\boldsymbol{\tau} - E_h^k(\boldsymbol{\tau})\|_{0,T} \leq C \, h_T^m \, |\boldsymbol{\tau}|_{m,T} \quad \forall \, T \in \mathcal{T}_h, \tag{4.19}$$

for each $\boldsymbol{\tau} \in [H^m(\Omega)]^2$, with $1 \leq m \leq k+1$,

$$\|\operatorname{div}\big(\boldsymbol{\tau} - E_h^k(\boldsymbol{\tau})\big)\|_{0,T} \leq C \, h_T^m \, |\operatorname{div} \boldsymbol{\tau}|_{m,T} \quad \forall \, T \in \mathcal{T}_h, \tag{4.20}$$

for each $\boldsymbol{\tau} \in [H^1(\Omega)]^2$ such that $\operatorname{div} \boldsymbol{\tau} \in H^m(\Omega)$, with $0 \leq m \leq k+1$, and

$$\|\boldsymbol{\tau} \cdot \nu - E_h^k(\boldsymbol{\tau}) \cdot \nu\|_{0,e} \leq C \, h_e^{1/2} \, \|\boldsymbol{\tau}\|_{1,T_e} \quad \forall \, \text{edge } e \in \mathcal{T}_h, \tag{4.21}$$

for each $\boldsymbol{\tau} \in [H^1(\Omega)]^2$, where $T_e \in \mathcal{T}_h$ contains e on its boundary. Note, in particular, that (4.20) follows directly from (4.17) and (4.18). In addition, it turns out (see, e.g. [34, Theorem 3.16]) that actually E_h^k can also be defined as a bounded linear operator from the larger space $[H^\delta(\Omega)]^2 \cap H(\operatorname{div};\Omega)$ into $M_{1,h}$ for all $\delta \in (0,1]$, and that in this case there holds

$$\|\boldsymbol{\tau} - E_h^k(\boldsymbol{\tau})\|_{0,T} \leq C \, h_T^\delta \Big\{ \|\boldsymbol{\tau}\|_{\delta,T} + \|\operatorname{div} \boldsymbol{\tau}\|_{0,T} \Big\} \quad \forall \, T \in \mathcal{T}_h.$$

In turn, we also need to consider the Clément interpolant $I_h : H^1(\Omega) \to X_h$ (cf. [16]), where

$$X_h := \Big\{ v_h \in C(\overline{\Omega}) : v_h|_T \in \mathbb{P}_1(T) \quad \forall \, T \in \mathcal{T}_h \Big\}.$$

It is well known (see [16]) that there exist constants $C_1, C_2 > 0$, independent of h, such that for all $v \in H^1(\Omega)$ there hold

$$\|v - I_h(v)\|_{0,T} \leq C_1 \, h_T \, \|v\|_{1,\Delta(T)} \quad \forall \, T \in \mathcal{T}_h, \tag{4.22}$$

and

$$\|v - I_h(v)\|_{0,e} \leq C_2 \, h_e^{1/2} \, \|v\|_{1,\Delta(e)} \quad \forall \, e \in \mathcal{E}_h, \tag{4.23}$$

where $\Delta(T)$ and $\Delta(e)$ are the union of all elements intersecting with T and e, respectively.

4.3.2 Reliability analysis

We first observe, from the fact that $\boldsymbol{\kappa} \in C^1(\Omega \times \mathbb{R}^+)$ and the assumptions (4.2), that $\mathbb{A}_1$ is Gâteaux differentiable and that $\{\mathcal{D}\mathbb{A}_1(\mathbf{x})\}_{\mathbf{x}\in X_1}$ is a family of uniformly bounded and uniformly elliptic bilinear forms on $X_1 \times X_1$. In particular, for the proof of the latter we refer to [3, Lemma 3]. Hence, a straightforward application of the abstract estimate (3.3) (cf. Theorem 3.1) gives

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\|_{\mathbf{X}} \leq C \Big\{ \|R_1\|_{X_1'} + \|R_2\|_{M_1'} + \|R_3\|_{M'} \Big\}, \tag{4.24}$$

where

$$R_1(\mathbf{s}) := -\int_\Omega \boldsymbol{\kappa}(\cdot, |\mathbf{t}_h|)\mathbf{t}_h \cdot \mathbf{s} + \int_\Omega \boldsymbol{\sigma}_h \cdot \mathbf{s} \qquad \forall\, \mathbf{s} \in X_1, \tag{4.25}$$

$$R_2(\boldsymbol{\tau}) := -\langle \boldsymbol{\tau} \cdot \nu \,, g\rangle + \int_\Omega \mathbf{t}_h \cdot \boldsymbol{\tau} + \int_\Omega u_h \operatorname{div} \boldsymbol{\tau} \qquad \forall\, \boldsymbol{\tau} \in M_1, \tag{4.26}$$

and

$$R_3(v) := \int_\Omega f v + \int_\Omega v \operatorname{div} \boldsymbol{\sigma}_h \qquad \forall\, v \in M. \tag{4.27}$$

It follows, using the Riesz Representation Theorem, that

$$\|R_1\|_{X_1'} = \|\boldsymbol{\sigma}_h - \boldsymbol{\kappa}(\cdot, |\mathbf{t}_h|)\mathbf{t}_h\|_{0,\Omega} \tag{4.28}$$

and

$$\|R_3\|_{M'} = \|f + \operatorname{div} \boldsymbol{\sigma}_h\|_{0,\Omega}. \tag{4.29}$$

Next, we proceed as in [29, Section 4.2] to derive an upper bound for $\|R_2\|_{M_1'}$. For this purpose, and in order to choose a suitable operator $\Pi_h : M_1 \to M_{1,h}$ to be utilized in (3.9), we consider a Helmholtz decomposition of M_1, which means that for each $\boldsymbol{\tau} \in M_1$ there exist $\varphi \in H^1(\Omega)$ with $\int_\Omega \varphi = 0$, and $z \in H^2(\Omega)$, such that

$$\boldsymbol{\tau} = \mathbf{curl}\, \varphi + \nabla z, \tag{4.30}$$

and

$$\|\varphi\|_{1,\Omega} + \|z\|_{2,\Omega} \leq C\, \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}. \tag{4.31}$$

Then, we let $\varphi_h := I_h(\varphi)$ and define

$$\Pi_h(\boldsymbol{\tau}) := \mathbf{curl}\, \varphi_h + E_h^k(\nabla z). \tag{4.32}$$

We refer to (4.32) as a discrete Helmholtz decomposition of $\boldsymbol{\tau}$. Then, employing the expressions (4.30) and (4.32), and noting, according to (4.9) and (4.17) and the fact that $\operatorname{div}(\nabla z) = \operatorname{div} \boldsymbol{\tau}$, that

$$\int_\Omega u_h \operatorname{div}(\nabla z - E_h^k(\nabla z)) = \int_\Omega u_h \Big\{ \operatorname{div}(\boldsymbol{\tau}) \,-\, \mathcal{P}_h^k\big(\operatorname{div}(\boldsymbol{\tau})\big) \Big\} = 0,$$

we deduce from (4.26) that

$$R_2(\boldsymbol{\tau}) = R_2(\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau})) = \widehat{R}_2(\varphi) + \widetilde{R}_2(z) \quad \forall \boldsymbol{\tau} \in M_1,$$

where

$$\widehat{R}_2\big(\varphi\big) \,:=\, R_2\big(\mathbf{curl}(\varphi-\varphi_h)\big) \,=\, -\langle \mathbf{curl}(\varphi-\varphi_h)\cdot\nu,\, g\rangle + \int_\Omega \mathbf{t}_h\cdot \mathbf{curl}(\varphi-\varphi_h) \tag{4.33}$$

and

$$\widetilde{R}_2(z) \,:=\, R_2\big(\nabla z - E_h^k(\nabla z)\big) \,=\, -\,\langle(\nabla z - E_h^k(\nabla z))\cdot\nu,\, g\rangle + \int_\Omega \mathbf{t}_h\cdot(\nabla z - E_h^k(\nabla z)). \tag{4.34}$$

The following two lemmas provide upper bounds for $|\widehat{R}_2(\varphi)|$ and $|\widetilde{R}_2(z)|$.

LEMMA 4.1 *Assume that $g\in H^1(\Gamma)$. Then, there exists $C>0$, independent of h, such that*

$$|\widehat{R}_2(\varphi)| \le C\left\{\sum_{T\in\mathcal{T}_h}\widehat{\theta}_{2,T}^2\right\}^{1/2}\|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}, \tag{4.35}$$

where

$$\widehat{\theta}_{2,T}^2 := h_T^2\,\|\mathrm{curl}\{\mathbf{t}_h\}\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega)} h_e\,\|[\mathbf{t}_h\cdot s]\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\left\|\frac{dg}{ds}-\mathbf{t}_h\cdot s\right\|_{0,e}^2.$$

Proof. We proceed analogously to the proof of [28, Lemma 4.3]. In fact, using that

$$\mathbf{curl}(\varphi-\varphi_h)\cdot\nu \,=\, \frac{d}{d\,s}(\varphi-\varphi_h)$$

and then integrating by parts on Γ, we find that

$$\langle\mathbf{curl}(\varphi-\varphi_h)\cdot\nu,\, g\rangle \,=\, -\langle\,\varphi-\varphi_h,\,\frac{dg}{ds}\rangle \,=\, -\sum_{e\in\mathcal{E}_h(\Gamma)}\int_e(\varphi-\varphi_h)\,\frac{dg}{ds}.$$

Now, integrating by parts on each $T\in\mathcal{T}_h$, we obtain that

$$\int_\Omega \mathbf{t}_h\cdot\mathbf{curl}(\varphi-\varphi_h) = \sum_{T\in\mathcal{T}_h}\left\{\int_T \mathrm{curl}(\mathbf{t}_h)\,(\varphi-\varphi_h) - \int_{\partial T}\mathbf{t}_h\cdot s\,(\varphi-\varphi_h)\right\}$$

$$= \sum_{T\in\mathcal{T}_h}\int_T \mathrm{curl}(\mathbf{t}_h)\,(\varphi-\varphi_h) - \sum_{e\in\mathcal{E}_h(\Omega)}[\mathbf{t}_h\cdot s]\,(\varphi-\varphi_h) - \sum_{e\in\mathcal{E}_h(\Gamma)}\mathbf{t}_h\cdot s\,(\varphi-\varphi_h)\,.$$

Then, replacing the above expressions into (4.33), we deduce that

$$\widehat{R}_2(\varphi) = \sum_{T\in\mathcal{T}_h}\int_T \mathrm{curl}(\mathbf{t}_h)\,(\varphi-\varphi_h) - \sum_{e\in\mathcal{E}_h(\Omega)}\int_e[\mathbf{t}_h\cdot s]\,(\varphi-\varphi_h) + \sum_{e\in\mathcal{E}_h(\Gamma)}\int_e\left\{\frac{dg}{ds}-\mathbf{t}_h\cdot s\right\}(\varphi-\varphi_h)\,.$$

In this way, applying the Cauchy-Schwarz inequality, the approximation properties of the Clément interpolant (cf. (4.22), (4.23)), the fact that the number of triangles in $\Delta(T)$ and $\Delta(e)$ are bounded, and finally the estimate (4.31), we arrive at the upper bound (4.35). $\square$

LEMMA 4.2 *There exists $C > 0$, independent of h, such that*

$$|\widetilde{R}_2(z)| \,\le\, C \left\{ \sum_{T\in\mathcal{T}_h} \widetilde{\theta}_{2,T}^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}\,, \tag{4.36}$$

where

$$\widetilde{\theta}_{2,T}^2 \,:=\, h_T^2\,\|\nabla u_h - \mathbf{t}_h\,\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\,\|g - u_h\,\|_{0,e}^2\,.$$

Proof. Since $u_h|_e \in \mathbb{P}_k(e)$ for each edge $e \in \mathcal{E}_h$ (in particular for each edge $e \in \mathcal{E}_h(\Gamma)$), the characterization identity (4.15) yields

$$\int_e \Big(\nabla z - E_h^k(\nabla z)\Big)\cdot \nu\; u_h = 0 \quad \forall\, e \in \mathcal{E}_h(\Gamma).$$

Similarly, the fact that $\nabla u_h|_T \in [\mathbb{P}_{k-1}(T)]^2$ for each $T \in \mathcal{T}_h$ and the identity (4.16) imply that

$$\int_T \Big(\nabla z - E_h^k(\nabla z)\Big)\cdot \nabla\, u_h = 0 \quad \forall\, T \in \mathcal{T}_h.$$

Then, introducing the above null expressions into the definition of $\widetilde{R}_2$ (cf. (4.33)), we obtain that

$$\widetilde{R}_2(z) = \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \Big(\nabla z - E_h^k(\nabla z)\Big)\cdot \nu\,(u_h\, -\, g) \,+\, \sum_{T\in\mathcal{T}_h}\int_T (\mathbf{t}_h - \nabla u_h)\cdot(\nabla z - E_h^k(\nabla z)).$$

Finally, applying the Cauchy-Schwarz inequality, the approximation properties of the Raviart-Thomas interpolation operator (cf. (4.19), (4.21)), and then the estimate (4.31), we get the upper bound (4.36). □

As a direct consequence of Lemmas 4.1 and 4.2, we deduce from (4.32) that

$$|R_2(\boldsymbol{\tau})| = |R_2(\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau}))| \,\le\, \left\{ \sum_{T\in\mathcal{T}_h} \Big(\hat{\theta}_{2,T}^2 \,+\, \widetilde{\theta}_{2,T}^2\Big) \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega} \quad \forall\,\boldsymbol{\tau} \in M_1,$$

which, using (3.9), gives an upper bound for $\|R_2\|_{M_1'}$.

In this way, according to our previous analysis, we can establish the following reliability result.

