A stabilized mixed method for generalized Stokes problem based on the velocity-pseudostress formulation: A priori error estimates and an optimal control problem*

Tomás P. Barrios[†], Rommel Bustinza[‡],

Galina C. García[§] and Erwin Hernández[¶]

Abstract

In this paper we present an augmented mixed formulation applied to generalized Stokes problem and uses it as state equation in an optimal control problem. The augmented scheme is obtained adding suitable least squares terms to the corresponding velocity-pseudostress formulation of the generalized Stokes problem. To ensure the existence and uniqueness of solution, at continuous and discrete levels, we prove coerciveness of the corresponding augmented bilinear form, and using approximation properties of the respective discrete subspaces, we deduce the optimal rate of convergence. As by product, and considering the associated optimal control problem, we derive error estimates for the approximated control unknown. Finally, we present several numerical examples confirming the theoretical properties of this approach.

Mathematics Subject Classifications (2010): 65N15, 65N30, 76D07, 49M25.

Key words: Mixed finite element, augmented formulation, Generalized Stokes problem, Optimal control problem.

*This research was partially supported by CONICYT-Chile through the FONDECYT Grants 1100490, 1080244, and 1080168, by Dirección de Investigación, Universidad Católica de la Santísima Concepción, by BASAL project CMM, Universidad de Chile, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, and by AM2V-Analysis and Mathematical Modelling Valparaíso, Univesidad Técnica Federico Santa María.

[†]Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile, email: tomas@ucsc.cl

[‡]CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: rbustinz@ing-mat.udec.cl

[§]Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile, email: galina.garcia@usach.cl

[¶]AM2V and Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile, e-mail: erwin.hernandez@usm.cl (corresponding author)

1 Introduction

It is very well known that the success of mixed finite element methods for the numerical solution of boundary value problems arising in continuum mechanics is mainly due to the possibility of the introduction of auxiliary unknowns. However, the fact that the discrete spaces used for approximation of different unknowns must satisfy the inf-sup condition leads to several difficulties of both theoretical and practical nature (see [3], [4], [10], [17], [20]). As a result, in the last time, the formulation of finite element methods that circumvent stability conditions, such as inf-sup condition, has become the subject of intensive research efforts.

The augmented mixed finite element method is a particular case of stabilization techniques, where usually least squares terms are added, locally or globally, to the dual mixed variational problems. Normally, these additional terms come from the equations that results when the second order equation is rewritten as first order system. The main advantage of these approaches is that it allows us to use any combination of the subspaces associated to different unknowns.As a consequence, this technique has been extended in different directions in the last years. In particular, an augmented mixed formulation applied to elliptic problems with mixed boundary conditions is presented and analyzed in [2], while in [19], [13], [12] and [11] the Darcy law, the elasticity problem, stationary Stokes as well as incompressible flows problems are studied.

We point out that the interest in studying the generalized Stokes problems is motivated by the fact that one has to deal with this kind of problem after applying a time discretization approach (e.g., Euler method) to the non steady Stokes problem. Additionally, it also plays a fundamental role in the numerical simulation of viscous incompressible flows (laminar and turbulent), since the most expensive part of the solution procedure for the time-dependent Navier-Stokes equations reduces to solve the generalized Stokes problem at each nonlinear iteration.

In order to describe the model of interest, we let Ω be a bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary Γ. Then, given the source term $\boldsymbol{f} \in [L^2(\Omega)]^2$ and $\boldsymbol{g} \in [H^{1/2}(\Gamma)]^2$, we look for the velocity $\boldsymbol{u}$ and the pressure p of the fluid occupying the region Ω, such that

$$\begin{aligned} \alpha\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad & \text{in} \quad \Omega, \\ \operatorname{div}(\boldsymbol{u}) = 0 \quad & \text{in} \quad \Omega, \\ \boldsymbol{u} = \boldsymbol{g} \quad & \text{on} \quad \Gamma, \end{aligned} \tag{1}$$

where ν is a positive constant called kinematic vistosity of the fluid, α is a positive parameter proportional to the inverse of the time-step and the datum $\boldsymbol{g}$ satisfies the compatibility condition $\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{\nu} = 0$, with $\boldsymbol{\nu}$ being the unit outward normal at Γ. In addition, and for uniqueness purposes, we assume that the pressure $p \in L^2_0(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q = 0\}$. We mention here that this problem has already been analyzed by different techniques, so the list of references on its approximation is quite large, and due to our current interest is the mixed formulation, we mention [6], where a dual-dual mixed approach is presented and analyzed. Up to the authors' knowledge, previous work for this problem using the velocity-pseudostress formulation, as developed in [8] (see also [14], [16]) for the stationary Stokes problem, are not available in the current literature.

On the other hand, we also consider the optimal control problem for the pressure

$$\text{minimize } \frac{1}{2}\|p - p_0\|^2_{L^2(\Omega)} + \frac{\eta}{2}\|\boldsymbol{z}_c\|^2_{[L^2(\Omega)]^2} \tag{2}$$

where

- the state of the system p is the solution of (1), called state equation, with external source term $\boldsymbol{f} + \boldsymbol{z}_c$,
- $\boldsymbol{z}_c$ is a finite control variable in the form $\boldsymbol{z}_c = \sum_{i=1}^m z_i \boldsymbol{f}_i$, for given functions $\boldsymbol{f}_i \in [L^2(\Omega)]^2$, $i = 1, \ldots, m$.

Note that p_0 is the desired pressure and $\eta > 0$ is a given regularization (or control cost) parameter. The optimal control problem (2) has been considered in the literature for the Stokes equations and Navier-Stokes equations, under different optimization approach, including state and control constraints, (see [5, 9, 21] and references therein), all of them use the adjoint state equation to solve the optimization problem. Here, we consider a classical scheme, proposed and analyzed in [18], which do not use properties for the adjoint equations.

Then the aims of this work are: to present and analyze the velocity-pseudostress formulation for (1) at least for moderately large value of parameter α; to study the corresponding stabilization scheme by applying augmented mixed method, obtaining optimal rate of convergence, and applied it to obtain error estimate for the control variable in the corresponding optimal control problem.

The rest of the paper is organized as follows. In Section 2, we present an analysis of velocity-pseudotress formulation. Its corresponding stabilization is developed in Section 3. In Section 4, the Galerkin schemes as well as the associated a priori error estimates are established. In Section 5, the optimal control problem concerning us is introduced and analyzed. An error estimate for the control variable is presented, too. Finally, several numerical examples are reported in Section 6.

We end this section with some notations to be used throughout the paper. Given any Hilbert space H, we denote by H^2 the space of vectors of order 2 with entries in H, and by $H^{2\times 2}$ the space of square tensors of order 2 with entries in H. In particular, given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we write, as usual, $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, $\operatorname{tr}(\boldsymbol{\tau}) := \tau_{11} + \tau_{22}$ and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^2 \tau_{ij}\,\zeta_{ij}$. We also use the standard notations for Sobolev spaces and norms. We denote by $[H_0^1(\Omega)]^2 := \{\, \boldsymbol{v} \in [H^1(\Omega)]^2 : \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma \,\}$, by $H(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2 \,\}$, and by $H_0 := \{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0\}$. Note that $H(\mathbf{div};\Omega) = H_0 \oplus \mathbb{R}\,\boldsymbol{I}$, that is for any $\boldsymbol{\tau} \in H$ there exist unique $\boldsymbol{\tau}_0 \in H_0$ and $d := \frac{1}{2|\Omega|}\int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \in \mathbb{R}$ such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix in $\mathbb{R}^{2\times 2}$. Finally, we use C or c, with or without subscripts, to denote generic constants, independent of the discretization parameters, which may take different values at different occurrences.

2 The dual-mixed formulation

We begin this section remarking that the dual-mixed formulation of (1) can be deduced from the ideas developed in [1], where the authors propose a displacement-pressure formulation for an anisotropic elasticity problem. In addition, studying least-squares methods for the numerical solution of linear, stationary and incompressible Newtonian fluid flow, in [7] a new variable was introduced instead of the stress tensor, the called pseudostress tensor. In the framework of mixed FEM, this approach also have been used recently in [8], [14], [16] and [15], where the stationary Stokes equations and the quasi-Newtonian fluid flow are studied.

With the aim to derive the dual-mixed formulation of (1), we first introduce the pseudostress $\boldsymbol{\sigma} := \nu\nabla\boldsymbol{u} - p\boldsymbol{I}$ in Ω. Using this new unknown, the first equation in (1) becomes

$$\alpha\boldsymbol{u} - \mathbf{div}(\boldsymbol{\sigma}) = \boldsymbol{f} \quad \text{in} \quad \Omega\,. \tag{3}$$

In addition, noting that $\mathrm{tr}(\boldsymbol{\sigma}) = \mathrm{tr}(\nu\nabla\boldsymbol{u} - p\boldsymbol{I}) = -2p$, then by the uniqueness condition $\int_\Omega p = 0$, we deduce that $\boldsymbol{\sigma} \in H_0$. Moreover, the deviator of tensor $\boldsymbol{\sigma}$ is denoted by $\boldsymbol{\sigma}^d := \boldsymbol{\sigma} - \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}$, which clearly belongs to H_0, and then the relation $\boldsymbol{\sigma} := \nu\nabla\boldsymbol{u} - p\boldsymbol{I}$ is rewriting as

$$\boldsymbol{\sigma}^d = \nu\nabla\boldsymbol{u} \quad \text{in} \quad \Omega. \tag{4}$$

Now, we are ready to introduce a new mixed formulation. Multiplying (3) and (4) by suitable test functions, and after integrating in Ω, we obtain the following dual-mixed formulation of (1): *Find* $(\boldsymbol{\sigma}, \boldsymbol{u}) \in H_0 \times [L^2(\Omega)]^2$ *such that*

$$\int_\Omega \boldsymbol{\sigma}^d : \boldsymbol{\tau}^d + \nu \int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \;=\; \nu\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{\tau\nu} \qquad \forall\boldsymbol{\tau} \in H_0\,, \tag{5}$$

$$\nu\int_\Omega \boldsymbol{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) - \alpha\nu\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} \;=\; -\nu\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad \forall\boldsymbol{v} \in [L^2(\Omega)]^2\,. \tag{6}$$

In order to prove the unique solvability of the variational formulation (5)-(6), we write it now as a system of operator equations with a saddle point structure. To this end, we first define the spaces $X := H_0$, $M := [L^2(\Omega)]^2$. Then, we introduce the operators and functionals $\boldsymbol{A} : X \to X'$, $\boldsymbol{B} : X \to M'$, $\boldsymbol{S} : M \to M'$, $G \in X'$ and $F \in M'$, as suggested by the structure of (5)-(6), so that this problem can be stated as: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in X \times M$ such that

$$\begin{aligned} [\boldsymbol{A}(\boldsymbol{\sigma}), \boldsymbol{\tau}] + [\boldsymbol{B}(\boldsymbol{\tau}), \boldsymbol{u}] &= \; [G, \boldsymbol{\tau}] \quad \forall\boldsymbol{\tau} \in X \\ [\boldsymbol{B}(\boldsymbol{\sigma}), \boldsymbol{v}] - [\boldsymbol{S}(\boldsymbol{u}), \boldsymbol{v}] &= \; [F, \boldsymbol{v}] \quad \forall\boldsymbol{v} \in M, \end{aligned} \tag{7}$$

where $[\cdot,\cdot]$ denotes the duality pairing induced by operators and functionals used in each case. The next lemma will be used to prove the well-posedness of (7), and its proof can be seen in Proposition 3.1 of Chapter IV in [4].

