ON THE HYPERBOLICITY OF CERTAIN MODELS OF POLYDISPERSE SEDIMENTATION

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Abstract. The sedimentation of a polydisperse suspension of small spherical particles dispersed in a viscous fluid, where particles belong to N species differing in size, can be described by a strongly coupled system of N scalar, nonlinear first-order conservation laws for the evolution of the volume fractions. The hyperbolicity of this system is a property of theoretical importance since it limits the range of validity of the model, and is of practical interest for the implementation of numerical methods. The present work, which extends the results of [R. Bürger, R. Donat, P. Mulet and C.A. Vega, *SIAM J. Appl. Math.* 70:2186–2213] is focused on the fluxes corresponding to the models by Batchelor and Wen, Höfler and Schwarzer, and Davis and Gecol, for which the Jacobian of the flux is a rank-3 or rank-4 perturbation of a diagonal matrix. Explicit estimates of the regions of hyperbolicity of these models are derived via the approach of the so-called secular equation [J. Anderson, *Lin. Alg. Appl.* 246:49–70, 1996], which identifies the eigenvalues of the Jacobian with the poles of a particular rational function. Hyperbolicity of the system is guaranteed if the coefficients of this function have the same sign. Sufficient conditions for this condition to be satisfied are established for each of the models considered. Some numerical examples are presented.

1. Introduction

1.1. **Scope.** We consider one-dimensional models of sedimentation of polydisperse suspensions of small solid particles dispersed in a viscous fluid. The particles are assumed to belong to N species that differ in size or density and may be treated as superimposed continuous phases, where species i is associated with the volume fraction (concentration) ϕ_i, the phase velocity v_i, size (diameter) d_i, and density ϱ_i. We assume that after suitable scaling, $d_1 = 1 \geq d_2 \geq \cdots \geq d_N$ and $d_i \neq d_j$ or $\varrho_i \neq \varrho_j$ for $i \neq j$. Since the velocities $v_1, \ldots, v_N$ are given functions of $\Phi := \Phi(x,t) := (\phi_1(x,t), \ldots, \phi_N(x,t))^{\mathrm{T}}$, we speak of a *kinematic* model. The continuity equations of the N solid species are given by the system of conservation laws

$$\partial_t \Phi + \partial_x \mathbf{f}(\Phi) = 0, \quad \mathbf{f}(\Phi) := \big(f_1(\Phi), \ldots, f_N(\Phi)\big)^{\mathrm{T}}, \quad f_i(\Phi) := \phi_i v_i(\Phi), \quad i = 1, \ldots, N, \tag{1.1}$$

where t is time and x is depth. We consider vectors $\Phi \in \bar{\mathcal{D}}_{\phi_{\max}}$, where $\bar{\mathcal{D}}_{\phi_{\max}}$ is the closure of the set $\mathcal{D}_{\phi_{\max}} := \{\Phi \in \mathbb{R}^N : \phi_1 > 0, \ldots, \phi_N > 0, \, \phi := \phi_1 + \cdots + \phi_N < \phi_{\max}\}$. We are interested in the hyperbolicity of (1.1) for arbitrary N under the assumption that $v_1, \ldots, v_N$ do not depend in an individual way on each component of Φ, but only on a small number $m \ll N$ of scalar functions of Φ, i.e.,

$$v_i = v_i(p_1, \ldots, p_m), \quad p_l = p_l(\Phi), \quad i = 1, \ldots, N, \quad l = 1, \ldots, m, \tag{1.2}$$

where we recall that (1.1) is called *hyperbolic* at a state Φ if all eigenvalues of the Jacobian $\mathcal{J}_{\mathbf{f}}(\Phi)$ are real, and *strictly hyperbolic* if these are moreover pairwise distinct. The latter occurs, for instance, if the eigenvalues

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are real and pairwise distinct. In the case (1.2), the entries $f_{ij}(\Phi) := \partial f_i(\Phi)/\partial \phi_j$ of $\mathcal{J}_{\mathbf{f}}(\Phi)$ are given by

$$f_{ij} = \frac{\partial(\phi_i v_i)}{\partial \phi_j} = v_i \delta_{ij} + \phi_i \sum_{l=1}^{m} \frac{\partial v_i}{\partial p_l} \frac{\partial p_l}{\partial \phi_j}, \quad i, j = 1, \dots, N, \tag{1.3}$$

i.e., $\mathcal{J}_{\mathbf{f}}(\Phi)$ is a rank-m perturbation of a diagonal matrix. Models of this type include those by Masliyah [1] and Lockett and Bassoon [2] ("MLB model"), Batchelor [3] and Batchelor and Wen [4] ("BW model"), Davis and Gecol [5] ("DG model"), and Höfler and Schwarzer [6, 7, 8] ("HS model"). Hyperbolicity is an important property for polydisperse sedimentation models since it is related to their range of validity and provides information required by numerical schemes for their simulation. However, determining the eigenvalues of $\mathcal{J}_{\mathbf{f}}(\Phi)$ by analyzing its characteristic polynomial is rarely an easy task for arbitrary N. Donat and Mulet [9] proved the hyperbolicity of the MLB model for equal-density spheres without explicitly computing $\det(\mathcal{J}_{\mathbf{f}}(\Phi) - \lambda \mathbf{I})$. Rather, they exploited the algebraic structure of $\mathcal{J}_{\mathbf{f}}(\Phi)$, and used that the eigenvalues of a rank-m perturbation of a diagonal matrix are the roots of the so-called secular equation [10]. The "secular approach" of hyperbolicity analysis is based on a rational function, $R(\lambda)$, that satisfies

$$\det\bigl(\mathcal{J}_{\mathbf{f}}(\Phi) - \lambda \mathbf{I}\bigr) = R(\lambda) \prod_{i=1}^{N} (v_i - \lambda)$$

for a fixed vector Φ, under appropriate circumstances. For (1.1), $R(\lambda)$ is of the form

$$R(\lambda) = 1 + \sum_{i=1}^{N} \frac{\gamma_i}{v_i - \lambda},$$

where $\gamma_1, \dots, \gamma_N$ can be calculated with acceptable effort for moderate values of m. If

$$\operatorname{sgn}(\gamma_1) = \operatorname{sgn}(\gamma_2) = \cdots = \operatorname{sgn}(\gamma_N), \tag{1.4}$$

then there exist N different eigenvalues of $\mathcal{J}_{\mathbf{f}}(\Phi)$, which can be localized easily since they interlace with $v_1, \dots, v_N$. This property is also important from the numerical point of view, since no explicit formulas for the eigenvalues are available, and they must always be computed by root finders, as was done in [11].

The "secular" approach has proven to be more convenient than the explicit computation of $\det(\mathcal{J}_{\mathbf{f}}(\Phi) - \lambda \mathbf{I})$ by successive row and column eliminations done e.g. in [12, 13, 14]. In [15] we extended the results of [9] to several variants of the MLB model (giving rise to (1.1), (1.2) with $m = 2$), and used the "secular approach" to estimate the region of hyperbolicity of certain particular cases of the BW and HS models, for which $m = 3$. (These cases are produced by setting to zero β_3, one of four possible parameters appearing in these models.) The results of [15], in particular, the easy access to the spectral decomposition of $\mathcal{J}_{\mathbf{f}}(\Phi)$, were employed in [11] for the characteristic-wise implementation of weighted essentially non-oscillatory (WENO) schemes.

It is the purpose of this paper to extend the results of [15], and in part those of [11], to the case $m = 4$ of the full BW and HS models (with $\beta_3 \neq 0$), and to the case $m = 3$ for the DG model, which was not handled in [15]. The DG model also emerges from the BW model (as does the HS model), but requires different techniques to estimate the hyperbolicity region. For all models the analysis is restricted to particles having the same density ($\varrho_1 = \cdots = \varrho_N$) differing in size only. We identify conditions on the smallest particle size, the maximum solids concentration and certain model parameters under which these models are strictly hyperbolic for arbitrary N. Some numerical simulations illustrate these models, and demonstrate how the hyperbolicity analysis provides characteristic information required by numerical schemes.

1.2. **Related work.** In [16] it was shown that loss of hyperbolicity, that is, the occurrence of pairs of complex-conjugate eigenvalues of $\mathcal{J}_{\mathbf{f}}(\Phi)$, is an instability criterion for polydisperse suspensions. For $N = 2$ or $N = 3$ this criterion can be evaluated conveniently by calculating a discriminant. In [16, 17, 18], instability regions for $N = 2, 3$ and different choices of $\mathbf{f}(\Phi)$ are determined, while in [12, 15] it is proven that for equal-density particles arbitrary N and $d_i \neq d_j$ for $i \neq j$, (1.1) with the MLB flux vector is strictly hyperbolic for all $\Phi \in \mathcal{D}_1$. (The MLB model has been studied intensively studied in a long list of papers including [9, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24], but will not be pursued in this work.)

The consequences of instability include the formation of blobs and "fingers" in bidisperse sedimentation and the formation of nonhomogeneous sediments [17]. These phenomena have been observed in experiments [25] under the circumstances predicted by the instability criterion. For one-dimensional kinematic models, loss of hyperbolicity sometimes leads to anomalous numerical solutions [26, 27]. Since instabilities have only been observed with particles of different densities [25], a sound model should be strictly hyperbolic for equal-density particles, at least if d_N is sufficiently close to one. Thus, there is interest in determining a region of guaranteed hyperbolicity of a given model in dependence of d_N and $\phi_{\max}$. This region should be independent of N, since only d_N can be controlled in real applications, for example by sieving. In [15] we outline a calculus that provides such a criterion for a number of models, which is based on the secular equation in the sense that we establish sufficient conditions for (1.4) to hold on $\mathcal{D}_{\phi_{\max}}$. As stated above, we here deal with cases of the BW and HS model and with the DG model that were not included in the previous analysis [15]. To highlight the relevance of the present analysis we mention that the BW, DG or HS model (or some close variant) is employed in some recent applicative studies including [28, 29, 30, 31, 32, 33]. Other multi-species kinematic flow models of the type (1.1), (1.2), which are amenable to a similar hyperbolicity analysis, include multi-class vehicular traffic (the multi-class Lighthill-Whitham-Richards (MCLWR) model [14, 22, 34, 35, 36]; see [15] for further references).

It is well known that high-resolution shock capturing schemes, such as the widely used weighted essentially non-oscillatory (WENO) schemes, can be applied to systems of conservation laws either in a component-wise or in a characteristic-wise (spectral) fashion. The latter requires a detailed knowledge of the spectral decomposition of the Jacobian matrix of the system, since the eigenstructure is used in a fundamental way in the design principles of the scheme. For multi-species kinematic flow models, however, eigenvalues are not available in closed form. In [34] and [11] it is shown (for the MCLWR model and the polydisperse sedimentation models examined in [15], respectively) that the "secular approach" of the hyperbolicity analysis provides good starting values for a root finder to identify the eigenvalues of the Jacobian, and furthermore explicit formulas for the corresponding eigenvectors. In these works it also demonstrated that spectral schemes require more computational effort, but are still more efficient in reducing discretization errors, than their component-wise counterparts. The spectral WENO scheme from [11] (see Sect. 2.3) will be employed herein for some numerical examples. We refer to [11] for further details, references and discussion.

1.3. **Outline of the paper.** The remainder of this paper is organized as follows. In Section 2 we outline basic results related to the secular equation (Sect. 2.1), the interlacing property and spectral decomposition (Sect. 2.2), and WENO schemes that make use of this information (Sect. 2.3), all of them for multi-species kinematic flow models of the type (1.1), (1.2). In Section 3 we introduce the models that are discussed in this work, namely the Batchelor and Wen model (BW model, Sect. 3.1), the Davis and Gecol model (DG model, Sect. 3.2) and the Höfler and Schwarzer model (HS model, Sect. 3.3). Sections 4 and 5 present the new results of hyperbolicity for the BW and HS and the DG models, respectively, and are at the core of this paper. Section 4 is subdivided into Sect. 4.1, where we introduce some preliminaries common to the BW and HS models, Sect. 4.2, which is dedicated to the estimate of the hyperbolicity region of the BW model for $\beta_3 < 0$ (i.e., $m = 4$) by establishing sufficient condition for (1.4) to hold, and Sect. 4.3, where an analogous analysis is conducted for the HS model. The DG model (with $\beta_3 = 0$, giving rise to $m = 3$) was not included in the analysis of [15], and the estimate of its hyperbolicity region requires different techniques, so the corresponding analysis is presented separately. Finally, in Section 6 we present some numerical examples illustrating the different models (Sects. 6.1–6.3) along with some conclusions (Sect. 6.4).

2. Preliminaries

2.1. **The secular equation.** For the present class of models, $\mathcal{J}_{\mathbf{f}}(\Phi)$ is a rank-m perturbation of the diagonal matrix $\mathbf{D} := \operatorname{diag}(v_1, \dots, v_N)$ of the form

$$\mathcal{J}_{\mathbf{f}} = \mathbf{D} + \mathbf{B}\mathbf{A}^{\mathrm{T}}, \quad \begin{cases} \mathbf{B} := (B_{il}) = (\phi_i \partial v_i / \partial p_l), \\ \mathbf{A} := (A_{jl}) = (\partial p_l / \partial \phi_j), \end{cases} \quad 1 \le i, j \le N, \quad 1 \le l \le m. \tag{2.1}$$

The corresponding hyperbolicity analysis is based on the following theorem, which can be found in [10], but we give here the form in [9], which provides the explicit formulas to be used in the applications.

Theorem 2.1 (The secular equation, [9, 10]). *Assume that $v_i > v_j$ for $i < j$, and that $\mathbf{A}$ and $\mathbf{B}$ have the formats specified in (2.1). We denote by S_r^p the set of all (ordered) subsets of r elements taken from a set of p elements. If $\mathbf{X}$ is an $m \times N$ matrix, $I := \{i_1 < \cdots < i_k\} \in S_k^N$ and $J := \{j_1 < \cdots < j_l\} \in S_l^m$, then we denote by $\mathbf{X}^{I,J}$ the $k \times l$ submatrix of $\mathbf{X}$ given by $(\mathbf{X}^{I,J})_{p,q} = X_{i_p,j_q}$. Let $\lambda \neq v_i$ for $i = 1, \dots, N$. Then λ is an eigenvalue of $\mathbf{D} + \mathbf{B}\mathbf{A}^{\mathrm{T}}$ if and only if it solves the so-called* secular equation [10]

$$R(\lambda) := \det\bigl(\mathbf{I} + \mathbf{A}^{\mathrm{T}}(\mathbf{D} - \lambda\mathbf{I})^{-1}\mathbf{B}\bigr) = 1 + \sum_{i=1}^{N} \frac{\gamma_i}{v_i - \lambda} = 0.$$

The coefficients γ_i, $i = 1, \dots, N$, are given by the following expression:

$$\gamma_i = \sum_{r=1}^{\min\{N,m\}} \sum_{i \in I \in S_r^N, J \in S_r^m} \frac{\det \mathbf{A}^{I,J} \det \mathbf{B}^{I,J}}{\prod_{l \in I, l \neq i}(v_l - v_i)}.$$

Assuming that $m < N$, with $\mathbf{A}$ and $\mathbf{B}$ defined in (2.1), we can write the following, where the notation should be self-explanatory:

$$\det \mathbf{A}^{I,J} = \det\left(\frac{\partial p_J}{\partial \phi_I}\right), \quad \det \mathbf{B}^{I,J} = \det\left(\frac{\partial v_I}{\partial p_J}\right) \prod_{l \in I} \phi_l,$$

$$\gamma_i = \phi_i \sum_{r=1}^{m} \gamma_{r,i}, \quad \gamma_{r,i} = \sum_{i \in I \in S_r^N} \prod_{l \in I, l \neq i} \frac{\phi_l}{v_l - v_i} \sum_{J \in S_r^m} \det\left(\frac{\partial v_I}{\partial p_J}\right) \det\left(\frac{\partial p_J}{\partial \phi_I}\right).$$

For $m \leq 2$, these quantities can be computed easily. For $m = 3$ or $m = 4$, the computations become more involved, but provide at least partial results concerning hyperbolicity, where the theoretical analysis of the characteristic polynomial of $\mathcal{J}_{\mathbf{f}}(\Phi)$ is essentially unfeasible.

2.2. **Interlacing property and spectral decomposition.** The following corollary follows from Theorem 2.1 by a discussion of the poles of $R(\lambda)$ and its asymptotic behaviour as $\lambda \to \pm\infty$.