THEOREM 4.4 *Let $(\mathbf{t},\boldsymbol{\sigma},u) \in \mathbf{X}$ and $(\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h) \in \mathbf{X}_h$ be the unique solutions of (4.3) and its associated Galerkin scheme, respectively, and assume that $g \in H^1(\Gamma)$. Then, there exists a positive constant $C_{\mathtt{rel}}$, independent of h, such that*

$$\|(\mathbf{t},\boldsymbol{\sigma},u) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h)\|_X \,\le\, C_{\mathtt{rel}}\,\boldsymbol{\theta}\,,$$

where $\boldsymbol{\theta}^2 \,:=\, \sum_{T\in\mathcal{T}_h} \theta_T^2$ *and*

$$\theta_T^2 \,:=\, \|\boldsymbol{\sigma}_h - \boldsymbol{\kappa}(\cdot,\,|\mathbf{t}_h|)\mathbf{t}_h\|_{0,T}^2 \,+\, \|f + \mathrm{div}\,\boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, h_T^2\,\|\mathrm{curl}\{\mathbf{t}_h\}\|_{0,T}^2 \,+\, h_T^2\,\|\nabla u_h - \mathbf{t}_h\|_{0,T}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega)} h_e\,\|[\mathbf{t}_h\cdot s]\|_{0,e}^2 \,+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e \left\{ \left\|\frac{dg}{ds} - \mathbf{t}_h\cdot s\right\|_{0,e}^2 +\, \|g\, -\, u_h\|_{0,e}^2 \right\}. \tag{4.37}$$

Proof. It follows from (4.24), (4.28), (4.29), and Lemmas 4.1 and 4.2. We omit further details. $\square$

4.3.3 Efficiency analysis

The purpose of this section is to prove the efficiency of our a posteriori error estimator $\boldsymbol{\theta}$, which means that there exists a positive constant $C_{\texttt{eff}}$ such that

$$C_{\texttt{eff}}\,\boldsymbol{\theta} \;+\; \text{h.o.t.} \;\le\; \|(\mathbf{t},\boldsymbol{\sigma},u)-(\mathbf{t}_h,\boldsymbol{\sigma}_h,u_h)\|_{\mathbf{X}}, \tag{4.38}$$

where h.o.t. stands for one or several terms of higher order. To this end, in what follows ve establish suitable upper bounds for the seven terms defining the local error indicator θ_T^2.

We begin by noticing, since $\operatorname{div}\boldsymbol{\sigma} \;=\; -f \quad \text{in} \quad \Omega$, that

$$\|f+\operatorname{div}\boldsymbol{\sigma}_h\|_{0,T} \;=\; \|\operatorname{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,T} \;\le\; \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\operatorname{div},T} \quad \forall\, T\in\mathcal{T}_h\,. \tag{4.39}$$

In addition, using that $\boldsymbol{\sigma} \;=\; \boldsymbol{\kappa}(\cdot,|\mathbf{t}|)\mathbf{t} \quad \text{in} \quad \Omega$, and applying the Lipschitz-continuity of $\mathbb{A}_1$, but restricted to each $T\in\mathcal{T}_h$, we find that

$$\begin{aligned}\|\boldsymbol{\sigma}_h-\boldsymbol{\kappa}(\cdot,|\mathbf{t}_h|)\mathbf{t}_h\|_{0,T} \;&\le\; \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}+\|\boldsymbol{\kappa}(\cdot,|\mathbf{t}|)\mathbf{t}-\boldsymbol{\kappa}(\cdot,|\mathbf{t}_h|)\mathbf{t}_h\|_{0,T}\\ &\le\; \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\operatorname{div},T} \;+\; C\;\|\mathbf{t}-\mathbf{t}_h\|_{0,T} \quad \forall\, T \,\in\, \mathcal{T}_h\,.\end{aligned} \tag{4.40}$$

We now proceed to bound the terms involving the mesh parameters h_T and h_e. For this purpose we make use of the general results and estimates already available for linear problems (see, e.g. [28, Section 4.2]), which are all derived by employing triangle-bubble and edge-bubble functions, together with extension operators and discrete trace and inverse inequalities. Further details on these tools and techniques can be found in [29, Lemma 4.7 and 4.8, eq (4.34)].

The estimates of the remaining five terms defining θ_T^2 (cf. (4.37)) are stated as follows.

Lemma 4.3 *There exist C_1, $C_2 \,>\, 0$, independent of h, such that*

$$h_T^2\,\|\operatorname{curl}\{\mathbf{t}_h\}\|_{0,T}^2 \;\le\; C_1\,\|\mathbf{t}-\mathbf{t}_h\|_{0,T}^2 \quad \forall\, T\in\,\mathcal{T}_h\,,$$

$$h_e\,\|[\mathbf{t}_h\cdot s]\|_{0,e}^2 \;\le\; C_2\,\|\mathbf{t}-\mathbf{t}_h\|_{0,w_e}^2 \quad \forall\, e\in\mathcal{E}_h(\Omega)\,,$$

where $w_e \,:=\, \cup\{T\in\mathcal{T}_h : \quad e\in\mathcal{E}(T)\}$.

Proof. It is a direct application of the general results provided in [28, Lemmas 4.9 and 4.10] to $\boldsymbol{\rho}_h \;=\; \mathbf{t}_h$ and $\boldsymbol{\rho} \;=\; \mathbf{t}$, noting that $\operatorname{curl}\{\boldsymbol{\rho}\} \;=\; \operatorname{curl}\{\nabla u\} \;=\; 0$ in Ω. $\square$

Lemma 4.4 *There exists $C_3>0$, independent of h, such that*

$$h_T^2\,\|\nabla u_h-\mathbf{t}_h\|_{0,T}^2 \;\le\; C_3\left\{\|u-u_h\|_{0,T}^2 \;+\; h_T^2\,\|\mathbf{t}-\mathbf{t}_h\|_{0,T}^2\right\} \quad \forall\, T \,\in\, \mathcal{T}_h\,.$$

Proof. It basically follows from the proof of [28, Lemma 4.13], which is a slight modification of the proof of [12, Lemma 6.3], by simply replacing the tensor utilized there by our vector $\mathbf{t}_h$, and then using that $\mathbf{t} \;=\; \nabla u \quad \text{in} \quad \Omega$. $\square$

Lemma 4.5 *Assume that g is piecewise polynomial. Then there exists $C_4 > 0$, independent of h, such that*

$$h_e \left\| \frac{dg}{ds} - \mathbf{t}_h \cdot s \right\|_{0.e}^2 \leq C_4 \|\mathbf{t} - \mathbf{t}_h\|_{0,T}^2 \quad \forall T \in \mathcal{E}_h(\Gamma)\,,$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. Similarly to the proof of Lemma 4.4, it follows from [28, Lemma 4.15] by replacing the tensor employed there by our vector $\mathbf{t}_h$, and then using that $\dfrac{dg}{ds} = \nabla u \cdot s = \mathbf{t} \cdot s \quad \text{on} \quad \Gamma$. □

Lemma 4.6 *There exists $C_5 > 0$, independent of h, such that*

$$h_e \|g - u_h\|_{0,e}^2 \leq C_5 \left\{ \|u - u_h\|_{0,T}^2 + h_T^2 \|\mathbf{t} - \mathbf{t}_h\|_{0,T}^2 \right\} \quad \forall e \in \mathcal{E}_h(\Gamma)\,,$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. As in the previous lemmas, it results from [28, Lemma 4.14] by replacing the tensor used there by our vector $\mathbf{t}_h$, and then using that $\nabla u = \mathbf{t}$ in Ω and $u = g$ on Γ. In addition, at the end of the proof, the efficiency estimate for $h_T^2 \|\nabla u_h - \mathbf{t}_h\|_{0,T}^2$ provided by Lemma 4.4 is also employed. □

It is important to observe here that if g were not piecewise polynomial, but sufficiently smooth, then higher order terms given by the errors arising from polynomial approximations of g would appear in the efficiency estimate given by Lemma 4.5, thus explaining the eventual expression h.o.t. in (4.38). Consequently, the efficiency of $\boldsymbol{\theta}$ (as defined by (4.38)) follows directly from estimates (4.39) and (4.40), together with Lemmas 4.3 throughout 4.6, after summing up over $T \in \mathcal{T}_h$ and applying that the number of triangles on each domain w_e (cf. Lemma 4.3) is actually bounded by two.

5 A problem from nonlinear elasticity

In this section we consider the pure displacement version of the hyperelasticity problem studied in [7]. More precisely, let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary Γ. Our goal is to determine the displacement $\mathbf{u}$ and stress $\boldsymbol{\sigma}$ of a hyperelastic material occupying the region Ω, which is subject to a volume force and has a known displacement on the boundary Γ. In other words, given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, the nonlinear boundary value problem reads as follows: Find a tensor field $\boldsymbol{\sigma}$ and a vector field $\mathbf{u}$ such that

$$\begin{aligned} \boldsymbol{\sigma} &= \widetilde{\lambda}(\|\mathbf{e}(\mathbf{u})^{\mathtt{d}}\|)\,(\operatorname{div}\mathbf{u})\,\mathbf{I} + \widetilde{\mu}(\|\mathbf{e}(\mathbf{u})^{\mathtt{d}}\|)\,\mathbf{e}(\mathbf{u}) \quad \text{in} \quad \Omega\,, \\ \mathbf{div}\,\boldsymbol{\sigma} &= -\mathbf{f} \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma, \end{aligned} \tag{5.1}$$

where $\widetilde{\lambda}, \widetilde{\mu} : \mathbb{R}^+ \to R$ are the nonlinear Lamé functions, $\mathbf{e}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\right)$ is the strain tensor of small deformations, $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^{2\times 2}$, the superscript $^{\mathtt{d}}$ denotes the corresponding deviatoric tensor, $\mathbf{div}$ is the usual divergence operator div acting along the rows of each tensor, and $\boldsymbol{\nu}$ stands for the unit outward normal to Γ. From now on we suppose that $\widetilde{\lambda}$, $\widetilde{\mu} \in C^1(\mathbb{R}^+)$ and that there exist $\kappa, \mu_0, \mu_1, \mu_2 > 0$ such that for all $\rho \geq 0$,

$$\begin{aligned} \widetilde{\lambda}(\rho) \,&=\, \kappa - \frac{1}{2}\widetilde{\mu}(\rho)\,, \\ \mu_0 \,&\leq\, \widetilde{\mu}(\rho) \,<\, 2\kappa, \\ \mu_1 \,&\leq\, \widetilde{\mu}(\rho) \,+\, \rho\,\widetilde{\mu}'(\rho) \,\leq\, \mu_2. \end{aligned} \tag{5.2}$$

5.1 The continuous twofold saddle point formulation

We first proceed as in [7] and derive the dual-mixed variational formulation of (5.1). In what follows, for each $\mathbf{r} \in [L^2(\Omega)]^{2\times 2}$ we define

$$\widehat{\lambda}(\mathbf{r}) \,:=\, \widetilde{\lambda}(\|\mathbf{r}^{\mathrm{d}}\|)$$

and

$$\widehat{\mu}(\mathbf{r}) \,:=\, \widetilde{\mu}(\|\mathbf{r}^{\mathrm{d}}\|)\,,$$

so that, introducing the new unknown $\mathbf{t} := \mathbf{e}(\mathbf{u})$, problem (5.1) adopts the equivalent form

$$\begin{gathered} \mathbf{t} \,=\, \mathbf{e}(\mathbf{u}) \quad \text{in} \quad \Omega\,, \quad \boldsymbol{\sigma} \,=\, \widehat{\lambda}(\mathbf{t})\,\mathrm{tr}(\mathbf{t})\,\mathbf{I} \,+\, \widehat{\mu}(\mathbf{t})\,\mathbf{t} \quad \text{in} \quad \Omega\,, \\ \mathbf{div}\,\boldsymbol{\sigma} \,=\, -\,\mathbf{f} \quad \text{in} \quad \Omega\,, \quad \mathbf{u} \,=\, \mathbf{g} \quad \text{on} \quad \Gamma\,. \end{gathered}$$

Let us now consider the space

$$H(\mathbf{div};\Omega) \,:=\, \Big\{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}\;\boldsymbol{\tau} \in [L^2(\Omega)]^2\Big\}\,,$$

with the inner product given for each $\boldsymbol{\zeta}$, $\boldsymbol{\tau} \in H(\mathbf{div};\Omega)$ by (cf. (4.4))

$$\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{\mathbf{div},\Omega} \,:=\, \sum_{i=1}^{2} \langle (\zeta_{i1}\,\zeta_{i2}), (\tau_{i1}\,\tau_{i2})\rangle_{\mathrm{div},\Omega}\,,$$

and the subspace $\mathcal{R}$ of $[L^2(\Omega)]^{2\times 2}$ defined as

$$\mathcal{R} \,:=\, \Big\{\boldsymbol{\eta} \in [L^2(\Omega)]^{2\times 2} : \quad \boldsymbol{\eta} \,+\, \boldsymbol{\eta}^{\mathrm{t}} \,=\, \mathbf{0}\Big\}\,,$$

equipped with its scalar product inherited from $[L^2(\Omega)]^{2\times 2}$. These tensor spaces become Hilbert spaces when endowed with the norms induced by such inner products, which will be denoted by $\|\cdot\|_{\mathbf{div},\Omega}$ and $\|\cdot\|_{\mathcal{R}}$, respectively. In addition, maintaining the notation $\langle\cdot,\cdot\rangle$ of Section 4.1 for the duality pairing of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$ - inner product, we use now $\langle\langle\cdot,\cdot\rangle\rangle$ to denote the corresponding duality pairing on $[H^{-1/2}(\Gamma)]^2 \times [H^{1/2}(\Gamma)]^2$. In other words, for each $\mathbf{\Phi} := (\Phi_1, \Phi_2) \in [H^{-1/2}(\Gamma)]^2$ and $\mathbf{\Psi} := (\Psi_1, \Psi_2) \in [H^{1/2}(\Gamma)]^2$ we set

$$\langle\langle \mathbf{\Phi}, \mathbf{\Psi}\rangle\rangle \,:=\, \sum_{j=1}^{2} \langle \Phi_j, \Psi_j\rangle.$$

Then, rewriting the identity $\mathbf{t} = \mathbf{e}(\mathbf{u})$ as

$$\mathbf{t} \,=\, \nabla\mathbf{u} \,-\, \boldsymbol{\gamma}\,,$$

where

$$\gamma := \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{t}}\right)$$

is an auxiliary unknown (named rotation) living in $\mathcal{R}$, and following the usual procedure (see e.g. [7]), we arrive at the problem: Find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u},\gamma) \in [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div};\Omega) \times [L^2(\Omega)]^2 \times \mathcal{R}$ such that

$$\begin{array}{lllll}
\displaystyle\int_\Omega \left(\widehat{\lambda}(\mathbf{t})\,\mathrm{tr}(\mathbf{t})\,\mathrm{tr}(\mathbf{s}) \,+\, \widehat{\mu}(\mathbf{t})\,\mathbf{t}:\mathbf{s}\right) & \displaystyle -\int_\Omega \boldsymbol{\sigma}:\mathbf{s} & & = & 0\,, \\
\displaystyle -\int_\Omega \mathbf{t}:\boldsymbol{\tau} & \displaystyle -\int_\Omega \mathbf{u}\cdot\mathbf{div}\boldsymbol{\tau} & \displaystyle -\int_\Omega \boldsymbol{\gamma}:\boldsymbol{\tau} & = & -\langle\langle\,\boldsymbol{\tau\nu}\,,\,\mathbf{g}\,\rangle\rangle\,, \\
 & \displaystyle -\int_\Omega \mathbf{v}\cdot\mathbf{div}\boldsymbol{\sigma} & \displaystyle -\int_\Omega \boldsymbol{\eta}:\boldsymbol{\sigma} & = & \displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v}\,,
\end{array} \tag{5.3}$$

for all $(\mathbf{s},\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}) \in [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div};\Omega) \times [L^2(\Omega)]^2 \times \mathcal{R}$.

