On the dual-mixed formulation for an exterior Stokes problem*

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Dedicated to Professor Wolfgang L. Wendland on the Occasion of his 75th Birthday

Abstract

This paper is concerned with a dual-mixed formulation for a three dimensional exterior Stokes problem via boundary integral equation methods. Here velocity, pressure and stress are the main unknowns. Following a similar analysis given recently for the Laplacian, we are able to extend the classical Johnson & Nédélec procedure to the present case, without assuming any restrictive smoothness requirement on the coupling boundary, but only Lipschitz-continuity. More precisely, after using the incompressibility condition to eliminate the pressure, we consider the resulting velocity-stress approach with a Neumann boundary condition on an annular bounded domain, and couple the underlying equations with only one boundary integral equation arising from the application of the normal trace to the Green representation formula in the exterior unbounded region. As a result, we obtain a saddle point operator equation, which is then analyzed by the well-known Babuška-Brezzi theory: in particular, the well-posedness of the formulation will be established.

Key words: Exterior Stokes problem, Dual-mixed formulation, Coupling procedure, Hydrodynamic potentials, Boundary integral equation, Sobolev space.
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1 Introduction

In this paper, we apply the classical Johnson & Nédélec coupling procedure [12] to exterior boundary value problems in $\mathbb{R}^3$ by introducing an auxiliary boundary which is only Lipschitz-continuous. Motivated from the recent works [15] and [17] on boundary value problems for the Laplacian (see also [14]), we analyze the reduced nonlocal problems and establish the existence and uniqueness of the solution of the nonlocal problem without the

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compactness of the corresponding double-layer boundary integral operators on the auxiliary boundary as employed for the proofs in [16] and [12] in the case of smooth coupling boundaries.

1.1 Exterior Stokes problems

In the following, for definiteness, let Ω_0 be a bounded domain in $\mathbb{R}^3$ with a Lipschitz boundary Γ_0. We denote by $\mathbf{u}$ and p respectively, the unknown velocity and pressure fields in terms of which the exterior Stokes boundary problem consists of the system of partial differential equations

$$\left.\begin{aligned} -\mu\,\Delta\mathbf{u} + \nabla p = \ & \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 & \end{aligned}\right\} \quad \text{in} \quad \Omega^c := \mathbb{R}^3 \setminus \overline{\Omega_0} \tag{1}$$

together with either Dirichlet or Neumann boundary condition on Γ_0 and the condition at infinity

$$\mathbf{u} = O(||x||^{-1}), \quad p_0 = O(||x||^{-2}) \quad \text{as} \quad ||x|| \to \infty, \tag{2}$$

where $\mathbf{f}$ is a prescribed function with compact support in Ω^c, and $\mu > 0$ the given viscosity, a constant. In order to apply the coupling procedure, we proceed as in the classical approaches (see, e.g. [4], [5], [7], [8], [9], [10]) and introduce an auxiliary surface Γ whose interior contains $\bar{\Omega}_0$. The main novelty here is that Γ is only assumed to be Lipschitz-continuous. Then we let Ω^- be the annular region bounded by Γ_0 and Γ, and $\Omega^+ := \mathbb{R}^3 \setminus \big(\bar{\Omega}_0 \cup \bar{\Omega}^-\big)$. Here and in the sequel $\boldsymbol{\nu}$ always denotes the outward unit normal pointing away from bounded domains (see Figure 1 below).

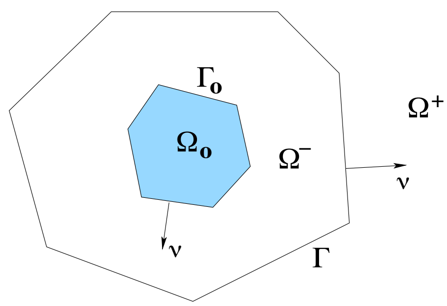

Figure 1: Geometry of the problem.

We first formulate the exterior problem in terms of the unknowns velocity $\mathbf{u}$, the pressure p and the stress tensor $\boldsymbol{\sigma}$ as a transmission problem for $\mathbf{u}$ and $\boldsymbol{\sigma}$ by eliminating the pressure p from the incompressibility condition. To this end, the following notations will be further adopted.

- Stress and strain tensors: $\boldsymbol{\sigma} = \Xi[\mathbf{u}, p] := -p\mathbf{I} + 2\,\mu\,\mathbf{e}(\mathbf{u}), \quad \mathbf{e}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right)$
- Trace of stress : $\operatorname{tr}\boldsymbol{\sigma} = -3p + 2\mu\operatorname{div}\mathbf{u} = -3p$, if $\operatorname{div}\mathbf{u} = 0$.
- Deviatoric stress: $\boldsymbol{\sigma}^{\mathrm{d}} = \boldsymbol{\sigma} + p\mathbf{I} = 2\,\mu\,\mathbf{e}(\mathbf{u})$.
  Note: $\operatorname{tr}\boldsymbol{\sigma}^{\mathrm{d}} = 0$ if $\operatorname{div}\mathbf{u} = 0$ and the pair of equations

$$\boldsymbol{\sigma} = -\,p\,\mathbf{I} + 2\mu\,\mathbf{e}(\mathbf{u}) \quad \text{and} \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega^c$$

can be rewritten, equivalently, as

$$\boldsymbol{\sigma}^{\mathrm{d}} = 2\mu\,\mathbf{e}(\mathbf{u}) \quad \text{and} \quad p + \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = 0 \quad \text{in} \quad \Omega^c$$