Lemma 2.1 *There exists* $c_1 \in (0, 1]$*, depending only on* Ω*, such that*

$$c_1\,\|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^2} \le \|\boldsymbol{\tau}^d\|^2_{[L^2(\Omega)]^2} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} \qquad \forall\boldsymbol{\tau} \in H_0. \tag{8}$$

Existence and uniqueness are establishes in the next theorem.

Theorem 2.1 *Problem* (7) *has a unique solution* $(\boldsymbol{\sigma}, \boldsymbol{u}) \in X \times M$. *Moreover, there exists a positive constant* $C(\nu, \alpha)$, *independent of the solution, such that*

$$||(\boldsymbol{\sigma}, \boldsymbol{u})||_{X\times M} \leq C(\nu, \alpha)(\|F\| + \|G\|). \tag{9}$$

Proof. Notice that the operators $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{S}$, as well as the functionals F and G, are all linear and bounded. In particular, it is easy to see that $||\boldsymbol{B}|| = \mathcal{O}(\nu)$, $||\boldsymbol{S}|| = \mathcal{O}(\alpha, \nu)$ and $||\boldsymbol{A}|| = \mathcal{O}(c_1)$.

Let us define

$$N(\boldsymbol{B}) := \mathrm{kernel}(\boldsymbol{B}) := \{\boldsymbol{\tau} \in X \;:\; \mathbf{div}(\boldsymbol{\tau}) = 0 \,\mathrm{in}\, \Omega\}. \tag{10}$$

Then, using Lemma 2.1 we deduce that $\boldsymbol{A}$ is such that

$$[\boldsymbol{A}(\boldsymbol{\tau}), \boldsymbol{\tau}] = \|\boldsymbol{\tau}^d\|^2_{[L^2(\Omega)]^{2\times 2}} \geq c_1 \|\boldsymbol{\tau}\|^2_X \qquad \forall \boldsymbol{\tau} \in N(\boldsymbol{B}). \tag{11}$$

In addition, since $\alpha \geq 0$ and $\nu \geq 0$, the linear operator $\boldsymbol{S}$ is positive semi-definite on M, that is

$$[\boldsymbol{S}(\boldsymbol{v}), \boldsymbol{v}] = \alpha\nu||\boldsymbol{v}||^2_M \geq 0 \qquad \forall \boldsymbol{v} \in M. \tag{12}$$

It only remains to show that $\boldsymbol{B}$ satisfy the corresponding inf-sup condition on $X \times M$. Indeed, given $\boldsymbol{v} \in M$ we get lower bounds for

$$\sup_{\boldsymbol{\tau}\in X\backslash\{0\}} \frac{[\boldsymbol{B}(\boldsymbol{\tau}), \boldsymbol{v}]}{||\boldsymbol{\tau}||_X} \geq \frac{\nu \displaystyle\int_\Omega \boldsymbol{v} \cdot \mathbf{div}(\tilde{\boldsymbol{\tau}})}{||\tilde{\boldsymbol{\tau}}||_X}$$

by introducing $\boldsymbol{z}$ weak solution of $-\Delta \boldsymbol{z} = \boldsymbol{v}$ in Ω with $\boldsymbol{z} = 0$ on Γ, and then setting $\tilde{\boldsymbol{\tau}} = -\nabla \boldsymbol{z}$, we conclude the inf-sup condition for $\boldsymbol{B}$. Finally, the result is consequence of Theorem II.1.2 in [4].

□

However, we remark that at first glance, this mixed formulation at the discrete level needs to satisfy an inf-sup condition, which implies that it not possible to use any pair of the subspaces in practice, we can only use a stable pair of discrete susbspaces for the standard Stokes problem for each row of the tensor. To circumvent this difficult we propose, in the next section, to use a stabilization method.

3 The stabilized mixed formulations

In this section we analyse two different approaches of the augmented mixed method for (5)-(6).

3.1 Velocity-pseudostress formulation

For the first approach, we proceed as in [2] and include the least-squares terms given by

$$\kappa_1 \int_\Omega (\nu\nabla\boldsymbol{u} - \boldsymbol{\sigma}^d) : (\nu\nabla\boldsymbol{v} + \boldsymbol{\tau}^d) \;=\; 0 \qquad \forall(\boldsymbol{\tau}, \boldsymbol{v}) \in H_0 \times [H^1(\Omega)]^2\,, \tag{13}$$

$$\kappa_2 \int_\Omega (\mathbf{div}\boldsymbol{\sigma} - \alpha\boldsymbol{u}) \cdot (\mathbf{div}\boldsymbol{\tau} + \alpha\boldsymbol{v}) \;=\; -\kappa_2 \int_\Omega \boldsymbol{f} \cdot (\mathbf{div}\boldsymbol{\tau} + \alpha\boldsymbol{v}) \qquad \forall(\boldsymbol{\tau}, v) \in H_0 \times [H^1(\Omega)]^2. \tag{14}$$

In this way, substracting the equations (5) and (6), and after adding (13) and (14), we derive the following augmented mixed scheme: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}_0 := H_0 \times [H^1(\Omega)]^2$ such that

$$a((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) = F_1(\boldsymbol{\tau}, \boldsymbol{v}) \qquad \forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{H}_0, \tag{15}$$

where the bilinear form $a : \boldsymbol{H}_0 \times \boldsymbol{H}_0 \to \mathbb{R}$, and the linear functional $F_1 : \boldsymbol{H}_0 \to \mathbb{R}$ are given by

$$\begin{aligned} a((\boldsymbol{\zeta}, \boldsymbol{w}), (\boldsymbol{\tau}, \boldsymbol{v})) &:= \int_\Omega \boldsymbol{\zeta}^d : \boldsymbol{\tau}^d + \nu \int_\Omega \boldsymbol{w} \cdot \mathbf{div}(\boldsymbol{\tau}) - \nu \int_\Omega \boldsymbol{v} \cdot \mathbf{div}(\boldsymbol{\zeta}) + \alpha\nu \int_\Omega \boldsymbol{w} \cdot \boldsymbol{v} \\ &+ \kappa_1 \int_\Omega (\nu \nabla \boldsymbol{w} - \boldsymbol{\zeta}^d) : (\nu \nabla \boldsymbol{v} + \boldsymbol{\tau}^d) + \kappa_2 \int_\Omega (\mathbf{div}\boldsymbol{\zeta} - \alpha \boldsymbol{w}) \cdot (\mathbf{div}\boldsymbol{\tau} + \alpha \boldsymbol{v}) \end{aligned} \tag{16}$$

and

$$F_1(\boldsymbol{\tau}, \boldsymbol{v}) := \nu \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \nu \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{\tau\nu} - \kappa_2 \int_\Omega \boldsymbol{f} \cdot (\alpha \boldsymbol{v} + \mathbf{div}\boldsymbol{\tau}), \tag{17}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\zeta}, \boldsymbol{w}) \in \boldsymbol{H}_0$, and κ_1 and κ_2 are real paremeters. At the begining, the idea is to choose these parameters such that we can satisfy the hypotheses of Lax-Milgram's Lemma. In addition, we remark that the mixed scheme developed in [6] is well posed for $\alpha \geq \nu$. This condition can be relaxed using the augmented mixed approach by choosing appropriately κ_1 and κ_2, as we present in the next lemma.

Lemma 3.1 *We assume $\kappa_1 = 1 - \kappa_2/2$.*

1. *If $\alpha \leq \nu$ and $\kappa_2 \in (0, 1)$ then there exists a positive constant $C(\alpha, \kappa_2) = \mathcal{O}(\alpha^2)$ such that*

$$a((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{v})) \geq C(\alpha, \kappa_2) ||(\boldsymbol{\tau}, \boldsymbol{v})||^2_{\boldsymbol{H}_0}.$$

   In particular for $\kappa_2 = 1/2$, we obtain $C(\alpha) = \min \left\{ \frac{c_1}{4}, \frac{\alpha^2}{2} \right\}$.

2. *If $\alpha > \nu$ and $\kappa_2 \in \left(0, \frac{\nu}{\alpha}\right)$ then there exists $C(\alpha, \nu, \kappa_2) > 0$ such that*

$$a((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{v})) \geq C(\alpha, \nu, \kappa_2) ||(\boldsymbol{\tau}, \boldsymbol{v})||^2_{\boldsymbol{H}_0}.$$

   In particular for $\kappa_2 = \frac{\nu}{2\alpha}$, we have $C(\alpha, \nu) = \min \left\{ \frac{c_1 \nu}{4\alpha}, \frac{\nu^2}{2} \right\}$.

where $c_1 \in (0, 1]$ is the constant of Lemma 2.1.

Proof. Let us first consider $\alpha \leq \nu$. Using (16), we obtain

$$a((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{v})) = (1 - \kappa_1) \int_\Omega |\boldsymbol{\tau}^d|^2 + (\alpha\nu - \kappa_2 \alpha^2) \int_\Omega |\boldsymbol{v}|^2 + \kappa_1 \nu^2 \int_\Omega |\nabla \boldsymbol{v}|^2 + \kappa_2 \int_\Omega |\mathbf{div}\boldsymbol{\tau}|^2,$$

which, after algebraic manipulation and considering $\kappa_1 = 1 - \kappa_2/2$, yields to

$$
\begin{aligned}
a((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\tau},\boldsymbol{v})) \;\; &\geq \;\; \frac{\kappa_2}{2}\left(\int_\Omega |\boldsymbol{\tau}^d|^2 + \int_\Omega |\mathbf{div}\boldsymbol{\tau}|^2\right) + \frac{\kappa_2}{2}||\mathbf{div}\boldsymbol{\tau}||^2_{[L^2(\Omega)]^2} \\
&\quad + (1-\kappa_2)\alpha^2 \int_\Omega |\boldsymbol{v}|^2 + \left(1 - \frac{\kappa_2}{2}\right)\nu^2 \int_\Omega |\nabla \boldsymbol{v}|^2.
\end{aligned}
$$

Now, thanks to Lemma 2.1 and $\alpha \leq \nu$, it follows that

$$
a((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\tau},\boldsymbol{v})) \;\; \geq \;\; \frac{c_1\kappa_2}{2}||\boldsymbol{\tau}||^2_H + C_*\alpha^2||\boldsymbol{v}||^2_{[H^1(\Omega)]^2}
$$

where $C_* = \min\left\{(1-\kappa_2),(1-\dfrac{\kappa_2}{2})\right\} = 1-\kappa_2$. Then, chosing $C(\alpha,\kappa_2) = \min\left\{\dfrac{c_1\kappa_2}{2}, C_*\alpha^2\right\}$, we prove the first case.