Corollary 2.1 ([15]). *With the notation of Theorem 2.1, assume that $\gamma_i \cdot \gamma_j > 0$ for $i, j = 1, \dots, N$. Then $\mathbf{D} + \mathbf{B}\mathbf{A}^{\mathrm{T}}$ is diagonalizable with real eigenvalues $\lambda_1, \dots, \lambda_N$. If $\gamma_1, \dots, \gamma_N < 0$, the* interlacing property

$$M_1 := v_N + \gamma_1 + \cdots + \gamma_N < \lambda_N < v_N < \lambda_{N-1} < \cdots < \lambda_1 < v_1 \tag{2.2}$$

holds, while for $\gamma_1, \dots, \gamma_N > 0$, the following analogous property holds:

$$v_N < \lambda_N < v_{N-1} < \lambda_{N-1} < \cdots < v_1 < \lambda_1 < M_2 := v_1 + \gamma_1 + \cdots + \gamma_N.$$

The analysis of [15] also provides an explicit expression of the spectral decomposition of $\mathcal{J}_{\mathbf{f}} = \mathcal{J}_{\mathbf{f}}(\Phi)$ needed for the implementation of spectral schemes. Namely, assume that λ is an eigenvalue of $\mathcal{J}_{\mathbf{f}}$ with $\lambda \neq v_i$ for all $i = 1, \dots, N$, and that $\boldsymbol{\xi} \neq \mathbf{0}$ is a solution of $[\mathbf{I} + \mathbf{A}^{\mathrm{T}}(\mathbf{D} - \lambda\mathbf{I})^{-1}\mathbf{B}]\boldsymbol{\xi} = \mathbf{0}$. Then $\mathbf{x} = -(\mathbf{D} - \lambda\mathbf{I})^{-1}\mathbf{B}\boldsymbol{\xi}$ is the corresponding right eigenvector of $\mathcal{J}_{\mathbf{f}}$. The same procedure may be employed to calculate the left eigenvectors of $\mathcal{J}_{\mathbf{f}}$, since they are the right eigenvectors of $\mathcal{J}_{\mathbf{f}}^{\mathrm{T}} = \mathbf{D} + \mathbf{A}\mathbf{B}^{\mathrm{T}}$, so the roles of $\mathbf{A}$ and $\mathbf{B}$ and corresponding columns just need to be interchanged. See [11, 15] for further details.

2.3. **Characteristic-wise WENO schemes.** The schemes that we use in this work are based on Shu and Osher's technique [37] of constructing finite difference conservative schemes with a high order of accuracy. We consider a spatial discretization of the spatial domain $[0, 1]$ (after normalization) into M cells of size $\Delta x = 1/M$ and cell center $x_j := (j + 1/2)\Delta x$, $j = 0, \dots, M - 1$. If we denote the cell interfaces by $x_{j+1/2} = (j + 1)\Delta x$, then the approximation to $\partial_x \mathbf{f}(x_j, t)$ is obtained as follows:

$$\partial_x \mathbf{f}(x_j, t) = \frac{1}{\Delta x}\bigl(\hat{\mathbf{f}}_{j+1/2} - \hat{\mathbf{f}}_{j-1/2}\bigr) + \mathcal{O}(\Delta x^r),$$

where the numerical fluxes $\hat{\mathbf{f}}_{j+1/2} = \hat{\mathbf{f}}_{j+1/2}(\Phi(x_{j-s}, t), \dots, \Phi(x_{j+s+1}, t))$ are computed as described below.

If we define the vector $\mathbf{\Phi} := (\Phi_{-s}, \Phi_{-s+1}, \dots, \Phi_{M+s-2}, \Phi_{M+s-1})^{\mathrm{T}}$ of approximations $\Phi_j(t) \approx \Phi(x_j, t)$, then this procedure yields the semi-discrete scheme (method of lines)

$$\frac{\mathrm{d}\Phi_j}{\mathrm{d}t} = -\frac{1}{\Delta x}\bigl(\hat{\mathbf{f}}_{j+1/2}(\Phi_{j-s}, \dots, \Phi_{j+s+1}) - \hat{\mathbf{f}}_{j-1/2}(\Phi_{j-s-1}, \dots, \Phi_{j+s})\bigr), \quad j = 0, \dots, M-1,$$

which is integrated by using the third-order Runge-Kutta strong stability preserving (SSP) ODE solver described in [37] to get a fully discrete conservative scheme

$$\mathbf{\Phi}_j^{n+1} = \mathbf{\Phi}_j^n - \frac{\Delta t}{\Delta x}\bigl(\tilde{\mathbf{f}}_{j+1/2} - \tilde{\mathbf{f}}_{j-1/2}\bigr), \tag{2.3}$$

where Δt, the time step to advance the solution from $t = t_n$ to $t = t_{n+1} = t_n + \Delta t$, is selected to comply with the CFL stability restriction and $\tilde{\mathbf{f}}_{j+1/2}$ is a convex combination of values of $\hat{\mathbf{f}}_{j+1/2}$ at the three stages of the ODE solver for $j = 0, \dots, M-2$, whereas we set $\hat{\mathbf{f}}_{-1/2} = \hat{\mathbf{f}}_{M-1/2} = \mathbf{0}$ to enforce zero-flux boundary conditions at the end points $x_{-1/2} = 0$ and $x_{M-1/2} = 1$.

To use local characteristic projections to improve the accuracy of the numerical fluxes $\hat{\mathbf{f}}_{j+1/2}$, we need the complete eigenstructure of $\mathcal{J}_{\mathbf{f}}(\mathbf{\Phi})$, which is provided by the results of the hyperbolicity analysis of Sections 4 and 5. We consider the normalized left eigenvectors $(\mathbf{l}_{j+1/2}^k)^{\mathrm{T}}$ and right eigenvectors $\mathbf{r}_{j+1/2}^k$, $k = 1, \dots, N$, of $\mathcal{J}_{\mathbf{f}}(\mathbf{\Phi}_{j+1/2})$, where we define $\mathbf{\Phi}_{j+1/2} := \frac{1}{2}(\mathbf{\Phi}_j + \mathbf{\Phi}_{j+1})$, forming the matrices of eigenvectors

$$\mathbf{R}_{j+1/2} = \bigl(\mathbf{r}_{j+1/2}^1, \dots, \mathbf{r}_{j+1/2}^N\bigr), \qquad \mathbf{R}_{j+1/2}^{-1} = \bigl(\mathbf{l}_{j+1/2}^1, \dots, \mathbf{l}_{j+1/2}^N\bigr)^{\mathrm{T}},$$

that are used in the following computation of local characteristic variables and fluxes at $x_{j+1/2}$:

$$\begin{aligned} &g_{j+1/2,i,k} := \bigl(\mathbf{l}_{j+1/2}^k\bigr)^{\mathrm{T}}\mathbf{f}(\mathbf{\Phi}_{j+i}), \quad g_{j+1/2,i,k}^{\pm} := \frac{1}{2}\bigl(\mathbf{l}_{j+1/2}^k\bigr)^{\mathrm{T}}\bigl(\mathbf{f}(\mathbf{\Phi}_{j+i}) \pm \alpha_{j+1/2}^k \mathbf{\Phi}_{j+i}\bigr), \\ &i = -2, \dots, 3, \quad j \in \mathbb{Z}, \quad k = 1, \dots, N. \end{aligned}$$

Here, $\alpha_{j+1/2}^k$, the local viscosity coefficient on the k-th characteristic field used for the local Lax-Friedrichs flux splitting, is an upper bound of $\max_{\mathbf{\Phi}\in\Gamma_j} |\lambda_k(\mathbf{\Phi})|$, where $\Gamma_j \subset \mathbb{R}^N$ is the straight line connecting $\mathbf{\Phi}_j$ and $\mathbf{\Phi}_{j+1}$. Since there is no closed formula for these eigenvalues we use the interlacing property (2.2) to effectively bound

$$\max_{\mathbf{\Phi}\in\Gamma_j} \bigl|\lambda_k(\mathbf{\Phi})\bigr| \leq \alpha_{j+1/2}^k := \max_{\mathbf{\Phi}\in\Gamma_j} \bigl\{\bigl|v_{k-1}(\mathbf{\Phi})\bigr|, \bigl|v_k(\mathbf{\Phi})\bigr|\bigr\}, \quad k = 1, \dots, N, \tag{2.4}$$

where we set $v_0 := M_1$, with M_1 being defined in (2.2). For the models under consideration here, this maximum is available in closed algebraic form, see [11] for details.

We compute the numerical fluxes as

$$\hat{\mathbf{f}}_{j+1/2} = \bigl(\hat{f}_{j+1/2,1}, \dots, \hat{f}_{j+1/2,N}\bigr)^{\mathrm{T}} = \mathbf{R}_{j+1/2}\hat{\mathbf{g}}_{j+1/2}, \quad j \in \mathbb{Z},$$

where $\hat{\mathbf{g}}_{j+1/2} = (\hat{g}_{j+1/2,1}, \dots, \hat{g}_{j+1/2,N})^{\mathrm{T}}$ is defined as follows. If $\lambda_j^k \cdot \lambda_{j+1}^k \leq 0$ (Case 1), we set

$$\hat{g}_{j+1/2,k} = \mathcal{R}^{+}\bigl(g_{j+1/2,-2,k}^{+}, \dots, g_{j+1/2,2,k}^{+}; x_{j+1/2}\bigr) + \mathcal{R}^{-}\bigl(g_{j+1/2,-1,k}^{-}, \dots, g_{j+1/2,3,k}^{-}; x_{j+1/2}\bigr),$$

while for $\lambda_j^k \cdot \lambda_{j+1}^k > 0$ (Case 2), we set

$$\hat{g}_{j+1/2,k} = \begin{cases} \mathcal{R}^{+}(g_{j+1/2,-2,k}, \dots, g_{j+1/2,2,k}; x_{j+1/2}) & \text{if } \lambda_j^k > 0 \text{ and } \lambda_{j+1}^k > 0, \\ \mathcal{R}^{-}(g_{j+1/2,-1,k}, \dots, g_{j+1/2,3,k}; x_{j+1/2}) & \text{if } \lambda_j^k < 0 \text{ and } \lambda_{j+1}^k < 0, \end{cases} \quad k = 1, \dots, N,$$

where $\mathcal{R}^{\pm}$ are upwind-biased reconstructions provided by the mapped WENO5 (WENO5M) reconstructions, proposed in [38], to avoid a possible loss of accuracy around extrema.

3. Models of polydisperse sedimentation

3.1. **The Batchelor and Wen (BW) model.** Batchelor [3] derived the following expression for the settling velocity v_i of spheres of species i, having diameter d_i, in a dilute suspension:

$$v_i(\Phi) = v_i(\mathbf{0})(1 + \mathbf{s}_i^{\mathrm{T}}\Phi), \quad i = 1, \dots, N. \tag{3.1}$$

Here, $v_i(\mathbf{0})$ is the settling velocity of a single sphere of species i in pure fluid, that is, $v_i(\mathbf{0})$ is the Stokes velocity $v_i(\mathbf{0}) = -d_i^2 \bar{\varrho}_i/(18\mu_{\mathrm{f}})$, and $\mathbf{s}_i^{\mathrm{T}} := (S_{i1}, \dots, S_{iN})$. The dimensionless sedimentation coefficients S_{ij} are negative functions of $\lambda_{ij} := d_j/d_i$ and $\varrho_{ij} := \bar{\varrho}_j/\bar{\varrho}_i$, of the Péclet number $\mathcal{P}_{ij} := (d_i + d_j)|v_j(\mathbf{0}) - v_i(\mathbf{0})|/(4\mathcal{D}_{ij})$, and of interparticle attractive-repulsive forces. Here, $\mathcal{D}_{ij} := (kT)(3\pi\mu_{\mathrm{f}})^{-1}(d_i^{-1} + d_j^{-1})$ is the so-called relative diffusivity, where T is temperature and k is the Boltzmann constant [3, 4]. The coefficients S_{ij} can be calculated from the pair distribution function, which represents the statistical structure of the suspension [3]. This was done numerically by Batchelor and Wen [4] for a range of values of $\lambda = \lambda_{ij}$ and $\varrho = \varrho_{ij}$, considering the limits of either a large ($\mathcal{P}_{ij} \gg 1$) or a small ($\mathcal{P}_{ij} \ll 1$) Péclet number, and neglecting Brownian diffusion.

The secular equation can be employed for the hyperbolicity analysis of several models based on Batchelor's approach with equal-density particles ($\varrho_{ij} \equiv 1$). In this case, after rescaling time, we may express (3.1) as

$$v_i(\Phi) = d_i^2(1 + \mathbf{s}_i^{\mathrm{T}}\Phi), \quad i = 1, \dots, N, \tag{3.2}$$

and the coefficients S_{ij} may be approximated by

$$S_{ij} = \sum_{l=0}^{3} \beta_l \left(\frac{d_j}{d_i}\right)^l, \quad 1 \le i, j \le N. \tag{3.3}$$

We will refer to (3.2), (3.3) as the Batchelor and Wen (BW) model.

Davis and Gecol [5] were the first to approximate the numerical values of S_{ij}, tabulated in [4] for $\varrho_{ij} = 1$ for eight different values of λ_{ij}, by an expression of the type (3.3). They obtained the coefficients

$$\boldsymbol{\beta}^{\mathrm{T}} := (\beta_0, \dots, \beta_3) = \begin{cases} (-3.5, -1.1, -1.02, -0.002) & \text{for large Péclet numbers } (\mathcal{P}_{ij} \gg 1), \\ (-3.42, -1.96, -1.21, -0.013) & \text{for small Péclet numbers } (\mathcal{P}_{ij} \ll 1). \end{cases} \tag{3.4}$$

We observe that in both cases, $\beta_i < 0$ for $i = 0, \dots, 2$, and that $|\beta_3|$ is very small. In fact, some authors utilize $\beta_3 = 0$ a priori; for example, Höfler and Schwarzer [8] fit the data from [4] for large Péclet numbers to a second-order polynomial corresponding to

$$\boldsymbol{\beta}^{\mathrm{T}} = (\beta_0, \dots, \beta_3) = (-3.52, -1.04, -1.03, 0). \tag{3.5}$$

That $|\beta_3|$ should be a small while $\beta_0, \beta_1, \beta_2 \le 0$ is also supported by a theoretical asymptotical result [3] stating that

$$S_{ij} + \varrho_{ij}(\lambda_{ij}^2 + 3\lambda_{ij} + 1) \to 0 \quad \text{as } \lambda_{ij} \to \infty, \tag{3.6}$$

which is relevant here only for $\varrho_{ij} = 1$. For a detailed discussion of the coefficients S_{ij} and further data we refer to [3, 4, 31, 39, 40, 41]. Our further analysis will indeed rely on the negativity of β_0, β_1 and β_2.

Setting $\beta_3 = 0$ simplified the computations the hyperbolicity analysis via the secular equation in [15]. We will here consider $\beta_3 \le 0$ so that the principle of extending the hyperbolicity computations of [15] is conserved. Nevertheless, since the coefficients $\boldsymbol{\beta}$ are usually determined by fitting tabulated data to a polynomial of the type (3.3), there is no physical principle that compels that $\beta_3 \le 0$. In fact, even though $\beta_3 \le 0$ holds for the original data of [4] (according to (3.4) and (3.5)), Wang and Wen [32] examine polydisperse particles with thin double layer at small Péclet number for which the asymptotics (3.6) remains valid. Setting $\beta_3 = 0$ for the data of [32, Table 1] one gets $\boldsymbol{\beta}^{\mathrm{T}} = (-3.9713, -1.9947, -1.0370, 0)$; otherwise one obtains

$$\boldsymbol{\beta}^{\mathrm{T}} = (-4.0139, -1.8780, -1.0818, 0.0039) \tag{3.7}$$

with $\beta_3 > 0$. Even though the BW, HS or DG models with the coefficients (3.7) are not covered by the present analysis, similar estimates of the hyperbolicity region can possibly be obtained for small positive values of β_3. This is supported by numerical experiments (see Figures 4 and 5 in Section 6), in which the HS model solved with $\boldsymbol{\beta}$ given by (3.7) produces numerical results that are free of anomalous behaviour.

3.2. **The Davis and Gecol (DG) model.** To overcome the limitation of (3.2), and the BW approach, to dilute suspensions, Davis and Gecol [5] proposed to replace (3.2) by

$$v_i(\Phi) = d_i^2(1 + \mathbf{s}_i^{\mathrm{T}}\Phi - S_{ii}\phi)(1-\phi)^{-S_{ii}}, \tag{3.8}$$

where from (3.3) we infer that $S_{ii} = \beta_0 + \beta_1 + \beta_2 + \beta_3$. Koo [33] reports good agreement of hindered settling velocities predicted by the DG model with those obtained from a large-scale particle-based simulation in the case $N = 2$ and $d_2 \geq 0.5$. Davis and Gecol claimed that (3.8) could be used for $d_N \geq 1/8$. However, in [16] it is shown that for $N = 2$ and $d_2 \approx 1/6$, the system (1.1) based on using (3.8) exhibits unphysical instability regions for equal-density spheres.

3.3. **The Höfler and Schwarzer (HS) model.** Another velocity equation that formally extends (3.2) to the whole range of concentrations was suggested by Höfler and Schwarzer [6, 7, 8]:

$$v_i(\Phi) = d_i^2 \exp(\mathbf{s}_i^{\mathrm{T}}\Phi + n\phi)(1-\phi)^n, \quad n \geq 0. \tag{3.9}$$

For $\Phi \to \mathbf{0}$, (3.8) and (3.9) have the same partial derivatives as (3.2), while for $\phi \to 1$, the velocities v_i given by (3.8) and (3.9) vanish. Moreover, for the HS model it is straightforward to verify (see [16]) that this model is strictly hyperbolic for $N = 2$ and arbitrary coefficients $S_{ij} \leq 0$. Furthermore, based on numerical tests, it was conjectured in [16] that the model based on (3.9) would be stable also for $N = 3$. The present work confirms this conjecture and shows that the model is stable for arbitrary N, provided that for a given vector of coefficients $\boldsymbol{\beta}$, the quantities d_N and Φ satisfy some mild conditions.

4. Hyperbolicity analysis of the BW and HS models for $\beta_3 < 0$

In [15] we studied the hyperbolicity for the BW and HS under the assumption $\beta_3 = 0$. This assumption allowed us to simplify many computations and put the main results into perspective. In this section we remove that assumption and we will see that it is still possible to obtain sufficient conditions for hyperbolicity.