1.2 A Transmission problem

We are now in a position to formulate the Transmission problem: Given $\mathbf{f}$ in Ω^- and either Dirichlet or Neumann boundary condition of $\mathbf{u}$ on Γ_0, find $\mathbf{u}, \boldsymbol{\sigma}$ (and p) satisfying the partial differential equations

$$\left.\begin{aligned} \boldsymbol{\sigma}^{\mathrm{d}} \;=\; 2\,\mu\,\mathbf{e}(\mathbf{u}) \quad \text{and} \quad p + \frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\sigma} \;=\; 0 \\ -\mathbf{div}\,\boldsymbol{\sigma} \;=\; \mathbf{f} \end{aligned}\right\} \quad \text{in} \quad \Omega^- \tag{3}$$

and

$$\left.\begin{aligned} -\mu\,\Delta\mathbf{u} + \nabla p = \;\mathbf{0} \quad \text{and} \quad \mathrm{div}\,\mathbf{u} \;=\; 0 \\ \boldsymbol{\sigma} \;=\; \Xi[\mathbf{u}, p] \;:=\; -p\mathbf{I} + 2\,\mu\,\mathbf{e}(\mathbf{u}) \end{aligned}\right\} \quad \text{in} \quad \Omega^+ \tag{4}$$

together with the transmission condition

$$\left.\begin{aligned} [\mathbf{u}] \;:=\; \mathbf{u}^- \;-\; \mathbf{u}^+ \;=\; \mathbf{0} \\ [\boldsymbol{\sigma}\boldsymbol{\nu}] \;:=\; (\boldsymbol{\sigma}\boldsymbol{\nu})^- \;-\; (\boldsymbol{\sigma}\boldsymbol{\nu})^+ \;=\; \mathbf{0} \end{aligned}\right\} \quad \text{on} \quad \Gamma \tag{5}$$

and the condition at infinity (2), namely

$$\mathbf{u}(\mathbf{x}) \;=\; O(\|\mathbf{x}\|^{-1}), \quad p(\mathbf{x}) \;=\; O(\|\mathbf{x}\|^{-2}) \quad \text{as} \quad \|\mathbf{x}\| \to +\infty\,.$$

The coupling procedure remains to reduce solutions in the exterior domain Ω^+ to appropriate boundary integral equations on the auxiliary boundary Γ via the transmission conditions. This refers to the nonlocal boundary conditions. The latter together with the solutions defined in the boundary domain Ω^- then forms an equivalent boundary value problem of the original exterior boundary value problem known as the nonlocal problem.

1.3 Boundary integral equation on Γ

The fundamental velocity tensor and its associated pressure vector for the Stokes equations are given respectively by

$$\begin{aligned} E(\mathbf{x}, \mathbf{y}) \;&:=\; \frac{1}{8\,\pi}\mu \left\{ \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\,\mathbf{I} \,+\, \frac{(\mathbf{x}-\mathbf{y})\,(\mathbf{x}-\mathbf{y})^{\mathrm{t}}}{\|\mathbf{x}-\mathbf{y}\|^3} \right\}, \quad \forall\,\mathbf{x} \neq \mathbf{y}\,, \\ Q(\mathbf{x}, \mathbf{y}) \;&:=\; \frac{1}{4\,\pi}\,\nabla_{\mathbf{y}} \left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|} \right), \quad \forall\,\mathbf{x} \neq \mathbf{y} \end{aligned}$$

in terms of which the hydrodynamic potentials in Ω^+ are defined as follows (see Hsiao and Wendland [11] or Kohr and Wendland [13])

- Simple layer hydrodynamic potentials for the velocity and pressure:

$$\begin{aligned} \mathbf{S}\,\boldsymbol{\rho}(\mathbf{x}) \;&:=\; \int_\Gamma E(\mathbf{x}, \mathbf{y})\,\boldsymbol{\rho}(\mathbf{y})\,d\mathbf{s}_{\mathbf{y}}, \;\forall\,\mathbf{x} \;\notin\; \Gamma, \\ \Phi\,\boldsymbol{\rho}(\mathbf{x}) \;&:=\; \int_\Gamma Q(\mathbf{x}, \mathbf{y}) \cdot \boldsymbol{\rho}(\mathbf{y})\,d\mathbf{s}_{\mathbf{y}}, \;\; \forall\,\mathbf{x} \;\notin\; \Gamma. \end{aligned}$$

- Double layer hydrodynamic potentials for the velocity and pressure:

$$\begin{aligned} \mathbf{D}\,\boldsymbol{\psi} \;&:=\; \left(\; \mathbf{D}_1\,\boldsymbol{\psi},\; \mathbf{D}_2\,\boldsymbol{\psi},\; \mathbf{D}_3\,\boldsymbol{\psi} \;\right)'\,, \\ \mathbf{D}_i\,\boldsymbol{\psi}(\mathbf{x}) \;&:=\; \int_\Gamma \left\{ \Xi[E_i(\mathbf{x}, \cdot), Q_i(\mathbf{x}, \cdot)](\mathbf{y})\,\boldsymbol{\nu}(\mathbf{y}) \right\}^{\mathrm{t}} \boldsymbol{\psi}(\mathbf{y})\,d\mathbf{s}_{\mathbf{y}}, \forall\,\mathbf{x} \;\notin\; \Gamma \\ \Pi\,\boldsymbol{\psi}(\mathbf{x}) \;&:=\; 2\mu \int_\Gamma \nabla_{\mathbf{y}}\,Q(\mathbf{x}, \mathbf{y})\,\boldsymbol{\nu}(\mathbf{y}) \cdot \boldsymbol{\psi}(\mathbf{y})\,d\mathbf{s}_{\mathbf{y}}, \forall\,\mathbf{x} \;\notin\; \Gamma \end{aligned}$$