It is clear that (5.3) has the form of the twofold saddle point problem (2.1), with the Hilbert spaces $X_1 := [L^2(\Omega)]^{2\times 2}$, $M_1 := H(\mathbf{div};\Omega)$, and $M := [L^2(\Omega)]^2 \times \mathcal{R}$, provided with the norms $\|\cdot\|_{0,\Omega}$, $\|\cdot\|_{\mathbf{div},\Omega}$ and $\|\cdot\|_M^2 := \|\cdot\|_{0,\Omega}^2 + \|\cdot\|_{\mathcal{R}}^2$, respectively, and the nonlinear operator $\mathbb{A}_1 : X_1 \to X_1'$, the bounded linear operators $\mathbb{B}_1 : X_1 \to M_1'$ and $\mathbb{B} : M_1 \to M'$, and the bounded linear functionals $\mathbb{H} \in X_1'$, $\mathbb{G} \in M_1'$ and $\mathbb{F} \in M'$, given for each $\mathbf{r},\mathbf{s} \in X_1$, $\boldsymbol{\zeta},\boldsymbol{\tau} \in M_1$ and $(\mathbf{v},\boldsymbol{\eta}) \in M$ as

$$\begin{aligned}
[\mathbb{A}_1(\mathbf{r}),\mathbf{s}] &:= \int_\Omega \left(\widehat{\lambda}(\mathbf{r})\,\mathrm{tr}(\mathbf{r})\,\mathrm{tr}(\mathbf{s}) \,+\, \widehat{\mu}(\mathbf{r})\,\mathbf{r}:\mathbf{s}\right), \\
[\mathbb{B}_1(\mathbf{r}),\boldsymbol{\tau}] &:= -\int_\Omega \mathbf{r}:\boldsymbol{\tau}, \\
[\mathbb{B}(\boldsymbol{\zeta}),(\mathbf{v},\boldsymbol{\eta})] &:= -\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\zeta} - \int_\Omega \boldsymbol{\zeta}:\boldsymbol{\eta},
\end{aligned} \tag{5.4}$$

$$[\mathbb{H},\mathbf{s}] := 0, \quad [\mathbb{G},\boldsymbol{\tau}] := -\langle\langle\,\boldsymbol{\tau}\mathbf{v}\,,\,\mathbf{g}\,\rangle\rangle, \quad \text{and} \quad [\mathbb{F},(\mathbf{v},\boldsymbol{\eta})] := \int_\Omega \mathbf{f}\cdot\mathbf{v}. \tag{5.5}$$

In view of the two first assumptions in (5.2), it is easy to see that $\mathbb{A}_1$ is well defined. In addition, the existence of a unique solution for (5.3), which follows from a straightforward application of Theorem 2.1, was previously stated in [7] (similarly as we did for Theorem 4.1). More precisely, we have the following result.

THEOREM 5.1 *There exists a unique* $(\mathbf{t},\boldsymbol{\sigma},(\mathbf{u},\boldsymbol{\gamma})) \in \mathbf{X}$ *solution of the nonlinear twofold saddle point problem* (5.3). *In addition, there exists* $C > 0$, *independent of the solution, such that*

$$\|(\mathbf{t},\boldsymbol{\sigma},(\mathbf{u},\boldsymbol{\gamma}))\|_{\mathbf{X}} \,\le\, C\left\{\,\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma}\,\right\}.$$

Proof. See [7, Theorem 4.5] for the existence of a unique solution, and [7, Lemmas 4.1, 4.3 and 4.4] together with Theorem 2.1 for that of $C > 0$. $\square$

5.2 The associated Galerkin scheme

As in Section 4.2, in what follows we suppose that Γ is a polygonal curve and that $\{\mathcal{T}_h\}_{h>0}$ is a regular family of triangulations of Ω, made up of triangles T of diameter h_T, such that $h := \sup\{h_T : T \in \mathcal{T}_h\}$ and $\overline{\Omega} := \bigcup\{T : T \in \mathcal{T}_h\}$. Also, given $T \in \mathcal{T}_h$, we let b_T be the triangle-bubble function defined as the unique polynomial in $\mathbb{P}_3(T)$ vanishing on ∂T with $\int_T b_T = 1$, and extended as 0 on $\Omega\backslash T$. Then, we introduce the finite element subspaces

$$X_{1,h} := \left\{ \boldsymbol{\tau}_h \in [L^2(\Omega)]^{2\times 2} : \quad \boldsymbol{\tau}_h|_T \in [\mathbb{P}_0(T)]^{2\times 2} \oplus [\mathbb{P}_0(T)\,(\mathbf{curl}\, b_T)^{\mathtt{t}}]^2 \quad \forall\, T \in \mathcal{T}_h \right\}, \tag{5.6}$$

$$M_{1,h} := \left\{ \boldsymbol{\tau}_h \in H(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_T \in [\mathbb{RT}_0(T)]^2 \oplus [\mathbb{P}_0(T)\,(\mathbf{curl}\, b_T)^{\mathtt{t}}]^2 \quad \forall\, T \in \mathcal{T}_h \right\}, \tag{5.7}$$

and

$$M_h := V_h \times \mathcal{R}_h\,, \tag{5.8}$$

where

$$V_h := \left\{ \mathbf{v}_h \in [L^2(\Omega)]^2 : \quad \mathbf{v}_h|_T \in [\mathbb{P}_0(T)]^2 \quad \forall\, T \in \mathcal{T}_h \right\}$$

and

$$\mathcal{R}_h := \left\{ \boldsymbol{\eta}_h \in [C(\Omega)]^{2\times 2} \cap \mathcal{R} : \quad \boldsymbol{\eta}_h|_T \in [\mathbb{P}_1(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h \right\}.$$

Note that $M_{1,h} \times M_h$ corresponds to the classical *PEERS*-space introduced originally in [4] for the linear elasticity problem.

Next, we recall from [7] that the Galerkin scheme associated with the continuous problem (5.3) is well-posed. We remark, however, that the present definition of the space $X_{1,h}$, which is motivated by the analysis provided in [29, Lemma 2.6, Section 2.4], simplifies the original definition given in [7, eq. (5.3), Section 5] in such a way that the well-posedness of the discrete scheme is still valid. More precisely, we can establish the following theorem.

THEOREM 5.2 *Let $X_{1,h}$, $M_{1,h}$ and M_h be the finite element subspaces given by* (5.6), (5.7) *and* (5.8). *Then, there exists a unique solution $(\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h)) \in \mathbf{X}_h := X_{1,h} \times M_{1,h} \times M_h$ of the discrete scheme* (2.4) *with the operators and functionals defined by* (5.4) *and* (5.5). *Furthermore, there exist positive constants c and C, independent of h, such that*

$$\|(\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h))\|_{\mathbf{X}} \le c \left\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \right\},$$

and

$$\|(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h))\|_{\mathbf{X}} \le C \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\eta}_h)) \\ \in \mathbf{X}_h}} \|(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) - (\mathbf{s}_h, \boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\eta}_h))\|_{\mathbf{X}}\,. \tag{5.9}$$

Proof. It follows from [7, Theorem 5.1], [7, Lemma 4.1] and Theorem 2.2. □

Thanks to the Céa estimate (5.9) and the approximation properties of the subspaces $X_{1,h}$, $M_{1,h}$ and M_h (see, e.g. [29, Section 2.4]), the rate of convergence of the Galerkin scheme associated with (5.3) is stated as follows.

THEOREM 5.3 *Assume that $X_{1,h}$, $M_{1,h}$ and M_h are the subspaces given by* (5.6), (5.7) *and* (5.8), *and that* $(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in \mathbf{X}$ *and* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h)) \in \mathbf{X}_h$ *are the unique solutions of* (5.3) *and its associated Galerkin scheme, respectively. Suppose, in addition, that* $\mathbf{t}, \boldsymbol{\sigma}, \boldsymbol{\gamma} \in [H^\delta(\Omega)]^{2\times 2}$ *and* $\mathbf{u}, \mathbf{div}\, \boldsymbol{\sigma} \in [H^\delta(\Omega)]^2$ *for some* $\delta \in (0,1]$. *Then, there exists* $C > 0$, *independent of* h, *such that*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h))\|_{\mathbf{X}} \le C\, h^\delta \Big\{ \|\mathbf{t}\|_{\delta,\Omega} + \|\boldsymbol{\sigma}\|_{\delta,\Omega} + \|\mathbf{div}\, \boldsymbol{\sigma}\|_{\delta,\Omega} + \|\mathbf{u}\|_{\delta,\Omega} + \|\boldsymbol{\gamma}\|_{\delta,\Omega} \Big\} \,.$$

5.3 The a posteriori error analysis

5.3.1 Reliability analysis

We now aim to apply the abstract estimate given by Theorem 3.1 to derive a reliable and efficient residual-based a posteriori error estimator for the Galerkin scheme associated with (5.3). For this purpose, we first verify that the hypotheses specified at the beginning of Section 3 are satisfied. Indeed, we have the following result.

LEMMA 5.1 *The nonlinear operator* $\mathbb{A}_1 : X_1 \to X_1'$ *defined in* (5.4) *is Gâteaux differentiable in* X_1. *Moreover, the family* $\{\mathcal{D}\mathbb{A}_1(\mathbf{x})\}_{\mathbf{x} \in X_1}$ *is uniformly bounded on* $X_1 \times X_1$ *and uniformly elliptic on* $V_1 \times V_1$, *where* $V_1 := \ker(\mathbb{B}_1)$.

Proof. We begin by observing, thanks to simple computations and the C^1-regularity of $\widetilde{\mu}$ and $\widetilde{\lambda}$, that for all $\mathbf{x}$, $\mathbf{r}$, $\mathbf{s} \in X_1$ there holds

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{[\mathbb{A}_1(\mathbf{x} + \epsilon\, \mathbf{r}) - \mathbb{A}_1(\mathbf{x}), \mathbf{s}]}{\epsilon} \;&=\; \int_\Omega \widetilde{\lambda}'(\|\mathbf{x}^{\mathrm{d}}\|) \frac{(\mathbf{x}^{\mathrm{d}} : \mathbf{r}^{\mathrm{d}})}{\|\mathbf{x}^{\mathrm{d}}\|} \operatorname{tr}(\mathbf{x}) \operatorname{tr}(\mathbf{s}) \\ + \int_\Omega \widetilde{\lambda}(\|\mathbf{x}^{\mathrm{d}}\|) \operatorname{tr}(\mathbf{r}) \operatorname{tr}(\mathbf{s}) \;&+\; \int_\Omega \widetilde{\mu}'(\|\mathbf{x}^{\mathrm{d}}\|) \frac{(\mathbf{x}^{\mathrm{d}} : \mathbf{r}^{\mathrm{d}})}{\|\mathbf{x}^{\mathrm{d}}\|}\, \mathbf{x} : \mathbf{s} \;+\; \int_\Omega \widetilde{\mu}(\|\mathbf{x}^{\mathrm{d}}\|)\, \mathbf{r} : \mathbf{s} \,, \end{aligned} \tag{5.10}$$

if $\mathbf{x}^{\mathrm{d}} \neq \mathbf{0}$, and

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{[\mathbb{A}_1(\mathbf{x} + \epsilon\, \mathbf{r}) - \mathbb{A}_1(\mathbf{x}), \mathbf{s}]}{\epsilon} \;&=\; \int_\Omega \widetilde{\lambda}'(0)\, \|\mathbf{r}^{\mathrm{d}}\| \operatorname{tr}(\mathbf{x}) \operatorname{tr}(\mathbf{s}) \\ + \int_\Omega \widetilde{\lambda}(0) \operatorname{tr}(\mathbf{r}) \operatorname{tr}(\mathbf{s}) \;&+\; \int_\Omega \widetilde{\mu}'(0)\, \|\mathbf{r}^{\mathrm{d}}\|\, \mathbf{x} : \mathbf{s} \;+\; \int_\Omega \widetilde{\mu}(0)\, \mathbf{r} : \mathbf{s} \,, \end{aligned} \tag{5.11}$$

if $\mathbf{x}^{\mathrm{d}} = \mathbf{0}$, which shows, in any case, that $\mathbb{A}_1$ is Gâteaux differentiable at $\mathbf{x}$. Moreover, $\mathcal{D}\mathbb{A}_1(\mathbf{x})$ is the bounded linear operator from X_1 into X_1' that can be identified with the bilinear form $\mathcal{D}\mathbb{A}_1(\mathbf{x}) : X_1 \times X_1 \to \mathbb{R}$ defined by

$$\mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r}, \mathbf{s}) \;:=\; \lim_{\epsilon \to 0} \frac{[\mathbb{A}_1(\mathbf{x} + \epsilon\, \mathbf{r}) - \mathbb{A}_1(\mathbf{x}), \mathbf{s}]}{\epsilon} \qquad \forall\, \mathbf{r},\, \mathbf{s} \in X_1 \,.$$