For the second one, $\alpha > \nu$, the proof follows the same ideas than in the previous case. We omit further details. □

The well posedness is established in the next theorem.

Theorem 3.1 *Under the same assumption of Lemma* 3.1*, the problem* (15) *has unique solution* $(\boldsymbol{\sigma},\boldsymbol{u}) \in \boldsymbol{H}_0$*, and there holds* $p = -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})$ *in* Ω*. Moreover, there exists a positive constant* $C = C(\alpha,\nu)$ *such that*

$$
||(\boldsymbol{\sigma},\boldsymbol{u})||_{\boldsymbol{H}} \leq C(\alpha,\nu)\bigg(\|\boldsymbol{f}\|_{[L^2(\Omega)]^2} + \|\boldsymbol{g}\|_{[H^{1/2}(\Gamma)]^2}\bigg). \tag{18}
$$

Proof. It is not difficult to see that the bilinear form $a(\cdot,\cdot)$ is continuous, that is there exists $M > 0$ such that

$$
|a((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v}))| \leq M||(\boldsymbol{\sigma},\boldsymbol{u})||_{\boldsymbol{H}_0}||(\boldsymbol{\tau},\boldsymbol{v})||_{\boldsymbol{H}_0}.
$$

Notice that $M = 1 + 2\nu + \alpha\nu + \kappa_1(1+\nu)^2 + \kappa_2(1+\alpha)^2$. Furthemore, from Lemma 3.1 the bilinear form is coercive in $\boldsymbol{H}_0$, then by Lax-Milgram's Lemma, the solution $(\boldsymbol{\sigma},\boldsymbol{u}) \in \boldsymbol{H}_0$ is unique and satisfy the stability condition (18). □

3.2 Velocity-pressure-pseudostress formulation

In this subsection, for the sake of completeness, we describe another stabilized formulation which allows us to approximate the velocity, the pressure and the pseudostress, simultaneously. To this end, we first introduce the next least-square type term. Given $\kappa_3 > 0$

$$
\kappa_3 \int_\Omega \left(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\right)\left(q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\right) = 0 \qquad \forall(\boldsymbol{\tau},q) \in \boldsymbol{H}_0 \times L^2(\Omega). \tag{19}
$$

Then, by adding (19) to (15), we deduce the new augmented mixed scheme: *Find* $(\boldsymbol{\sigma},\boldsymbol{u},p) \in \tilde{\boldsymbol{H}}_0 := \boldsymbol{H}_0 \times L^2(\Omega)$ *such that*

$$
b((\boldsymbol{\sigma},\boldsymbol{u},p),(\boldsymbol{\tau},\boldsymbol{v},q)) = F_2(\boldsymbol{\tau},\boldsymbol{v},q) \qquad \forall(\boldsymbol{\tau},\boldsymbol{v},q) \in \tilde{\boldsymbol{H}}_0, \tag{20}
$$

where the bilinear form $b : \tilde{\boldsymbol{H}}_0 \times \tilde{\boldsymbol{H}}_0 \to \mathbb{R}$, and the functional $F_2 : \tilde{\boldsymbol{H}}_0 \to \mathbb{R}$ are given by

$$b((\boldsymbol{\zeta}, \boldsymbol{w}, r), (\boldsymbol{\tau}, \boldsymbol{v}, q)) = a((\boldsymbol{\zeta}, \boldsymbol{w}), (\boldsymbol{\tau}, \boldsymbol{v})) + \kappa_3 \int_\Omega \left(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\right)\left(q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\right), \tag{21}$$

and

$$F_2(\boldsymbol{\tau}, \boldsymbol{v}, q) := F_1(\boldsymbol{\tau}, \boldsymbol{v}), \tag{22}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}, q), (\boldsymbol{\zeta}, \boldsymbol{w}, r) \in \tilde{\boldsymbol{H}}_0$. Here $a(\cdot, \cdot)$ and $F_1(\cdot)$ are defined in (16) and (17), respectively.

Lemma 3.2 *We assume $\kappa_1 = 1 - \kappa_2/2$ and $0 < \kappa_3 < \kappa_2 c_1$.*

1. *If $\alpha \leq \nu$ and $\kappa_2 \in (0, 1)$, then there exists a positive constant $C(\alpha, \kappa_2, \kappa_3) = \mathcal{O}(\alpha^2)$ such that*

$$b((\boldsymbol{\tau}, \boldsymbol{v}, q), (\boldsymbol{\tau}, \boldsymbol{v}, q)) \geq C(\alpha, \kappa_2, \kappa_3)||(\boldsymbol{\tau}, \boldsymbol{v}, q)||^2_{\tilde{\boldsymbol{H}}_0}.$$

In particular for $\kappa_2 = 1/2$ and $\kappa_3 = c_1/4$, we obtain $C(\alpha) = \min\left\{\frac{c_1}{8}, \frac{\alpha^2}{2}\right\}$.

2. *If $\alpha > \nu$ and $\kappa_2 \in \left(0, \frac{\nu}{\alpha}\right)$, then there exists $C(\alpha, \nu, \kappa_2, \kappa_3) > 0$ such that*

$$b((\boldsymbol{\tau}, \boldsymbol{v}, q), (\boldsymbol{\tau}, \boldsymbol{v}, q)) \geq C(\alpha, \nu, \kappa_2, \kappa_3)||(\boldsymbol{\tau}, \boldsymbol{v}, q)||^2_{\tilde{\boldsymbol{H}}_0}.$$

In particular for $\kappa_2 = \frac{\nu}{2\alpha}$ and $\kappa_3 = \frac{\kappa_2 c_1}{2}$, we have $C(\alpha, \nu) = \min\left\{\frac{\nu c_1}{8\alpha}, \frac{\nu^2}{2}\right\}$.

Proof. Noting that $\|q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\|^2_{L^2(\Omega)} \geq \frac{1}{2}\|q\|^2_{L^2(\Omega)} - \|\frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\|^2_{L^2(\Omega)} \geq \frac{1}{2}\|q\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^{2\times 2}}$ for all $(q, \boldsymbol{\tau}) \in L^2(\Omega) \times [L^2(\Omega)]^{2\times 2}$. The proof of this lemma follows the same arguments of Lemma 2.1. We omit further details. □

Existence and uniqueness of the solution is presented in the next theorem.

Theorem 3.2 *Under the same assumptions of Lemma 3.2, problem (20) has unique solution $(\boldsymbol{\sigma}, \boldsymbol{u}, p) \in \tilde{\boldsymbol{H}}$. Moreover, there exists a positive constant $C = C(\alpha, \nu)$ such that*

$$||(\boldsymbol{\sigma}, \boldsymbol{u}, p)||_{\tilde{\boldsymbol{H}}_0} \leq C(\alpha, \nu)\|\boldsymbol{f}||_{[L^2(\Omega)]^2}. \tag{23}$$

Proof. It is consequence of Lax-Milgram's Lemma. □

In what follows, we develop the necessary tools to study the well-posedness and convergence of the corresponding discrete approximations of these formulations. However, since in the first formulation the approximation of the pressure can be computed by mean of the relation $p = -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})$ in Ω, in our opinion, this is the most favourable approach. Therefore, throughout the rest of the paper, we will just concentrate on the problem (15), the corresponding extension to the problem (20) should be derived straightforwardly.

4 The Galerkin schemes

In this section, we describe the Galerkin scheme associated to the continuous problem (15). Hereafter, we assume that the parameters κ_1 and κ_2 satisfy the assumptions of Lemma 3.1. In addition, we suppose that Ω is a polygonal region and h is a positive parameter. We consider finite element subspaces $H_{0,h}^{\boldsymbol{\sigma}} \subset H_0$ and $H_h^{\boldsymbol{u}} \subset [H^1(\Omega)]^2$. Then, a Galerkin scheme associated to the variational problem (15) reads: Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{H}_{0,h} := H_{0,h}^{\boldsymbol{\sigma}} \times H_h^{\boldsymbol{u}}$ such that

$$a((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) \;=\; F_1(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \qquad \forall\, (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \boldsymbol{H}_{0,h}\,. \tag{24}$$

The well-posedness of this discrete problem follows from the Lax-Milgram's Lemma for any subspace $\boldsymbol{H}_{0,h} \subseteq \boldsymbol{H}_0$.

To describe a particular case of finite element subspaces, let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$. We assume that $\bar{\Omega} \;=\; \cup\{\, T : \quad T \in \mathcal{T}_h \,\}$. Given a triangle $T \in \mathcal{T}_h$, we denote by h_T its diameter and define the mesh size $h := \max\{\, h_T : \quad T \in \mathcal{T}_h \,\}$. In addition, given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathbb{P}_\ell(S)$ the space of polynomials in two variables defined in S of total degree at most ℓ, and for each $T \in \mathcal{T}_h$, we define the local Raviart-Thomas space of order zero

$$\mathbb{RT}_0(T) \;:=\; \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} \;\subseteq\; [\mathbb{P}_1(T)]^2\,,$$

where $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a generic vector of $\mathbb{R}^2$. Then defining,

$$H_h^{\boldsymbol{\sigma}} := \left\{\, \boldsymbol{\tau}_h \in H : \quad \boldsymbol{\tau}_h|_T \;\in\; [\mathbb{RT}_0(T)^{\mathtt{t}}]^2 \qquad \forall\, T \;\in\; \mathcal{T}_h \,\right\}\,,$$

$$H_{0,h}^{\boldsymbol{\sigma}} \;:=\; \left\{ \boldsymbol{\tau}_h \;\in\; H_h^{\boldsymbol{\sigma}} : \quad \int_\Omega \text{tr}(\boldsymbol{\tau}_h) \;=\; 0 \right\}\,,$$

$$X_h \;:=\; \left\{\, v_h \;\in\; C(\bar{\Omega}) : \quad v_h|_T \;\in\; \mathbb{P}_1(T) \quad \forall\, T \;\in\; \mathcal{T}_h \,\right\}\,,$$

and

$$H_h^{\boldsymbol{u}} \;:=\; X_h \times X_h\,,$$

we take

$$\boldsymbol{H}_{0,h} \;:=\; H_{0,h}^{\boldsymbol{\sigma}} \times H_h^{\boldsymbol{u}}. \tag{25}$$

Next, we give the rate of convergence of the Galerkin schemes (24) when the finite element subspaces (25) is used.