for appropriate density and moment vector-valued functions $\boldsymbol{\rho}$ and $\boldsymbol{\psi}$ to be specified later. Here $E_i(\mathbf{x},\mathbf{y})$ is the i-th column of $E(\mathbf{x},\mathbf{y})$ and $Q_i(\mathbf{x},\mathbf{y})$ is the i-th component of $Q(\mathbf{x},\mathbf{y})$. From Green's formula we have the representation of $(\mathbf{u},\, p)$ in Ω^+:

$$\mathbf{u} \,=\, \mathbf{D}\,\gamma^+(\mathbf{u}) \,-\, \mathbf{S}\,\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma}), \quad p \,=\, \Pi\,\gamma^+(\mathbf{u}) - \Phi\,\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega^+\,, \tag{6}$$

in terms of which

$$\boldsymbol{\sigma} \,=\, \Xi[\mathbf{u},p] := -p\mathbf{I} + 2\mu\,\mathbf{e}(\mathbf{u}) \quad \text{in} \quad \Omega^+. \tag{7}$$

In the representation (6), the traces $(\mathbf{u}^+, \boldsymbol{\sigma}^+\boldsymbol{\nu})$ on Γ from Ω^+ are denoted by $\gamma^+(\mathbf{u})$ and $\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma})$, respectively. They are the Cauchy data for the solutions of Stokes equations. Using the standard jump relations lead to the two basic system of boundary integral equations on Γ for the traces

$$\gamma^+(\mathbf{u}) \;=\; \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\gamma^+(\mathbf{u}) - \mathbf{V}\,\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) \quad \text{on} \quad \Gamma\,, \tag{8}$$

$$\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) \;=\; -\,\mathbf{W}\,\gamma^+(\mathbf{u}) + \left(\frac{1}{2}\mathbf{I} - \mathbf{K}^{\mathtt{t}}\right)\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma}) \quad \text{on} \quad \Gamma\,. \tag{9}$$

Here $\mathbf{V}, \mathbf{K}, \mathbf{K}^{\mathtt{t}}$ and $\mathbf{W}$ are the corresponding four basic hydrodynamic boundary integral operators as in the case for the Laplacian (see the precise definitions in Hsiao and Wendland [11]). The boundary integral equations (8) and (9) may serve as the desirable nonlocal condition for the relevant traces via the transmission condition (5).

Finally, the nonlocal problem is to find the unknowns $\mathbf{u}, \boldsymbol{\sigma}$ (and p) in Ω^- governing the partial differential equations (3) and the unknown traces $\gamma^+(\mathbf{u})$ and/or $\gamma^+_{\boldsymbol{\nu}}(\boldsymbol{\sigma})$ satisfying (8) and/or (9) via the transmission condition (5) depending on the variational form for the solutions in Ω^-.

2 Variational Formulations

We begin with some relevant function spaces which will be needed for the variational formulations. Here and in the sequel, we utilize the standard terminology for Sobolev spaces and norms. For simplifying the presentation, some notations will also be adopted.

- Given a domain $\mathcal{O}$, a closed Lipschitz curve Σ, and $r \in \mathbb{R}$, we simplify notations and define

$$\mathbf{H}^r(\mathcal{O}) \,:=\, [H^r(\mathcal{O})]^3\,, \quad \mathbb{H}^r(\mathcal{O}) \,:=\, [H^r(\mathcal{O})]^{3\times 3}$$
$$\text{and} \quad \mathbf{H}^r(\Sigma) \,:=\, [H^r(\Sigma)]^3,$$

  where $H^r(\mathcal{O})$ and $H^r(\Sigma)$ are the usual Sobolev spaces. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\Sigma}$ (for $H^r(\Sigma)$ and $\mathbf{H}^r(\Sigma)$).

- Hilbert space $\mathbb{H}(\mathbf{div}\,;\mathcal{O}) \,:=\, \left\{\boldsymbol{\tau} \,\in\, \mathbb{L}^2(\mathcal{O}) : \quad \mathbf{div}\,\boldsymbol{\tau} \,\in\, \mathbf{L}^2(\mathcal{O})\right\}$, equipped with the norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}\,;\mathcal{O}} \,:=\, \left\{\, \|\boldsymbol{\tau}\|^2_{0,\mathcal{O}} \,+\, \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\mathcal{O}} \,\right\}^{1/2} \qquad \forall\, \boldsymbol{\tau} \,\in\, \mathbb{H}(\mathbf{div}\,;\mathcal{O})\,.$$

- The space of skew symmetric strain tensors

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega^-) \,:=\, \left\{\, \boldsymbol{\eta} \,\in\, \mathbb{L}^2(\Omega^-) : \quad \boldsymbol{\eta}^{\mathtt{t}} \,=\, -\,\boldsymbol{\eta} \,\right\}.$$