Let us now prove that the family $\{\mathcal{D}\mathbb{A}_1(\mathbf{x})\}_{\mathbf{x} \in X_1}$ is both uniformly bounded and uniformly elliptic on $X_1 \times X_1$. In fact, taking into account the relationship between $\widetilde{\lambda}$ and $\widetilde{\mu}$ (cf. (5.2)), we find from (5.10) that for all $\mathbf{x},\ \mathbf{r},\ \mathbf{s} \in X_1$, with $\mathbf{x}^{\mathrm{d}} \neq \mathbf{0}$, there holds

$$\mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r}, \mathbf{s}) \;=\; \int_\Omega \widetilde{\mu}'(\|\mathbf{x}^{\mathrm{d}}\|)\, \frac{(\mathbf{x}^{\mathrm{d}} : \mathbf{r}^{\mathrm{d}})}{\|\mathbf{x}^{\mathrm{d}}\|}\, \mathbf{x}^{\mathrm{d}} : \mathbf{s} \;+\; \int_\Omega \widetilde{\mu}(\|\mathbf{x}^{\mathrm{d}}\|)\, \mathbf{r}^{\mathrm{d}} : \mathbf{s} \;+\; \int_\Omega \kappa \operatorname{tr}(\mathbf{r}) \operatorname{tr}(\mathbf{s}) \,, \tag{5.12}$$

which, employing the Cauchy-Schwarz inequality and the fact that $\|\mathbf{r}^{\mathtt{d}}\| \le \|\mathbf{r}\|$, yields

$$\left| \mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r},\mathbf{s}) \right| \;\le\; \int_\Omega \Big\{ |\widetilde{\mu}'(\|\mathbf{x}^{\mathtt{d}}\|)|\,\|\mathbf{x}^{\mathtt{d}}\| \,+\, \widetilde{\mu}(\|\mathbf{x}^{\mathtt{d}}\|) \,+\, 2\,\kappa \Big\}\, \|\mathbf{r}\|\,\|\mathbf{s}\|\,.$$

If $\widetilde{\mu}'(\|\mathbf{x}^{\mathtt{d}}\|) \,\ge\, 0$, the above inequality and the third equation in (5.2) imply that

$$\left| \mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r},\mathbf{s}) \right| \;\le\; \big(\mu_2 \,+\, 2\,\kappa\big)\, \|\mathbf{r}\|_{X_1}\, \|\mathbf{s}\|_{X_1}\,,$$

whereas if $\widetilde{\mu}'(\|\mathbf{x}^{\mathtt{d}}\|) \,<\, 0$, we can write

$$\left| \mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r},\mathbf{s}) \right| \;\le\; \int_\Omega \Big\{ \,-\, \widetilde{\mu}'(\|\mathbf{x}^{\mathtt{d}}\|)\,\|\mathbf{x}^{\mathtt{d}}\| \,-\, \widetilde{\mu}(\|\mathbf{x}^{\mathtt{d}}\|) \,+\, 2\,\widetilde{\mu}(\|\mathbf{x}^{\mathtt{d}}\|) \,+\, 2\,\kappa \Big\}\, \|\mathbf{r}\|\,\|\mathbf{s}\|\,,$$

from which, using the second and third equations in (5.2), we deduce that

$$\left| \mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r},\mathbf{s}) \right| \;\le\; 6\,\kappa\,\|\mathbf{r}\|_{X_1}\,\|\mathbf{s}\|_{X_1}\,.$$

On the other hand, for all $\mathbf{x}$, $\mathbf{r} \,\in\, X_1$, with $\mathbf{x}^{\mathtt{d}} \,\ne\, \mathbf{0}$, we have from (5.12) that

$$\mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r},\mathbf{r}) \;=\; \int_\Omega \widetilde{\mu}'(\|\mathbf{x}^{\mathtt{d}}\|)\,\frac{(\mathbf{x}^{\mathtt{d}} : \mathbf{r}^{\mathtt{d}})^2}{\|\mathbf{x}^{\mathtt{d}}\|} \;+\; \int_\Omega \widetilde{\mu}(\|\mathbf{x}^{\mathtt{d}}\|)\,\|\mathbf{r}^{\mathtt{d}}\|^2 \;+\; \int_\Omega \kappa\,\big(\mathrm{tr}(\mathbf{r})\big)^2\,. \tag{5.13}$$

If $\widetilde{\mu}'(\|\mathbf{x}^{\mathtt{d}}\|) \,\ge\, 0$, we use the second equation in (5.2) and find that

$$\mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r},\mathbf{r}) \;\ge\; \int_\Omega \Big\{ \widetilde{\mu}(\|\mathbf{x}^{\mathtt{d}}\|)\,\|\mathbf{r}^{\mathtt{d}}\|^2 \;+\; \kappa\,\big(\mathrm{tr}(\mathbf{r})\big)^2 \Big\} \;\ge\; \min\{\mu_0, 2\kappa\}\,\|\mathbf{r}\|_{X_1}^2\,,$$

whereas if $\widetilde{\mu}'(\|\mathbf{x}^{\mathtt{d}}\|) \,<\, 0$, we deduce from (5.13) and the third equation in (5.2) that

$$\begin{aligned}
\mathcal{D}\mathbb{A}_1(\mathbf{x})(\mathbf{r},\mathbf{r}) \;&\ge\; \int_\Omega \Big\{ \,-\, |\widetilde{\mu}'(\|\mathbf{x}^{\mathtt{d}}\|)|\,\|\mathbf{x}^{\mathtt{d}}\|\,\|\mathbf{r}^{\mathtt{d}}\|^2 \;+\; \widetilde{\mu}(\|\mathbf{x}^{\mathtt{d}}\|)\,\|\mathbf{r}^{\mathtt{d}}\|^2 \,+\, \kappa\,\big(\mathrm{tr}(\mathbf{r})\big)^2 \Big\} \\
&=\; \int_\Omega \Big\{ \widetilde{\mu}'(\|\mathbf{x}^{\mathtt{d}}\|)\,\|\mathbf{x}^{\mathtt{d}}\|\,\|\mathbf{r}^{\mathtt{d}}\|^2 \;+\; \widetilde{\mu}(\|\mathbf{x}^{\mathtt{d}}\|)\,\|\mathbf{r}^{\mathtt{d}}\|^2 \,+\, \kappa\,\big(\mathrm{tr}(\mathbf{r})\big)^2 \Big\} \\
&\ge\; \min\{\mu_1, 2\kappa\}\,\|\mathbf{r}\|_{X_1}^2\,.
\end{aligned}$$

The case $\mathbf{x}^{\mathtt{d}} \,=\, \mathbf{0}$ proceeds similarly for both properties of $\mathcal{D}\mathbb{A}_1$. We omit further details. $\square$

We are now in a position to make use again of Theorem 3.1. More specifically, from the estimate (3.3) we deduce the existence of $C > 0$ such that

$$\|(\mathbf{t},\boldsymbol{\sigma},(\mathbf{u},\boldsymbol{\gamma})) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,(\mathbf{u}_h,\boldsymbol{\gamma}_h))\|_{\mathbf{X}} \,\le\, C\,\Big\{\|R_1\|_{X_1'} + \|R_2\|_{M_1'} + \|R_3\|_{M'}\Big\}, \tag{5.14}$$

where for all $(\mathbf{s},\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta})) \,\in\, \mathbf{X}$ we have

$$R_1(\mathbf{s}) \;:=\; -\,\int_\Omega \widehat{\lambda}(\mathbf{t}_h)\;\mathrm{tr}(\mathbf{t}_h)\;\mathbf{I} : \mathbf{s} \;-\; \int_\Omega \widehat{\mu}(\mathbf{t}_h)\;\mathbf{t}_h : \mathbf{s} \,+\, \int_\Omega \boldsymbol{\sigma}_h : \mathbf{s}\,,$$

$$R_2(\boldsymbol{\tau}) \;:=\; -\,\langle\langle \boldsymbol{\tau}\boldsymbol{\nu},\mathbf{g}\rangle\rangle \;+\; \int_\Omega \mathbf{u}_h\cdot\mathbf{div}\,\boldsymbol{\tau} \;+\; \int_\Omega (\mathbf{t}_h + \boldsymbol{\gamma}_h) : \boldsymbol{\tau}\,,$$

and

$$R_3(\mathbf{v},\boldsymbol{\eta}) \,:=\, \int_\Omega \mathbf{f}\cdot\mathbf{v} \,+\, \int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}_h \,+\, \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\eta}\,.$$

It is straightforward to see that

$$\|R_1\|_{X_1'} \,=\, \|\boldsymbol{\sigma}_h - \widehat{\lambda}(\mathbf{t}_h)\,\operatorname{tr}(\mathbf{t}_h)\,\mathbf{I} - \widehat{\mu}(\mathbf{t}_h)\,\mathbf{t}_h\|_{0,\Omega}\,. \tag{5.15}$$

It is also an easy matter to arrive at the estimate

$$\|R_3\|_{M'} \,\le\, \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \frac{1}{2}\,\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,\Omega}\,, \tag{5.16}$$

which follows from the Cauchy-Schwarz inequality and the fact, because of the skew-symmetry of $\boldsymbol{\eta} \in \mathcal{R}$, that $\displaystyle\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\eta} \,=\, \frac{1}{2}\int_\Omega \big(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\big) : \boldsymbol{\eta}$.

In what follows, given vector and tensor fields $\boldsymbol{\varphi} := (\varphi_1,\varphi_2)$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, and having in mind (4.13) and (4.14), we let $\underline{\mathbf{curl}}\,\boldsymbol{\varphi}$ and $\underline{\operatorname{curl}}(\boldsymbol{\tau})$ be the tensor and vector fields given by

$$\underline{\mathbf{curl}}\,\boldsymbol{\varphi} \,:=\, \begin{pmatrix} \mathbf{curl}(\varphi_1)^{\mathtt{t}} \\ \mathbf{curl}(\varphi_2)^{\mathtt{t}} \end{pmatrix} \,=\, \begin{pmatrix} \frac{\partial\varphi_1}{\partial x_2} & -\frac{\partial\varphi_1}{\partial x_1} \\ \frac{\partial\varphi_2}{\partial x_2} & -\frac{\partial\varphi_2}{\partial x_1} \end{pmatrix}\,, \tag{5.17}$$

and

$$\underline{\operatorname{curl}}(\boldsymbol{\tau}) \,:=\, \begin{pmatrix} \operatorname{curl}(\tau_{11},\tau_{12}) \\ \operatorname{curl}(\tau_{21},\tau_{22}) \end{pmatrix} \,=\, \begin{pmatrix} \frac{\partial\tau_{12}}{\partial x_1} - \frac{\partial\tau_{11}}{\partial x_2} \\ \frac{\partial\tau_{22}}{\partial x_1} - \frac{\partial\tau_{21}}{\partial x_2} \end{pmatrix}\,. \tag{5.18}$$

We now proceed as in Section 4.3.2 (see also [29, Section 4.2]) to derive an upper bound for $\|R_2\|_{M_1'}$. In fact, in order to define a suitable operator $\Pi_h : M_1 \to M_{1,h}$ to be employed in the corresponding abstract estimate (3.9), we first observe that in this case the Helmholtz decomposition of $M_1 := H(\mathbf{div};\Omega)$ ensures that for all $\boldsymbol{\tau} \in M_1$ there exist $\boldsymbol{\varphi} \in [H^1(\Omega)]^2$, with $\displaystyle\int_\Omega \boldsymbol{\varphi} = \mathbf{0}$, and $\mathbf{z} \in [H^2(\Omega)]^2$, satisfying

$$\boldsymbol{\tau} = \underline{\mathbf{curl}}\,\boldsymbol{\varphi} + \nabla\mathbf{z},$$

and

$$\|\boldsymbol{\varphi}\|_{1,\Omega} + \|\mathbf{z}\|_{2,\Omega} \le C\,\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,. \tag{5.19}$$