4.1. **Preliminaries for the BW and HS models.** First, we can write the settling velocity for the BW and HS models ((3.2) and (3.9), respectively) as

$$v_i(\Phi) = v_i(p_1,\dots,p_4) = d_i^2\varphi\bigl((\beta_0+n)p_1 + \beta_1 d_i^{-1}p_2 + \beta_2 d_i^{-2}p_3 + \beta_3 d_i^{-3}p_4\bigr)(1-p_1)^n, \quad i = 1,\dots,N,$$

where $\varphi(z) = 1 + z$, $n = 0$ for the BW model and $\varphi(z) = \exp(z)$, $n \geq 0$, arbitrary for the HS model. We define $\eta_i := \varphi(\mathbf{s}_i^{\mathrm{T}}\Phi + n\phi)$ and $\eta_i' := \varphi'(\mathbf{s}_i^{\mathrm{T}}\Phi + n\phi)$ for $i = 1,\dots,N$, where $\varphi'(z) := d\varphi(z)/dz$. The entries of matrix $\mathbf{A}$ are $\alpha_i^k = d_i^{k-1}$, $k = 1,\dots,4$, $i = 1,\dots,N$. Those of $\mathbf{B}$ are given by

$$\begin{aligned}
\beta_i^1 &= d_i^2\phi_i(1-\phi)^{n-1}\bigl((1-\phi)(\beta_0+n)\eta_i' - n\eta_i\bigr),\\
\beta_i^k &= d_i^{3-k}\phi_i(1-\phi)^n\beta_{k-1}\eta_i', \quad k = 2,3,4; \quad i = 1,\dots,N.
\end{aligned}$$

We now calculate the determinants $\alpha_I^J := \det \mathbf{A}^{I,J}$ and $\beta_I^J := \det \mathbf{B}^{I,J}$ for the case of $m = 4$,

$$\begin{aligned}
\gamma_i = {}& \alpha_i^1\beta_i^1 + \alpha_i^2\beta_i^2 + \alpha_i^3\beta_i^3 + \alpha_i^4\beta_i^4 + \sum_{\substack{j=1\\ j\neq i}}^N \frac{\alpha_{ij}^{12}\beta_{ij}^{12} + \alpha_{ij}^{13}\beta_{ij}^{13} + \alpha_{ij}^{23}\beta_{ij}^{23} + \alpha_{ij}^{14}\beta_{ij}^{14} + \alpha_{ij}^{24}\beta_{ij}^{24} + \alpha_{ij}^{34}\beta_{ij}^{34}}{v_j - v_i}\\
&+ \sum_{\substack{j,k=1\\ i\neq j<k\neq i}}^N \frac{\alpha_{ijk}^{123}\beta_{ijk}^{123} + \alpha_{ijk}^{234}\beta_{ijk}^{234} + \alpha_{ijk}^{134}\beta_{ijk}^{134} + \alpha_{ijk}^{124}\beta_{ijk}^{124}}{(v_k - v_i)(v_j - v_i)} + \sum_{\substack{j,k,l=1\\ j<k<l\\ j,k,l\neq i}}^N \frac{\alpha_{ijkl}^{1234}\beta_{ijkl}^{1234}}{(v_k - v_i)(v_j - v_i)(v_l - v_i)}.
\end{aligned} \tag{4.1}$$

We recall that sums over a void index range are zero, and utilize the following auxiliary notation:

$$\begin{aligned}
&\sigma_{ijk} := d_i + d_j + d_k, && \pi_{ijk} := (d_j - d_i)(d_k - d_i)(d_k - d_j),\\
&\tilde{\sigma}_{ijk} := d_id_j + d_id_k + d_jd_k, && \pi_{ijkl} := (d_j - d_i)(d_k - d_i)(d_l - d_i)(d_l - d_j)(d_l - d_k)(d_k - d_j).
\end{aligned} \tag{4.2}$$

We then obtain

$$
\begin{aligned}
&\alpha_i^1 = 1, \quad \alpha_i^4 = d_i^3, \quad \alpha_{ij}^{23} = d_i d_j (d_j - d_i), \quad \alpha_{ij}^{34} = d_i^2 d_j^2 (d_j - d_i), \quad \alpha_{ijk}^{134} = \tilde{\sigma}_{ijk}\pi_{ijk},\\
&\alpha_i^2 = d_i, \quad \alpha_{ij}^{12} = d_j - d_i, \quad \alpha_{ij}^{14} = d_j^3 - d_i^3, \quad \alpha_{ijk}^{123} = \pi_{ijk}, \quad \alpha_{ijk}^{124} = \sigma_{ijk}\pi_{ijk},\\
&\alpha_i^3 = d_i^2, \quad \alpha_{ij}^{13} = d_j^2 - d_i^2, \quad \alpha_{ij}^{24} = d_i d_j \big(d_j^2 - d_i^2\big), \quad \alpha_{ijk}^{234} = d_i d_j d_k \pi_{ijk}, \quad \alpha_{ijkl}^{1234} = \pi_{ijkl},
\end{aligned} \tag{4.3a}
$$

$$
\begin{aligned}
\beta_{ij}^{12} &= \phi_i\phi_j(1-\phi)^{2n-1} d_i d_j \beta_1 \big[(1-\phi)(\beta_0+n)\eta_i'\eta_j'(d_i-d_j) - n\big(\eta_i\eta_j' d_i - \eta_i'\eta_j d_j\big)\big],\\
\beta_{ij}^{13} &= \phi_i\phi_j(1-\phi)^{2n-1}\beta_2\big((1-\phi)(\beta_0+n)\eta_i'\eta_j'(d_i^2-d_j^2) - n(\eta_i\eta_j' d_i^2 - \eta_i'\eta_j d_j^2)\big),\\
\beta_{ij}^{14} &= \phi_i\phi_j(1-\phi)^{2n-1} d_i^{-1} d_j^{-1}\beta_3\big((1-\phi)(\beta_0+n)\eta_i'\eta_j'(d_i^3-d_j^3) - n(\eta_i\eta_j' d_i^3 - \eta_i'\eta_j d_j^3)\big),\\
\beta_{ij}^{23} &= \phi_i\phi_j(1-\phi)^{2n}\beta_1\beta_2\eta_i'\eta_j'(d_i-d_j),\\
\beta_{ij}^{24} &= \phi_i\phi_j(1-\phi)^{2n}\beta_1\beta_3\eta_i'\eta_j' d_i^{-1} d_j^{-1}\big(d_i^2-d_j^2\big),\\
\beta_{ij}^{34} &= \phi_i\phi_j(1-\phi)^{2n}\beta_2\beta_3\eta_i'\eta_j' d_i^{-1} d_j^{-1}\big(d_i-d_j\big),
\end{aligned} \tag{4.3b}
$$

$$
\begin{aligned}
\beta_{ijk}^{123} &= -(1-\phi)^{3n-1}\phi_i\phi_j\phi_k\beta_1\beta_2\big\{(1-\phi)(\beta_0+n)\pi_{ijk}\eta_i'\eta_j'\eta_k'\\
&\qquad + n\big[d_i^2(d_j-d_k)\eta_i\eta_j'\eta_k' - d_j^2(d_i-d_k)\eta_i'\eta_j\eta_k' + d_k^2(d_i-d_j)\eta_i'\eta_j'\eta_k\big]\big\},\\
\beta_{ijk}^{124} &= -(1-\phi)^{3n-1}\phi_i\phi_j\phi_k\beta_1\beta_3\big\{(1-\phi)(\beta_0+n)d_i^{-1}d_j^{-1}d_k^{-1}\tilde{\sigma}_{ijk}\pi_{ijk}\eta_i'\eta_j'\eta_k'\\
&\qquad + n\big[d_i^2 d_j^{-1}d_k^{-1}\big(d_j^2-d_k^2\big)\eta_i\eta_j'\eta_k' - d_j^2 d_i^{-1}d_k^{-1}\big(d_i^2-d_k^2\big)\eta_i'\eta_j\eta_k' + d_k^2 d_i^{-1}d_j^{-1}\big(d_i^2-d_j^2\big)\eta_i'\eta_j'\eta_k\big]\big\},\\
\beta_{ijk}^{134} &= -(1-\phi)^{3n-1}\phi_i\phi_j\phi_k\beta_2\beta_3\big\{(1-\phi)(\beta_0+n)d_i^{-1}d_j^{-1}d_k^{-1}\sigma_{ijk}\pi_{ijk}\eta_i'\eta_j'\eta_k'\\
&\qquad + n\big[d_i^2 d_j^{-1}d_k^{-1}(d_j-d_k)\eta_i\eta_j'\eta_k' - d_j^2 d_i^{-1}d_k^{-1}(d_i-d_k)\eta_i'\eta_j\eta_k' + d_k^2 d_i^{-1}d_j^{-1}(d_i-d_j)\eta_i'\eta_j'\eta_k\big]\big\},\\
\beta_{ijk}^{234} &= -(1-\phi)^{3n}\phi_i\phi_j\phi_k\beta_1\beta_2\beta_3 d_i^{-1}d_j^{-1}d_k^{-1}\pi_{ijk}\eta_i'\eta_j'\eta_k',
\end{aligned} \tag{4.3c}
$$

$$
\begin{aligned}
\beta_{ijkl}^{1234} &= (1-\phi)^{4n-1}\phi_i\phi_j\phi_k\phi_l\beta_1\beta_2\beta_3\bigg[(1-\phi)(\beta_0+n)\frac{\pi_{ijkl}}{d_i d_j d_k d_l}\eta_i'\eta_j'\eta_k'\eta_l'\\
&\quad + n\bigg(\frac{d_i^2\pi_{jkl}}{d_j d_k d_l}\eta_i\eta_j'\eta_k'\eta_l' - \frac{d_j^2\pi_{ikl}}{d_i d_k d_l}\eta_i'\eta_j\eta_k'\eta_l' + \frac{d_k^2\pi_{ijl}}{d_i d_j d_l}\eta_i'\eta_j'\eta_k\eta_l' - \frac{d_l^2\pi_{ijk}}{d_i d_j d_k}\eta_i'\eta_j'\eta_k'\eta_l\bigg)\bigg].
\end{aligned} \tag{4.3d}
$$

Substantial simplifications in the expressions β_I^J occur for the BW model, for whiich $\eta_i' = 1$ and $n = 0$, and for the HS model, where $\eta_i = \eta_i'$.

4.2. **The Batchelor and Wen (BW) model.** The coefficients γ_i, $i = 1,\dots,N$ can now be rewritten as $\gamma_i = \phi_i(\mathcal{S}_{1,i} + \mathcal{S}_{2,i} + \mathcal{S}_{3,i} + \mathcal{S}_{4,i})$. Inserting (4.2) and (4.3) into (4.1) and defining $\hat{\eta}_i := 1 + \mathbf{s}_i^{\mathrm{T}}\Phi$, we obtain

$$
\begin{aligned}
\mathcal{S}_{1,i} &= d_i^2(\beta_0+\beta_1+\beta_2+\beta_3),\\
\mathcal{S}_{2,i} &= \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\phi_j}{d_j^2\hat{\eta}_j - d_i^2\hat{\eta}_i}\bigg(-\beta_0\beta_1 d_i d_j(d_j-d_i)^2 - \beta_0\beta_2(d_j^2-d_i^2)^2 - \beta_1\beta_2 d_i d_j(d_j-d_i)^2\\
&\qquad - \beta_1\beta_3(d_j^2-d_i^2)^2 - \beta_0\beta_3\frac{(d_j^3-d_i^3)^2}{d_i d_j} - \beta_2\beta_3 d_i d_j (d_j-d_i)^2\bigg),\\
\mathcal{S}_{3,i} &= \sum_{\substack{j,k=1\\ i\neq j<k\neq i}}^{N} \frac{\phi_j\phi_k\pi_{ijk}^2}{(d_k^2\hat{\eta}_k - d_i^2\hat{\eta}_i)(d_j^2\hat{\eta}_j - d_i^2\hat{\eta}_i)}\bigg[-\beta_0\bigg(\beta_1\beta_2 + (\beta_1\beta_3+\beta_2\beta_3)\frac{\sigma_{ijk}\tilde{\sigma}_{ijk}}{d_i d_j d_k}\bigg) - \beta_1\beta_2\beta_3\bigg],\\
\mathcal{S}_{4,i} &= \sum_{\substack{j,k,l=1\\ j<k<l\\ j,k,l\neq i}}^{N} \frac{\phi_j\phi_k\phi_l\pi_{ijkl}^2\beta_0\beta_1\beta_2\beta_3}{(d_j^2\hat{\eta}_j - d_i^2\hat{\eta}_i)(d_k^2\hat{\eta}_k - d_i^2\hat{\eta}_i)(d_l^2\hat{\eta}_l - d_i^2\hat{\eta}_i) d_i d_j d_k d_l}.
\end{aligned} \tag{4.4}
$$

Clearly, $\mathcal{S}_{1,i} < 0$ for $\phi_i > 0$; in addition, $\mathcal{S}_{1,i}$ is independent of Φ or N. Related to the other terms, we suppose that there is a constant $\theta \geq 1$ such that

$$-\mathbf{s}_N^{\mathrm{T}}\Phi = \sum_{j=1}^{N}\left(-\sum_{\nu=0}^{3}\frac{\beta_\nu d_j^\nu}{d_N^\nu}\right)\phi_j \leq \frac{1}{1+\theta}. \tag{4.5}$$

Let us define $\mathbf{d}_\nu := (d_1^\nu, \dots, d_N^\nu)^{\mathrm{T}}$, $\nu = 0, \dots, 3$. Then (4.5) implies that

$$0 < \frac{1}{d_i^2\hat{\eta}_i - d_j^2\hat{\eta}_j} \leq \frac{1}{(1+\mathbf{s}_j^{\mathrm{T}}\Phi)(d_i^2 - d_j^2)} \leq \left(-\theta(d_i^2 - d_j^2)\sum_{\nu=0}^{3}\frac{\beta_\nu}{d_j^\nu}\mathbf{d}_\nu^{\mathrm{T}}\Phi\right)^{-1} \quad \text{for } i < j. \tag{4.6}$$

Of course, we can estimate the last term in (4.6) by omitting some of the summands. This result is the main tool for proving the next lemma.

Lemma 4.1. *The quantities $\mathcal{S}_{p,i}$, $p = 1, \dots, 4$ defined in* (4.4) *satisfy the following inequalities:*

$$\mathcal{S}_{2,i} \leq -\frac{d_i^2}{\theta}\left[2\beta_0 + \beta_1 + \beta_2 + \left(7 + \frac{3}{d_N}\right)\beta_3\right], \tag{4.7}$$

$$\mathcal{S}_{3,i} \leq -\frac{d_i^2}{\theta^2}\left[2\beta_0 + \beta_3\left(2 + \frac{9}{2d_N} + \frac{6}{d_N^3}\right)\right], \tag{4.8}$$

$$\mathcal{S}_{4,i} \leq -\frac{2d_i^2\beta_1}{\theta^3}. \tag{4.9}$$

Proof. Since $\hat{\eta}_i > \hat{\eta}_j$ for $i < j$, the summands of $\mathcal{S}_{2,i}$ with $j < i$ and $j > i$ are negative and positive, respectively; let us denote the corresponding partial sums by $\mathcal{S}_{2,i}^- \leq 0$ and $\mathcal{S}_{2,i}^+ \geq 0$, with $\mathcal{S}_{2,i} = \mathcal{S}_{2,i}^- + \mathcal{S}_{2,i}^+$. We start finding a bound $\mathcal{S}_{2,i}^+$ in such a way that this quantity is compensated by the terms of $\mathcal{S}_1$.

Let us now turn to $\mathcal{S}_{2,i}^+$. We here get

$$\begin{aligned}\mathcal{S}_{2,i}^+ \leq -\frac{1}{\theta}\sum_{j=i+1}^{N}\Biggl(&\frac{\beta_0\beta_1 d_i(d_i-d_j)^2 d_j^2\phi_j}{(d_i^2-d_j^2)\beta_1\mathbf{d}_1^{\mathrm{T}}\Phi} + \frac{\beta_0\beta_2(d_i+d_j)^2(d_i-d_j)^2 d_j^2\phi_j}{(d_i^2-d_j^2)\beta_2\mathbf{d}_2^{\mathrm{T}}\Phi} + \frac{\beta_1\beta_2 d_i(d_i-d_j)^2 d_j^2\phi_j}{\beta_1(d_i^2-d_j^2)\mathbf{d}_1^{\mathrm{T}}\Phi}\\ &+\frac{\beta_1\beta_3(d_j^2-d_i^2)^2 d_j^3\phi_j}{\beta_3(d_i^2-d_j^2)\mathbf{d}_3^{\mathrm{T}}\Phi} + \frac{\beta_0\beta_3(d_j^3-d_i^3)^2\phi_j}{\beta_0 d_i d_j(d_i^2-d_j^2)\phi} + \frac{\beta_2\beta_3 d_i(d_i-d_j)^2 d_j^3\phi_j}{\beta_2(d_i^2-d_j^2)\mathbf{d}_2^{\mathrm{T}}\Phi}\Biggr).\end{aligned} \tag{4.10}$$

To deal with the fifth term in the summands of (4.10), we note that

$$\frac{(d_j^3-d_i^3)^2}{(d_i^2-d_j^2)d_id_j} = \frac{(d_i-d_j)^2(d_i^2+d_id_j+d_j^2)^2}{(d_i^2-d_j^2)d_id_j} \leq (d_i^2+d_id_j+d_j^2)\left(1+\frac{d_i}{d_j}+\frac{d_j}{d_i}\right) \leq 3d_i^2\left(2+\frac{1}{d_N}\right).$$

Consequently, we obtain from (4.10) the following inequality, which implies (4.7):

$$\mathcal{S}_{2,i}^+ \leq -d_i^2\theta^{-1}\sum_{j=i+1}^{N}\left[\beta_0\left(\frac{d_j\phi_j}{\mathbf{d}_1^{\mathrm{T}}\Phi} + \frac{d_j^2\phi_j}{\mathbf{d}_2^{\mathrm{T}}\Phi}\right) + \beta_2\frac{d_j\phi_j}{\mathbf{d}_1^{\mathrm{T}}\Phi} + \beta_1\frac{d_j^3\phi_j}{\mathbf{d}_3^{\mathrm{T}}\Phi} + 3\bigl(2+d_N^{-1}\bigr)\beta_3\frac{\phi_j}{\phi} + \beta_3\frac{d_j^2\phi_j}{\mathbf{d}_2^{\mathrm{T}}\Phi}\right].$$

Since only those summands of $\mathcal{S}_{3,i}$ are positive for which either $i < j$ and $i < k$ or $i > j$ and $i > k$, we rewrite $\mathcal{S}_{3,i}$ as $\mathcal{S}_{3,i} = \mathcal{S}_{3,i}^- + \mathcal{S}_{3,i}^{+,1} + \mathcal{S}_{3,i}^{+,2}$, where $\mathcal{S}_{3,i}^- < 0$, $\mathcal{S}_{3,i}^{+,1} > 0$ and $\mathcal{S}_{3,i}^{+,2} > 0$, and $\mathcal{S}_{3,i}^{+,1}$ and $\mathcal{S}_{3,i}^{+,2}$ are the partial sums of $\mathcal{S}_{3,i}$ for which $j > i$, $k > i$ and $k \neq j$ and $j < i$, $k < i$ and $k \neq j$, respectively.