2.1 The dual-mixed formulation in Ω^-

We proceed here similarly as for the linear elasticity problem (see, e.g. [1], [18]). Indeed, we first notice that denoting

$$\boldsymbol{\chi} := \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\right) \in \mathbb{L}^2_{\mathtt{skew}}(\Omega^-)$$

the constitutive equation relating $\mathbf{u}$ and $\boldsymbol{\sigma}$ in Ω^- becomes

$$\boldsymbol{\sigma}^{\mathtt{d}} = 2\mu\,\mathbf{e}(\mathbf{u}) = 2\,\mu\left(\nabla\mathbf{u} - \boldsymbol{\chi}\right) \quad \text{in} \quad \Omega^- .$$

In what follows, without loss of generality, we set $2\,\mu = 1$. Then the governing equation (3) is

$$p + \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma}^{\mathtt{d}} = \nabla\mathbf{u} - \boldsymbol{\chi}, \quad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega^- \tag{10}$$

We note that since the pressure p can be computed in terms of the stress from the first relation in (10), we focus mainly on the approaches that do not include p as an explicit unknown but only as part of $\boldsymbol{\sigma}$ in the variational formulation below.

For the dual-mixed formulation, multiplying (tensor product :) 2nd equation in (10) by $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega^-)$ and integrating by parts, yields

$$\int_{\Omega^-} \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} - \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \boldsymbol{\varphi}\rangle_\Gamma + \int_{\Omega^-} \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_{\Omega^-} \boldsymbol{\chi} : \boldsymbol{\tau} + \langle \gamma^-_{\boldsymbol{\nu}}(\boldsymbol{\tau}), \gamma^-(\mathbf{u})\rangle_{\Gamma_0} = 0\,, \tag{11}$$

where

$$\boldsymbol{\varphi} := \gamma^-(\mathbf{u}) = \gamma^+(\mathbf{u}) \in \mathbf{H}^{1/2}(\Gamma)$$

is an additional unknown. We note that for the dual-mixed formulation, Dirichlet data will be the nature boundary data. Here $\gamma^\pm : \mathbf{H}^1(\Omega^\pm) \to \mathbf{H}^{1/2}(\Gamma)$ and $\gamma^\pm_{\boldsymbol{\nu}} : \mathbb{H}(\mathbf{div}\,;\Omega^\pm) \to \mathbf{H}^{-1/2}(\Gamma)$ are the usual trace and normal trace operators, respectively, on Γ. On the other hand, incorporating the equilibrium relation in Ω^- and the symmetry of the stress tensor $\boldsymbol{\sigma}$ in a weak sense, we arrive at

$$\int_{\Omega^-} \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} + \int_{\Omega^-} \boldsymbol{\eta} : \boldsymbol{\sigma} = -\int_{\Omega^-} \mathbf{f}\cdot\mathbf{v}, \quad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in \mathbf{L}^2(\Omega^-) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega^-)\,. \tag{12}$$

Equations (11) and (12) are the variational equation for the weak solution pairs $(\mathbf{u},\boldsymbol{\sigma})$ in Ω^-. In fact, the variational equation (11) holds for all test functions $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbb{H}(\mathbf{div}\,;\Omega^-) \times \mathbf{H}^{1/2}(\Gamma)$. However, its final form depends on the boundary conditions on Γ_0. We summarize the situations according to boundary conditions (BCs) on Γ_0 as follows:

- Dirichlet BC (the natural BC): On $\Gamma_0, \gamma^-(\mathbf{u})$ is known. In this case let

  $$\mathbf{X}_D := \mathbb{H}(\mathbf{div}\,;\Omega^-) \times \mathbf{H}^{1/2}(\Gamma), \quad \mathbf{Y}_D := \mathbf{L}^2(\Omega^-) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega^-).$$

  We have total 4 unknowns $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \in \mathbf{X}_D \times \mathbf{Y}_D$

- Neumann BC (the essential BC): On $\Gamma_0, \gamma^-(\mathbf{u})$ is unknown. We need to introduce the further unknown $\boldsymbol{\lambda} := \gamma^-(\mathbf{u}) \in \mathbf{H}^{1/2}(\Gamma_0)$. In this case let

  $$\mathbf{X}_N := \mathbf{X}_D, \quad \mathbf{Y}_N := \mathbf{L}^2(\Omega^-) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega^-) \times \mathbf{H}^{1/2}(\Gamma_0).$$

  We have total 5 unknowns $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\lambda})) \in \mathbf{X}_N \times \mathbf{Y}_N$

Remark: $((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta})) \in \mathbf{X}_D \times \mathbf{Y}_D$ and $((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta},\boldsymbol{\xi})) \in \mathbf{X}_N \times \mathbf{Y}_N$ are the corresponding test functions, respectively.

2.2 A coupled variational formulation

It is clear that in order to consider the complete variational formulation for the nonlocal problem we need to supplement the variation equations (11) and (12) with those from the nonlocal conditions (8) and/or (9). In this short communication, we shall confine ourselves to the homogeneous Neumann BC on Γ_0 and employee only one of the two boundary integral equations. The approach here carries over to other BCs on Γ_0 including the Dirichlet BC and with two boundary integral equations (see the forthcoming paper [6] for further details).