Theorem 4.1 *Let $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{H}_0$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{H}_{0,h}$ be the unique solutions of the continuous and discrete augmented mixed formulations* (15) *and* (24)*, respectively. Assume that $\boldsymbol{\sigma} \in [H^r(\Omega)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^r(\Omega)]^2$, and $\boldsymbol{u} \in [H^{r+1}(\Omega)]^2$ for some $r \in (0, 1]$. Then there exists $C_1 > 0$, independent of h, such that*

$$\|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_{\boldsymbol{H}_0} \leq \; C_1\, h^r \;\left\{\, \|\boldsymbol{\sigma}\|_{[H^r(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{[H^r(\Omega)]^2} + \|\boldsymbol{u}\|_{[H^{r+1}(\Omega)]^2} \right\}\,.$$

Proof. The proof follow the classical Cea's estimate and the corresponding approximation properties of the subspaces. We note that the behaviour of the constant with respect to the parameter α is similar to the one describe by the constant C in Theorem 3.1. $\square$

Remark. We note that the behaviour of the constant in Theorem 4.1 is given by $C_1 = \mathcal{O}(\alpha^2)$ if $\alpha \gg \nu$ with $\kappa_2 = \frac{\nu}{2\alpha}$, and by $C_1 = \mathcal{O}(\frac{1}{\alpha^2})$ if $\alpha \ll \nu$ with $\kappa_2 = \frac{1}{2}$. This constitutes a light improvement with respect to the results given in [6] where the behaviour of the constant only is described when $\alpha \geq \nu$ and its order is like $\mathcal{O}(\alpha^3/\nu)$ $\square$

5 Optimal control problem

In this section, we describe the optimization scheme used to solve the optimal control problem (2). Since the solution of the state equation (1) has been discretized in the previous section, by considering an augmented mixed Galerkin scheme associated to the velocity-pseudostress formulation (15), the state of the system should be given by $p_h = -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)$. Then, the solution of our optimal control problem depends on the discrete parameter h. In fact, given the desired pressure p_0, the optimal control problem reads:

$$\text{minimize } \frac{1}{2}\|p_h - p_0\|^2_{L^2(\Omega)} + \frac{\eta}{2}\|\boldsymbol{z}_c^h\|^2_{L^2(\Omega)} \tag{26}$$

where

- the state of the system p_h is obtained from the solution of (24), called state equation, with external source term $\boldsymbol{f} + \boldsymbol{z}_c$,
- $\boldsymbol{z}_c$ is a finite control variable in the form $\boldsymbol{z}_c = \sum_{i=1}^m z_i \boldsymbol{f}_i$, for given functions $\boldsymbol{f}_i \in [L^2(\Omega)]^2$, $i = 1, \ldots, m$.

On the other hand, from the linearity of (24), we can write the solution of the state equations as

$$(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) = (\boldsymbol{\sigma}_{0h}, \boldsymbol{u}_{0h}) + \sum_{i=1}^m z_i^h (\boldsymbol{\sigma}_{ih}, \boldsymbol{u}_{ih}),$$

where $(\boldsymbol{\sigma}_{0h}, \boldsymbol{u}_{0h})$ is the solution of problem (24) without control, considering only the external force $\boldsymbol{f}$ and $(\boldsymbol{\sigma}_{ih}, \boldsymbol{u}_{ih})$ is the solution (24), considering as external force $\boldsymbol{f}_i \in [L^2(\Omega)]^2$.

The global state of the system p_h is written in terms of the control variable $\boldsymbol{Z}_h \in \mathbb{R}^m$, $\boldsymbol{Z}_h = (z_1^h, z_2^h, \ldots, z_m^h)$ and the pressures $p_{ih} = -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_{ih})$ in the following way:

$$p_h := p_{0h} + \sum_{i=1}^m z_i^h p_{ih}.$$

In order to compute the numerical approximation of the optimal control, i.e., $\boldsymbol{Z}_h^{op}$, we follow the classical scheme used in [18], where the necessary and sufficient condition is given by the Euler equality, which in our case reads

$$(\boldsymbol{P}_h + \eta \boldsymbol{I})\boldsymbol{Z}_h^{op} = -\boldsymbol{b}, \tag{27}$$

with $\boldsymbol{I}$ denoting the identity matrix of order m and $\boldsymbol{P}_h \in \mathbb{R}^{m\times m}$ and $\boldsymbol{b} \in \mathbb{R}^m$ are defined by

$$
\begin{aligned}
(\boldsymbol{P}_h)_{ij} &:= \Big(\int_\Omega p_{ih} p_{jh}\Big), \qquad i,j = 1,\dots,m,\\
(\boldsymbol{b}_h)_i &:= \Big(\int_\Omega p_{ih} p_{0h}\Big), \qquad i = 1,\dots,m\,.
\end{aligned}
$$

Notice that the optimal control $\boldsymbol{Z}_h^{op}$ is expected to be a good approximation of the optimal control $\boldsymbol{Z}^{op}$. This will be addressed in the numerical examples.

Theorem 5.1 *Let $\boldsymbol{Z}^{op}$ and $\boldsymbol{Z}_h^{op}$ the optimal solutions of problems* (2) *and* (26)*, respectively. Then there exists $C > 0$, independent of h, such that*

$$
\|\boldsymbol{Z}_h^{op} - \boldsymbol{Z}^{op}\| \le C h^r\,,
$$

with $\|\cdot\|$ denoting the Euclidean norm and $r \in (0,1]$ is the additional regularity of the exact solution.

Proof. We point out that the numerical scheme (26), proposed to solve (2), belongs to the optimal control abstract setting described in [18] . Then, according with Theorem 3.9 in [18] and using the result of Theorem 4.1, we have

$$
\begin{aligned}
\|\boldsymbol{Z}_h^{op} - \boldsymbol{Z}^{op}\| &\le C\|p - p_h\|_{L^2(\Omega)}\\
&\le \frac{C}{2}\|-\mathrm{tr}(\boldsymbol{\sigma}) + \mathrm{tr}(\boldsymbol{\sigma}_h)\|_{L^2(\Omega)}\\
&\le \frac{C}{2}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(\Omega)]^{2\times 2}}\\
&\le C h^r\,,
\end{aligned}
$$

which completes the proof. □

6 Numerical results

We begin this section by remarking that, for implementation purposes, it is very hard to find a suitable basis of $H_{0,h}^{\boldsymbol{\sigma}}$ due to the null media condition required by its elements. We circumvent this search by imposing this requirement through a Lagrange multiplier, which yields to the following auxiliar discrete scheme: Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \varphi_h) \in \boldsymbol{H}_h := H_h^{\boldsymbol{\sigma}} \times H_h^{\boldsymbol{u}} \times \mathbb{R}$ such that

$$
\begin{aligned}
a((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) + \varphi_h \int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) &= F_1(\boldsymbol{\tau}_h, \boldsymbol{v}_h)\,,\\
\psi \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h) &= 0\,,
\end{aligned}
\tag{28}
$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}, \psi) \in \boldsymbol{H}_h$. The next theorem establishes the equivalence between the variational problems (24) and (28).

Theorem 6.1

i) Let $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{H}_{0,h}$ *be the solution of* (24). *Then* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, 0)$ *is a solution of* (28).

ii) Let $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \varphi_h) \in \boldsymbol{H}_h$ *be a solution of* (28). *Then* $\varphi_h = 0$ *and* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ *is the solution of* (24).

Proof. We adapt the proof of Theorem 4.3 in [13]. We first observe, according to the definition of $a(\cdot,\cdot)$, that for each $(\boldsymbol{\tau}, \boldsymbol{v}) \in H(\mathbf{div};\, \Omega) \times [H^1(\Omega)]^2$ there holds

$$a((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{I}, 0)) = 0 \qquad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in H(\mathbf{div};\, \Omega) \times [H^1(\Omega)]^2 \,. \tag{29}$$

Now, let $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{H}_{0,h}$ be the solution of (24), and let $(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \boldsymbol{H}_h^{\boldsymbol{\sigma}} \times H_h^{\boldsymbol{u}}$. We write $\boldsymbol{\tau} = \boldsymbol{\tau}_{0,h} + d_h \boldsymbol{I}$, with $\boldsymbol{\tau}_{0,h} \in \boldsymbol{H}_{0,h}^{\boldsymbol{\sigma}}$ and $d_h \in \mathbb{R}$, and observe that $(\boldsymbol{\tau}_{0,h}, \boldsymbol{v}_h) \in \boldsymbol{H}_{0,h}$, whence the definition of F_1, (24) and (29) yield

$$F_1(\boldsymbol{\tau}_h, \boldsymbol{v}_h) = F_1(\boldsymbol{\tau}_{0,h}, \boldsymbol{v}_h) = a((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_{0,h}, \boldsymbol{v}_h)) = a((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h))\,.$$

This identity and the fact that $\boldsymbol{\sigma}_h$ clearly satisfies the second equation of (28), show that $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, 0)$ is indeed a solution of (28).

Conversely, let $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \varphi_h) \in \boldsymbol{H}_h$ be a solution of (28). Then taking $(\boldsymbol{\tau}_h, \boldsymbol{v}_h) = (\boldsymbol{I}, \boldsymbol{0})$ in the first equation of (28) and using the definition of F_1 and (29), we find that $\varphi_h = 0$, whence $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ becomes the solution of (24). □

We now specify the data of the two examples to be presented here. We take Ω as either the square $]-1,1[^2$ (for Example 1) or the circular section $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 :\, x_1^2 + x_2^2 < 1\} \setminus [0,1] \times [-1,0]$ (for Example 2). In both examples, the data $\boldsymbol{f}$ and $\boldsymbol{g}$ are chosen so that the exact solutions $\boldsymbol{u}$ and p are the ones shown in Table 4.1, where $s = \sqrt{(x_1-2)^2 + (x_2-2)^2}$. We remind that in all cases, $\mathrm{div}(\boldsymbol{u}) = 0$ in Ω and $\boldsymbol{\sigma} = \nu \nabla \boldsymbol{u} - p\boldsymbol{I}$ in Ω. We emphasize that the solution $(\boldsymbol{u}, p)$ of Example 1 is smooth, while the exact pressure p in the solution of Example 2 lives in $H^{1+2/3}(\Omega)$, since their derivatives are singular at $(0,0)$. This implies that $\mathbf{div}(\boldsymbol{\sigma}) \in [H^{2/3}(\Omega)]^2$ only, which, according to Theorem 4.1, yields $2/3$ as the expected rate of convergence for the uniform refinement.