Applying several versions of (4.6) to both factors in the denominator of the summands of $\mathcal{S}_{3,i}^{+,1}$, we obtain

$$\begin{aligned}\mathcal{S}_{3,i}^{+,1} &\leq -\frac{1}{\theta^2}\sum_{\substack{j,k=i+1\\ j<k}}^{N}\frac{(\beta_0+\beta_3)\pi_{ijk}^2 d_j\phi_j d_k^2\phi_k}{(d_k^2-d_i^2)(d_j^2-d_i^2)\mathbf{d}_1^{\mathrm{T}}\Phi\mathbf{d}_2^{\mathrm{T}}\Phi} - \frac{\beta_3}{\theta^2}\sum_{\substack{j,k=i+1\\ j<k}}^{N}\frac{\phi_k\sigma_{ijk}\tilde{\sigma}_{ijk}\pi_{ijk}^2}{(d_k^2-d_i^2)(d_j^2-d_i^2)d_id_jd_k\phi}\left(\frac{d_j\phi_j}{\mathbf{d}_1^{\mathrm{T}}\Phi} + \frac{d_j^2\phi_j}{\mathbf{d}_2^{\mathrm{T}}\Phi}\right)\\ &\leq -(\beta_0+\beta_3)\frac{d_i^2}{\theta^2} - \frac{\beta_3}{\theta^2}\sum_{\substack{j,k=i+1\\ j<k}}^{N}\frac{\phi_k\sigma_{ijk}\tilde{\sigma}_{ijk}(d_i-d_j)(d_i-d_k)(d_j-d_k)^2}{(d_i+d_k)(d_i+d_j)d_id_jd_k\phi}\left(\frac{d_j\phi_j}{\mathbf{d}_1^{\mathrm{T}}\Phi} + \frac{d_j^2\phi_j}{\mathbf{d}_2^{\mathrm{T}}\Phi}\right).\end{aligned}$$

Noting that for $j,k>i$, we have that $\sigma_{ijk}\leq 3d_i$ and

$$\frac{\tilde{\sigma}_{ijk}}{(d_i+d_j)(d_i+d_k)}\leq\frac{3}{4},\quad (d_i-d_j)(d_i-d_k)\leq d_i^2,\quad \frac{(d_j-d_k)^2}{d_id_jd_k}\leq\frac{1}{d_N},\tag{4.11}$$

we finally obtain the inequality

$$\mathcal{S}_{3,i}^{+,1}\leq-\frac{d_i^2}{\theta^2}\left[\beta_0+\beta_3\left(1+\frac{9d_i}{2d_N}\right)\right]\leq-\frac{d_i^2}{\theta^2}\left[\beta_0+\beta_3\left(1+\frac{9}{2d_N}\right)\right].\tag{4.12}$$

Furthermore, using the order $d_j>d_k>d_i$, and the fact that $\sigma_{ijk}\leq 3d_j$ for $j<k<i$ and the version of (4.6) with the roles of i and j interchanged we have that

$$\begin{aligned}\mathcal{S}_{3,i}^{+,2}&\leq-\frac{1}{\theta^2}\sum_{\substack{j,k=1\\j<k}}^{i-1}\frac{(\beta_0+\beta_3)\pi_{ijk}^2d_i\phi_jd_i^2\phi_k}{(d_k^2-d_i^2)(d_j^2-d_i^2)\mathbf{d}_1^{\mathrm{T}}\Phi\mathbf{d}_2^{\mathrm{T}}\Phi}-\frac{\beta_3}{\theta^2}\sum_{\substack{j,k=1\\j<k}}^{i-1}\frac{\sigma_{ijk}\tilde{\sigma}_{ijk}\pi_{ijk}^2\phi_j}{(d_k^2-d_i^2)(d_j^2-d_i^2)d_id_jd_k\phi}\left(\frac{d_i\phi_k}{\mathbf{d}_1^{\mathrm{T}}\Phi}+\frac{d_i^2\phi_k}{\mathbf{d}_2^{\mathrm{T}}\Phi}\right)\\&\leq-\frac{\beta_0+\beta_3}{\theta^2\mathbf{d}_1^{\mathrm{T}}\Phi\mathbf{d}_2^{\mathrm{T}}\Phi}\sum_{\substack{j,k=1\\j<k}}^{i-1}(d_k-d_j)^2d_i\phi_jd_i^2\phi_k-\frac{\beta_3}{\theta^2}\sum_{\substack{j,k=1\\j<k}}^{i-1}\frac{\sigma_{ijk}(d_j-d_i)(d_j-d_k)\phi_j}{d_i\phi}\left(\frac{d_i\phi_k}{\mathbf{d}_1^{\mathrm{T}}\Phi}+\frac{d_i^2\phi_k}{\mathbf{d}_2^{\mathrm{T}}\Phi}\right)\\&\leq-\frac{d_i^2(\beta_0+\beta_3)}{\theta^2\mathbf{d}_1^{\mathrm{T}}\Phi\mathbf{d}_2^{\mathrm{T}}\Phi}\sum_{\substack{j,k=1\\j<k}}^{i-1}d_j^2\phi_jd_k\phi_k-\frac{3\beta_3}{\theta^2}\sum_{\substack{j,k=1\\j<k}}^{i-1}\frac{d_j^2d_j\phi_j}{d_i\phi}\left(\frac{d_i\phi_k}{\mathbf{d}_1^{\mathrm{T}}\Phi}+\frac{d_i^2\phi_k}{\mathbf{d}_2^{\mathrm{T}}\Phi}\right)\leq-\frac{d_i^2}{\theta^2}\left[\beta_0+\beta_3\left(1+\frac{6}{d_N^3}\right)\right].\end{aligned}$$

Combining this with (4.12) we obtain (4.8).

Finally, we rewrite $\mathcal{S}_{4,i}$ as $\mathcal{S}_{4,i}=\mathcal{S}_{4,i}^-+\mathcal{S}_{4,i}^{+,1}+\mathcal{S}_{4,i}^{+,2}$, where $\mathcal{S}_{4,i}^{+,1}$ is the sum of all summands of $\mathcal{S}_{4,i}$ for which exactly one factor in the denominator is positive, i.e., i is the second largest species, and $\mathcal{S}_{4,i}^{+,2}$ is the sum of all summands of $\mathcal{S}_{4,i}$ for which all three factors in the denominator are positive, i.e., $i>j$, $i>k$ and $i>l$, that is, i is the smallest species. To estimate $\mathcal{S}_{4,i}^{+,1}$, we first note that

$$\begin{aligned}\mathcal{S}_{4,i}^{+,1}&=\sum_{\substack{j,k,l=1\\j<i<k<l}}^{N}\frac{\phi_j\phi_k\phi_l\pi_{ijkl}^2\beta_0\beta_1\beta_2\beta_3}{(d_j^2\hat{\eta}_j-d_i^2\hat{\eta}_i)(d_k^2\hat{\eta}_k-d_i^2\hat{\eta}_i)(d_l^2\hat{\eta}_l-d_i^2\hat{\eta}_i)d_id_jd_kd_l}\\&\leq-\frac{\beta_1}{\theta^3\phi\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_3^{\mathrm{T}}\Phi}\sum_{\substack{j,k,l=1\\j<i<k<l}}^{N}\frac{\phi_j\phi_k\phi_l(d_k-d_j)^2(d_l-d_j)^2(d_l-d_k)^2d_k^2d_l^3}{d_id_jd_kd_l}\\&\leq-\frac{\beta_1}{\theta^3\phi\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_3^{\mathrm{T}}\Phi}\sum_{\substack{j,k,l=1\\j<i<k<l}}^{N}\frac{\phi_j\phi_k\phi_ld_j^4d_k^2d_k^2d_l^3}{d_id_jd_kd_l}\leq-\frac{\beta_1}{\theta^3\phi\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_3^{\mathrm{T}}\Phi}\sum_{\substack{j,k,l=1\\j<i<k<l}}^{N}\frac{\phi_j\phi_k\phi_ld_j^3d_k^3d_l^2}{d_i}\\&\leq-\frac{\beta_1d_i^2}{\theta^3\phi\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_3^{\mathrm{T}}\Phi}\sum_{\substack{j,k,l=1\\j<i<k<l}}^{N}\phi_j\phi_k\phi_ld_j^3d_k^2\leq-\frac{\beta_1d_i^2}{\theta^3}.\end{aligned}\tag{4.13}$$

By similar arguments we obtain the following estimate for $\mathcal{S}_{4,i}^{+,2}$:

$$\begin{aligned}\mathcal{S}_{4,i}^{+,2}&=\sum_{\substack{j,k,l=1\\j<k<l}}^{i-1}\frac{\phi_j\phi_k\phi_l\pi_{ijkl}^2\beta_0\beta_1\beta_2\beta_3}{(d_j^2\hat{\eta}_j-d_i^2\hat{\eta}_i)(d_k^2\hat{\eta}_k-d_i^2\hat{\eta}_i)(d_l^2\hat{\eta}_l-d_i^2\hat{\eta}_i)d_id_jd_kd_l}\\&\leq-\frac{\beta_1d_i^2}{\theta^3\phi\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_3^{\mathrm{T}}\Phi}\sum_{\substack{j,k,l=1\\j<k<l}}^{i-1}\frac{\phi_j\phi_k\phi_ld_i^2d_j^3d_k}{d_l}\leq-\frac{\beta_1d_i^2}{\theta^3\phi\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_3^{\mathrm{T}}\Phi}\sum_{\substack{j,k,l=1\\j<k<l}}^{i-1}\phi_j\phi_k\phi_ld_j^3d_k^2\leq-\frac{\beta_1d_i^2}{\theta^3}.\end{aligned}\tag{4.14}$$

Inequality (4.9) is now a consequence of (4.13) and (4.14). □

Corollary 4.1. *For the BW model, the following inequality is valid:*

$$\mathcal{S}_{1,i}+\mathcal{S}_{2,i}+\mathcal{S}_{3,i}+\mathcal{S}_{4,i}\leq d_i^2M(\theta,\boldsymbol{\beta},d_N),\tag{4.15}$$

where we define the function

$$\begin{aligned} M(\theta,\boldsymbol{\beta},d_N) := \left(1-\frac{2}{\theta}-\frac{2}{\theta^2}\right)\beta_0 + \left(1-\frac{1}{\theta}-\frac{2}{\theta^3}\right)\beta_1 + \left(1-\frac{1}{\theta}\right)\beta_2 \\ + \left[1-\frac{7}{\theta}-\frac{3}{d_N\theta}-\frac{1}{\theta^2}\left(2+\frac{9}{2d_N}+\frac{6}{d_N^3}\right)\right]\beta_3. \end{aligned} \tag{4.16}$$

Proof. Combining (4.7), (4.8) and (4.9) we obtain (4.15) and (4.16). Each of the inequalities (4.7), (4.8) and (4.9) estimates a non-negative sum from above, and therefore remains valid if the respective sum runs over a void index range, and is therefore zero. Consequently, (4.15) and (4.16) hold for arbitrary N. ☐

We have proved the following theorem.

Theorem 4.1. *Assume that θ is chosen such that for the smallest given particle size $d_N > 0$, the inequality*

$$M(\theta,\boldsymbol{\beta},d_N) < 0 \tag{4.17}$$

is satisfied, where $M(\theta,\boldsymbol{\beta},d_N)$ is defined in (4.16). *If the maximum solids concentration $\phi_{\max}$ is chosen such that the inequality* (4.5) *is satisfied for all $\Phi \in \mathcal{D}_{\phi_{\max}}$ for this value of θ, then $\gamma_i < 0$ for $i = 1,\dots,4$ and $\Phi \in \mathcal{D}_{\phi_{\max}}$, i.e., the model equations are strictly hyperbolic on $\mathcal{D}_{\phi_{\max}}$.*

We remark first that for a given value of d_N, it is always possible to make all coefficients of $\beta_0,\dots,\beta_3$ in (4.16) positive, and thereby to ensure that (4.17) holds, by choosing $\theta > 1$ large enough. On the other hand, the particular way in which d_N^{-1} appears in the coefficient of β_3 in (4.16) implies that in the case $\beta_3 < 0$, as we increase the particle size ratio, i.e. consider $d_N \to 0$, the smaller the set of admissible values of θ (that is, values of θ for which (4.17) holds) will become. Suppose that we choose an admissible value of θ, then (4.5) can hold either for a dilute suspension, i.e. ϕ is small, but for a large range of coefficients $\boldsymbol{\beta}$, or we consider relatively small (in absolute value) coefficients $\boldsymbol{\beta}$ and obtain a hyperbolicity (stability) result valid up to relatively large concentrations.

Furthermore, the strategy that has led to (4.16) has been motivated by the observation that $\beta_3 \leq 0$, but $|\beta_3| \ll 1$. In fact, we have performed the term cancellations and estimations in such a way that $1/d_N$, a potentially large number, appears only as a coefficient of β_3. We stress that in the case $\beta_3 = 0$, the set of admissible values of θ is independent of (the smallness of) d_N, see [15]. The present analysis also shows that for $N = 3$ species, $\mathcal{S}_{4,i} = 0$ and the terms in which we divide by θ^3 in (4.16) do not appear; for $N = 2$, we additionally have $\mathcal{S}_{3,i} = 0$ and the terms in which we divide by θ^2 are zero.

Since $M(\theta,\boldsymbol{\beta},d_N)$ is a strictly decreasing function of θ, we may uniquely solve $M(\theta,\boldsymbol{\beta},d_N) = 0$ for θ. Let us denote this solution by $\theta_{\min}$. Then $M(\theta,\boldsymbol{\beta},d_N) < 0$ for $\theta > \theta_{\min}$, but unless $\beta_3 = 0$ (see [15] for that case), $M(\theta,\boldsymbol{\beta},d_N)$, and therefore $\theta_{\min}$ depend on d_N, which we denote by $\theta_{\min} = \theta_{\min}(d_N)$. Therefore, for the purpose of determining the largest value ϕ^* of the total concentration ϕ up to which we can guarantee hyperbolicity, we can rewrite the left-hand side of (4.5) as $\sigma_1\phi_1 + \dots + \sigma_N\phi_N$, where we define

$$\sigma_j := -\beta_0 - \beta_1\frac{d_j}{d_N} - \beta_2\frac{d_j^2}{d_N^2} - \beta_3\frac{d_j^3}{d_N^3}.$$

Then the sought concentration ϕ^* solves the problem "minimize ϕ subject to $\sigma_1\phi_1 + \dots + \sigma_N\phi_N = (1 + \theta_{\min}(d_N))^{-1}$". Expressing ϕ_1 in terms of $\phi_2,\dots,\phi_N$ and ϕ, we can rewrite this equation as

$$\phi = \left(1-\frac{\sigma_2}{\sigma_1}\right)\phi_2 + \dots + \left(1-\frac{\sigma_N}{\sigma_1}\right)\phi_N + \frac{1}{\sigma_1(1+\theta_{\min}(d_N))}.$$

Since $\sigma_1 > \sigma_2 > \dots > \sigma_N$, the coefficients of $\phi_2,\dots,\phi_N$ on the right-hand side of this equation are all positive, and the minimum ϕ^* of ϕ is attained for $\phi_2 = \dots = \phi_N = 0$. Consequently, the value ϕ^* is given here by $\phi^* = (\sigma_1(\beta,d_N)(1+\theta_{\min}(d_N)))^{-1}$, where $M(\theta_{\min}(d_N),\boldsymbol{\beta},d_N) = 0$. As a numerical example, we consider the parameter vectors $\boldsymbol{\beta}$ given by (3.4). Figure 1 shows plots of ϕ^* as a function of d_N for the cases of large and small Péclet numbers.