As we mentioned, since the Neumann boundary condition is an essential BC in the dual-mixed formulation, we have total 5 unknowns $((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi}, \boldsymbol{\lambda})) \in \mathbf{X}_N \times \mathbf{Y}_N$ However, for the given homogeneous Neumann BC, namely

$$\gamma_{\boldsymbol{\nu}}^{-}(\boldsymbol{\sigma}) \;=\; \mathbf{g}_{_N} = \mathbf{0} \,\in\, \mathbf{H}^{-1/2}(\Gamma_0)\,,$$

there is no need of including the additional unknown $\boldsymbol{\lambda} = \gamma^{-}(\mathbf{u})$ on Γ_0 in the variational equation (11). In fact, it reduces to 4 unknowns $((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi})) \in \mathbf{X}_N^0 \times \mathbf{Y}_D$, if we restrict $(\boldsymbol{\sigma}, \boldsymbol{\varphi})$ in the subspace of $\mathbf{X}_N^0$ of $\mathbf{X}_N$ such that

$$\mathbf{X}_N^0 := \mathbb{H}_0(\mathbf{div}\,;\Omega^-) \times \mathbf{H}^{1/2}(\Gamma),$$

where

$$\mathbb{H}_0(\mathbf{div}\,;\Omega^-) \;:=\; \Big\{ \boldsymbol{\tau} \,\in\, \mathbb{H}(\mathbf{div}\,;\Omega^-) :\; \gamma_{\boldsymbol{\nu}}^{-}(\boldsymbol{\tau}) \;=\; \mathbf{0} \quad \text{on} \quad \Gamma_0 \Big\}.$$

The variational equation (12) remains the same, and in addition we need to add the variational equation of the boundary integral equation (12) in the form

$$\langle \mathbf{W}\,\boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_\Gamma \,+\, \left\langle \left(\frac{1}{2}\mathbf{I} \,+\, \mathbf{K}^{\mathrm{t}}\right) \gamma_{\boldsymbol{\nu}}^{-}(\boldsymbol{\sigma}), \boldsymbol{\psi} \right\rangle_\Gamma \;=\; 0,\; \forall\, \boldsymbol{\psi} \,\in \mathbf{H}^{1/2}(\Gamma), \tag{13}$$

where we have tacitly employee the transmission conditions:

$$\gamma^{+}(\mathbf{u}) = \boldsymbol{\varphi}, \qquad \gamma_{\boldsymbol{\nu}}^{+}(\boldsymbol{\sigma}) = \gamma_{\boldsymbol{\nu}}^{-}(\boldsymbol{\sigma}) \quad \text{on} \quad \Gamma.$$

The coupled variational formulation now reads: *Find* $((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\chi})) \,\in\, \mathbf{X}_N^0 \times \mathbf{Y}_D$*such that*

$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\boldsymbol{\tau}, \boldsymbol{\psi})) + \mathbf{b}_N((\boldsymbol{\tau}, \boldsymbol{\psi}), (\mathbf{u}, \boldsymbol{\chi})) = \mathbf{F}_N(\boldsymbol{\tau}, \boldsymbol{\psi}), \forall\, (\boldsymbol{\tau}, \boldsymbol{\psi}) \in \mathbf{X}_N^0, \\ \mathbf{b}_N((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\mathbf{v}, \boldsymbol{\eta})) = \mathbf{G}_N(\mathbf{v}, \boldsymbol{\eta}), \forall\, (\mathbf{v}, \boldsymbol{\eta}) \,\in\, \mathbf{Y}_D \end{aligned} \tag{14}$$

In the formulation (14), $\mathbf{a}_N : \mathbf{X}_N^0 \times \mathbf{X}_N^0 \to \mathbb{R}$ and $\mathbf{b}_N : \mathbf{X}_N^0 \times \mathbf{Y}_D \to \mathbb{R}$ are the bounded bilinear forms defined, respectively by

$$\begin{aligned} \mathbf{a}_N((\boldsymbol{\sigma}, \boldsymbol{\varphi}), (\boldsymbol{\tau}, \boldsymbol{\psi})) := \int_{\Omega^-} \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} - \langle \gamma_{\boldsymbol{\nu}}^{-}(\boldsymbol{\tau}), \boldsymbol{\varphi}\rangle_\Gamma + \langle \mathbf{W}\,\boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_\Gamma \\ + \left\langle \left(\frac{1}{2}\mathbf{I} + \mathbf{K}^{\mathrm{t}}\right) \gamma_{\boldsymbol{\nu}}^{-}(\boldsymbol{\sigma}), \boldsymbol{\psi} \right\rangle_\Gamma, \end{aligned} \tag{15}$$

$$\mathbf{b}_N((\boldsymbol{\tau}, \boldsymbol{\psi}), (\mathbf{v}, \boldsymbol{\eta})) \;:=\; \int_{\Omega^-} \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\tau} \,+\, \int_{\Omega^-} \boldsymbol{\eta} : \boldsymbol{\tau}\,, \tag{16}$$

$\mathbf{F}_N : \mathbf{X}_N^0 \to \mathbb{R}$ and $\mathbf{G}_N : \mathbf{Y}_D \to \mathbb{R}$ are the bounded linear functionals given by

$$\mathbf{F}_N(\boldsymbol{\tau}, \boldsymbol{\psi}) \;:=\; 0\,, \quad \mathbf{G}_N(\mathbf{v}, \boldsymbol{\eta}) \;:=\; -\int_{\Omega^-} \mathbf{f} \cdot \mathbf{v}\,,$$

for all $((\boldsymbol{\tau}, \boldsymbol{\psi}), (\mathbf{v}, \boldsymbol{\eta})) \,\in\, \mathbf{X}_N^0 \times \mathbf{Y}_D$.