Then, denoting by $\mathbf{I}_h$ and $\mathbf{E}_h$ the tensor versions of the Clément interpolant I_h and the Raviart-Thomas interpolation operator E_h^0, respectively, we define

$$\Pi_h(\boldsymbol{\tau}) := \underline{\mathbf{curl}}\,\boldsymbol{\varphi}_h + \mathbf{E}_h(\nabla\mathbf{z})\,,$$

where $\boldsymbol{\varphi}_h := \mathbf{I}_h(\boldsymbol{\varphi})$. It is easy to see that $\mathbf{I}_h$ and $\mathbf{E}_h$ satisfy analog properties to those given by (4.15), (4.16) and (4.18)-(4.23). In addition, noting that $\mathbf{u}_h \in V_h$, there also holds

$$\int_\Omega \mathbf{u}_h\cdot\mathbf{div}\,(\nabla\mathbf{z} - \mathbf{E}_h(\nabla\mathbf{z})) \,=\, \int_\Omega \mathbf{u}_h\cdot\{\mathbf{div}\,\boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_h(\mathbf{div}\,\boldsymbol{\tau})\} = 0 \qquad \forall\,\boldsymbol{\tau} \in M_1\,,$$

where $\boldsymbol{\mathcal{P}}_h : [L^2(\Omega)]^2 \to V_h$ is the orthogonal projector. This identity, together with the fact that

$$R_2(\boldsymbol{\tau}) = R_2(\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau})),$$

allows us to express R_2 in the equivalent way

$$R_2(\boldsymbol{\tau}) \;=\; \widehat{R}_2(\boldsymbol{\varphi}) \,+\, \widetilde{R}_2(\mathbf{z}) \qquad \forall\, \boldsymbol{\tau} \,\in\, M_1\,, \tag{5.20}$$

with

$$\begin{aligned} \widehat{R}_2(\boldsymbol{\varphi}) \;&:=\; R_2(\underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)) \\ &=\; -\langle\langle \underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\,\boldsymbol{\nu}\,,\, g\rangle\rangle \,+\, \int_\Omega (\mathbf{t}_h+\boldsymbol{\gamma}_h) : \underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_{\mathbf{h}}) \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} \widetilde{R}_2(\mathbf{z}) \;&:=\; R_2\big(\nabla\mathbf{z} - \mathbf{E}_h(\nabla\mathbf{z})\big) \\ &=\; -\,\langle\langle (\nabla\mathbf{z} - \mathbf{E}_h(\nabla\mathbf{z}))\,\boldsymbol{\nu}\,,\, g\rangle\rangle \,+\, \int_\Omega (\mathbf{t}_h+\boldsymbol{\gamma}_h) : (\nabla\mathbf{z} - \mathbf{E}_h(\nabla\mathbf{z}))\,. \end{aligned} \tag{5.22}$$

Before stating respective upper bounds for $|\widehat{R}_2(\boldsymbol{\varphi})|$ and $|\widetilde{R}_2(\mathbf{z})|$, we extend to the tensorial framework the concept of jump across an edge. Thus, given $e \in \mathcal{E}_h$, we fix a normal vector $\boldsymbol{\nu}_e = (\nu_1,\nu_2)^{\mathbf{t}}$ and take $\mathbf{s}_e := (-\nu_2,\nu_1)^{\mathbf{t}}$, that is, the corresponding tangential vector along e. Then, for $e \in \mathcal{E}_h(\Omega)$ and $\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$ such that $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h$, $[\boldsymbol{\tau}\,\mathbf{s}_e]$ stands for the corresponding jump across e, i.e. $\quad [\boldsymbol{\tau}\,\mathbf{s}_e] \;:=\; (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e\,\mathbf{s}_e$, T and T' being the unique triangles of $\mathcal{T}_h$ having e as a common edge. When there is no cause for confusion, we will write $\mathbf{s}$ and $\boldsymbol{\tau}$ instead of $\mathbf{s}_e$ and $\boldsymbol{\tau}_e$, respectively.

LEMMA 5.2 *Assume that* $\mathbf{g} \in [H^1(\Gamma)]^2$. *Then there exists* $C > 0$, *independent of* h, *such that*

$$|\widehat{R}_2(\boldsymbol{\varphi})| \;\le\; C \left\{ \sum_{T\in\mathcal{T}_h} \widehat{\theta}^2_{2,T} \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,,$$

where

$$\begin{aligned} \widehat{\theta}^2_{2,T} \;&:= h_T^2\, \|\underline{\mathrm{curl}}(\mathbf{t}_h+\boldsymbol{\gamma}_h)\|^2_{0,T} \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega)} h_e\, \|[(\mathbf{t}_h+\boldsymbol{\gamma}_h)\,\mathbf{s}]\|^2_{0,\,e} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e \left\| \frac{d\mathbf{g}}{d\mathbf{s}} - (\mathbf{t}_h+\boldsymbol{\gamma}_h)\,\mathbf{s} \right\|^2_{0,e}\,. \end{aligned}$$

Proof. We proceed analogously to the proof of Lemma 4.1 (see also [28, Lemma 4.3]). For the first summand in (5.21) we integrate by parts on Γ, noticing that

$$\underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\,\boldsymbol{\nu} \;=\; \frac{d}{d\,\mathbf{s}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_{\mathbf{h}})\,,$$

which yields

$$\langle\langle \underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\,\boldsymbol{\nu},\, \mathbf{g}\rangle\rangle \;=\; -\langle\langle \boldsymbol{\varphi}-\boldsymbol{\varphi}_h,\, \frac{d\mathbf{g}}{d\mathbf{s}}\rangle\rangle \;=\; -\sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \frac{d\mathbf{g}}{d\mathbf{s}}\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\,.$$

For the second one we also integrate by parts, now on each $T \in \mathcal{T}_h$, arriving at

$$\int_\Omega (\mathbf{t}_h+\boldsymbol{\gamma}_h) : \underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) = \sum_{T\in\mathcal{T}_h} \left\{ \int_T \underline{\mathrm{curl}}(\mathbf{t}_h+\boldsymbol{\gamma}_h)\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) - \int_{\partial T} (\mathbf{t}_h+\boldsymbol{\gamma}_h)\,\mathbf{s}\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) \right\}$$

$$= \sum_{T\in\mathcal{T}_h} \int_T \underline{\mathrm{curl}}(\mathbf{t}_h + \boldsymbol{\gamma}_h) \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) - \sum_{e\in\mathcal{E}_h(\Omega)} [(\mathbf{t}_h + \boldsymbol{\gamma}_h)\,\mathbf{s}] \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) - \sum_{e\in\mathcal{E}_h(\Gamma)} (\mathbf{t}_h + \boldsymbol{\gamma})\,\mathbf{s} \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\,.$$

Finally, it follows from the above expressions and the definition of $\widehat{R}_2$ (cf. (5.21)) that

$$\begin{aligned}\widehat{R}_2(\boldsymbol{\varphi}) \;&= \sum_{T\in\mathcal{T}_h} \int_T \underline{\mathrm{curl}}(\mathbf{t}_h \,+\, \boldsymbol{\gamma}_h) \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \\ &- \sum_{e\in\mathcal{E}_h(\Omega)} \int_e [(\mathbf{t}_h + \boldsymbol{\gamma}_h)\,\mathbf{s}] \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \,+ \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \left\{ \frac{d\mathbf{g}}{d\mathbf{s}} - (\mathbf{t}_h + \boldsymbol{\gamma}_h)\,\mathbf{s} \right\} \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\,,\end{aligned}$$

and all we have to do next is to apply the Cauchy-Schwarz inequality, the approximation properties of the operator $\mathbf{I}_h$ (cf. (4.22) and (4.23)), the fact that the number of triangles in $\Delta(T)$ and $\Delta(e)$ are bounded, and the estimate (5.19). We omit further details. $\square$

LEMMA 5.3 *There exists $C > 0$, independent of h, such that*

$$|\widetilde{R}_2(\mathbf{z})| \,\le\, C \left\{ \sum_{T\in\mathcal{T}_h} \widetilde{\theta}_{2,T}^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,,$$

where

$$\widetilde{\theta}_{2,T}^2 \,:=\, h_T^2\,\|\,\mathbf{t}_h + \boldsymbol{\gamma}_h\,\|_{0,T}^2 \;+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\,\|\mathbf{g} - \mathbf{u}_h\,\|_{0,e}^2\,.$$

Proof. Since $\mathbf{u}_h|_e \in [\mathbb{P}_0(e)]^2 \quad \forall\, e \in \mathcal{E}_h$, the tensor version of the identity (4.15) gives

$$\int_e \Big(\nabla\mathbf{z} - \mathbf{E}_h(\nabla\mathbf{z})\Big)\,\boldsymbol{\nu}\cdot\mathbf{u}_h \,=\, 0 \quad \forall\, e \,\in\, \mathcal{E}_h(\Gamma)\,,$$

and hence $\widetilde{R}_2$ (cf. (5.22)) becomes

$$\widetilde{R}_2(\mathbf{z}) \,=\, \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \Big(\nabla\mathbf{z} - \mathbf{E}_h(\nabla\mathbf{z})\Big)\,\boldsymbol{\nu}\cdot(\mathbf{u}_h \,-\, \mathbf{g}) \,+\, \sum_{T\in\mathcal{T}_h} \int_T (\mathbf{t}_h + \boldsymbol{\gamma}_h) : (\nabla\mathbf{z} - \mathbf{E}_h(\nabla\mathbf{z}))\,.$$

Hence, for the rest of the proof it suffices to apply the Cauchy-Schwarz inequality, the approximation properties of the operator $\mathbf{E}_h$ (cf. (4.19) and (4.21)), and the estimate (5.19). $\square$

Finally, Lemmas 5.2 and 5.3, together with the identities (5.14), (5.15), (5.16), (5.20), and (3.9), provide the following reliability estimate.

THEOREM 5.4 *Let $(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in \mathbf{X}$ and $(\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma})) \in \mathbf{X}_h$ be the unique solutions of the saddle point problem (4.3) and its associated Galerkin scheme, respectively, and assume that $\mathbf{g} \in [H^1(\Gamma)]^2$. Then, there exists a positive constant $C_{\mathtt{rel}}$, independent of h, such that*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h))\|_X \,\le\, C_{\mathtt{rel}}\,\boldsymbol{\theta}\,,$$

where $\boldsymbol{\theta}^2 := \sum\limits_{T \in \mathcal{T}_h} \theta_T^2$ *and*

$$
\begin{aligned}
\theta_T^2 \;\; := \;\; & \|\boldsymbol{\sigma}_h - \widehat{\lambda}(\mathbf{t}_h)\,\mathrm{tr}(\mathbf{t}_h)\,\mathbf{I} - \widehat{\mu}(\mathbf{t}_h)\,\mathbf{t}_h\|_{0,T}^2 \; + \; \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T}^2 \; + \; \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,T}^2 \\
+ \;\; & h_T^2\,\|\underline{\mathrm{curl}}(\mathbf{t}_h + \boldsymbol{\gamma}_h)\|_{0,T}^2 \; + \; h_T^2\,\|\,\mathbf{t}_h + \boldsymbol{\gamma}_h\,\|_{0,T}^2 \; + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} h_e\,\|[(\mathbf{t}_h + \boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2 \\
+ \;\; & \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \left\{ \left\| \frac{d\mathbf{g}}{d\mathbf{s}} - (\mathbf{t}_h + \boldsymbol{\gamma}_h)\,\mathbf{s} \right\|_{0,e}^2 \; + \; \|\mathbf{g} \, - \, \mathbf{u}_h\|_{0,e}^2 \right\} .
\end{aligned}
\tag{5.23}
$$

5.3.2 Efficiency analysis

We now study the efficiency of our a posteriori error estimator $\boldsymbol{\theta}$, along the lines of Section 4.3.3. More precisely, we aim to prove the existence of a positive constant $C_{\mathtt{eff}}$ such that

$$
C_{\mathtt{eff}}\,\boldsymbol{\theta} \; + \; \text{h.o.t.} \; \leq \; \|(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h - \boldsymbol{\gamma}_h))\|_{\mathbf{X}} \, , \tag{5.24}
$$

where h.o.t. stands for one or several terms of higher order. To this end, in what follows we establish suitable upper bounds for each one of the eight terms defining θ_T^2. We begin with the first three of them, whose corresponding estimates are pretty straightforward.

On one hand, since $\boldsymbol{\sigma} \; = \; \widehat{\lambda}(\mathbf{t})\,\mathrm{tr}(\mathbf{t})\,\mathbf{I} \, + \, \widehat{\mu}(\mathbf{t})\,\mathbf{t}$, the Lipschitz-continuity of $\mathbb{A}_1$, restricted to each $T \, \in \, \mathcal{T}_h$, implies that

$$
\begin{aligned}
& \|\boldsymbol{\sigma}_h - \, \widehat{\lambda}(\mathbf{t}_h)\,\mathrm{tr}(\mathbf{t}_h)\,\mathbf{I} \, - \, \widehat{\mu}(\mathbf{t}_h)\,\mathbf{t}_h\|_{0,T} \\
& \qquad \leq \quad \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} \, + \, \|\widehat{\lambda}(\mathbf{t})\,\mathrm{tr}(\mathbf{t})\,\mathbf{I} \, + \, \widehat{\mu}(\mathbf{t})\,\mathbf{t} \, - \, \widehat{\lambda}(\mathbf{t}_h)\,\mathrm{tr}(\mathbf{t}_h)\,\mathbf{I} \, - \, \widehat{\mu}(\mathbf{t}_h)\,\mathbf{t}_h\|_{0,T} \\
& \qquad \leq \quad C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},T} \; + \; \|\mathbf{t} - \mathbf{t}_h\|_{0,T} \right\} \qquad \forall\, T \, \in \, \mathcal{T}_h \, .
\end{aligned}
$$

Next, since $\mathbf{div}\,\boldsymbol{\sigma} \; = \; -\,\mathbf{f}$ in Ω, we have that

$$
\|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T} \; = \; \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,T} \; \leq \; \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},T} \qquad \forall\, T \in \mathcal{T}_h \, .
$$

On the other hand, taking into account the symmetry of $\boldsymbol{\sigma}$, we easily find that

$$
\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,T} \; \leq \; \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} \, + \, \|\boldsymbol{\sigma}^{\mathtt{t}} - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,T} \; \leq \; 2\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} \quad \forall\, T \, \in \, \mathcal{T}_h \, .
$$

The upper bounds for the remaining five terms, being the analogue of the estimates provided by Lemmas 4.3, 4.4, 4.5, and 4.6, respectively, are established next by applying also some results from [28].