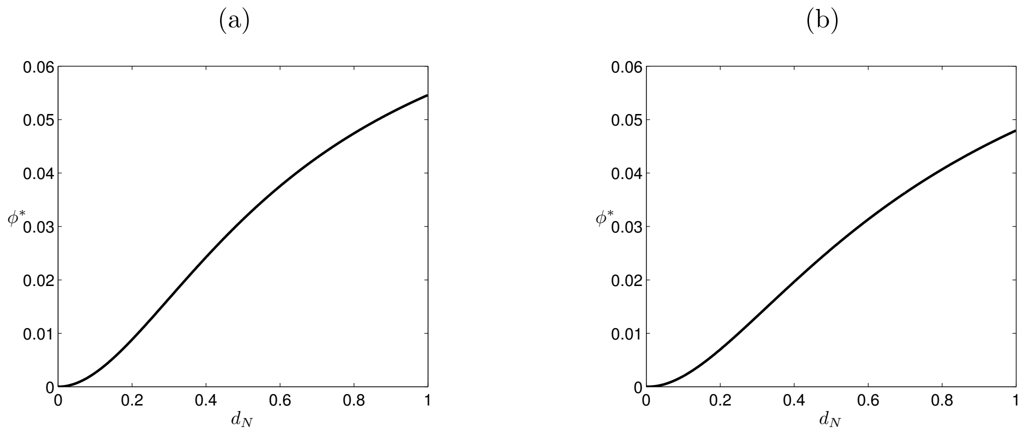

FIGURE 1. Maximum total concentrations ϕ^* for which hyperbolicity of the BW model is ensured for the coefficients (3.4) (a) for large Péclet numbers, (b) for small Péclet numbers.

4.3. **The Höfler and Schwarzer (HS) model.** Since $\eta_i = \eta_i'$ for this model, the coefficients γ_i of the secular equation given by (4.1) can be expressed as $\gamma_i = \phi_i(1-\phi)^n\eta_i(\mathcal{S}_{1,i}+\mathcal{S}_{2,i}+\mathcal{S}_{3,i}+\mathcal{S}_{4,i})$, where in terms of $\tilde{\eta}_i := \exp(\mathbf{s}_i^{\mathrm{T}}\Phi)$ we obtain for the HS model

$$\begin{aligned}
\mathcal{S}_{1,i} &= d_i^2(\tilde{\beta}_0+\beta_1+\beta_2+\beta_3),\\
\mathcal{S}_{2,i} &= \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\phi_j\tilde{\eta}_j}{d_j^2\tilde{\eta}_j - d_i^2\tilde{\eta}_i}\Biggl(-(d_i-d_j)^2\tilde{\beta}_0\bigl(\beta_1 d_i d_j + \beta_2(d_i+d_j)^2\bigr) - \tilde{\beta}_0\beta_3\frac{(d_i^3-d_j^3)^2}{d_i d_j}\\
&\qquad - \bigl(\beta_1\beta_2 d_i d_j (d_i-d_j)^2 + \beta_1\beta_3(d_i^2-d_j^2)^2 + \beta_2\beta_3 d_i d_j (d_i-d_j)^2\bigr)\Biggr),\\
\mathcal{S}_{3,i} &= \sum_{\substack{j,k=1\\ j<k,\, j,k\neq i}}^{N} \frac{\phi_j\phi_k\tilde{\eta}_j\tilde{\eta}_k\pi_{ijk}^2}{(d_k^2\tilde{\eta}_k - d_i^2\tilde{\eta}_i)(d_j^2\tilde{\eta}_j - d_i^2\tilde{\eta}_i)}\left[-\tilde{\beta}_0\left(\beta_1\beta_2 + (\beta_1\beta_3+\beta_2\beta_3)\frac{\sigma_{ijk}\tilde{\sigma}_{ijk}}{d_i d_j d_k}\right) - \beta_1\beta_2\beta_3\right],\\
\mathcal{S}_{4,i} &= \sum_{\substack{j,k,l=1\\ j<k<l,\, j,k,l\neq i}} \frac{\phi_j\phi_k\phi_l\tilde{\eta}_j\tilde{\eta}_k\tilde{\eta}_l\pi_{ijkl}^2\tilde{\beta}_0\beta_1\beta_2\beta_3}{(d_j^2\tilde{\eta}_j - d_i^2\tilde{\eta}_i)(d_k^2\tilde{\eta}_k - d_i^2\tilde{\eta}_i)(d_l^2\tilde{\eta}_l - d_i^2\tilde{\eta}_i)d_i d_j d_k d_l},
\end{aligned}$$

where we define

$$\tilde{\beta}_0 := \beta_0 - \frac{n\phi}{1-\phi}.$$

Next, we will prove some algebraic results which employ repeatedly the following inequality valid for $i<j$,

$$\frac{\tilde{\eta}_j}{d_i^2\tilde{\eta}_i - d_j^2\tilde{\eta}_j} \leq -\frac{1}{\mathrm{e}(d_i^2-d_j^2)}\left(\sum_{s=0}^{3}\beta_s\frac{d_i^s-d_j^s}{d_i^s d_j^s}\mathbf{d}_s^{\mathrm{T}}\Phi\right)^{-1}. \tag{4.18}$$

We first note that $\mathcal{S}_{1,i}<0$. Then we analyze the positive and negative parts of $\mathcal{S}_{2,i}$, $\mathcal{S}_{3,i}$ and $\mathcal{S}_{4,i}$ separately, and show that we eventually obtain $\gamma_i<0$.

Lemma 4.2. *Let us rewrite $\mathcal{S}_{2,i}$ as $\mathcal{S}_{2,i} = \mathcal{S}_{2,i}^+ + \mathcal{S}_{2,i}^-$, where $\mathcal{S}_{2,i}^+$ and $\mathcal{S}_{2,i}^-$ correspond to the summands of $\mathcal{S}_{2,i}$ with $j>i$ and $j<i$, respectively. Then $\mathcal{S}_{2,i}^-\leq 0$, and the following inequality holds:*

$$\mathcal{S}_{2,i}^+ \leq -\frac{d_i^2}{\mathrm{e}}\left[\left(1+3\frac{\beta_3}{d_N\beta_2}\right)\tilde{\beta}_0+\beta_1+\beta_2+\beta_3\right]. \tag{4.19}$$

Proof. Since $\exp(\mathbf{s}_i^{\mathrm{T}}\Phi) > \exp(\mathbf{s}_j^{\mathrm{T}}\Phi)$ for $i < j$ and $\exp(\mathbf{s}_i^{\mathrm{T}}\Phi) < \exp(\mathbf{s}_j^{\mathrm{T}}\Phi)$ for $i > j$, the factor multiplying $\{\dots\}$ in the summands of $\mathcal{S}_{2,i}^-$ is always positive, while $\{\dots\} < 0$. This confirms that $\mathcal{S}_{2,i}^- \leq 0$ (note that for $i = 1$, the sum is void, i.e. $\mathcal{S}_{2,i}^- = 0$). To estimate $\mathcal{S}_{2,i}^+$, note first that from (4.18) we may conclude that

$$\begin{aligned}
\mathcal{S}_{2,i}^+ &\leq -\frac{\tilde{\beta}_0}{\mathrm{e}}\sum_{j=i+1}^{N}\frac{(\beta_1 d_i d_j + \beta_2(d_i+d_j)^2)(d_i-d_j)^2\phi_j}{(d_i^2-d_j^2)\Bigl[\beta_1\dfrac{d_i-d_j}{d_id_j}\mathbf{d}_1^{\mathrm{T}}\Phi + \beta_2\dfrac{d_i^2-d_j^2}{d_i^2d_j^2}\mathbf{d}_2^{\mathrm{T}}\Phi\Bigr]} - \frac{\tilde{\beta}_0\beta_3}{\beta_2\mathrm{e}}\sum_{j=i+1}^{N}\frac{(d_i^3-d_j^3)^2 d_i^2d_j^2\phi_j}{d_id_j(d_i^2-d_j^2)^2\mathbf{d}_2^{\mathrm{T}}\Phi} \\
&\quad -\frac{1}{\mathrm{e}}\sum_{j=i+1}^{N}\left(\frac{\beta_2(d_i-d_j)^2d_i^2d_j^2\phi_j}{(d_i^2-d_j^2)(d_i-d_j)\mathbf{d}_1^{\mathrm{T}}\Phi} + \frac{\beta_1(d_i^2-d_j^2)^2d_i^3d_j^3\phi_j}{(d_i^2-d_j^2)(d_i^3-d_j^3)\mathbf{d}_3^{\mathrm{T}}\Phi} + \frac{\beta_3 d_id_j(d_i-d_j)^2d_i^2d_j^2\phi_j}{(d_i^2-d_j^2)^2\mathbf{d}_2^{\mathrm{T}}\Phi}\right) \\
&\leq -\frac{\tilde{\beta}_0}{\mathrm{e}}\left(d_i^2\sum_{j=i+1}^{N}\frac{d_j^2(\beta_1d_id_j+\beta_2(d_i+d_j)^2)\phi_j}{\beta_1d_id_j(d_i+d_j)\mathbf{d}_1^{\mathrm{T}}\Phi + \beta_2(d_i+d_j)^2\mathbf{d}_2^{\mathrm{T}}\Phi} + \frac{\beta_3}{\beta_2}\sum_{j=i+1}^{N}\frac{d_id_j(d_i^2+d_id_j+d_j^2)^2\phi_j}{(d_i+d_j)^2\mathbf{d}_2^{\mathrm{T}}\Phi}\right) \\
&\quad -\frac{d_i^2}{\mathrm{e}}\left(\sum_{j=i+1}^{N}\frac{\beta_2d_j^2\phi_j}{(d_i+d_j)\mathbf{d}_1^{\mathrm{T}}\Phi} + \sum_{j=i+1}^{N}\frac{\beta_1(d_i+d_j)d_id_j^3\phi_j}{(d_i^2+d_id_j+d_j^2)\mathbf{d}_3^{\mathrm{T}}\Phi} + \sum_{j=i+1}^{N}\frac{\beta_3d_id_j^3\phi_j}{(d_i+d_j)^2\mathbf{d}_2^{\mathrm{T}}\Phi}\right)
\end{aligned}$$

We may then continue estimating $\mathcal{S}_{2,i}^+$ as follows:

$$\mathcal{S}_{2,i}^+ \leq -\frac{d_i^2\tilde{\beta}_0}{\mathrm{e}}\sum_{j=i+1}^{N}\left(\frac{(\beta_1d_id_j+\beta_2(d_i+d_j)^2)d_j^2\phi_j}{(\beta_1d_id_j+\beta_2(d_i+d_j)^2)(d_{i+1}^2\phi_{i+1}+\dots+d_N^2\phi_N)} + \frac{3\beta_3}{d_N\beta_2}\frac{d_j^2\phi_j}{\mathbf{d}_2^{\mathrm{T}}\Phi}\right) - \frac{d_i^2}{\mathrm{e}}(\beta_1+\beta_2+\beta_3),$$

which implies (4.19). □

Lemma 4.3. *Let us we rewrite $\mathcal{S}_{3,i}$ as $\mathcal{S}_{3,i} = \mathcal{S}_{3,i}^- + \mathcal{S}_{3,i}^{+,1} + \mathcal{S}_{3,i}^{+,2}$, where $\mathcal{S}_{3,i}^{+,1}$ and $\mathcal{S}_{3,i}^{+;2}$ are the sums over all summands for which $j > i$, $k > i$ and $k \neq j$ and $j < i$, $k < i$ and $k \neq j$, respectively. Then we have $\mathcal{S}_{3,i}^- < 0$, $\mathcal{S}_{3,i}^{+,1} > 0$ and $\mathcal{S}_{3,i}^{+;2} > 0$. Furthermore, the following inequality holds:*

$$\mathcal{S}_{3,i}^{+,1} \leq -\frac{d_i^2}{\mathrm{e}^2}\left[\left(1+\frac{3\beta_3}{2\beta_1}+\frac{3\beta_3}{2\beta_2}\right)\tilde{\beta}_0 + \beta_3\right]. \tag{4.20}$$

Finally, let us assume that the parameters $\boldsymbol{\beta}$ are related to the vector of sizes $\mathbf{d}_1$ via the condition

$$\forall 1 \leq j < i \leq N: \quad \forall\phi\in[0,\phi_{\max}]: \quad \tilde{H}_{ij}(\phi,\boldsymbol{\beta}) < 0, \tag{4.21}$$

where we define the functions

$$\begin{aligned}
\tilde{H}_{ij}(\phi;\boldsymbol{\beta}) &:= -\tilde{\beta}_0\left(\beta_1d_id_j + \beta_2(d_i+d_j)^2 + \beta_3\frac{(d_i^2+d_id_j+d_j^2)^2}{d_id_j}\right) \\
&\qquad - \bigl(\beta_2(\beta_1+\beta_3)d_id_j + \beta_1\beta_3(d_i+d_j)^2\bigr) - \phi G_{ij}(\phi,\boldsymbol{\beta}), \\
G_{ij}(\phi,\boldsymbol{\beta}) &:= (d_j-d_i)^2\left\{\tilde{\beta}_0\left[\beta_1\beta_2 + (\beta_1\beta_3+\beta_2\beta_3)\left(1+2\frac{d_j}{d_i}\right)^2\right] + \beta_1\beta_2\beta_3\right\}.
\end{aligned} \tag{4.22}$$

Then

$$\mathcal{S}_{2,i}^- + \mathcal{S}_{3,i}^{+;2} \leq 0. \tag{4.23}$$

Proof. The inequalities $\mathcal{S}_{3,i}^- < 0$, $\mathcal{S}_{3,i}^{+,1} > 0$ and $\mathcal{S}_{3,i}^{+;2} > 0$ are a simple consequence of the fact that only those summands of $\mathcal{S}_{3,i}$ are positive for which either $i < j$ and $i < k$ or $i > j$ and $i > k$, according to the ordering $d_1 > d_2 > \dots > d_N$. To deal with

$$\mathcal{S}_{3,i}^{+,1} = \sum_{\substack{j,k=i+1 \\ j<k}}^{N}\frac{\phi_j\phi_k\tilde{\eta}_j\tilde{\eta}_k\pi_{ijk}^2}{(d_k^2\tilde{\eta}_k - d_i^2\tilde{\eta}_i)(d_j^2\tilde{\eta}_k - d_i^2\tilde{\eta}_i)}\left[-\tilde{\beta}_0\left(\beta_1\beta_2 + (\beta_1\beta_3+\beta_2\beta_3)\frac{\sigma_{ijk}\tilde{\sigma}_{ijk}}{d_id_jd_k}\right) - \beta_1\beta_2\beta_3\right],$$

note first that based on formulas similar to (4.18), we get

$$
\begin{aligned}
-\sum_{\substack{j,k=i+1\\ j<k}}^{N} \frac{\phi_j\phi_k\tilde{\eta}_j\tilde{\eta}_k\pi_{ijk}^2\tilde{\beta}_0\beta_1\beta_2}{(d_k^2\tilde{\eta}_k-d_i^2\tilde{\eta}_i)(d_j^2\tilde{\eta}_j-d_i^2\tilde{\eta}_i)} &\leq -\frac{d_i^2\tilde{\beta}_0}{\mathrm{e}^2}\sum_{\substack{j,k=i+1\\ j\neq k}}^{N} \frac{\phi_j\phi_k d_i^3 d_j d_k^2(d_i-d_j)^2(d_j-d_k)^2(d_k-d_i)^2}{(d_i+d_j)(d_i-d_j)^2(d_i+d_k)^2(d_i-d_k)^2\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_1^{\mathrm{T}}\Phi} \\
&\leq -\frac{d_i^2\tilde{\beta}_0}{\mathrm{e}^2}\sum_{\substack{j,k=i+1\\ j\neq k}}^{N} \frac{\phi_j\phi_k d_j d_k^2}{\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_1^{\mathrm{T}}\Phi} \leq -\frac{d_i^2\tilde{\beta}_0}{\mathrm{e}^2}.
\end{aligned}
$$

We may estimate the other terms in $\mathcal{S}_{3,i}^{+,1}$ as follows:

$$
\begin{aligned}
\mathcal{S}_{3,i}^{+,1,1} &:= -\sum_{\substack{j,k=i+1\\ j<k}}^{N} \frac{\phi_j\phi_k\tilde{\eta}_j\tilde{\eta}_k\tilde{\beta}_0\pi_{ijk}^2(\beta_1+\beta_2)\beta_3\sigma_{ijk}\tilde{\sigma}_{ijk}}{(d_k^2\tilde{\eta}_k-d_i^2\tilde{\eta}_i)(d_j^2\tilde{\eta}_j-d_i^2\tilde{\eta}_i)d_i d_j d_k} \\
&\leq -\frac{\tilde{\beta}_0(\beta_1+\beta_2)\beta_3}{\mathrm{e}^2\beta_1\beta_2\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_1^{\mathrm{T}}\Phi}\sum_{\substack{j,k=i+1\\ j<k}}^{N} \frac{d_i^3 d_k^2 d_j\sigma_{ijk}\tilde{\sigma}_{ijk}\pi_{ijk}^2\phi_j\phi_k}{(d_i^2-d_j^2)(d_i^2-d_k^2)d_i d_j d_k(d_i^2-d_k^2)(d_i-d_j)} \\
&= -\left(\frac{\beta_3}{\beta_1}+\frac{\beta_3}{\beta_2}\right)\frac{d_i^2\tilde{\beta}_0}{\mathrm{e}^2\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_1^{\mathrm{T}}\Phi}\sum_{\substack{j,k=i+1\\ j<k}}^{N} \frac{d_k\sigma_{ijk}\tilde{\sigma}_{ijk}(d_k-d_j)^2\phi_j\phi_k}{(d_i+d_j)(d_i+d_k)^2}.
\end{aligned}
$$