3 Existence and Uniqueness Results

We begin with some preliminary results for the bilinear forms $\mathbf{a}_N$ and $\mathbf{b}_N$. Recall that

$$\mathbf{X}_N^0 := \mathbb{H}_0(\mathbf{div}\,;\Omega^-) \times \mathbf{H}^{1/2}(\Gamma)\,, \quad \mathbf{Y}_D := \mathbf{L}^2(\Omega^-) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega^-)$$

and note that the bounded linear operator induced by $\mathbf{b}_N$

$$\mathbf{B}_N : \mathbf{X}_N^0 \to \mathbf{Y}_D$$

is given by $\mathbf{B}_N((\boldsymbol{\tau},\boldsymbol{\psi})) := \big(\mathbf{div}\,\boldsymbol{\tau}, \frac{1}{2}(\boldsymbol{\tau} - \boldsymbol{\tau}^{\mathtt{t}})\big)$ for any $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N^0$. Moreover, the kern of $\mathbf{B}_N$ is a closed subspace of $\mathbf{X}_N^0$ defined by

$$\mathbf{V}_N := \Big\{ (\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N^0 : \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \text{ and } \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \in \Omega^- \Big\}.$$

The following lemmas, which establish a positiveness property of $\mathbf{a}_N$ on $\mathbf{V}_N$ and an inf-sup condition for $\mathbf{b}_N$, are crucial for the analysis of the dual-mixed formulation.

LEMMA 1 *There holds*

$$\mathbf{a}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})) \geq \frac{1}{2}\Big\{ \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma \Big\}$$

for all $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{V}_N$.

Proof. Given $(\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{V}_N$, we have from (15)

$$\begin{aligned}\mathbf{a}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})) &= \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma + \Big\langle \Big(-\frac{1}{2}\mathbf{I} + \mathbf{K}^{\mathtt{t}}\Big)\,\gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}),\boldsymbol{\psi}\Big\rangle_\Gamma \\ &= \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma + \Big\langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}), \Big(-\frac{1}{2}\mathbf{I} + \mathbf{K}\Big)\,\boldsymbol{\psi}\Big\rangle_\Gamma\,,\end{aligned}$$

which can be written as

$$a_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})) = \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma + \big\langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}), \gamma^-(\mathbf{D}\,\boldsymbol{\psi})\big\rangle_\Gamma\,.$$

Hence, integrating by parts in Ω^- and using that $\gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}) = \mathbf{0}$ on Γ_0, we find that

$$\begin{aligned}\langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}), \gamma^-(\mathbf{D}\,\boldsymbol{\psi})\rangle_\Gamma &= \int_{\Omega^-} \Big\{ \nabla \mathbf{D}\,\boldsymbol{\psi} : \boldsymbol{\tau} + \mathbf{D}\,\boldsymbol{\psi}\cdot \mathbf{div}\,\boldsymbol{\tau} \Big\} \\ &= \int_{\Omega^-} \mathbf{e}\big(\mathbf{D}\,\boldsymbol{\psi}\big) : \boldsymbol{\tau} = \int_{\Omega^-} \mathbf{e}\big(\mathbf{D}\,\boldsymbol{\psi}\big) : \boldsymbol{\tau}^{\mathtt{d}}\,,\end{aligned}$$

where the free-divergence and symmetry properties of $\boldsymbol{\tau}$, together with the incompressibility condition satisfied by $\mathbf{D}\,\boldsymbol{\psi}$, have been utilized in the last two equalities. Consequently, by the Cauchy-Schwarz's inequality and the identity

$$\langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma = \|\mathbf{e}(\mathbf{D}\,\boldsymbol{\psi})\|^2_{0,\mathbb{R}^3\setminus\Gamma} \qquad \forall\,\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma)\,,$$

we see that

$$\begin{aligned}a_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\boldsymbol{\tau},\boldsymbol{\psi})) &= \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma + \int_{\Omega^-} \mathbf{e}\big(\mathbf{D}\,\boldsymbol{\psi}\big) : \boldsymbol{\tau}^{\mathtt{d}} \\ &\geq \frac{1}{2}\,\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma - \frac{1}{2}\,\|\mathbf{e}(\mathbf{D}\,\boldsymbol{\psi}\|^2_{0,\Omega^-} \\ &\geq \frac{1}{2}\,\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma - \frac{1}{2}\,\|\mathbf{e}(\mathbf{D}\,\boldsymbol{\psi}\|^2_{0,\mathbb{R}^3\setminus\Gamma} \\ &= \frac{1}{2}\,\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega^-} + \frac{1}{2}\,\langle \mathbf{W}\,\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma\,.\end{aligned}$$

This completes the proof. $\square$

It is worth mentioning that in the proof, the compactness property of the boundary integral operator $\mathbf{K}$ is not needed. In fact, for the Lipschitz boundary Γ, $\mathbf{K}$ is not compact. We further remark that $\mathbf{W}$ is not $\mathbf{H}^{1/2}(\Gamma)$ but $\mathbf{H}_0^{1/2}(\Gamma)$-elliptic, where

$$\mathbf{H}_0^{1/2}(\Gamma) \;:=\; \mathbf{H}^{1/2}(\Gamma)/\mathbb{RM}(\Gamma) \;\equiv\; \Big\{ \boldsymbol{\psi} \,\in\, \mathbf{H}^{1/2}(\Gamma) : \langle \mathbf{r}, \boldsymbol{\psi}\rangle_{1/2,\Gamma} \;=\; 0,\; \forall\, \mathbf{r} \,\in\, \mathbb{RM}(\Gamma) \Big\},$$
$$\mathbb{RM}(\mathcal{O}) \;:=\; \big\{ \mathbf{z} : \quad \mathbf{z}(\mathbf{x}) \;=\; \mathbf{c} \,+\, \mathbf{d}\times\mathbf{x} \quad \forall\, \mathbf{x}\in\mathcal{O}\,; \quad \mathbf{c},\, \mathbf{d}\in\mathbb{R}^3 \big\}\,.$$