Lemma 5.4 *There exist C_1, $C_2 \, > \, 0$, independent of h, such that*

$$
\begin{aligned}
& h_T^2\,\|\underline{\mathrm{curl}}(\mathbf{t}_h + \boldsymbol{\gamma}_h))\|_{0,T}^2 \; \leq \; C_1 \left\{ \|\mathbf{t} - \mathbf{t}_h\|_{0,T}^2 \, + \, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\} \qquad \forall\, T \, \in \, \mathcal{T}_h \, , \\
& h_e\,\|[(\mathbf{t}_h + \boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2 \; \leq \; C_2 \left\{ \|\mathbf{t} - \mathbf{t}_h\|_{0,w_e}^2 \, + \, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,w_e}^2 \right\} \qquad \forall\, e \, \in \, \mathcal{E}_h(\Omega) \, ,
\end{aligned}
$$

where $w_e := \cup \big\{ T \in \mathcal{T}_h : \quad e \in \mathcal{E}(T) \big\}$.

Proof. It suffices to apply [28, Lemmas 4.9 and 4.10] to $\boldsymbol{\rho}_h = \mathbf{t}_h + \boldsymbol{\gamma}_h$ and $\boldsymbol{\rho} = \mathbf{t} + \boldsymbol{\gamma}$, making use of the fact that $\underline{\mathrm{curl}}(\boldsymbol{\rho}) = \underline{\mathrm{curl}}(\nabla \mathbf{u}) = \mathbf{0}$ in Ω. □

LEMMA 5.5 *There exists $C_3 > 0$, independent of h, such that*

$$h_T^2 \, \|\mathbf{t}_h + \boldsymbol{\gamma}_h\|_{0,T}^2 \,\leq\, C_3 \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \,+\, h_T^2 \, \|\mathbf{t} - \mathbf{t}_h\|_{0,T}^2 \,+\, h_T^2 \, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\} \qquad \forall\, T \,\in\, \mathcal{T}_h \,.$$

Proof. It follows from a slight modification of [28, Lemma 4.13], by replacing the tensor utilized there by $\mathbf{t}_h + \boldsymbol{\gamma}_h$, using in this case that $\nabla \mathbf{u}_h$ vanishes, and recalling that $\nabla \mathbf{u} = \mathbf{t} + \boldsymbol{\gamma}$ in Ω. We omit further details. □

LEMMA 5.6 *Assume that $\mathbf{g}$ is piecewise polynomial. Then there exists $C_4 > 0$, independent of h, such that*

$$h_e \left\| \frac{d\mathbf{g}}{d\mathbf{s}} \,-\, (\mathbf{t}_h + \boldsymbol{\gamma}_h)\, s \right\|_{0,e}^2 \,\leq\, C_4 \left\{ \|\mathbf{t} - \mathbf{t}_h\|_{0,T}^2 \,+\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\} \qquad \forall\, e \,\in\, \mathcal{E}_h(\Gamma) \,,$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. It suffices to modify the proof of [28, Lemma 4.15], by using $\dfrac{d\mathbf{g}}{d\mathbf{s}} - (\mathbf{t}_h + \boldsymbol{\gamma}_h)\,\mathbf{s}$ instead of $\dfrac{d\mathbf{g}}{d\mathbf{s}} - \dfrac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{t}}\,\mathbf{s}$, and noting in the present case that $\dfrac{d\mathbf{g}}{d\mathbf{s}} = (\nabla \mathbf{u})\,\mathbf{s} = (\mathbf{t} + \boldsymbol{\gamma})\,\mathbf{s}$ on Γ. □

LEMMA 5.7 *There exists $C_5 > 0$, independent of h, such that*

$$h_e \, \|\mathbf{g} - \mathbf{u}_h\|_{0,e}^2 \,\leq\, C_5 \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \,+\, h_T^2 \, \|\mathbf{t} - \mathbf{t}_h\|_{0,T}^2 \,+\, h_T^2 \, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\} \qquad \forall\, e \,\in\, \mathcal{E}_h(\Gamma) \,,$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. It follows as in the proof of [28, Lemma 4.14] by taking now $\chi_T := \mathbf{t}_h + \boldsymbol{\gamma}_h$, and then using that $\nabla \mathbf{u} = \mathbf{t} + \boldsymbol{\gamma}$ in Ω and $\mathbf{u} = \mathbf{g}$ on Γ. At the end, the efficiency estimate for $h_T^2 \, \|\mathbf{t}_h + \boldsymbol{\gamma}_h\|_{0,T}^2$ given by Lemma 5.5 is also utilized. □

At this point we observe that the same remark provided at the end of Section 4.3.3, which concerns an eventual non-polynomial $\mathbf{g}$ and the consequent appearing of higher order terms (h.o.t.) in the upper bound given by Lemma 5.6, is also valid here. In this way, the efficiency of $\boldsymbol{\theta}$ (as defined by (5.24)) follows directly from the three simple estimates derived at the beginning of this section, together with Lemmas 5.4 throughout 5.7, after summing up over $T \in \mathcal{T}_h$ and applying that the number of triangles on each domain w_e (cf. Lemma 5.4) is bounded by 2.

6 Numerical results

In this section we present numerical examples illustrating the performance of the Galerkin schemes associated with (4.3) and (5.3), confirming the reliability and efficiency of the respective a posteriori error estimators $\boldsymbol{\theta}$ derived in Sections 4.3 and 5.3, and showing the behaviour of the associated adaptive algorithms. We consider the finite element subspaces $X_{1,h}$, $M_{1,h}$, and M_h given by (4.7), (4.8), and (4.9) with $k = 0$ for the problem from heat conduction, and the specific finite element subspaces $X_{1,h}$, $M_{1,h}$, and M_h given by (5.6), (5.7), and (5.8) for the problem from nonlinear elasticity. All the nonlinear algebraic systems arising from both Galerkin schemes are solved by the Newton method with a tolerance of 1E-05 and taking as initial iteration the solution of the associated linear problems with κ, $\widetilde{\lambda}$ and $\widetilde{\mu}$ constant.

In what follows, N stands for the total number of degrees of freedom (unknowns) of each Galerkin scheme. In turn, the individual and total errors for the problem from heat conduction are given by

$$\mathtt{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}\,, \quad \mathtt{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{div},\Omega}\,, \quad \mathtt{e}(u) := \|u - u_h\|_{0,\Omega}\,,$$

and

$$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},u) := \Big\{(\mathtt{e}(\mathbf{t}))^2 + (\mathtt{e}(\boldsymbol{\sigma}))^2 + (\mathtt{e}(u))^2\Big\}^{1/2}\,,$$

whereas the effectivity index with respect to $\boldsymbol{\theta}$ is defined by

$$\mathtt{eff}(\boldsymbol{\theta}) := \mathtt{e}(\mathbf{t},\boldsymbol{\sigma},u)/\boldsymbol{\theta}\,.$$

Then, we introduce the experimental rates of convergence

$$\mathtt{r}(\mathbf{t}) := \frac{\log(\mathtt{e}(\mathbf{t})/\mathtt{e}'(\mathbf{t}))}{\log(h/h')}\,, \quad \mathtt{r}(\boldsymbol{\sigma}) := \frac{\log(\mathtt{e}(\boldsymbol{\sigma})/\mathtt{e}'(\boldsymbol{\sigma}))}{\log(h/h')}\,, \quad \mathtt{r}(u) := \frac{\log(\mathtt{e}(u)/\mathtt{e}'(u))}{\log(h/h')}\,,$$

and

$$\mathtt{r}(\mathbf{t},\boldsymbol{\sigma},u) := \frac{\log(\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},u)/\mathtt{e}'(\mathbf{t},\boldsymbol{\sigma},u))}{\log(h/h')}\,,$$

where $\mathtt{e}$ and $\mathtt{e}'$ denote the corresponding errors at two consecutive triangulations with mesh sizes h and h', respectively. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation. Similar notations to the above, whose meanings become clear from the tables shown below, are used for the nonlinear elasticity problem.

The examples to be considered in this section are described next. Examples 1 and 2 are employed to illustrate the performance of the discrete schemes and to confirm the reliability and efficiency of the a posteriori error estimator $\boldsymbol{\theta}$ when a sequence of quasi-uniform meshes is considered. Then, Examples 3 and 4 are utilized to show the behavior of the associated adaptive algorithms, which apply the following procedure from [40]:

1) Start with a coarse mesh $\mathcal{T}_h$.

2) Solve the discrete problem for the actual mesh $\mathcal{T}_h$.

3) Compute θ_T (cf. (4.37) and (5.23)) for each triangle $T \in \mathcal{T}_h$.

4) Evaluate stopping criterion and decide to finish or go to next step.

5) Use *blue-green* procedure to refine each $T' \in \mathcal{T}_h$ whose indicator $\theta_{T'}$ satisfies

$$\theta_{T'} \ge \frac{1}{2}\max\Big\{\theta_T : \quad T \in \mathcal{T}_h\Big\}$$

6) Define resulting mesh as actual mesh $\mathcal{T}_h$ and go to step 2.

Examples 1 and 3 deal with the problem from heat conduction and take $\kappa : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ given by

$$\kappa(\mathbf{x}, \rho) \;:=\; 2 + \frac{1}{1+\rho} \qquad \forall\, (\mathbf{x}, \rho) \,\in\, \Omega \times \mathbb{R}^+ \,,$$

which is easily shown to satisfy (4.2). In these examples we consider $\Omega =]0,1[^2$ and the L-shaped domain $\Omega =]-1,1[^2 \setminus [0,1]^2$, and choose the data f and g so that the exact solutions are given, respectively, for each $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \,\in\, \Omega$ by

$$u(\mathbf{x}) \;:=\; \sin x_1 \cos x_2 \exp(x_1 x_2)\,,$$

and

$$u(\mathbf{x}) \;:=\; \left(x_1^2 + x_2^2\right)^{5/6} \sin\left(\frac{2\theta - \pi}{3}\right)\,, \quad \text{with} \quad \theta = \operatorname{Arctan}\left(\frac{x_2}{x_1}\right)\,.$$

Note that the partial derivatives of the solution of Example 3 are singular at the origin, which is the middle corner of the L.

In turn, Examples 2 and 4 refer to the problem from nonlinear elasticity and consider the Lamé functions $\widetilde{\lambda}$, $\widetilde{\mu} : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\widetilde{\lambda}(\rho) \;:=\; \kappa - \frac{1}{2}\,\widetilde{\mu}(\rho) \quad \text{and} \quad \widetilde{\mu}(\rho) \;:=\; \kappa_0 \,+\, \kappa_1\,(1+\rho^2)^{(\beta-2)/2} \qquad \forall\, \rho \,\in\, \mathbb{R}^+\,,$$

with the parameters $\kappa = 1$, $\kappa_0 = \kappa_1 = 0.5$, and $\beta = 1.5$, which are easily shown to verify the assumptions (5.2). In these examples we set $\Omega =]0,1[^2$ and the T-shaped domain $\Omega =]-1,1[^2 \setminus \big([-1,-0.25] \times [-1,0.5] \cup [0.25,1] \times [-1,0.5]\big)$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solutions are given, respectively, for each $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \,\in\, \Omega$ by

$$\mathbf{u}(\mathbf{x}) \;:=\; \begin{pmatrix} \sin x_1 \cos x_2 \exp(x_1 x_2) \\[2ex] \cos x_1 \sin x_2 \exp(-x_1 x_2) \end{pmatrix}\,,$$

and

$$\mathbf{u}(\mathbf{x}) \;:=\; \left(\big\|\mathbf{x} - (-0.25, 0.5)\big\|^{5/3} \sin\left(\frac{2\theta_1 + \pi}{3}\right), \big\|\mathbf{x} - (0.25, 0.5)\big\|^{5/3} \sin\left(\frac{2\theta_2}{3}\right) \right)^{\mathtt{t}}$$

with

$$\theta_1 = \operatorname{Arctan}\left(\frac{x_2 - 0.50}{x_1 + 0.25}\right) \quad \text{and} \quad \theta_2 = \operatorname{Arctan}\left(\frac{x_2 - 0.50}{x_1 - 0.25}\right)\,.$$

Note now that the partial derivatives of the solution of Example 4 are singular at the points $(-0.25, 0.5)$ and $(0.25, 0.5)$, which are the middle corners of the T.

In Tables 6.1 and 6.2 we summarize the convergence history of the mixed finite element schemes associated with (4.3) and (5.3) as applied to Examples 1 and 2, respectively, for sequences of quasi-uniform triangulations of the domains. The number of Newton iterations required, for the tolerance given, ranges between 3 and 5 for Example 1, and between 1 and 3 for Example 2. We observe in these tables, looking at the corresponding experimental rates

N	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(u)$	$\mathtt{r}(u)$	$\mathtt{eff}(\boldsymbol{\theta})$
2336	1/16	3.508E−02	–	1.234E−01	–	1.808E−02	–	0.5403
3640	1/20	2.814E−02	0.989	9.884E−02	0.994	1.446E−02	1.000	0.5386
5232	1/24	2.349E−02	0.992	8.244E−02	0.996	1.205E−02	1.000	0.5376
7112	1/28	2.015E−02	0.994	7.070E−02	0.997	1.033E−02	1.000	0.5369
9280	1/32	1.764E−02	0.998	6.188E−02	0.997	9.040E−03	1.000	0.5365
11736	1/36	1.569E−02	0.996	5.502E−02	0.998	8.035E−03	1.000	0.5361
20832	1/48	1.178E−02	0.997	4.128E−02	0.999	6.027E−03	1.000	0.5355
36992	1/64	8.841E−03	0.998	3.097E−02	0.999	4.520E−03	1.000	0.5349
83136	1/96	5.897E−03	0.999	2.065E−02	1.000	3.013E−03	1.000	0.5346
147712	1/128	4.424E−03	0.999	1.549E−02	1.000	2.260E−03	1.000	0.5344
230720	1/160	3.540E−03	0.998	1.239E−02	1.000	1.808E−03	1.000	0.5343
452032	1/224	2.531E−03	0.998	8.854E−03	0.999	1.292E−03	1.000	0.5343
922240	1/320	1.774E−03	0.996	6.200E−03	0.999	9.042E−04	1.000	0.5343
1327872	1/384	1.475E−03	1.014	5.165E−03	1.001	7.533E−04	1.001	0.5343

Table 6.1: EXAMPLE 1, quasi–uniform scheme

of convergence, that the $O(h)$ predicted by Theorems 4.3 and 5.3 (with $\delta = 1$ in both cases) is attained by all the unknowns. In particular, as observed in the tenth column of Table 6.2, the convergence of $\boldsymbol{\gamma}_h$ is a bit faster than expected (around 1.4), which could mean either a superconvergence phenomenon or a special behavior of the particular solutions involved. We will investigate this issue in a separate work. On the other hand, we notice that the effectivity indexes $\mathtt{eff}(\boldsymbol{\theta})$ remain bounded in both examples (they lie in neighborhoods of 0.53 and 0.34), which illustrates, in these cases of regular solutions, the reliability and efficiency of $\boldsymbol{\theta}$.