Now, taking into account (4.11) and that $\sigma_{ijk}/(d_i+d_k)\leq 2$ for $i<j,k$, we get

$$
\mathcal{S}_{3,i}^{+,1,1} \leq -\frac{3}{2\mathrm{e}^2}\left(\frac{\beta_3}{\beta_1}+\frac{\beta_3}{\beta_2}\right)\frac{d_i^2\tilde{\beta}_0}{\mathbf{d}_2^{\mathrm{T}}\Phi\mathbf{d}_1^{\mathrm{T}}\Phi}\sum_{\substack{j,k=i+1\\ j<k}}^{N} d_k d_j^2\phi_j\phi_k \leq -\frac{3d_i^2\tilde{\beta}_0}{2\mathrm{e}^2}\left(\frac{\beta_3}{\beta_1}+\frac{\beta_3}{\beta_2}\right). \tag{4.24}
$$

Inequality (4.20) now follows from (4.24) and

$$
\mathcal{S}_{3,i}^{+,1,2} \leq -\frac{\beta_3}{\mathrm{e}^2}\sum_{\substack{j,k=i+1\\ j\neq k}}^{N} \frac{\phi_j\phi_k\pi_{ijk}^2 d_i^3 d_k d_j^2}{(d_i^2-d_j^2)(d_i^2-d_k^2)(d_i-d_k)(d_i^2-d_j^2)\mathbf{d}_1^{\mathrm{T}}\Phi\mathbf{d}_2^{\mathrm{T}}\Phi} \leq -\frac{\beta_3 d_i^2}{\mathrm{e}^2},
$$

where

$$
\mathcal{S}_{3,i}^{+,1,2} = -\sum_{\substack{j,k=i+1\\ j<k}}^{N} \frac{\phi_j\phi_k\tilde{\eta}_j\tilde{\eta}_k\pi_{ijk}^2\beta_1\beta_2\beta_3}{(d_k^2\tilde{\eta}_k-d_i^2\tilde{\eta}_i)(d_j^2\tilde{\eta}_j-d_i^2\tilde{\eta}_i)}.
$$

Next, we analyze

$$
\mathcal{S}_{3,i}^{+,2} := \sum_{\substack{j,k=1\\ j<k}}^{i-1} \frac{\phi_j\phi_k\tilde{\eta}_j\tilde{\eta}_k\pi_{ijk}^2}{(d_k^2\tilde{\eta}_k-d_i^2\tilde{\eta}_i)(d_j^2\tilde{\eta}_j-d_i^2\tilde{\eta}_i)}\left[-\tilde{\beta}_0\left(\beta_1\beta_2+(\beta_1\beta_3+\beta_2\beta_3)\frac{\sigma_{ijk}\tilde{\sigma}_{ijk}}{d_i d_j d_k}\right)-\beta_1\beta_2\beta_3\right].
$$

As in the case of $\beta_3=0$, this term cannot be estimated easily and therefore will compensate it with $\mathcal{S}_{2,i}^{-}$, as expressed in (4.23). Notice that in order to ensure that our hyperbolicity result is also valid for $N=3$, $\mathcal{S}_{3,i}^{+,1}$ should be compensated by one of the terms that have arisen *earlier* in our analysis. Observe now that

$$
\mathcal{S}_{2,i}^{-}+\mathcal{S}_{3,i}^{+,2} = \sum_{j=1}^{i-1}\frac{\phi_j\tilde{\eta}_j(d_i-d_j)^2}{d_j^2\tilde{\eta}_j-d_i^2\tilde{\eta}_i}\mathcal{R}_{ij}, \tag{4.25}
$$

where we define

$$
\begin{aligned}
\mathcal{R}_{ij} &:= -\tilde{\beta}_0\left(\beta_1 d_i d_j+\beta_2(d_i+d_j)^2+\beta_3\frac{(d_i^2+d_i d_j+d_j^2)^2}{d_i d_j}\right)-\big(\beta_2(\beta_1+\beta_3)d_i d_j+\beta_1\beta_3(d_i+d_j)^2\big)+\tilde{\mathcal{R}}_{ij}, \\
\tilde{\mathcal{R}}_{ij} &:= -\sum_{k=j+1}^{i-1}\frac{\phi_k(d_k-d_i)^2(d_k-d_j)^2\tilde{\eta}_k}{d_k^2\tilde{\eta}_k-d_i^2\tilde{\eta}_i}\left[\tilde{\beta}_0\left(\beta_1\beta_2+(\beta_1\beta_3+\beta_2\beta_3)\frac{\sigma_{ijk}\tilde{\sigma}_{ijk}}{d_i d_j d_k}\right)+\beta_1\beta_2\beta_3\right].
\end{aligned}
$$

Since $d_i < d_j$ and $d_i < d_k$ in these summands, and the factor multiplying $\mathcal{R}_{ij}$ in (4.25) is positive, we will satisfy (4.23) by achieving that $\mathcal{R}_{ij} < 0$. Noting that for $j < k < i$

$$\begin{aligned}
&\frac{(d_k - d_i)^2 (d_k - d_j)^2 \tilde{\eta}_k}{d_k^2 \tilde{\eta}_k - d_i^2 \tilde{\eta}_i} = \frac{(d_k - d_i)^2 (d_k - d_j)^2}{d_k^2 - d_i^2 \exp((\mathbf{s}_i^{\mathrm{T}} - \mathbf{s}_k^{\mathrm{T}})\Phi)} \leq \frac{(d_k - d_j)^2 (d_k - d_i)}{d_k + d_i} \leq (d_j - d_i)^2, \\
&\frac{\sigma_{ijk} \tilde{\sigma}_{ijk}}{d_i d_j d_k} \leq \frac{(d_i + 2d_j)(d_i d_j + 2d_j^2)}{d_i^2 d_j} = \left(1 + 2\frac{d_j}{d_i}\right)^2,
\end{aligned}$$

and using the function $G_{ij}(\phi, \boldsymbol{\beta})$ we have that $\tilde{\mathcal{R}}_{ij} \leq -G_{ij}(\phi, \boldsymbol{\beta})(\phi_{j+1} + \phi_{j+2} + \cdots + \phi_{i-1})$. Thus, (4.23) holds if (4.21) is satisfied, where $\tilde{H}_{ij} := \tilde{H}_{ij}(\phi; \boldsymbol{\beta})$ is defined in (4.22). ☐

Lemma 4.4. *Let us rewrite $\mathcal{S}_{4,i}$ as $\mathcal{S}_{4,i} = \mathcal{S}_{4,i}^- + \mathcal{S}_{4,i}^{+,1} + \mathcal{S}_{4,i}^{+,2}$, where $\mathcal{S}_{4,i}^{+,1}$ is the sum of all summands of $\mathcal{S}_{4,i}$ for which $j < i < k < l$, and $\mathcal{S}_{4,i}^{+,2}$ is the sum of all summands of $\mathcal{S}_{4,i}$ for which $i > j$, $i > k$ and $i > l$. Then we have $\mathcal{S}_{4,i}^- \leq 0$, $\mathcal{S}_{4,i}^{+,1} \geq 0$ and $\mathcal{S}_{4,i}^{+,2} \geq 0$, and the following inequalities hold:*

$$\mathcal{S}_{4,i}^{+,1} \leq \frac{d_i^2 \tilde{\beta}_0 \beta_3 \phi}{\mathrm{e}^2 d_N}, \quad \mathcal{S}_{4,i}^{+,2} \leq \frac{4\tilde{\beta}_0 \beta_1 \beta_2 \beta_3 d_i^2}{27 d_N^4} \phi^3. \tag{4.26}$$

Proof. Utilizing the inequality (4.18) and performing cancellations and using the ordering $d_j > d_i > d_l > d_k$ in the summands, we get

$$\begin{aligned}
\mathcal{S}_{4,i}^{+,1} &\leq \frac{\tilde{\beta}_0 \beta_3}{\mathrm{e}^2 \mathbf{d}_1^{\mathrm{T}} \Phi \mathbf{d}_2^{\mathrm{T}} \Phi} \sum_{\substack{j,k,l=1 \\ j<i<k<l}}^{N} \frac{\phi_j \phi_k \phi_l \pi_{ijkl}^2 d_i d_k d_i^2 d_l^2}{(d_j^2 - d_i^2)(d_k^2 - d_i^2)(d_l^2 - d_i^2) d_i d_j d_k d_l (d_k - d_i)(d_l^2 - d_i^2)} \\
&\leq \frac{\tilde{\beta}_0 \beta_3 d_i^2}{\mathrm{e}^2 \mathbf{d}_1^{\mathrm{T}} \Phi \mathbf{d}_2^{\mathrm{T}} \Phi} \sum_{\substack{j,k,l=1 \\ j<i<k<l}}^{N} \frac{\phi_j \phi_k \phi_l d_l (d_j - d_i)(d_l - d_j)^2 (d_l - d_k)^2 (d_k - d_j)^2}{d_j (d_j + d_i)(d_k + d_i)(d_l + d_i)^2} \\
&\leq \frac{\tilde{\beta}_0 \beta_3 d_i^2}{\mathrm{e}^2 d_N \mathbf{d}_1^{\mathrm{T}} \Phi \mathbf{d}_2^{\mathrm{T}} \Phi} \sum_{\substack{j,k,l=1 \\ j<i<k<l}}^{N} \phi_j d_j^2 \phi_k \phi_l d_l,
\end{aligned}$$

which implies the first inequality in (4.26). Next, we employ the fact that

$$\frac{\tilde{\eta}_j}{d_j^2 \tilde{\eta}_j - d_i^2 \tilde{\eta}_i} \leq \frac{1}{d_j^2 - d_i^2} \quad \text{for } i > j$$

for calculating that

$$\begin{aligned}
\mathcal{S}_{4,i}^{+,2} &= \sum_{\substack{j,k,l=1 \\ j<k<l}}^{i-1} \frac{\phi_j \phi_k \phi_l \tilde{\eta}_j \tilde{\eta}_k \tilde{\eta}_l \pi_{ijkl}^2 \tilde{\beta}_0 \beta_1 \beta_2 \beta_3}{(d_j^2 \tilde{\eta}_j - d_i^2 \tilde{\eta}_i)(d_k^2 \tilde{\eta}_k - d_i^2 \tilde{\eta}_i)(d_l^2 \tilde{\eta}_l - d_i^2 \tilde{\eta}_i) d_i d_j d_k d_l} \leq \sum_{\substack{j,k,l=1 \\ j<k<l}}^{i-1} \frac{\phi_j \phi_k \phi_l \pi_{ijkl}^2 \tilde{\beta}_0 \beta_1 \beta_2 \beta_3}{(d_j^2 - d_i^2)(d_k^2 - d_i^2)(d_l^2 - d_i^2) d_i d_j d_k d_l} \\
&\leq \sum_{\substack{j,k,l=1 \\ j<k<l}}^{i-1} \frac{\phi_j \phi_k \phi_l (d_j - d_k)^2 (d_j - d_l)(d_k - d_l) \tilde{\beta}_0 \beta_1 \beta_2 \beta_3}{d_i d_l} \leq \frac{4}{27} \frac{\tilde{\beta}_0 \beta_1 \beta_2 \beta_3 d_i^2}{d_N^4} \sum_{\substack{j,k,l=1 \\ j<k<l}}^{i-1} \phi_j \phi_k \phi_l,
\end{aligned}$$

where the factor $4/27$ comes from a discussion of the maximum of the function $(d_j, d_j, d_l) \mapsto (d_j - d_k)^2 (d_j - d_l)(d_k - d_l)$ for $1 \geq d_j > d_k > d_l > 0$. This proves the second inequality in (4.26). ☐

Summarizing, and collecting the inequalities for the various terms, we see that

$$\begin{aligned}
\mathcal{S}_{1,i} + \mathcal{S}_{2,i} + \mathcal{S}_{3,i} + \mathcal{S}_{4,i} &= \mathcal{S}_{1,i} + \mathcal{S}_{2,i}^- + \mathcal{S}_{2,i}^+ + \mathcal{S}_{3,i}^- + \mathcal{S}_{3,i}^{+,1} + \mathcal{S}_{3,i}^{+,2} + \mathcal{S}_{4,i}^- + \mathcal{S}_{4,i}^{+,1} + \mathcal{S}_{4,i}^{+,2} \\
&< \mathcal{S}_{1,i} + \mathcal{S}_{2,i}^+ + \mathcal{S}_{2,i}^- + \mathcal{S}_{3,i}^{+,2} + \mathcal{S}_{3,i}^{+,1} + \mathcal{S}_{4,i}^{+,1} + \mathcal{S}_{4,i}^{+,2} \leq d_i^2 M(\phi, \boldsymbol{\beta}, d_N),
\end{aligned}$$

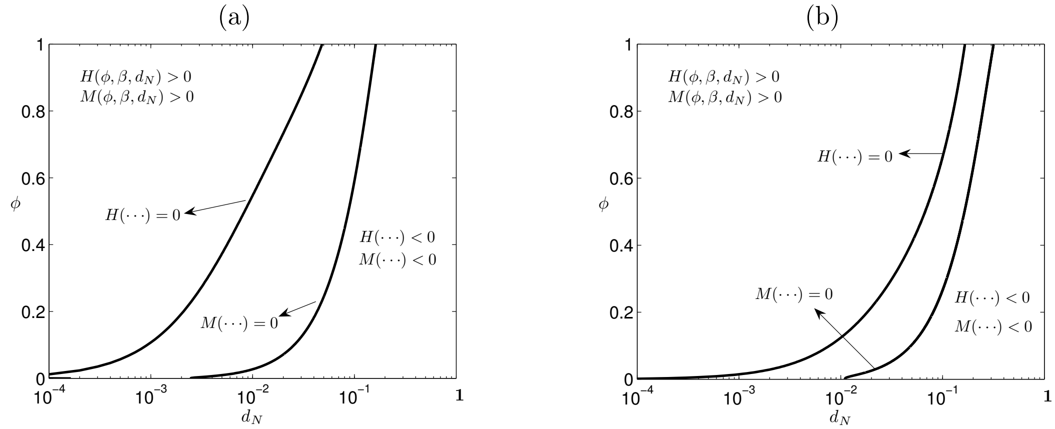

FIGURE 2. Regions of hyperbolicity ($H(\phi,\boldsymbol{\beta},d_N)<0$ and $M(\phi,\boldsymbol{\beta},d_N)<0$) for the HS model with the coefficients (3.4) (a) for large Péclet numbers, (b) for small Péclet numbers.

where we define the function

$$\begin{aligned} M(\phi,\boldsymbol{\beta},d_N) := \biggl[1+\frac{4\beta_1\beta_2\beta_3\phi^3}{27d_N^4}-\frac{1}{\mathrm{e}}\biggl(1+\frac{3\beta_3}{d_N\beta_2}\biggr)-\frac{1}{\mathrm{e}2}\biggl(1+\frac{3\beta_3}{2\beta_1}+\frac{3\beta_3}{2\beta_2}-\frac{\phi\beta_3}{d_N}\biggr)\biggr]\tilde{\beta}_0 \\ +\biggl(1-\frac{1}{\mathrm{e}}\biggr)(\beta_1+\beta_2)+\biggl(1-\frac{1}{\mathrm{e}}-\frac{1}{\mathrm{e}^2}\biggr)\beta_3. \end{aligned} \tag{4.27}$$

Instead of using (4.21) directly, which is not practical for large N, we provide a sufficient condition for (4.21) to be satisfied. To this end, we fix a pair $i>j$, define $\delta := d_i/d_j$, and divide (4.22) by d_j^2 to obtain

$$\begin{aligned} \tilde{H}_{ij} = -\tilde{\beta}_0\biggl[\beta_1\delta+\beta_2(1+\delta)^2+\beta_3(1+\delta+\delta^2)\biggl(1+\delta+\frac{1}{\delta}\biggr)\biggr]-\bigl(\beta_2(\beta_1+\beta_3)\delta+\beta_1\beta_3(1+\delta)^2\bigr) \\ -\phi(1-\delta)^2\biggl\{\tilde{\beta}_0\biggl[\beta_1\beta_2+(\beta_1+\beta_2)\beta_3\biggl(1+\frac{2}{\delta}\biggr)\biggr]+\beta_1\beta_2\beta_3\biggr\}. \end{aligned} \tag{4.28}$$

Since $\delta\in(d_N,1]$ a sufficient condition for (4.21) to be satisfied is given by

$$\forall\phi\in[0,\phi_{\max}]:\quad H(\phi,\boldsymbol{\beta},d_N)<0, \tag{4.29}$$

where the following definition of $H(\phi,\boldsymbol{\beta},d_N)$ is derived from the observation that the two terms in the first line of (4.28) are non-positive, while the term in the second line is non-negative:

$$\begin{aligned} H(\phi,\boldsymbol{\beta},d_N) := -\tilde{\beta}_0\bigl(\beta_1 d_N+\beta_2(1+d_N)^2+\beta_3(1+d_N+d_N^2)(2+d_N)\bigr) \\ -\bigl(\beta_2(\beta_1+\beta_3)d_N+\beta_1\beta_3(1+d_N)^2\bigr) \\ -\phi(1-d_N)^2\biggl\{\tilde{\beta}_0\biggl[\beta_1\beta_2+(\beta_1+\beta_2)\beta_3\biggl(1+\frac{2}{d_N}\biggr)\biggr]+\beta_1\beta_2\beta_3\biggr\}. \end{aligned} \tag{4.30}$$

Theorem 4.2. *Assume that the parameters $\boldsymbol{\beta}$, the maximum solids concentration $\phi_{\max}$ and the width of the particle size distribution given by $d_N\in(0,1]$ are chosen such that* (4.29) *is satisfied, where $H(\phi,\boldsymbol{\beta},d_N)$ is defined by* (4.30)*, and that $M(\phi,\boldsymbol{\beta},d_N)<0$ for all $\phi\in[0,\phi_{\max}]$, where the function $M(\phi,\boldsymbol{\beta},d_N)$ is defined in* (4.27)*. Then $\gamma_i<0$ for $i=1,\dots,N$, i.e., the HS model is strictly hyperbolic for $\Phi\in\mathcal{D}_{\phi_{\max}}$.*

As an example of the case $\beta_3<0$, consider the parameter vectors $\boldsymbol{\beta}$ given by (3.4); let us focus first on the case of large Péclet numbers. Figure 2 (a) shows in a ϕ versus d_N plot the curves $H(\phi,\boldsymbol{\beta},d_N)=0$ and $M(\phi,\boldsymbol{\beta},d_N)=0$. The region $H(\dots)<0, M(\dots)<0$, where the model is strictly hyperbolic, is located to the right of the curve $M(\phi,\boldsymbol{\beta},d_N)=0$. Here we employ a logarithmic scale since the term d_N^{-1} in (4.30) becomes singular. Solving $M(1,\boldsymbol{\beta},d_N)=0$ for d_N yields here that $M(1,\boldsymbol{\beta},d_N)<0$ for $d_N>d_N^*:=0.164092$, which means that for these values of d_N, the HS model is strictly hyperbolic on $\mathcal{D}_{\phi_{\max}}$ for all $\phi_{\max}\in(0,1]$. The behaviour of the curve $M(\phi,\boldsymbol{\beta},d_N)=0$ indicates that this property remains valid for slightly smaller values of d_N provided that $\phi_{\max}$ is chosen sufficiently small.