Lemma 2 *There exists $\beta > 0$ such that for any $(\mathbf{v},\boldsymbol{\eta}) \,\in\, \mathbf{Y}_D$ there holds*

$$\sup_{(\boldsymbol{\tau},\boldsymbol{\psi})\in\mathbf{X}_N^0\backslash\{\mathbf{0}\}} \frac{\mathbf{b}_N((\boldsymbol{\tau},\boldsymbol{\psi}),(\mathbf{v},\boldsymbol{\eta}))}{\|(\boldsymbol{\tau},\boldsymbol{\psi})\|_{\mathbf{X}_N^0}} \;\geq\; \beta\,\|(\mathbf{v},\boldsymbol{\eta})\|_{\mathbf{Y}_D}\,.$$

Proof. It suffices to show that the operator $\mathbf{B}_N$ is surjective. Given $(\mathbf{v},\boldsymbol{\eta}) \,\in\, \mathbf{Y}_D$, we let $\mathbf{z}$ be the unique element in $\mathbf{H}^1_\Gamma(\Omega^-) \;:=\; \Big\{ \mathbf{w} \,\in\, \mathbf{H}^1(\Omega^-) : \quad \mathbf{w} \;=\; \mathbf{0} \quad \text{on} \quad \Gamma \Big\}$, whose existence is guaranteed by the Korn inequality and the Lax-Milgram lemma, such that

$$\int_{\Omega^-} \mathbf{e}(\mathbf{z}) : \mathbf{e}(\mathbf{w}) \;=\; -\int_{\Omega^-} \mathbf{v}\cdot\mathbf{w} \;-\; \int_{\Omega^-} \boldsymbol{\eta} : \nabla\mathbf{w} \qquad \forall\,\mathbf{w} \,\in\, \mathbf{H}^1_\Gamma(\Omega^-)\,.$$

Hence, defining $\widehat{\boldsymbol{\tau}} \;:=\; \mathbf{e}(\mathbf{z}) + \boldsymbol{\eta} \,\in\, \mathbb{L}^2(\Omega^-)$, we deduce from the above formulation that $\mathbf{div}\,\widehat{\boldsymbol{\tau}} \;=\; \mathbf{v}$ in Ω^-, which shows that $\widehat{\boldsymbol{\tau}} \,\in\, \mathbb{H}(\mathbf{div}\,;\Omega^-)$, and then that $\gamma_{\boldsymbol{\nu}}^-(\widehat{\boldsymbol{\tau}}) \;=\; \mathbf{0}$ on Γ_0. In this way, $\widehat{\boldsymbol{\tau}} \,\in\, \mathbb{H}_0(\mathbf{div}\,;\Omega^-)$ and it is easy to see that $\mathbf{B}_N((\widehat{\boldsymbol{\tau}},\mathbf{0})) \;=\; (\mathbf{v},\boldsymbol{\eta})$, which ends the proof. $\square$

For the solvability analysis of of the coupled variational problem (14), we consider the associated homogeneous problem. We have the following lemma.

Lemma 3 *The set of solutions of the corresponding homogeneous coupled variational problem is given by*

$$\Big\{ \big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})\big) \;:=\; \big((\mathbf{0},\mathbf{z}|_\Gamma),(\mathbf{z},\nabla\mathbf{z})\big) : \;\; \mathbf{z} \,\in\, \mathbb{RM}(\Omega^-) \Big\}\,.$$

Proof. Let $\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})\big) \,\in\, \mathbf{X}_N^0 \times \mathbf{Y}_D$ be a solution of (14) with $\mathbf{f} \;=\; \mathbf{0}$. It is clear from the second equation that $(\boldsymbol{\sigma},\boldsymbol{\varphi}) \,\in\, \mathbf{V}_N$, that is $\boldsymbol{\sigma} \;=\; \boldsymbol{\sigma}^{\mathtt{t}}$ and $\mathbf{div}\,\boldsymbol{\sigma} \;=\; \mathbf{0}$ in Ω^-. Then, taking in particular $(\boldsymbol{\tau},\boldsymbol{\psi}) \;=\; (\boldsymbol{\sigma},\boldsymbol{\varphi})$ in the first equation, and then applying the Lemma 1, we find that in view of the $\mathbf{H}_0^{1/2}(\Gamma)$-elliptic property of $\mathbf{W}$, we find

$$0 \;=\; \mathbf{a}_N\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\boldsymbol{\sigma},\boldsymbol{\varphi})\big) \;\geq\; \frac{1}{2}\Big\{ \|\boldsymbol{\sigma}^{\mathtt{d}}\|^2_{0,\Omega^-} \;+\; \langle \mathbf{W}\,\boldsymbol{\varphi},\boldsymbol{\varphi}\rangle_\Gamma \Big\} \;\geq\; \frac{1}{2}\,\|\boldsymbol{\sigma}^{\mathtt{d}}\|^2_{0,\Omega^-} \;+\; \frac{\tilde{\alpha}_2}{2}\,\|\boldsymbol{\varphi}_0\|^2_{1/2,\Gamma}\,,$$