In what follows, N stands for the total number of degrees of freedom (unknowns) of (28), that is, N = 2*(Numbers of vertexes of $\mathcal{T}_h$) + 2*(Number of edges $\mathcal{T}_h$) +1, which leads asymptotically to 4 unknowns per triangle, which reflects the low computational cost, almost the same than the required by considering the $\mathbb{P}_1-$iso$\mathbb{P}_1$ elements for the standard velocity-pressure formulation, whose degrees of freedom are asymptotically 4.5 (unknowns) per triangle. In addition, by setting $p_h := -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)$, we obtain a reasonable piecewise-linear approximation of the pressure $p := -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})$. Hereafter, the individual and total errors are denoted as follows

$$\boldsymbol{\epsilon}(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{[H^1(\Omega)]^2}\,, \quad \boldsymbol{\epsilon}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_H\,, \quad \boldsymbol{\epsilon} := \Big([\boldsymbol{\epsilon}(\boldsymbol{u})]^2 \,+\, [\boldsymbol{\epsilon}(\boldsymbol{\sigma})]^2 \Big)^{1/2}\,,$$

$$\boldsymbol{\epsilon}_0(p) := \|p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\|_{L^2(\Omega)}\,, \quad \boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d) := \|\boldsymbol{\sigma}^d - \boldsymbol{\sigma}_h^d\|_{[L^2(\Omega)]^{2\times 2}}$$

$$\text{and} \quad \boldsymbol{\epsilon}_0(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{[L^2(\Omega)]^2}\,,$$

where $(\boldsymbol{\sigma}, \boldsymbol{u}) \in H_0 \times [H^1(\Omega)]^2$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in H_{0,h} \times H_h^{\boldsymbol{u}}$ are the unique solutions of the continuous and discrete formulations, respectively. In addition, if $\boldsymbol{\epsilon}$ and $\tilde{\boldsymbol{\epsilon}}$ stand for the errors at two consecutive triangulations with N and $\tilde{N}$ degrees of freedom, respectively, then the experimental rate of convergence is given by $r := -2\dfrac{\log(\boldsymbol{\epsilon}/\tilde{\boldsymbol{\epsilon}})}{\log(N/\tilde{N})}$. The definitions of $r(\boldsymbol{u})$, $r(\boldsymbol{\sigma})$, $r_0(\boldsymbol{\sigma}^d)$, $r_0(\boldsymbol{u})$ and $r_0(p)$ are given in analogous way.

Table 4.1. Summary of data for the three examples.

Ex.	ν	α	Solution $\boldsymbol{u}$	Solution p
1	1.0	10^{-6}	$\begin{pmatrix} -e^{x_1}(x_2\cos(x_2)+\sin(x_2)) \\ e^{x_1}x_2\sin(x_2) \end{pmatrix}$	$2e^{x_1}\sin(x_2)$
	1.0	10^{-4}		
	1.0	1.0		
	1.0	100		
	1.0	1000		
	1.0	10^4		
	1.0	10^6		
2	0.5	10^{-4}	$\dfrac{1}{8\pi\nu}\left\{-\ln(s)\begin{pmatrix}1\\0\end{pmatrix}+\dfrac{1}{s^2}\begin{pmatrix}(x_1-2)^2\\(x_1-2)(x_2-1)\end{pmatrix}\right\}$	$r^{2/3}\sin\left(\dfrac{2}{3}\theta\right)-\dfrac{3}{2\pi}$
	1.0	1.0		
	1.0	100		
	0.5	1000		
	0.5	10^4		
	0.5	10^6		

In addition, with the aim of showing the good behaviour of the augmented method with respect to the parameters α and ν, and thus for the parameters (κ_1, κ_2), included in the definition of the bilinear form $a(\cdot,\cdot)$ (cf. (16)) as well as the linear functional $F_1(\cdot)$ (cf. (17)), we choose different values, in agreement with the feasible choices described in Lemma 3.1. All the considered choices are summary in Table 4.1.

In Tables 4.2-4.14 we give the individual and global errors and the corresponding experimental rates of convergence for the uniform refinements as applied to Examples 1 and 2. Hereafter, uniform refinement means that, given a uniform initial triangulation, each subsequent mesh is obtained from the previous one by dividing each triangle into the four ones arising when connecting the midpoints of its sides. We remark that the errors are computed on each triangle using a 7 point-Gaussian quadrature rule. Additionaly, using a strightforward postprocessing, we have computed the $L^2(\Omega)$-error behaviour for approximations of $\boldsymbol{\sigma}^d$ and p, which are in relations with the $\nabla\boldsymbol{u}$ and the pressure. We note that, for these errors, we expect at least the same behaviour as $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2(\Omega)}$.

Table 4.2. Example 1 with $\alpha = 10^{-6}$ and $\nu = 1$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
19	4.608e+00	—-	6.503e+00	——	7.970e+00	—-
51	3.000e+00	0.8691	4.236e+00	0.8682	5.191e+00	0.8685
163	1.654e+00	1.0254	2.145e+00	1.1712	2.709e+00	1.1197
579	8.492e-01	1.0515	1.063e+00	1.1082	1.360e+00	1.0866
2179	4.278e-01	1.0346	5.290e-01	1.0529	6.803e-01	1.0458
8451	2.144e-01	1.0196	2.641e-01	1.0248	3.402e-01	1.0227
33283	1.072e-01	1.0104	1.320e-01	1.0118	1.701e-01	1.0113
132099	5.363e-02	1.0054	6.601e-02	1.0058	8.505e-02	1.0056
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
19	4.544e+00	—-	3.290e+00	——	1.615e+00	—-
51	3.433e+00	0.5675	1.667e+00	1.3768	6.260e-01	1.9198
163	1.772e+00	1.1381	8.406e-01	1.1788	1.764e-01	2.1799
579	8.823e-01	1.1008	4.172e-01	1.1055	4.564e-02	2.1335
2179	4.395e-01	1.0515	2.079e-01	1.0511	1.152e-02	2.0770
8451	2.195e-01	1.0244	1.038e-01	1.0242	2.890e-03	2.0411
33283	1.097e-01	1.0117	5.191e-02	1.0117	7.231e-04	2.0213
132099	5.486e-02	1.0057	2.595e-02	1.0058	1.808e-04	2.0109

Table 4.3. Example 1 with $\alpha = 10^{-4}$ and $\nu = 1$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
19	4.608e+00	—-	6.503e+00	——	7.970e+00	—-
51	3.000e+00	0.8691	4.236e+00	0.8682	5.191e+00	0.8685
163	1.654e+00	1.0254	2.145e+00	1.1712	2.709e+00	1.1197
579	8.492e-01	1.0515	1.063e+00	1.1082	1.360e+00	1.0866
2179	4.278e-01	1.0346	5.290e-01	1.0529	6.803e-01	1.0458
8451	2.144e-01	1.0196	2.641e-01	1.0248	3.402e-01	1.0227
33283	1.072e-01	1.0104	1.320e-01	1.0118	1.701e-01	1.0113
132099	5.363e-02	1.0054	6.601e-02	1.0058	8.505e-02	1.0056
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
19	4.544e+00	—-	3.290e+00	——	1.615e+00	—-
51	3.433e+00	0.5675	1.667e+00	1.3768	6.260e-01	1.9198
163	1.772e+00	1.1381	8.406e-01	1.1788	1.764e-01	2.1799
579	8.823e-01	1.1008	4.172e-01	1.1055	4.564e-02	2.1335
2179	4.395e-01	1.0515	2.079e-01	1.0511	1.152e-02	2.0770
8451	2.195e-01	1.0244	1.038e-01	1.0242	2.890e-03	2.0411
33283	1.097e-01	1.0117	5.191e-02	1.0117	7.231e-04	2.0213
132099	5.486e-02	1.0057	2.595e-02	1.0058	1.808e-04	2.0109

Table 4.4. Example 1 with $\alpha = 1$ and $\nu = 1$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
19	4.4767e+00	—	6.5849e+00	—	7.9625	—
51	2.9998e+00	0.8109	4.1794e+00	0.9209	5.1445	0.8848
163	1.6540e+00	1.0247	2.1395e+00	1.1525	2.7043	1.1069
579	8.4925e-01	1.0518	1.0622e+00	1.1049	1.3600	1.0846
2179	4.2781e-01	1.0347	5.2890e-01	1.0523	0.6803	1.0454
8451	2.1436e-01	1.0196	2.6412e-01	1.0246	0.3402	1.0226
33283	1.0725e-01	1.0104	1.3202e-01	1.0118	0.1701	1.0112
132099	5.3632e-02	1.0054	6.6005e-02	1.0057	0.0850	1.0056
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
19	4.395e+00	—	3.4210e+00	—	1.5358e+00	—
51	3.421e+00	0.5078	1.6733e+00	1.4485	6.4107e-01	1.7697
163	1.772e+00	1.1326	8.4519e-01	1.1756	1.8363e-01	2.1519
579	8.822e-01	1.1001	4.1797e-01	1.1111	4.7742e-02	2.1256
2179	4.395e-01	1.0514	2.0800e-01	1.0531	1.2069e-02	2.0752
8451	2.195e-01	1.0244	1.0386e-01	1.0248	3.0274e-03	2.0406
33283	1.097e-01	1.0117	5.1909e-02	1.0119	7.5759e-04	2.0212
132099	5.486e-02	1.0057	2.5952e-02	1.0058	1.8945e-04	2.0109

Table 4.5. Example 1 with $\alpha = 100$ and $\nu = 1$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
19	4.132e+00	—	2.208e+02	—	2.209e+02	—
51	3.928e+00	0.1027	1.463e+02	0.8337	1.464e+02	0.8333
163	1.842e+00	1.3036	2.299e+01	3.1860	2.306e+01	3.1811
579	8.823e-01	1.1611	4.829e+00	2.4620	4.909e+00	2.4411
2179	4.330e-01	1.0742	1.259e+00	2.0292	1.331e+00	1.9695
8451	2.151e-01	1.0322	3.888e-01	1.7331	4.444e-01	1.6187
33283	1.074e-01	1.0141	1.502e-01	1.3879	1.846e-01	1.2816
132099	5.365e-02	1.0064	6.841e-02	1.1409	8.694e-02	1.0926
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
19	2.101e+02	—	1.538e+01	—	1.240e+00	—
51	1.284e+02	0.9981	3.242e+01	—	7.607e-01	0.9897
163	1.926e+01	3.2648	3.595e+00	3.7853	1.612e-01	2.6708
579	4.124e+00	2.4321	7.982e-01	2.3745	3.543e-02	2.3908
2179	1.086e+00	2.0134	2.693e-01	1.6397	8.354e-03	2.1802
8451	3.323e-01	1.7476	1.126e-01	1.2862	2.051e-03	2.0726
33283	1.263e-01	1.4109	5.311e-02	1.0968	5.102e-04	2.0297
132099	5.707e-02	1.1531	2.611e-02	1.0301	1.274e-04	2.0131