For the parameters given by (3.4) for small Péclet numbers, the behaviour is similar, as can be seen from Figure 2 (b), but the hyperbolicity region is smaller. We obtain unconditional hyperbolicity for $d_N>d_N^*:=0.328981$; this number is the solution of $M(1,\boldsymbol{\beta},d_N)=0$.

5. Hyperbolicity analysis of the Davis and Gecol model

We analyze the DG model under the assumption that $\beta_3=0$. For $i=1,\dots,N$ we define $\eta_i=1+\mathbf{s}_i^{\mathrm{T}}\Phi+n\phi$ with $n=-S_{ii}$. For this model the matrices $\mathbf{A}=(\alpha_i^k)$ and $\mathbf{B}=(\beta_i^k)$ are given by

$$\alpha_i^k=d_i^{k-1},\quad \beta_i^k=\begin{cases}d_i^2\phi_i(1-\phi)^{n-1}\bigl((1-\phi)(\beta_0+n)-n\eta_i\bigr) & \text{for } k=1,\\ d_i^{3-k}\phi_i(1-\phi)^n\beta_{k-1}, & \text{for } k=2,3.\end{cases}$$

Now, taking into account that for this model,

$$v_j-v_i=(1-\phi)^n\bigl(d_j^2\eta_j-d_i^2\eta_i\bigr)=(1-\phi)^n\bigl(d_j^2-d_i^2\bigr)\left(1-(\beta_1+\beta_2)\phi+\frac{\beta_1\mathbf{d}_1^{\mathrm{T}}\Phi}{d_i+d_j}\right),$$

we obtain the following coefficients for the secular equation

$$\begin{aligned}\gamma_i&=\alpha_i^1\beta_i^1+\alpha_i^2\beta_i^2+\alpha_i^3\beta_i^3+\sum_{\substack{j=1\\ j\neq i}}^N\frac{\alpha_{ij}^{12}\beta_{ij}^{12}+\alpha_{ij}^{13}\beta_{ij}^{13}+\alpha_{ij}^{23}\beta_{ij}^{23}}{v_j-v_i}+\sum_{\substack{j,k=1\\ i\neq j<k\neq i}}^N\frac{\alpha_{ijk}^{123}\beta_{ijk}^{123}}{(v_k-v_i)(v_j-v_i)}\\ &=\phi_i(1-\phi)^{n-1}(\mathcal{S}_{1,i}+\mathcal{S}_{2,i}+\mathcal{S}_{3,i}),\quad i=1,\dots,N,\end{aligned}$$

where

$$\begin{aligned}\mathcal{S}_{1,i}&=-nd_i^2\eta_i=d_i^2(\beta_0+\beta_1+\beta_2)\left(1-(\beta_1+\beta_2)\phi+\frac{\beta_1}{d_i}\mathbf{d}_1^{\mathrm{T}}\Phi+\frac{\beta_2}{d_i^2}\mathbf{d}_2^{\mathrm{T}}\Phi\right),\\ \mathcal{S}_{2,i}&=\sum_{\substack{j=1\\ j\neq i}}^N\frac{\phi_j(d_i-d_j)}{(d_i+d_j)\left[1-(\beta_1+\beta_2)\phi+\dfrac{\beta_1\mathbf{d}_1^{\mathrm{T}}\Phi}{d_i+d_j}\right]}\mathcal{S}_{2,i,j},\\ \mathcal{S}_{2,i,j}&:=\bigl(\beta_1d_id_j+\beta_2(d_i+d_j)^2\bigr)\bigl[(1-\phi)(\beta_0+n)-n\bigl(1-(\beta_1+\beta_2)\phi\bigr)\bigr]\\ &\quad+\beta_1\beta_2\bigl[n\bigl(\mathbf{d}_2^{\mathrm{T}}\Phi-(d_i+d_j)\mathbf{d}_1^{\mathrm{T}}\Phi\bigr)+(1-\phi)d_id_j\bigr],\\ \mathcal{S}_{3,i}&=\sum_{\substack{j,k=1\\ i\neq j<k\neq i}}^N-\frac{\phi_j\phi_k\pi_{ijk}^2\beta_1\beta_2\left[\beta_0(1-\phi)-n\phi(1-(\beta_1+\beta_2))\right]}{(d_k^2-d_i^2)(d_j^2-d_i^2)\left[1-(\beta_1+\beta_2)\phi+\dfrac{\beta_1\mathbf{d}_1^{\mathrm{T}}\Phi}{d_i+d_k}\right]\left[1-(\beta_1+\beta_2)\phi+\dfrac{\beta_1\mathbf{d}_1^{\mathrm{T}}\Phi}{d_i+d_j}\right]}.\end{aligned}$$

Although this model does not allow for term cancellations as for the BW and HS models, it is still possible to deduce that the model is strictly hyperbolic on $\mathcal{D}_{\phi_{\max}}$ for realistically large values of $\phi_{\max}$ provided that d_N is sufficiently close to one. Our analysis leads here to a narrow size distribution only. The salient point is, however, that our bounds for $d_N<1$ are independent of N. Here, we can prove the following result.

Theorem 5.1. *Assume that the parameters d_N, $\phi_{\max}$ and $\boldsymbol{\beta}=(\beta_0,\beta_1,\beta_2)^{\mathrm{T}}$ satisfy*

$$d_N>1/2, \tag{5.1}$$

$$1+\left[\beta_1\left(\frac{1}{d_N}-1\right)+\beta_2\left(\frac{1}{d_N^2}-1\right)\right]\phi_{\max}>0. \tag{5.2}$$

Then the model is strictly hyperbolic for all $\phi\in\mathcal{D}_{\phi_{\max}}$ provided that

$$\mathcal{S}(\phi,d_N;\boldsymbol{\beta})<0\quad\text{for all }\phi\in(0,\phi_{\max}], \tag{5.3}$$

where we define

$$\mathcal{S}(\phi, d_N; \boldsymbol{\beta}) := -n\left\{1+\left[\beta_1\left(\frac{1}{d_N}-1\right)+\beta_2\left(\frac{1}{d_N^2}-1\right)\right]\phi\right)+\frac{1-d_N}{2d_N}C_1\phi+\frac{(1-d_N)^4}{4d_N^4}C_2\phi^2. \tag{5.4}$$

The constants C_1 and C_2 are given by

$$C_1 := (\beta_1+4\beta_2)\big(\beta_0+(\beta_1+\beta_2)\phi_{\max}\big)+n\phi_{\max}\big[\beta_1(\beta_1+\beta_2)+4\beta_2^2\big]+\beta_1\beta_2\left\{\phi_{\max}\left[n\left(4+\frac{1}{d_N^2}\right)-1\right]+1\right\},$$

$$C_2 := \beta_1\beta_2\big(\beta_0-n\phi_{\max}(1-(\beta_1+\beta_2))\big).$$

Proof. We first note that for all $i=1,\dots,N$ and all $\Phi\in\mathcal{D}_{\phi_{\max}}$ the following inequality holds:

$$\eta_i = 1+\mathbf{s}_i^{\mathrm{T}}\Phi+n\phi = 1+\sum_{j=1}^{N}\sum_{k=1}^{2}\beta_k\left(\frac{d_j^k}{d_i^k}-1\right)\phi_j \geq 1+\left[\beta_1\left(\frac{1}{d_N}-1\right)+\beta_2\left(\frac{1}{d_N^2}-1\right)\right]\phi_{\max},$$

so (5.2) ensures that always $\eta_i>0$, and therefore $\mathcal{S}_{1,i}<0$. Observe that (5.2) holds if d_N is chosen sufficiently close to one, or $\phi_{\max}$ is sufficiently small. Next, a straightfoward calculation, and utilizing that

$$(1-\phi)(\beta_0+n)-n\big(1-(\beta_1+\beta_2)\phi\big)=\beta_0(1-\phi)-n\phi(1-(\beta_1+\beta_2)),$$

yields $\mathcal{S}_{2,i,j}=d_i^2\tilde{\mathcal{S}}_{2,i,j}=d_i^2\tilde{\mathcal{S}}_{2,i,j}(\boldsymbol{\beta},\mathbf{d},\Phi)$, where

$$\begin{aligned}\tilde{\mathcal{S}}_{2,i,j} = {} & \left[\beta_1\frac{d_j}{d_i}+\beta_2\left(1+\frac{d_j}{d_i}\right)^2\right]\big(\beta_0(1-\phi)-n\phi\big)+n\phi\left[\beta_1(\beta_1+\beta_2)\frac{d_j}{d_i}+\beta_2^2\left(1+\frac{d_j}{d_i}\right)^2\right]\\ & +\beta_1\beta_2\left\{n\left(1+\frac{d_j}{d_i}\right)\left[\phi\left(1+\frac{d_j}{d_i}\right)-\frac{\mathbf{d}_1^{\mathrm{T}}\Phi}{d_i}\right]+n\frac{\mathbf{d}_2^{\mathrm{T}}\Phi}{d_i^2}+(1-\phi)\frac{d_j}{d_i}\right\}.\end{aligned} \tag{5.5}$$

A sufficient condition for $\tilde{\mathcal{S}}_{2,i,j}>0$ to hold for all Φ, and without further restrictions on β_0, β_1 and β_2, is that the expression in the curled bracket is positive, i.e.,

$$\phi\left(1+\frac{d_j}{d_i}\right)-\frac{\mathbf{d}_1^{\mathrm{T}}\Phi}{d_i}=\sum_{l=1}^{N}\left(1+\frac{d_j-d_l}{d_i}\right)\phi_l>0. \tag{5.6}$$

A sufficient condition for (5.6) to hold for all vectors Φ is that the coefficients of ϕ_l for all $i,j,l\in\{1,\dots,N\}$ are positive. This occurs if and only if $1-(1-d_N)/d_N>0$, or equivalently, (5.1) is satisfied.

Assume now that $\mathcal{S}_{2,i,j}>0$, and note that for $d_N>1/2$, we have that

$$1-(\beta_1+\beta_2)\phi+\frac{\beta_1\mathbf{d}_1^{\mathrm{T}}\Phi}{d_i+d_j}>1+\left[\beta_1\left(\frac{1}{2d_N}-1\right)-\beta_2\right]\phi>1 \quad \text{for } \phi\in[0,\phi_{\max}].$$

Then we need to estimate $\mathcal{S}_{2,i}^{+}$, which (as in the BW and HS models) is the partial sum of all positive summands of $\mathcal{S}_{2,i}$, that is,

$$\mathcal{S}_{2,i}^{+}=d_i^2\sum_{j=i+1}^{N}\frac{\phi_j(d_i-d_j)\tilde{S}_{2,i,j}}{(d_i+d_j)\left(1-(\beta_1+\beta_2)\phi+\dfrac{\beta_1\mathbf{d}_1^{\mathrm{T}}\Phi}{d_i+d_j}\right)}.$$

In light of our previous assumptions and considerations, we obtain

$$\mathcal{S}_{2,i}^{+}\leq\frac{d_i^2(1-d_N)\phi}{2d_N}\max_{i<j\leq N}\tilde{S}_{2,i,j}.$$

However, from (5.5) and (5.1) we get that

$$\max_{i<j\leq N}\tilde{\mathcal{S}}_{2,i,j}\leq(\beta_1+4\beta_2)\big(\beta_0+(\beta_1+\beta_2)\phi\big)+n\phi\big(\beta_1(\beta_1+\beta_2)+4\beta_2^2\big)+\beta_1\beta_2\left[n\phi\left(4+\frac{1}{d_N^2}\right)+1-\phi\right]\leq C_1.$$

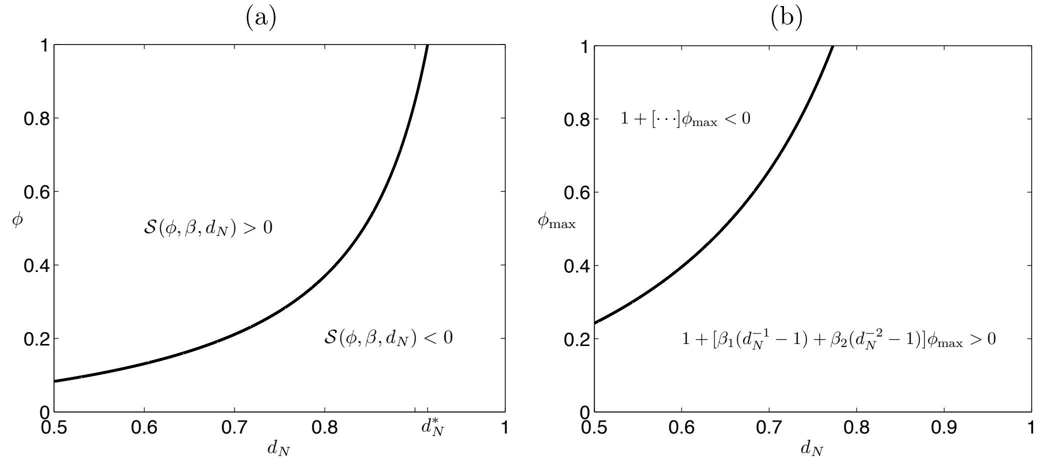

FIGURE 3. (a) Region of hyperbolicity ($\mathcal{S}(\phi,\boldsymbol{\beta},d_N)<0$) for the DG model and (b) the diagram with the region where the condition (5.2) holds with $\boldsymbol{\beta}$ given by (3.5).

Finally, similar considerations for $\mathcal{S}_{3,i}$ and noting that

$$\frac{\pi_{ijk}^2}{(d_j^2-d_i^2)(d_k^2-d_i^2)}=d_i^2\frac{(d_j-d_i)(d_k-d_i)(d_k-d_j)^2}{d_i^2(d_i+d_j)(d_i+d_k)}\leq d_i^2\frac{(1-d_N)^4}{4d_N^4}$$

lead to

$$\mathcal{S}_{3,i}^+\leq-\frac{d_i^2(1-d_N)^4\phi_{\max}^2C_2}{4d_N^4}.$$

Summarizing, we see that $\mathcal{S}_{1,i}+\mathcal{S}_{2,i}+\mathcal{S}_{3,i}\leq d_i^2\mathcal{S}(\phi,d_N;\boldsymbol{\beta})$, where $\mathcal{S}(\phi,d_N;\boldsymbol{\beta})$ is defined in (5.4). Thus, we conclude that for given parameters d_N, $\phi_{\max}$ and $\boldsymbol{\beta}$ we have $\gamma_i<0$ for all $i=1,\dots,N$ on $\mathcal{D}_{\phi_{\max}}$, and therefore hyperbolicity, provided that $\mathcal{S}(\phi,d_N;\boldsymbol{\beta})<0$ for all $\phi\in(0,\phi_{\max}]$. □

Figure 3 (a) illustrates the hyperbolicity region defined by (5.3) for this model. We limit the discussion here to $d_N>1/2$, and it can be verified straightfowardly that for all pairs $(d_N,\phi_{\max})$ that lie in the displayed region $\mathcal{S}(\phi,\boldsymbol{\beta},d_N)<0$, also (5.2) is satisfied, as can be noticed in Figure 3 (b). We observe that the larger $\phi=\phi_{\max}$ is chosen, the closer d_N needs to be chosen near one, i.e., the narrower the size distribution must be to ensure hyperbolicity. In the most extreme case, for $\phi_{\max}=1$, hyperbolicity can be observed only for $d_N>d_N^*=0.914022$; this value is the revelant root of $\mathcal{S}(1,\boldsymbol{\beta},d_N)=0$. Consequently, hyperbolicity, and therefore stability, can be ensured for the DG model only if the suspension is nearly monodisperse, a result that sharply contrasts with the HS model. This result is, however, independent of the number of species N.