where $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \mathbf{r}$ with $\boldsymbol{\varphi}_0 \in \mathbf{H}_0^{1/2}(\Gamma)$ and $\mathbf{r} \,\in\, \mathbb{RM}(\Gamma)$. As a consequence, $\boldsymbol{\sigma}^{\mathtt{d}} \;=\; \mathbf{0}$ in Ω^- and $\boldsymbol{\varphi}_0 \;=\; 0$ on Γ, that is $\boldsymbol{\varphi} \;=\; \mathbf{z}|_\Gamma$, with $\mathbf{z} \,\in\, \mathbb{RM}(\Omega^-)$. In turn, the conditions satisfied by $\boldsymbol{\sigma}$, namely $\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{0}$ and $\boldsymbol{\sigma}^{\mathtt{d}} = \mathbf{0}$ in Ω^-, together with the fact that $\gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\sigma}) = \mathbf{0}$ on Γ_0 imply that $\boldsymbol{\sigma} \;=\; \mathbf{0}$. Next, taking $\boldsymbol{\psi} = \mathbf{0}$ in the first equation of our homogeneous problem, and then integrating by parts in Ω^-, we obtain that for any $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega^-)$ there holds

$$\begin{aligned} 0 \;&=\; \mathbf{b}_N((\boldsymbol{\tau},\mathbf{0}),(\mathbf{u},\boldsymbol{\chi})) \;-\; \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}),\boldsymbol{\varphi}\rangle_\Gamma \;=\; \mathbf{b}_N((\boldsymbol{\tau},\mathbf{0}),(\mathbf{u},\boldsymbol{\chi})) \;-\; \langle \gamma_{\boldsymbol{\nu}}^-(\boldsymbol{\tau}),\mathbf{z}\rangle_\Gamma \\ &=\; \mathbf{b}_N((\boldsymbol{\tau},\mathbf{0}),(\mathbf{u},\boldsymbol{\chi})) \;-\; \int_{\Omega^-} \mathbf{z}\cdot\mathbf{div}\,\boldsymbol{\tau} \;-\; \int_{\Omega^-} \nabla\mathbf{z} : \boldsymbol{\tau} \;=\; \mathbf{b}_N((\boldsymbol{\tau},\mathbf{0}),(\mathbf{u}-\mathbf{z},\boldsymbol{\chi}-\nabla\mathbf{z}))\,, \end{aligned}$$

which, in view of the inf-sup condition in Lemma 2, gives $(\mathbf{u},\boldsymbol{\chi}) = (\mathbf{z},\nabla\mathbf{z})$. Conversely, it is easy to see, in particular using that $\ker(\mathbf{W}) = \mathbb{RM}(\Gamma)$, that for any $\mathbf{z} \in \mathbb{RM}(\Omega^-)$ the element $\big((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})\big) := \big((\mathbf{0},\mathbf{z}|_\Gamma),(\mathbf{z},\nabla\mathbf{z})\big)$ solves the homogeneous version of (14). □

This lemma shows that the corresponding homogeneous coupled variational problem has nontrivial solutions. In order to apply the well-known Babuška-Brezzi theory (see, e.g. [2], [3]) for the existence proof of the solution to the coupled variational problem (14), we modify the space $\mathbf{X}_N^0$ and let

$$\mathbf{X}_N^{00} := \mathbb{H}_0(\mathbf{div}\,;\Omega^-) \times \mathbf{H}_0^{1/2}(\Gamma).$$

Instead of solving (14) in $\mathbf{X}_N^0 \times \mathbf{Y}_D$, we now solve (14) in $\mathbf{X}_N^{00} \times \mathbf{Y}_D$. We note that the kernel of the operator induced by $\mathbf{b}_N : \mathbf{X}_N^{00} \to \mathbf{Y}_D$ becomes now

$$\mathbf{V}_N^0 := \Big\{ (\boldsymbol{\tau},\boldsymbol{\psi}) \in \mathbf{X}_N^{00} : \ \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \Omega^- \Big\}.$$

For the existence proof, it is not difficult to verify that all the hypotheses of the classical Babuška-Brezzi theory hold. In fact, we have the following

THEOREM 1 *Given* $\mathbf{f} \in \mathbf{L}^2(\Omega^-)$*, there exists a unique* $((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi})) \in X_N^{00} \times \mathbf{Y}_D$ *solution of the coupled variational problem (14) with* $X_N^0 \times \mathbf{Y}_D$ *replaced by* $X_N^{00} \times \mathbf{Y}_D$*. Moreover, there exists a constant* $c > 0$ *such that*

$$\|((\boldsymbol{\sigma},\boldsymbol{\varphi}),(\mathbf{u},\boldsymbol{\chi}))\|_{\mathbf{X}_N^{00}\times\mathbf{Y}_D} \ \leq \ c\,\|\mathbf{f}\|_{0,\Omega^-}\,.$$

We end this paper by remarking that the application of other coupling procedures to the exterior Stokes problem in 3D, with Dirichlet and Neumann boundary conditions, and including the analysis of the corresponding Galerkin schemes, will be provided in [6].

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