Table 4.6. Example 1 with $\alpha = 1000$ and $\nu = 1$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
19	4.151e+00	——	2.211e+03	——	2.211e+03	——
51	4.029e+00	0.0601	1.414e+03	0.9060	1.414e+03	0.9059
163	1.876e+00	1.3155	2.116e+02	3.2694	2.116e+02	3.2694
579	8.958e-01	1.1668	3.966e+01	2.6417	3.967e+01	2.6414
2179	4.367e-01	1.0840	8.734e+00	2.2835	8.745e+00	2.2820
8451	2.158e-01	1.0404	2.115e+00	2.0928	2.126e+00	2.0870
33283	1.074e-01	1.0174	5.339e-01	2.0085	5.446e-01	1.9870
132099	5.366e-02	1.0074	1.449e-01	1.8920	1.545e-01	1.8276
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
19	2.108e+03	——	1.280e+02	——	1.236e+00	——
51	1.235e+03	1.0832	2.922e+02	——	7.849e-01	0.9198
163	1.691e+02	3.4221	2.900e+01	3.9760	1.661e-01	2.6729
579	3.172e+01	2.6407	3.804e+00	3.2052	3.647e-02	2.3922
2179	7.081e+00	2.2628	6.524e-01	2.6605	8.395e-03	2.2167
8451	1.734e+00	2.0758	1.659e-01	2.0205	2.015e-03	2.1059
33283	4.389e-01	2.0049	5.993e-02	1.4855	4.968e-04	2.0428
132099	1.194e-01	1.8887	2.698e-02	1.1579	1.237e-04	2.0166

Table 4.7. Example 1 with $\alpha = 10^4$ and $\nu = 1$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
19	4.153e+00	—-	2.212e+04	——	2.212e+04	—-
51	4.041e+00	0.0555	1.409e+04	0.9136	1.409e+04	0.9136
163	1.882e+00	1.3156	2.098e+03	3.2783	2.098e+03	3.2783
579	8.999e-01	1.1639	3.856e+02	2.6723	3.856e+02	2.6723
2179	4.394e-01	1.0820	8.199e+01	2.3364	8.200e+01	2.3364
8451	2.169e-01	1.0416	1.870e+01	2.1813	1.870e+01	2.1813
33283	1.077e-01	1.0216	4.494e+00	2.0801	4.495e+00	2.0797
132099	5.370e-02	1.0096	1.107e+00	2.0321	1.109e+00	2.0308
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
19	2.108e+04	—-	1.256e+03	——	1.236e+00	—-
51	1.230e+04	1.0923	2.887e+03	—-	7.876e-01	0.9125
163	1.664e+03	3.4424	2.840e+02	3.9915	1.669e-01	2.6713
579	3.021e+02	2.6924	3.457e+01	3.3228	3.684e-02	2.3835
2179	6.408e+01	2.3398	4.571e+00	3.0534	8.506e-03	2.2120
8451	1.443e+01	2.1998	6.747e-01	2.8230	2.031e-03	2.1134
33283	3.455e+00	2.0858	1.238e-01	2.4736	4.949e-04	2.0600
132099	8.495e-01	2.0355	3.495e-02	1.8354	1.225e-04	2.0258

Table 4.8. Example 1 with $\alpha = 10^6$ and $\nu = 1$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
19	4.154e+00	—-	2.212e+06	——	2.212e+06	—-
51	4.042e+00	0.0550	1.408e+06	0.9145	1.408e+06	0.9145
163	1.882e+00	1.3155	2.096e+05	3.2793	2.096e+05	3.2793
579	9.005e-01	1.1634	3.844e+04	2.6760	3.844e+04	2.6760
2179	4.399e-01	1.0810	8.129e+03	2.3444	8.129e+03	2.3444
8451	2.174e-01	1.0400	1.829e+03	2.2010	1.829e+03	2.2010
33283	1.080e-01	1.0211	4.312e+02	2.1083	4.312e+02	2.1083
132099	5.381e-02	1.0106	1.042e+02	2.0610	1.042e+02	2.0610
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
19	2.108e+06	—-	1.254e+05	——	1.236e+00	—-
51	1.229e+06	1.0933	2.883e+05	—-	7.879e-01	0.9117
163	1.661e+05	3.4447	2.833e+04	3.9931	1.669e-01	2.6711
579	3.003e+04	2.6991	3.421e+03	3.3356	3.689e-02	2.3821
2179	6.310e+03	2.3540	4.374e+02	3.1039	8.531e-03	2.2095
8451	1.387e+03	2.2357	5.619e+01	3.0283	2.042e-03	2.1095
33283	3.211e+02	2.1345	7.163e+00	3.0053	4.979e-04	2.0594
132099	7.649e+01	2.0815	9.044e-01	3.0024	1.228e-04	2.0307

Table 4.9. Example 2 with $\alpha = 10^{-4}$ and $\nu = 0.5$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
43	6.607e-02	—-	4.105e-01	——	4.158e-01	—-
131	2.035e-02	2.1145	2.466e-01	0.9148	2.475e-01	0.9316
451	6.298e-03	1.8971	1.442e-01	0.8679	1.444e-01	0.8719
1667	1.965e-03	1.7820	8.539e-02	0.8019	8.541e-02	0.8029
6403	6.765e-04	1.5844	5.138e-02	0.7548	5.139e-02	0.7551
25091	2.724e-04	1.3322	3.136e-02	0.7232	3.136e-02	0.7232
99331	1.242e-04	1.1412	1.934e-02	0.7025	1.934e-02	0.7025
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
43	2.431e-01	—-	1.589e-01	——	3.522e-02	—-
131	1.374e-01	1.0247	8.307e-02	1.1639	9.877e-03	2.2828
451	7.176e-02	1.0505	4.131e-02	1.1300	2.670e-03	2.1160
1667	3.646e-02	1.0360	2.061e-02	1.0639	6.988e-04	2.0509
6403	1.833e-02	1.0219	1.030e-02	1.0314	1.792e-04	2.0229
25091	9.182e-03	1.0123	5.147e-03	1.0154	4.544e-05	2.0089
99331	4.594e-03	1.0066	2.573e-03	1.0076	1.146e-05	2.0022

Table 4.10. Example 2 with $\alpha = 1$ and $\nu = 1$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
43	4.9985e-02	——	4.0842e-01	——	0.4115	——
131	1.8825e-02	1.7532	2.4606e-01	0.9097	0.2468	0.9179
451	5.6672e-03	1.9422	1.4416e-01	0.8649	0.1443	0.8684
1667	1.8120e-03	1.7445	8.5378e-02	0.8014	0.0854	0.8022
6403	6.4212e-04	1.5417	5.1380e-02	0.7547	0.0514	0.7550
25091	2.6557e-04	1.2929	3.1357e-02	0.7231	0.0314	0.7232
99331	1.2298e-04	1.1190	1.9339e-02	0.7025	0.0193	0.7025
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
43	2.337e-01	——	1.6309e-01	——	1.5922e-02	——
131	1.362e-01	0.9689	8.3149e-02	1.2095	4.7291e-03	2.1795
451	7.160e-02	1.0407	4.1347e-02	1.1302	1.2712e-03	2.1254
1667	3.643e-02	1.0338	2.0618e-02	1.0646	3.2995e-04	2.0635
6403	1.832e-02	1.0212	1.0298e-02	1.0317	8.4043e-05	2.0325
25091	9.181e-03	1.0121	5.1470e-03	1.0156	2.1211e-05	2.0162
99331	4.594e-03	1.0065	2.5732e-03	1.0077	5.3291e-06	2.0078

Table 4.11. Example 2 with $\alpha = 100$ and $\nu = 1$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
43	8.140e-03	——	5.438e-01	——	5.438e-01	——
131	4.132e-03	1.2173	2.542e-01	1.3653	2.542e-01	1.3653
451	2.010e-03	1.1658	1.456e-01	0.9009	1.457e-01	0.9010
1667	9.801e-04	1.0987	8.558e-02	0.8134	8.559e-02	0.8135
6403	4.788e-04	1.0647	5.141e-02	0.7574	5.141e-02	0.7574
25091	2.379e-04	1.0242	3.136e-02	0.7238	3.136e-02	0.7238
99331	1.187e-04	1.0105	1.934e-02	0.7027	1.934e-02	0.7027
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
43	3.744e-01	——	2.096e-01	——	1.241e-03	——
131	1.513e-01	1.6271	8.131e-02	1.7001	3.096e-04	2.4933
451	7.472e-02	1.1410	4.103e-02	1.1066	7.728e-05	2.2450
1667	3.694e-02	1.0776	2.057e-02	1.0564	1.789e-05	2.2382
6403	1.842e-02	1.0342	1.029e-02	1.0292	4.214e-06	2.1489
25091	9.199e-03	1.0168	5.146e-03	1.0148	1.063e-06	2.0168
99331	4.597e-03	1.0083	2.573e-03	1.0074	2.734e-07	1.9740

Table 4.12. Example 2 with $\alpha = 1000$ and $\nu = 0.5$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
43	9.085e-03	—-	5.631e+00	——	5.631e+00	—-
131	4.103e-03	1.4270	5.791e-01	4.0834	5.792e-01	4.0834
451	2.031e-03	1.1375	2.200e-01	1.5662	2.200e-01	1.5661
1667	9.925e-04	1.0956	1.009e-01	1.1926	1.009e-01	1.1926
6403	4.860e-04	1.0611	5.391e-02	0.9312	5.391e-02	0.9312
25091	2.400e-04	1.0333	3.174e-02	0.7755	3.175e-02	0.7755
99331	1.193e-04	1.0164	1.940e-02	0.7155	1.940e-02	0.7155
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
43	4.701e+00	—-	1.646e+00	——	2.766e-03	—-
131	4.300e-01	4.2937	1.193e-01	4.7125	5.807e-04	2.8025
451	1.594e-01	1.6051	4.565e-02	1.5538	1.427e-04	2.2705
1667	6.206e-02	1.4437	2.100e-02	1.1879	3.223e-05	2.2761
6403	2.412e-02	1.4044	1.034e-02	1.0528	7.544e-06	2.1582
25091	1.036e-02	1.2371	5.154e-03	1.0197	1.856e-06	2.0536
99331	4.849e-03	1.1041	2.575e-03	1.0087	4.648e-07	2.0126