Next, in Tables 6.3, 6.4, 6.5, and 6.6, we provide the convergence history of the quasi-uniform and adaptive schemes as applied to Examples 3 and 4. The number of Newton iterations required ranges between 5 and 9, and between 3 and 7, respectively. We notice, as expected, that the errors of the adaptive methods decrease faster than those obtained by the quasi-uniform ones. This fact is better illustrated in Figures 6.1 and 6.3 where we display the total errors $\mathtt{e}(\mathbf{t}, \boldsymbol{\sigma}, u)$ and $\mathtt{e}(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, \gamma)$ vs. the degrees of freedom N for both refinements. Note that these figures include additional data on the quasi-uniform refinements that are not shown in the corresponding tables. Furthermore, the effectivity indexes remain again bounded from above and below, which confirms the reliability and efficiency of $\boldsymbol{\theta}$ in these cases of non-smooth solutions, as well. Some intermediate meshes obtained with the adaptive algorithm are displayed in Figures 6.2 and 6.4. It is important to observe here that the adapted meshes concentrate the refinements around the origin in Example 3, and around the points $(-0.25, 0.5)$ and $(0.25, 0.5)$ in Example 4, which confirms that the method is able to recognize the singularity regions of the solutions.

Finally, in order to illustrate the accurateness of the Galerkin methods and their associated adaptive algorithms, in Figures 6.5, 6.6, 6.7, and 6.8, we display some components of the approximate (left) and exact (right) solutions for all the examples.

References

[1] M. AINSWORTH AND J.T. ODEN, *A posteriori error estimation in finite element analysis.* Computer Methods in Applied Mechanics and Engineering, vol. 142, pp. 1-88, (1997).

[2] A. ALONSO, *Error estimators for a mixed method.* Numerische Mathematik, vol. 74, pp. 385-395, (1996).

N	h	e(**t**)	r(**t**)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(**u**)	r(**u**)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)	eff($\boldsymbol{\theta}$)
7009	1/16	3.808E−02	–	7.034E−02	–	2.003E−02	–	1.472E−02	–	0.3418
10921	1/20	3.047E−02	1.000	5.628E−02	1.000	1.602E−02	1.000	1.106E−02	1.294	0.3412
15697	1/24	2.539E−02	1.001	4.690E−02	1.000	1.335E−02	1.000	8.693E−03	1.327	0.3409
21337	1/28	2.176E−02	1.002	4.020E−02	1.000	1.145E−02	1.000	7.065E−03	1.351	0.3407
27841	1/32	1.903E−02	1.002	3.517E−02	1.000	1.001E−02	1.000	5.887E−03	1.370	0.3407
35209	1/36	1.691E−02	1.002	3.126E−02	1.000	8.902E−03	1.000	5.003E−03	1.385	0.3406
62497	1/48	1.268E−02	1.002	2.344E−02	1.000	6.676E−03	1.000	3.342E−03	1.407	0.3407
110977	1/64	9.502E−03	1.002	1.758E−02	1.000	5.007E−03	1.000	2.217E−03	1.432	0.3408
173281	1/80	7.598E−03	1.002	1.406E−02	1.000	4.006E−03	1.000	1.607E−03	1.443	0.3409
249409	1/96	6.330E−03	1.002	1.172E−02	1.000	3.338E−03	1.000	1.233E−03	1.453	0.3409
443137	1/128	4.746E−03	1.000	8.790E−03	0.999	2.504E−03	1.000	8.165E−04	1.403	0.3411
692161	1/160	3.796E−03	1.000	7.032E−03	1.000	2.003E−03	1.000	5.909E−04	1.448	0.3412
996481	1/192	3.164E−03	1.000	5.861E−03	1.000	1.669E−03	1.000	4.545E−04	1.440	0.3413

Table 6.2: Example 2, quasi–uniform scheme

N	h	e(**t**)	e($\boldsymbol{\sigma}$)	e(u)	e($\mathbf{t}, \boldsymbol{\sigma}, u$)	r($\mathbf{t}, \boldsymbol{\sigma}, u$)	eff($\boldsymbol{\theta}$)
31	1/1	5.636E−01	2.452E−00	4.687E−01	2.559E−00	–	0.6448
399	1/3	1.604E−01	1.034E−00	1.147E−01	1.053E−00	0.659	0.8097
1082	1/5	9.855E−02	7.960E−01	7.043E−02	8.052E−01	0.179	0.8673
2278	1/7	6.725E−02	6.268E−01	4.723E−02	6.322E−01	0.522	0.8949
3654	1/9	5.313E−02	5.333E−01	3.744E−02	5.373E−01	0.176	0.9075
5552	1/11	4.338E−02	4.685E−01	2.998E−02	4.715E−01	0.158	0.9191
7891	1/13	3.625E−02	4.241E−01	2.511E−02	4.263E−01	0.230	0.9304
10266	1/15	3.194E−02	3.925E−01	2.225E−02	3.944E−01	1.314	0.9362
13262	1/17	2.786E−02	3.569E−01	1.950E−02	3.585E−01	0.612	0.9401
18656	1/20	2.370E−02	3.270E−01	1.648E−02	3.283E−01	0.606	0.9481
29359	1/25	1.878E−02	2.808E−01	1.304E−02	2.817E−01	0.704	0.9557
56678	1/35	1.361E−02	2.359E−01	9.424E−03	2.365E−01	0.278	0.9664
114662	1/50	9.532E−03	1.856E−01	6.630E−03	1.859E−01	0.869	0.9731
184770	1/63	7.495E−03	1.527E−01	5.208E−03	1.530E−01	0.706	0.9753
297995	1/80	5.916E−03	1.376E−01	4.116E−03	1.378E−01	0.189	0.9809
460480	1/100	4.756E−03	1.170E−01	3.300E−03	1.172E−01	0.411	0.9830
909848	1/140	3.382E−03	9.395E−02	2.348E−03	9.404E−02	0.582	0.9866
1185751	1/160	2.967E−03	8.094E−02	2.056E−03	8.102E−02	1.116	0.9861

Table 6.3: Example 3, quasi–uniform scheme

N	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{e}(u)$	$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},u)$	$\mathtt{r}(\mathbf{t},\boldsymbol{\sigma},u)$	$\mathtt{eff}(\boldsymbol{\theta})$
31	1.000	5.636E−01	2.452E−00	4.687E−01	2.559E−00	–	0.6448
116	0.707	3.101E−01	1.764E−00	2.393E−01	1.807E−00	0.528	0.7420
344	0.707	1.851E−01	1.244E−00	1.448E−01	1.266E−00	0.654	0.7727
613	0.500	1.486E−01	9.065E−01	1.141E−01	9.257E−01	1.084	0.7500
971	0.500	1.325E−01	7.161E−01	1.002E−01	7.352E−01	1.002	0.7150
1439	0.354	1.093E−01	5.544E−01	7.864E−02	5.706E−01	1.289	0.6969
1992	0.250	8.809E−02	4.759E−01	6.360E−02	4.881E−01	0.960	0.7189
3093	0.250	7.049E−02	3.828E−01	5.170E−02	3.926E−01	0.989	0.7183
4622	0.177	6.109E−02	2.985E−01	4.418E−02	3.079E−01	1.211	0.6802
7705	0.125	4.412E−02	2.372E−01	3.224E−02	2.435E−01	0.919	0.7098
12071	0.125	3.512E−02	1.935E−01	2.560E−02	1.983E−01	0.914	0.7192
17208	0.088	3.139E−02	1.545E−01	2.268E−02	1.593E−01	1.235	0.6796
29560	0.063	2.246E−02	1.226E−01	1.627E−02	1.256E−01	0.877	0.7135
47566	0.063	1.766E−02	9.796E−02	1.281E−02	1.004E−01	0.945	0.7177
67440	0.044	1.592E−02	7.840E−02	1.145E−02	8.082E−02	1.241	0.6769
118541	0.044	1.129E−02	6.189E−02	8.180E−03	6.344E−02	0.858	0.7123
189949	0.031	8.853E−03	4.947E−02	6.400E−03	5.066E−02	0.954	0.7179
265901	0.022	8.003E−03	3.990E−02	5.752E−03	4.110E−02	1.243	0.6792
471990	0.022	5.663E−03	3.137E−02	4.102E−03	3.214E−02	0.857	0.7144
754889	0.016	4.441E−03	2.505E−02	3.202E−03	2.565E−02	0.962	0.7198
1052262	0.011	4.018E−03	2.024E−02	2.884E−03	2.083E−02	1.252	0.6817

Table 6.4: EXAMPLE 3, adaptive scheme

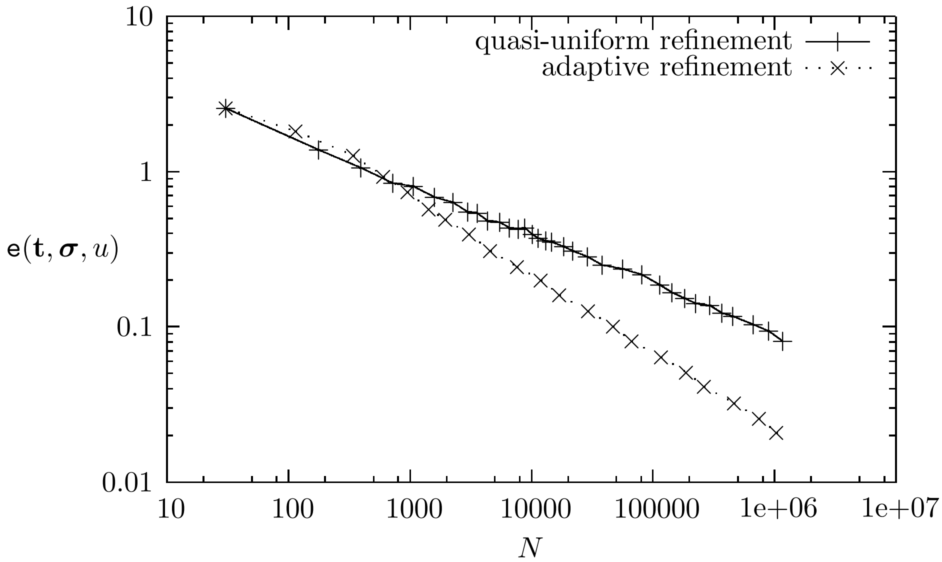

Figure 6.1: EXAMPLE 3, $\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},u)$ vs. N

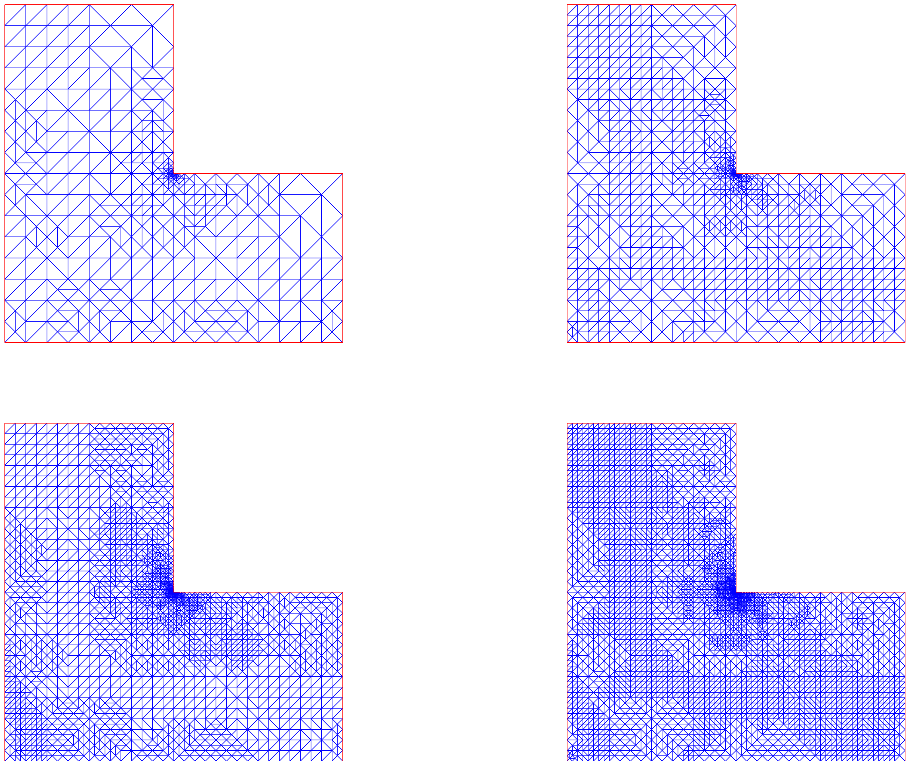

Figure 6.2: EXAMPLE 3, adapted meshes with 3093, 7705, 17208, and 29560 degrees of freedom