6. NUMERICAL EXAMPLES AND CONCLUSIONS

In all numerical experiments the number of cells has been set to $M=800$ and the time step in (2.3) to advance the solution from t_n to $t_{n+1}=t_n+\Delta t$ has been selected as

$$\Delta t=0.5\frac{\Delta x}{\max_{j,k}|\alpha_{j+1/2}^k|},$$

where $\alpha_{j+1/2}^k$ is given by (2.4) based on the numerical solution $\{\Phi_j^n\}_{j\in\mathbb{Z}}$ at time t_n.

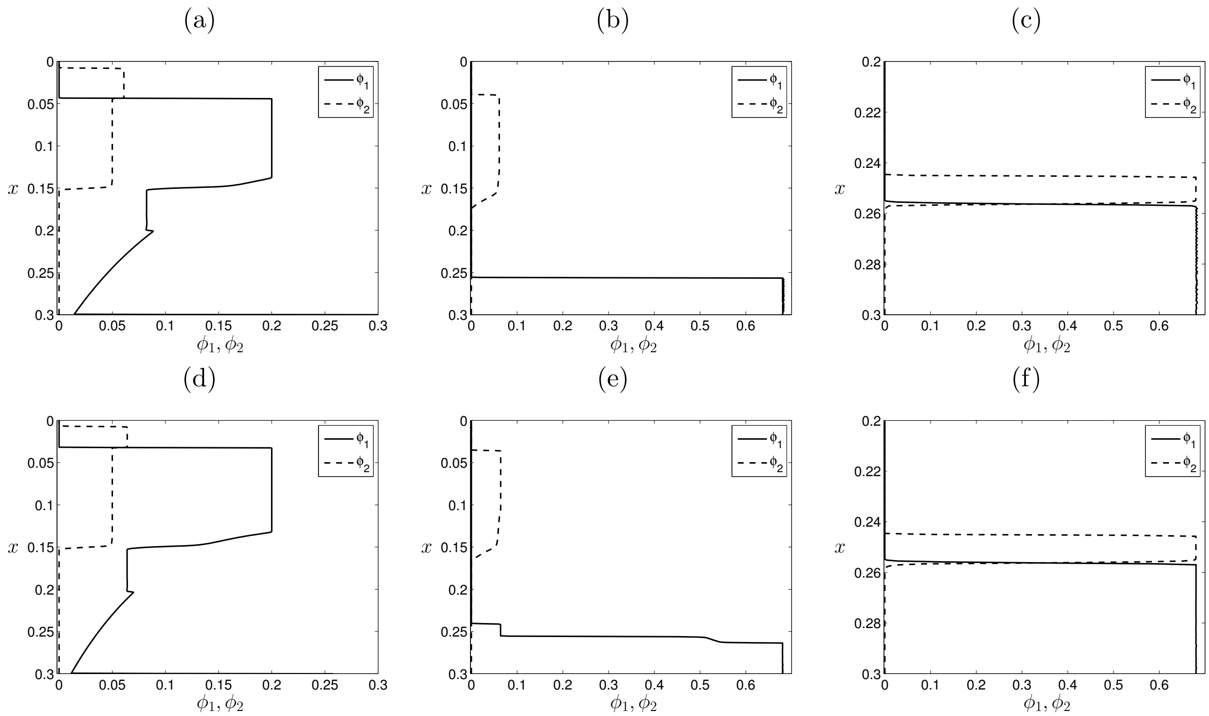

FIGURE 4. Example 1 (HS model, $N = 2$): numerical solution at (a, d) $t = 20\,\mathrm{s}$, (b, e) $t = 100\,\mathrm{s}$ and (c, f) $t = 800\,\mathrm{s}$ with $\boldsymbol{\beta}$ given by (a, b, c) (3.4) (for large Péclet numbers) and (d, e, f) (3.7).

6.1. **Example 1.** In this example we use the HS model with two sets of parameters $\boldsymbol{\beta}$, namely (3.4) (for large Péclet numbers) and (3.7), for which $\beta_3 < 0$ and $\beta_3 > 0$, respectively. We consider the initial datum

$$\Phi(x,0) = \begin{cases} \Phi_0 & \text{if } x \le 0.15, \\ 0 & \text{if } x > 0.15 \end{cases} \tag{6.1}$$

defined for a settling column of unnormalized depth 0.3 m. For this example we take $\phi_{\max} = 0.68$ and $\Phi_0 = (0.2, 0.05)^{\mathrm{T}}$, $\mathbf{d_2} = (1, 0.06351213)^{\mathrm{T}}$ for $N = 2$ and $\Phi_0 = (0.05, 0.05, 0.05, 0.05)^{\mathrm{T}}$, $\mathbf{d_2} = (1, 0.64, 0.36, 0.16)^{\mathrm{T}}$ and $\phi_{\max} = 0.6$ for $N = 4$. The value of δ_2 for $N = 2$, Φ_0, and the depth of the settling column have been chosen according to experimental data by Schneider et al. [42] so that results may be compared. As can be seen in Figures 4 and 5, the results obtained with the numerical schemes have sharp profiles and do not show any oscillatory behavior. Moreover, although some monor differences between the solutions with different choices of $\boldsymbol{\beta}$ are visible, results are very similar.

6.2. **Example 2.** In this example we compare the predictions by the DG, BW and HS models using small data (coresponding to an initially dilute suspension) in order to guarantee that the three models are well defined and hyperbolic. For this test we take $\phi_{\max} = 0.04$, $\Phi_0 = (0.005, 0.005, 0.005, 0.005)^{\mathrm{T}}$, $\mathbf{d_2} = (1, 0.75, 0.5, 0.26)^{\mathrm{T}}$ and the initial datum (6.1). The parameters $\boldsymbol{\beta}$ are given by (3.5). From the results displayed in Figure 6 we notice the close agreement of the three models in the early stages of sedimentation, where the suspension is still quite dilute. Note that the plots of Figure 6 are strongly enlarged views of the numerical solution. The increase of the concentration beyond the initial one in the three layers that form between the bulk suspension at the initial composition and the clear liquid is a well-known phenomenon, the so-called Smith effect [43]. See e.g. [6] for further details.

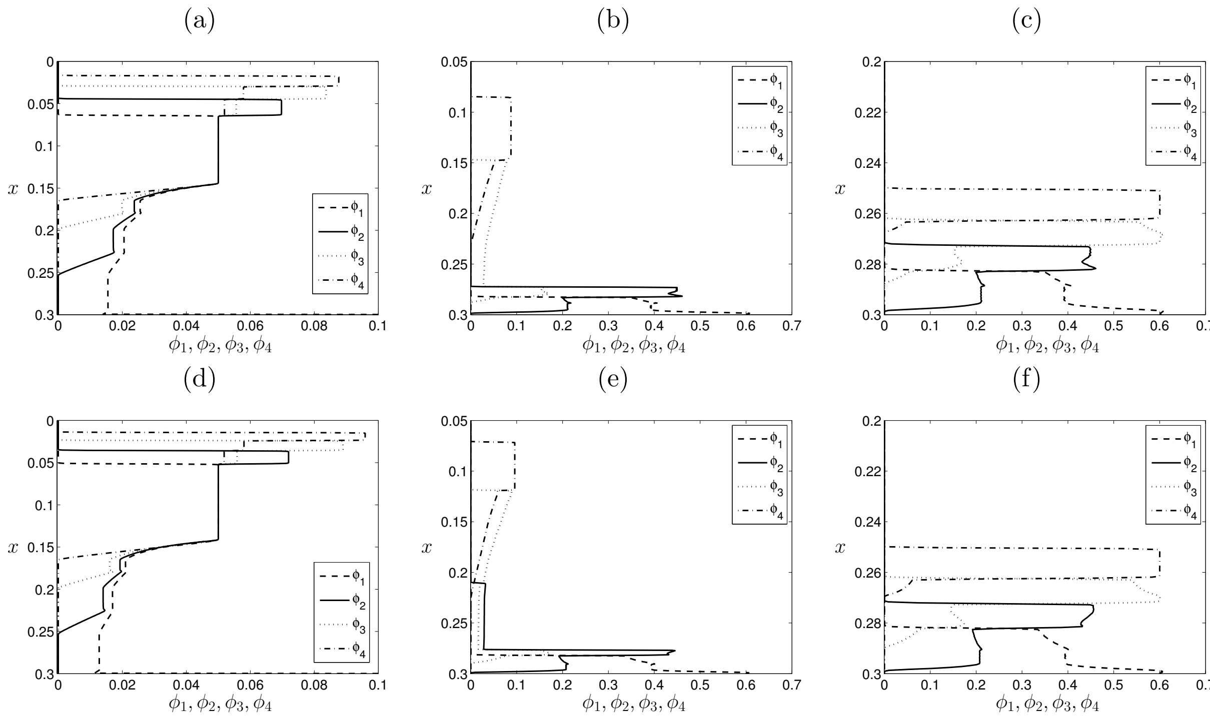

FIGURE 5. Example 1 (HS model, $N = 4$): numerical solution at (a, d) $t = 20\,\mathrm{s}$, (b, e) $t = 100\,\mathrm{s}$ and (c, f) $t = 800\,\mathrm{s}$ with $\boldsymbol{\beta}$ given by (a, b, c) (3.4) (for large Péclet numbers) and (d, e, f) (3.7).

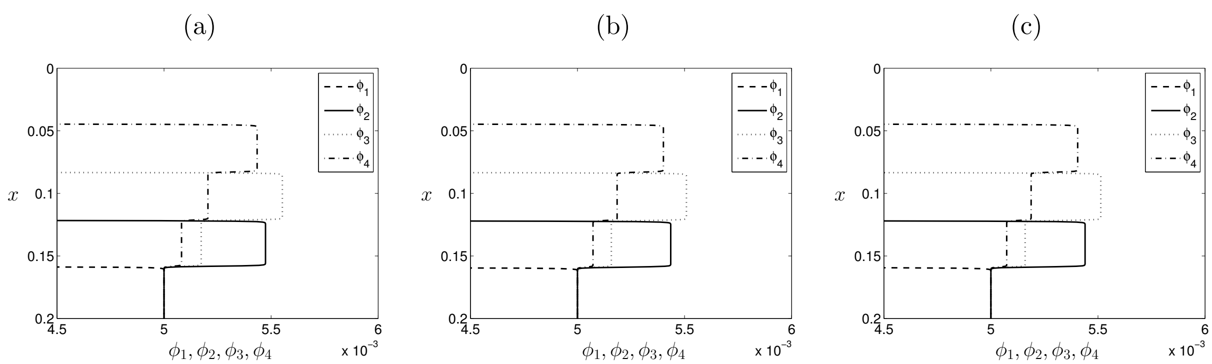

FIGURE 6. Example 2 ($N = 4$): enlarged view of the numerical solution for the (a) BW, (b) HS and (c) DG models at $t = 20\,\mathrm{s}$.

6.3. **Example 3.** In this last test we compare the DG and HS models with data at a non-dilute regime. The parameters $\boldsymbol{\beta}$ are given by (3.5) and we consider $N = 4$, $\phi_{\max} = 0.6$, $\Phi_0 = (0.1, 0.1, 0.1, 0.1)^{\mathrm{T}}$, $\mathbf{d_2} = (1, 0.75, 0.50, 0.26)^{\mathrm{T}}$ and the initial datum $\Phi(x, 0) = \Phi_0$ for $0 \leq x \leq L = 0.3\,\mathrm{m}$. The results are displayed in Figure 7. Some differences between the numerical solutions of both models become apparent.

6.4. **Conclusions.** In this work we extend the hyperbolicity analysis developed in [15] for the BW and HS models to the more general case in which $\beta_3 < 0$. We also obtain sufficient conditions for the hyperbolicity of the DG model for $\beta_3 = 0$. All these sufficient conditions involve the key design parameters $\boldsymbol{\beta}$, $\phi_{\max}$ and

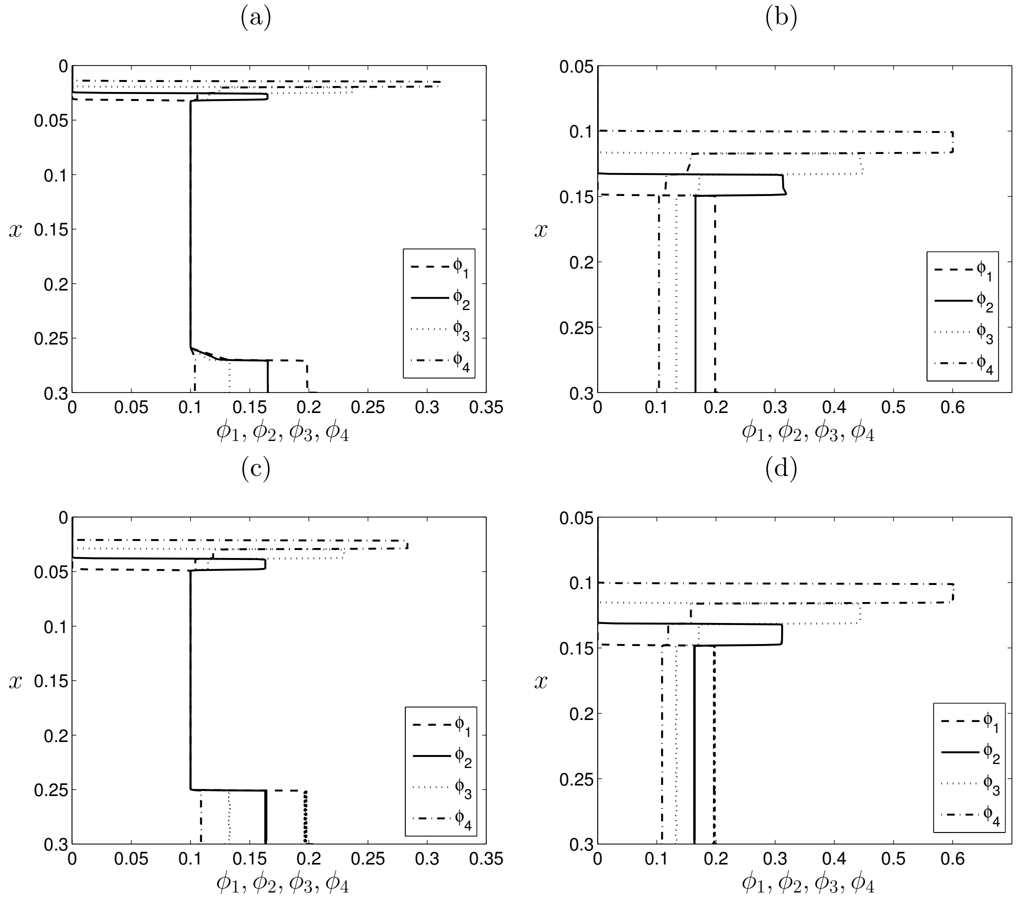

FIGURE 7. Example 3 (DG and HS models $N = 2$): numerical solution of (a, b) the DG model and (c, d) the HS model at (a, c) $t = 50\,\mathrm{s}$ and (b, d) $t = 500\,\mathrm{s}$.

d_N, but are independent of N. This means in particular that the present results could also be applied to the situation where a continuous particle size distribution of a "real-world" suspension is approximated by a finite distribution but with a fairly large number N of size classes, and where the parameter d_N is controlled by suitable sieving.

The basic tool for obtaining these results is the so-called secular equation [10], whose numerical solution permits the development of robust characteristic-wise numerical methods for the models that have been considered. In previous work including [9, 14, 15, 34] this formula was applied to establish the hyperbolicity of MCLWR or polydisperse sedimentation models that give rise to systems of conservation laws of the form (1.1), (1.2) where the $N \times N$ Jacobian $\mathcal{J}_{\mathbf{f}}(\Phi)$ is a perturbation of a diagonal matrix of rank $m \leq 3$ (cf. (1.3)). To our knowledge, this is the first time that this result is applied for $m = 4$. To underline the significance of these results, let us emphasize that we have shown elsewhere [11, 34] that for the present class of problems, the efficiency of spectral WENO methods is superior to that of their component-wise counterparts. However, these methods can only be implemented if it is guaranteed that the system under study is hyperbolic, in which case the required spectral decomposition of $\mathcal{J}_{\mathbf{f}}(\Phi)$ can be extracted with moderate effort.

Further limitations and possible extensions of the "secular approach" for polydisperse sedimentation models are broadly discussed in [15]. We recall here that the main results of our work, Theorems 4.1, 4.2 and 5.1 state in which regions hyperbolicity is *ensured*, that is, where we can guarantee that $\gamma_i \cdot \gamma_j > 0$. However, the models may well be hyperbolic in other sub-regions of parameter space, but with $\gamma_i \cdot \gamma_j \leq 0$ for some choices of i and j. While this is an intrinsic limitation of the secular equation, our analysis of the HS model suggests that slightly larger hyperbolicity regions could be obtained for a given set of particle sizes $d_1, \dots, d_N$ if the functions $\tilde{H}_{ij}$ given by (4.22) (rather than the single function $H(\phi, \boldsymbol{\beta}, d_N)$) are evaluated. Also, further realism can be added if the phase space is not simply limited by a hyperplane $\phi = \phi_{\max}$, but by a curved surface in $\bar{\mathcal{D}}_1$ which takes into account that mixtures of small and large particles permit denser packings than monodisperse sediments of any of the species involved.

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