Table 4.13. Example 2 with $\alpha = 10^4$ and $\nu = 0.5$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
43	9.127e-03	—-	5.561e+01	——	5.561e+01	—-
131	4.105e-03	1.4344	5.196e+00	4.2558	5.196e+00	4.2558
451	2.047e-03	1.1259	1.613e+00	1.8925	1.613e+00	1.8925
1667	1.015e-03	1.0732	5.109e-01	1.7588	5.109e-01	1.7588
6403	5.018e-04	1.0467	1.541e-01	1.7811	1.541e-01	1.7811
25091	2.451e-04	1.0496	5.213e-02	1.5875	5.213e-02	1.5875
99331	1.208e-04	1.0282	2.288e-02	1.1969	2.288e-02	1.1969
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
43	4.662e+01	—-	1.584e+01	——	2.785e-03	—-
131	4.021e+00	4.3994	8.576e-01	5.2351	5.812e-04	2.8129
451	1.362e+00	1.7519	1.918e-01	2.4226	1.442e-04	2.2550
1667	4.676e-01	1.6350	4.250e-02	2.3055	3.296e-05	2.2578
6403	1.398e-01	1.7946	1.253e-02	1.8152	7.671e-06	2.1667
25091	4.126e-02	1.7870	5.376e-03	1.2392	1.855e-06	2.0791
99331	1.278e-02	1.7039	2.606e-03	1.0527	4.581e-07	2.0326

Table 4.14. Example 2 with $\alpha = 10^6$ and $\nu = 0.5$: uniform refinement.

N	$\boldsymbol{\epsilon}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\boldsymbol{\epsilon}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{\epsilon}$	r
43	9.132e-03	—-	5.555e+03	—–	5.555e+03	—-
131	4.106e-03	1.4352	5.188e+02	4.2566	5.188e+02	4.2566
451	2.050e-03	1.1238	1.602e+02	1.9013	1.602e+02	1.9013
1667	1.023e-03	1.0632	5.004e+01	1.7799	5.004e+01	1.7799
6403	5.236e-04	0.9954	1.419e+01	1.8731	1.419e+01	1.8731
25091	2.724e-04	0.9568	3.918e+00	1.8846	3.918e+00	1.8846
99331	1.330e-04	1.0424	1.095e+00	1.8533	1.095e+00	1.8533
N	$\boldsymbol{\epsilon}_0(\boldsymbol{\sigma}^d)$	$r_0(\boldsymbol{\sigma}^d)$	$\boldsymbol{\epsilon}_0(p)$	$r_0(p)$	$\boldsymbol{\epsilon}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$
43	4.658e+03	—-	1.577e+03	—–	2.787e-03	—-
131	4.017e+02	4.3997	8.514e+01	5.2411	5.813e-04	2.8140
451	1.354e+02	1.7595	1.858e+01	2.4625	1.444e-04	2.2529
1667	4.624e+01	1.6437	3.652e+00	2.4886	3.314e-05	2.2520
6403	1.349e+01	1.8306	6.780e-01	2.5028	7.773e-06	2.1549
25091	3.768e+00	1.8677	1.357e-01	2.3554	1.877e-06	2.0811
99331	1.062e+00	1.8405	3.075e-02	2.1582	4.595e-07	2.0451

We notice that the orders of convergence predicted by the theory are achieved for all examples. Indeed, for Example 1, since we have a smooth solution in a convex region, the global orders behave as $\mathcal{O}(h)$. In Example 2, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^{2/3}(\Omega)]^2$, which in accordance with Theorem 4.1, allows us to expect $\mathcal{O}(h^{2/3})$ for its velocity of convergence, and thus for the total error. We also note a quadratic convergence for the error $\boldsymbol{\epsilon}_0(\boldsymbol{u})$, whose theoretical proof should be deduced from the standard duality argument. Furthermore, similarly to the examples described in [6], the numerical examples presented here behave much better than what the previous theoretical results insinuated. In particular, the order of the constant obtained in Theorem 4.1 indicates that the rates of convergence are affected for large values of α, which, nevertheless, was not too severe in the examples. The above observations yield the conjecture that these constants are overestimated and they could be improved.

6.1 Numerical examples of the associated control problem

Here we show the numerical results obtained using (27) to solve the control problem (26). We consider one example defined in the convex domain $\Omega := (0,1)^2$, The desired state, which we intend to get, is the pressure p_0 with the "controllers" functions p_1, p_2, p_3 and p_4, which is given in Table 4.15. The external control forces $\boldsymbol{f}_j \in [L^2(\Omega)]^2$, for $j = 1,2,3$ and 4, are obtained such that the pairs $(\boldsymbol{u}_j, p_j)$ in Table 4.15 are the solution of (1) with right hand side $\boldsymbol{f}_j \in [L^2(\Omega)]^2$, for $j = 1,2,3$ and 4. Hereafter, $\bar{p}$ denote the mean value of p in Ω. We present here two examples using $\boldsymbol{z}_c$ as a finite control variable of the form $\boldsymbol{z}_c = \sum_{i=1}^{k} z_i \boldsymbol{f}_i$, $k \in \{3,4\}$, where the amplitudes z_i are the unknown quantities of the system, which should allow us to get the desired pressure p_0.

Table 4.15. Example 1: Summary of data for the control problem.

j	SOLUTION $\boldsymbol{u}_j$	SOLUTION p_j
0	$\frac{1}{8\pi\nu}\left\{-\ln(s)\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{s^2}\begin{pmatrix}(x_1-2)^2\\(x_1-2)(x_2-1)\end{pmatrix}\right\}$	$\frac{1}{4\pi\nu}\frac{x_1-2}{(x_1-2)^2+(x_2-2)^2}-\frac{1}{\nu}\bar{p}_0$
1	$\begin{pmatrix}-(2.1-x_1-x_2)^{-1/3}\\(2.1-x_1-x_2)^{-1/3}\end{pmatrix}$	$2e^{2x_1-1}\sin(2x_2-1)$
2	$\left[(x_1+0.1)^2+(x_2+0.1)^2\right]^{-1/2}\begin{pmatrix}x_2+0.1\\-(x_1+0.1)\end{pmatrix}$	$\frac{1}{1.1-x_1}-\log(11)$
3	$\begin{pmatrix}\sin(2\pi x_1)\cos(2\pi x_2)\\-\cos(2\pi x_1)\sin(2\pi x_2)\end{pmatrix}$	$x_1^2+x_2^2-2/3$
4	$\begin{pmatrix}-e^{x_1}(x_2\cos(x_2)+\sin(x_2))\\e^{x_1}x_2\sin(x_2)\end{pmatrix}$	$(2-x_1+x_2)^{1/2}-\bar{p}_4$

As in the previous section, N stands for the number of degrees of freedom, while $\boldsymbol{\epsilon}_{C,k}$, $k\in\{3,4\}$, denotes the control error with k-controllers, measured in the Euclidean norm. For this purpose, we consider as exact solution the one obtained in the finest mesh. We denote by $r_{C,k}$ the respective experimental rates of convergence associated to $\boldsymbol{\epsilon}_{C,k}$, $k\in\{3,4\}$. We remark that by triangle inequality, we have

$$\|\boldsymbol{Z}_H^{op}-\boldsymbol{Z}^{op}\|\le\|\boldsymbol{Z}_h^{op}-\boldsymbol{Z}^{op}\|+\|\boldsymbol{Z}_H^{op}-\boldsymbol{Z}_h^{op}\|\,, \tag{30}$$

where $\boldsymbol{Z}_H^{op}$ and $\boldsymbol{Z}_h^{op}$ denote discrete solutions of (26) using different meshes. Then, since the exact solution is unknown, and in accordance with (30), we expect that the behaviour of the experimental order of convergence $r_{C,k}$, $k\in\{3,4\}$ will be at least as described in Theorem 5.1. In Tables 4.16-4.27, we exhibit the numerical results for different values of the parameters α, ν and η. In all the cases, we note that the rate of convergence is better than the expected $\mathcal{O}(h)$, which in virtue of (30) and Theorem 5.1, allow us to deduce the expected $\mathcal{O}(h)$, since $\|\boldsymbol{Z}_h^{op}-\boldsymbol{Z}^{op}\|\le C\|p-p_h\|_{L^2(\Omega)}$ and the behaviour of the error, as we had seen in the above subsection, is $\|p-p_h\|_{L^2(\Omega)}=\mathcal{O}(h)$ for this example. In addition, we can see the smooth effect of the control cost when η is increased for α fixed and large.

Table 4.16. Control example 1 (with 4 and 3 controllers) with $\alpha = 1$, $\nu = 1$ and $\eta = 1$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	2.733e-03	——	2.727e-03	——
83	2.658e-04	4.1501	2.212e-04	4.4733
291	6.226e-05	2.3143	5.817e-05	2.1298
1091	1.653e-05	2.0068	1.677e-05	1.8820
4227	4.319e-06	1.9820	4.664e-06	1.8898
16643	1.026e-06	2.0971	1.143e-06	2.0527
66051	2.016e-07	2.3610	2.256e-07	2.3540

Table 4.17. Control example 1 (with 4 and 3 controllers) with $\alpha = 1$, $\nu = 1$ and $\eta = 10$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	6.155e-04	——	7.054e-04	——
83	1.630e-04	2.3660	1.694e-04	2.5406
291	4.777e-05	1.9570	4.829e-05	2.0008
1091	1.240e-05	2.0411	1.247e-05	2.0489
4227	3.096e-06	2.0491	3.103e-06	2.0539
16643	7.374e-07	2.0938	7.384e-07	2.0951
66051	1.470e-07	2.3399	1.473e-07	2.3391

Table 4.18. Control example 1 (with 4 and 3 controllers) with $\alpha = 1$, $\nu = 1$ and $\eta = 100$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	7.146e-05	——	8.508e-05	——
83	2.200e-05	2.0979	2.490e-05	2.1882
291	6.873e-06	1.8553	7.680e-06	1.8754
1091	1.790e-06	2.0358	1.999e-06	2.0373
4227	4.450e-07	2.0557	4.972e-07	2.0543
16643	1.061e-07	2.0922	1.186e-07	2.0911
66051	2.123e-08	2.3341	2.376e-08	2.3335