N	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{e}(\gamma)$	$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u},\gamma)$	$\mathtt{r}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u},\gamma)$	$\mathtt{eff}(\boldsymbol{\theta})$
184	1/1	4.559E−01	1.038E−00	2.681E−01	6.425E−01	1.331E−00	–	0.3549
745	1/3	1.955E−01	5.934E−01	1.018E−01	1.364E−01	6.475E−01	0.949	0.5292
2212	1/5	1.116E−01	4.096E−01	6.021E−02	6.107E−02	4.331E−01	0.750	0.6055
3817	1/7	8.701E−02	3.638E−01	4.502E−02	4.496E−02	3.794E−01	0.557	0.6561
6256	1/9	6.572E−02	2.981E−01	3.485E−02	2.810E−02	3.085E−01	0.662	0.6897
9211	1/11	5.351E−02	2.519E−01	2.917E−02	2.267E−02	2.602E−01	0.620	0.6999
13297	1/13	4.484E−02	2.330E−01	2.433E−02	2.004E−02	2.393E−01	0.855	0.7310
17953	1/15	3.905E−02	2.133E−01	2.091E−02	1.478E−02	2.184E−01	1.180	0.7530
23281	1/17	3.399E−02	1.904E−01	1.830E−02	1.295E−02	1.946E−01	1.344	0.7592
31873	1/20	2.916E−02	1.745E−01	1.577E−02	1.082E−02	1.780E−01	0.617	0.7795
50620	1/25	2.303E−02	1.505E−01	1.249E−02	8.023E−03	1.530E−01	0.661	0.8052
98572	1/35	1.647E−02	1.245E−01	8.849E−03	4.925E−03	1.260E−01	0.412	0.8446
203521	1/50	1.144E−02	9.795E−02	6.166E−03	3.000E−03	9.886E−02	0.643	0.8732
321172	1/63	9.091E−03	8.151E−02	4.905E−03	2.233E−03	8.219E−02	1.133	0.8824
519349	1/80	7.151E−03	7.079E−02	3.851E−03	1.667E−03	7.127E−02	0.955	0.9005
813916	1/100	5.696E−03	5.672E−02	3.080E−03	1.258E−03	5.710E−02	1.206	0.9016
1168369	1/120	4.761E−03	5.340E−02	2.568E−03	1.022E−03	5.368E−02	0.339	0.9201
1599502	1/140	4.067E−03	5.008E−02	2.200E−03	8.364E−04	5.030E−02	0.422	0.9322

Table 6.5: EXAMPLE 4, quasi–uniform scheme

N	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{e}(\boldsymbol{\gamma})$	$\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$	$\mathtt{r}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$	$\mathtt{eff}(\boldsymbol{\theta})$
184	1.000	4.559E−01	1.038E−00	2.681E−01	6.425E−01	1.331E−00	–	0.3549
478	0.707	2.437E−01	7.089E−01	1.513E−01	2.229E−01	7.966E−01	1.075	0.4503
1315	0.451	1.543E−01	4.896E−01	9.529E−02	1.374E−01	5.399E−01	0.769	0.4692
2521	0.375	1.145E−01	4.091E−01	7.060E−02	9.900E−02	4.419E−01	0.616	0.5079
4750	0.250	8.519E−02	3.021E−01	4.982E−02	6.101E−02	3.235E−01	0.984	0.5280
10390	0.188	5.581E−02	2.173E−01	3.519E−02	4.198E−02	2.309E−01	0.862	0.5383
17332	0.125	4.600E−02	1.705E−01	2.647E−02	3.096E−02	1.812E−01	0.948	0.5450
27646	0.125	3.562E−02	1.361E−01	2.115E−02	2.359E−02	1.442E−01	0.977	0.5520
39226	0.094	2.937E−02	1.128E−01	1.828E−02	1.990E−02	1.197E−01	1.066	0.5449
66862	0.088	2.319E−02	8.580E−02	1.383E−02	1.378E−02	9.100E−02	1.028	0.5420
108520	0.063	1.797E−02	6.903E−02	1.054E−02	1.100E−02	7.294E−02	0.913	0.5567
157513	0.047	1.470E−02	5.630E−02	9.191E−03	8.916E−03	5.958E−02	1.087	0.5482
265285	0.044	1.159E−02	4.353E−02	6.988E−03	6.373E−03	4.603E−02	0.990	0.5480
426394	0.031	9.028E−03	3.491E−02	5.352E−03	4.804E−03	3.677E−02	0.947	0.5618
596311	0.023	7.521E−03	2.915E−02	4.671E−03	3.945E−03	3.072E−02	1.072	0.5580
899497	0.023	6.386E−03	2.445E−02	3.814E−03	4.002E−03	2.587E−02	0.836	0.5546
1183669	0.016	5.539E−03	2.099E−02	3.313E−03	3.010E−03	2.216E−02	1.127	0.5568
1594771	0.016	4.752E−03	1.814E−02	2.825E−03	2.628E−03	1.914E−02	0.982	0.5598

Table 6.6: Example 4, adaptive scheme

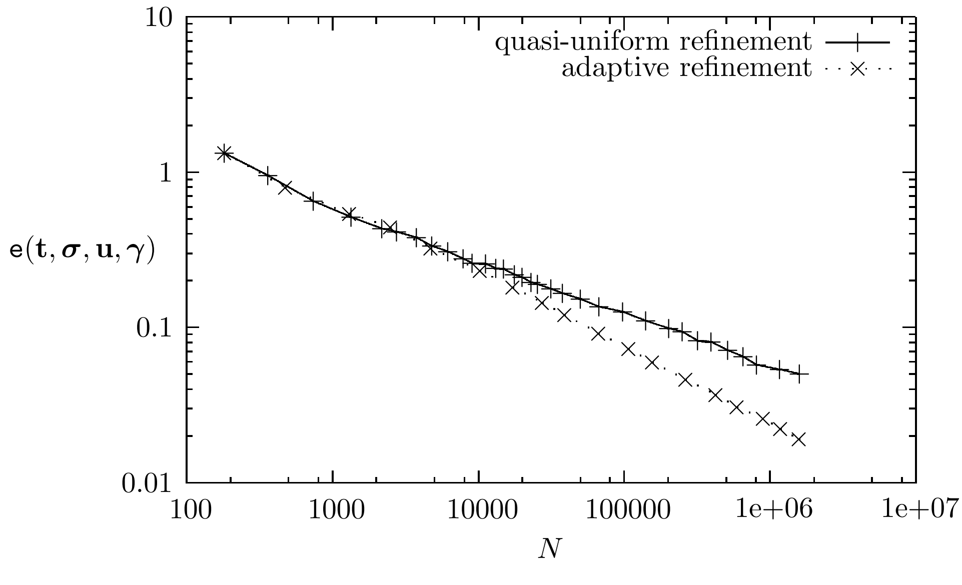

Figure 6.3: Example 4, $\mathtt{e}(\mathbf{t},\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$ vs. N

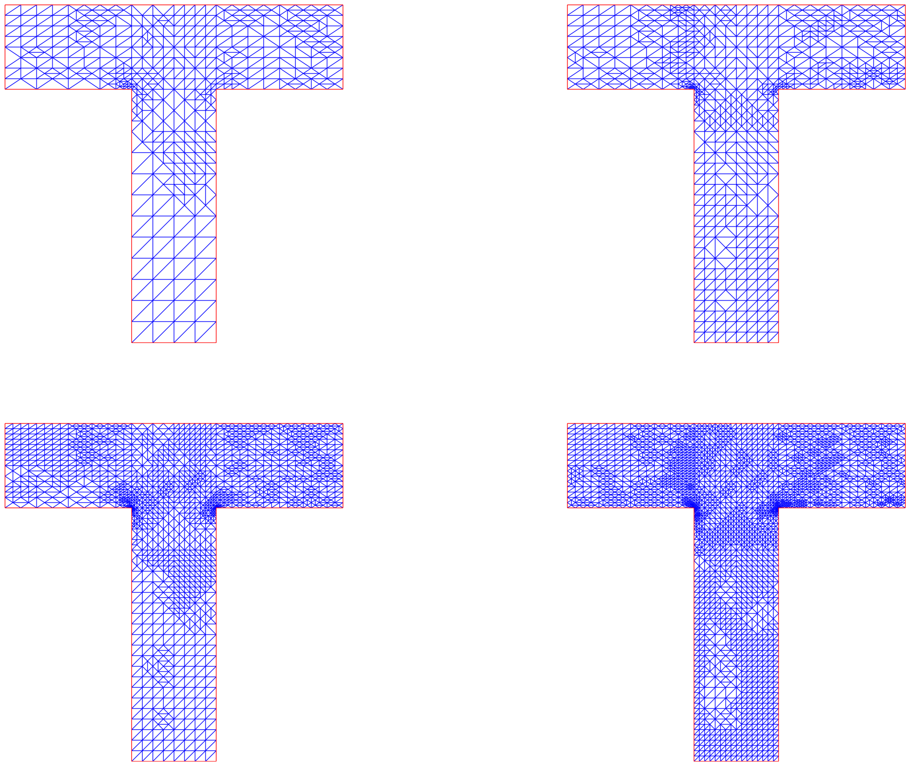

Figure 6.4: EXAMPLE 4, adapted meshes with 10390, 17332, 39226, and 66862 degrees of freedom

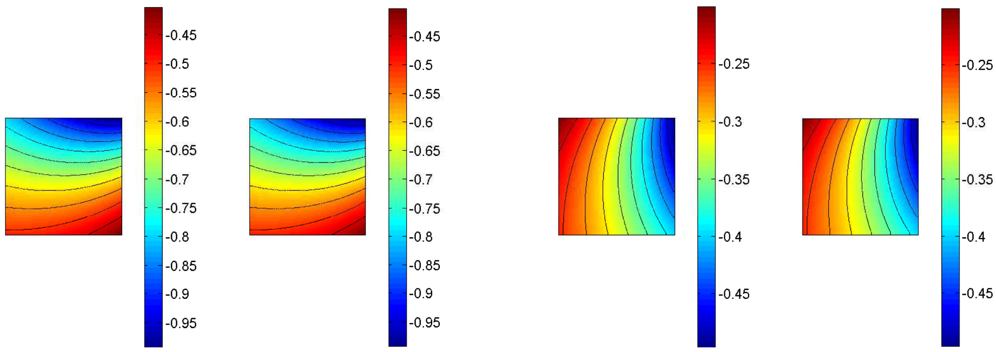

Figure 6.5: EXAMPLE 1, approximate and exact σ_1 and t_2 ($N = 36992$)

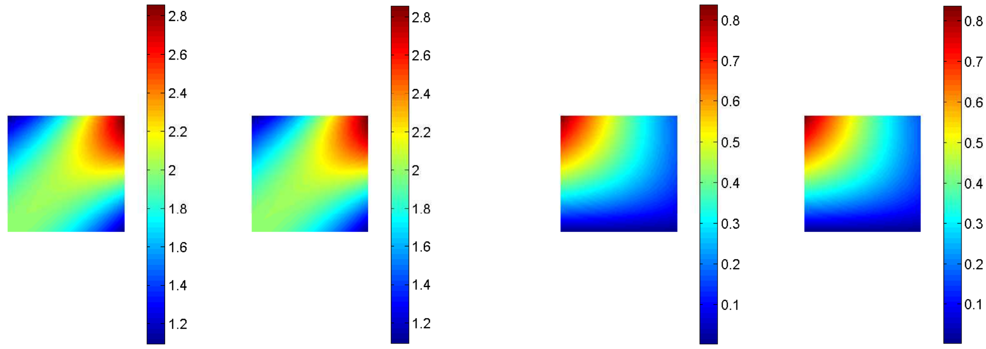

Figure 6.6: EXAMPLE 2, approximate and exact σ_{11} and u_2 $(N = 173281)$

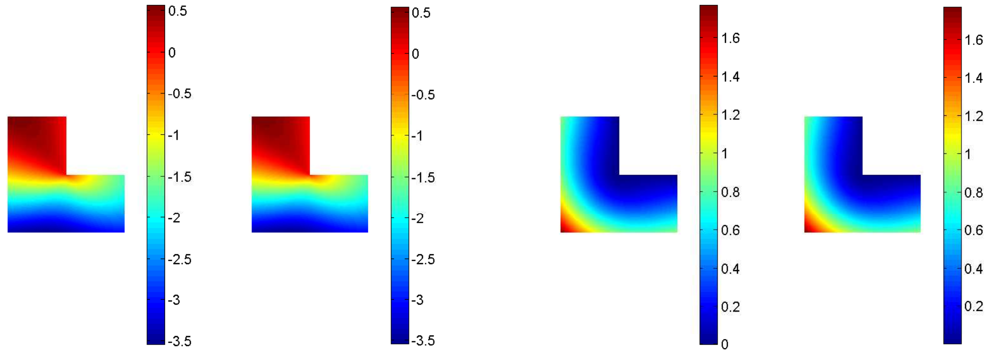

Figure 6.7: EXAMPLE 3, approximate and exact σ_2 and u $(N = 189949)$ for adaptive scheme

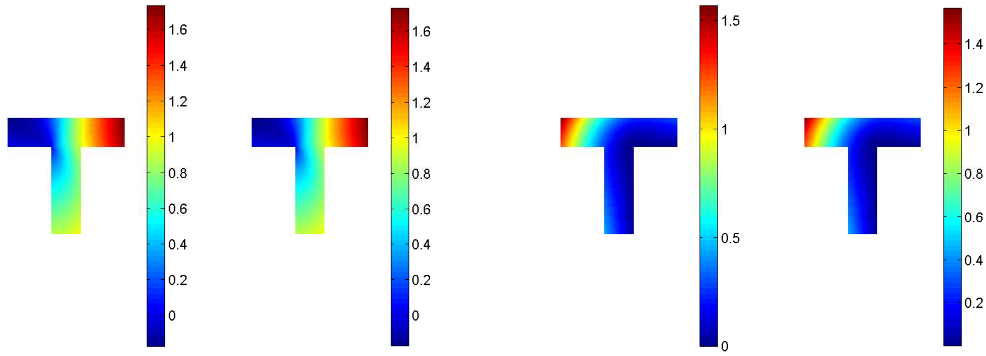

Figure 6.8: EXAMPLE 4, approximate and exact t_{11} and u_2 $(N = 108520)$ for adaptive scheme

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