Table 4.19. Control example 1 (with 4 and 3 controllers) with $\alpha = 100$, $\nu = 0.5$ and $\eta = 1$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	3.841e-03	—	2.136e-03	—
83	8.301e-04	2.7285	8.381e-04	1.6662
291	5.573e-04	0.6353	4.703e-05	4.5921
1091	2.406e-04	1.2712	4.074e-05	0.2173
4227	2.291e-05	3.4723	3.638e-06	3.5675
16643	3.088e-06	2.9247	2.648e-07	3.8237
66051	4.135e-07	2.9174	8.361e-08	1.6725

Table 4.20. Control example 1 (with 4 and 3 controllers) with $\alpha = 100$, $\nu = 0.5$ and $\eta = 10$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	8.553e-04	—	1.369e-03	—
83	4.265e-04	1.2394	2.839e-04	2.8023
291	1.156e-04	2.0817	4.086e-05	3.0907
1091	2.849e-05	2.1192	1.139e-05	1.9330
4227	2.299e-06	3.7168	1.318e-06	3.1846
16643	3.682e-07	2.6729	3.369e-07	1.9910
66051	5.927e-08	2.6502	7.553e-08	2.1692

Table 4.21. Control example 1 (with 4 and 3 controllers) with $\alpha = 100$, $\nu = 0.5$ and $\eta = 100$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	1.411e-04	—	4.348e-04	—
83	7.165e-05	1.2072	8.570e-05	2.8923
291	2.114e-05	1.9463	3.237e-05	1.5523
1091	2.191e-06	3.4302	3.964e-06	3.1781
4227	2.175e-07	3.4110	5.550e-07	2.9030
16643	4.802e-08	2.2047	1.161e-07	2.2828
66051	9.481e-09	2.3538	2.236e-08	2.3905

Table 4.22. Control example 1 (with 4 and 3 controllers) with $\alpha = 1000$, $\nu = 0.5$ and $\eta = 1$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	1.413e-03	——	2.715e-03	——
83	1.463e-03	——	1.451e-03	1.1156
291	1.956e-04	3.2078	2.284e-05	6.6193
1091	1.579e-04	0.3239	3.122e-05	——
4227	9.828e-05	0.7005	2.262e-05	0.4759
16643	1.025e-05	3.2987	2.442e-06	3.2485
66051	1.382e-06	2.9071	3.321e-07	2.8946

Table 4.23. Control example 1 (with 4 and 3 controllers) with $\alpha = 1000$, $\nu = 0.5$ and $\eta = 10$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	1.061e-03	——	2.162e-03	——
83	1.457e-03	——	1.402e-03	0.7711
291	1.407e-04	3.7265	2.831e-05	6.2214
1091	1.031e-04	0.4711	1.740e-05	0.7368
4227	5.269e-05	0.9907	1.108e-05	0.6662
16643	4.131e-06	3.7154	8.309e-07	3.7804
66051	5.327e-07	2.9720	1.045e-07	3.0080

Table 4.24. Control example 1 (with 4 and 3 controllers) with $\alpha = 1000$, $\nu = 0.5$ and $\eta = 100$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	4.230e-04	——	1.349e-03	——
83	1.185e-03	——	1.091e-03	0.3776
291	6.294e-05	4.6793	4.283e-05	5.1615
1091	2.631e-05	1.3199	2.830e-06	4.1119
4227	4.783e-06	2.5177	6.882e-07	2.0878
16643	1.771e-07	4.8106	1.901e-07	1.8777
66051	2.155e-08	3.0555	3.385e-08	2.5034

Table 4.25. Control example 1 (with 4 and 3 controllers) with $\alpha = 10^{-4}$, $\nu = 0.5$ and $\eta = 1$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	2.790e-03	——	2.835e-03	——
83	3.132e-04	3.8952	2.935e-04	4.0391
291	7.999e-05	2.1758	7.523e-05	2.1704
1091	2.023e-05	2.0806	2.092e-05	1.9369
4227	5.198e-06	2.0067	5.517e-06	1.9681
16643	1.248e-06	2.0814	1.351e-06	2.0536
66051	2.491e-07	2.3386	2.731e-07	2.3193

Table 4.26. Control example 1 (with 4 and 3 controllers) with $\alpha = 10^{-4}$, $\nu = 0.5$ and $\eta = 10$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	5.923e-04	——	6.853e-04	——
83	1.341e-04	2.6452	1.421e-04	2.8020
291	4.213e-05	1.8461	4.373e-05	1.8790
1091	1.101e-05	2.0305	1.136e-05	2.0398
4227	2.769e-06	2.0389	2.849e-06	2.0427
16643	6.639e-07	2.0839	6.827e-07	2.0848
66051	1.331e-07	2.3316	1.369e-07	2.3313

Table 4.27. Control example 1 (with 4 and 3 controllers) with $\alpha = 10^{-4}$, $\nu = 0.5$ and $\eta = 100$: uniform refinement.

N	$\epsilon_{C,4}$	$r_{C,4}$	$\epsilon_{C,3}$	$r_{C,3}$
27	6.785e-05	——	8.215e-05	——
83	1.736e-05	2.4277	2.011e-05	2.5060
291	5.626e-06	1.7963	6.438e-06	1.8162
1091	1.509e-06	1.9920	1.717e-06	1.9999
4227	3.821e-07	2.0277	4.341e-07	2.0309
16643	9.188e-08	2.0800	1.043e-07	2.0810
66051	1.845e-08	2.3289	2.094e-08	2.3292

7 Conclusions

We have analized the applicability of the velocity-pseudostress formulation, previously introduced in [8] for the stationary Stokes problem, for the aproximation of solution of the generalized Stokes problem. Furthermore, with the aim of circumventing the inf-sup condition, we have studied the corresponding augmented mixed scheme by applying a stabilization technique. We have obtained optimal rate of convergence, with an improvement of the constant of the a-priori estimate with respect to the results given in [6], at least for moderately large value of parameter α. Additionaly, we have applied it to obtain an error estimate for the control variable in the optimal control problem that we have considered.

Several numerical results are presented, which are in agreement with the theoretical results developed by us. We remark that they show also the robustness of the scheme, for small and large values of parameter α. Finally, we point out that the a posteriori error analysis, as well as the corresponding adaptivity algorithm, will be reported in a separate work.

References

[1] D.N. Arnold and R.S. Falk, *Well-posedness of the fundamental boundary value problems for constrained anisotropic elastic materials.* Archive for Rational Mechanics and Analysis, vol. 98, 2, pp. 143-165, (1987).

[2] T.P. Barrios and G.N. Gatica, *An augmented mixed finite element method with Lagrange multipliers: A priori and a posteriori error analyses.* Journal of Computational and Applied Mathematics, vol. 200, pp. 653-676, (2007).

[3] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods. Springer Verlag, 1994.

[4] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. Springer-Verlag, 1991.

[5] P. Bochev and M. Gunzburger, *Least-squares finite-element methods for optimization and control problems for the Stokes equations*, Computational & Applied Mathematics, vol. 48, pp. 1035-1057, (2004).

[6] R. Bustinza, G.N. Gatica and M. González, *A mixed finite element method for the generalized Stokes problem.* International Journal for Numerical Methods in Fluids, vol. 49, pp. 877-903, (2005).

[7] Z. Cai, B. Lee and P. Wang, *Least-squares methods for incompressible Newtonian fluid flow: Linear stationary problems.* SIAM Journal of Numerical Analysis, vol. 42, pp. 843-859, (2004).

[8] Z. Cai, C. Tong, P.S. Vassilevski and C. Wang, *Mixed finite element methods for incompressible flow: Stationary Stokes equations.* Numerical Methods for Partial Diferential Equations, vol. 26, pp. 957-978, (2010).

[9] J. C. De Los Reyes, C. Meyer, and B. Vexler, *Finite element error analysis for state-constrained optimal control of the Stokes equations.* Control and Cybernetics, Vol. 37, pp. 251-284, (2008).

[10] A. Ern and J.-L. Guermond, Theory and Practice of Finite Element. Springer-Verlag, 2004.

[11] L. Figueroa, G.N. Gatica and N. Heuer, *A priori and a posteriori error analysis of an augmented mixed finite element method for incompressible fluid flows.* Computer Methods in Applied Mechanics and Engineering, vol. 198, 2, pp. 280-291, (2008).

[12] L. Figueroa, G.N. Gatica and A. Marquez, *Augmented mixed finite element methods for the stationary Stokes equations.* SIAM Journal of Scientific Computing, vol. 31, 2, pp. 1082-1119, (2008).

[13] G.N. Gatica, *Analysis of a new augmented mixed finite element method for linear elasticity allowing $RT_0 - P_1 - P_0$ approximations.* ESAIM: Mathematical Modelling and Numerical Analysis, vol. 40, 1, pp. 1-28, (2006).

[14] G.N. Gatica, A. Márquez and M.A. Sánchez, *Analysis of a velocity-pressure-pseudostress formulation for the stationary Stokes equations.* Computer Methods in Applied Mechanics and Engineering, vol. 199, 17-20, pp. 1064-1079, (2010).

[15] G.N. Gatica, A. Márquez and M.A. Sánchez, *A priori and a posteriori error analyses of a velocity-pseudostress formulation for a class of quasi-Newtonian Stokes flows.* Computer Methods in Applied Mechanics and Engineering, vol. 200, 17-20, pp. 1619-1636, (2011).

[16] G.N. Gatica, A. Márquez and M.A. Sánchez, *Pseudostress-based mixed finite element methods for the Stokes problem in R^n with Dirichlet boundary conditions I: A priori error analysis.* Communications in Computational Physics, to appear.

[17] V. Girault and P.A. Raviart, Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms. Springer Series in Computational Mathematics, 1986.

[18] P. Gamallo and E. Hernández, *Error estimates for the approximation of a class of optimal control systems governed by linear PDEs.* Numerical Functional Analysis and Optimization, vol. 30, 5, pp. 523-547, (2009).

[19] A. Masud and T.J.R. Hughes, *A stabilized mixed finite element method for Darcy flow.* Computer Methods in Applied Mechanics and Engineering, vol. 191, 39-40, pp. 4341-4370, (2002).

[20] J.E. Roberts and J.-M. Thomas, *Mixed and Hybrid Methods.* In: Handbook of Numerical Analysis, edited by P.G. Ciarlet and J.L. Lions, vol. II, Finite Element Methods (Part 1), 1991, North-Holland, Amsterdam.

[21] A. Rösch and B. Vexler, *Optimal control of the Stokes equations: A priori error analysis for finite element discretization with postprocessing*, SIAM Journal on Numerical Analysis, vol. 44, 5, pp. 1903-1920, (2006).