Analysis of the coupling of Lagrange and Arnold-Falk-Winther finite elements for a fluid-solid interaction problem in 3D*

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Abstract

We introduce and analyze a new finite element method for a three-dimensional fluid-solid interaction problem. The media are governed by the acoustic and elastodynamic equations in time-harmonic regime, and the transmission conditions are given by the equilibrium of forces and the equality of the corresponding normal displacements. We employ a dual-mixed variational formulation in the solid, in which the Cauchy stress tensor and the rotation are the only unknowns, and mantain the usual primal formulation in the fluid. The main novelty of our method, with respect to previous approaches for a 2D version of this problem, consists of the introduction of the first transmission condition as part of the definition of the space to which the stress of the solid and the pressure of the fluid belong. As a consequence, and since the second transmission condition becomes natural, no Lagrange multipliers on the coupling boundary are needed, which certainly leads to a much simpler variational formulation. We show that a suitable decomposition of the space of stresses and pressures allows the application of the Babuška-Brezzi theory and the Fredholm alternative for concluding the solvability of the whole coupled problem. The unknowns of the fluid and the solid are then approximated, respectively, by Lagrange and Arnold-Falk-Winther finite element subspaces of order 1, which yields a conforming Galerkin scheme. In this way, the stability and convergence of the discrete method relies on a stable decomposition of the finite element space used to approximate the stress and the pressure variables, and also on a classical result on projection methods for Fredholm operators of index zero.

Key words: mixed finite elements, Helmholtz equation, elastodynamic equation

Mathematics subject classifications (1991): 65N30, 65N12, 65N15, 74F10, 74B05, 35J05

1 Introduction

The development of suitable numerical methods for fluid-solid interaction problems, specially for those modelled by the acoustic and elastodynamic equations in time-harmonic regime, has become a subject of increasing interest during the last two decades. For instance, several approaches relying on a primal formulation in the solid, in which the displacement becomes the only unknown in this medium, have been studied in [9], [21], [22], [23], [24], [27], and [28]. More recently, in order to avoid the locking

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phenomenon that arises in the nearly incompressible case, or motivated by the need of obtaining direct finite element approximations of the stresses, dual-mixed formulations in the solid have begun to be considered as well (see e.g. [15] and [16]).

More precisely, the interaction problem studied in [15] and [16] consists of an elastic body that is subject to a given incident wave that travels in the fluid surrounding it. The transmission conditions hold on the boundary of the solid and they are given by the equilibrium of forces and the equality of the normal displacements from both media. Actually, in [15] we simplify a bit the original model and assume that the fluid occupies an annular region, whence a Robin boundary condition imitating the behavior of the scattered field at infinity is imposed on its exterior boundary, which is located far from the obstacle. Then, we employ a dual-mixed approach for plane elasticity in the solid, in which the elastodynamic equation is used to eliminate the displacement unknown, and keeps the usual primal method in the fluid region. In addition, since the first transmission condition mentioned above becomes essential, it is enforced weakly by means of a Lagrange multiplier. In this way, the stress tensor in the solid and the pressure in the fluid, which solves the Helmholtz equation, constitute the main unknowns of the resulting formulation. Next, we show that a judicious decomposition of the space of stresses renders suitable the application of a Fredholm alternative for the analysis of the whole coupled problem. The associated discrete scheme is defined with PEERS elements in the obstacle and the traditional first order Lagrange finite elements in the fluid domain. The stability and convergence of this Galerkin method also relies on a stable decomposition of the finite element space used to approximate the stress variable. In [16] we modify the strategy from [15] and, instead of considering a Robin condition on the exterior boundary, we follow the approach from [28] and introduce a non-local absorbing boundary condition based on boundary integral equations. This implies, in particular, that the exterior boundary can be chosen as any parametrizable smooth closed curve containing the solid, which, in order to minimize the size of the computational domain, is adjusted as sharply as possible to the shape of the obstacle. In this way, the discretization procedure proposed in [16] couples the primal/dual-mixed finite element scheme from [15] with a suitable boundary element method arising from a combined double and single layer potential representation of the scattered wave (see [13]). The rest of the analysis for the continuous and discrete formulations of [16] follows very closely the techniques and arguments developed in [15].

On the other hand, new stable mixed finite element methods for linear elasticity in 2D and 3D, including strong symmetry and weakly imposed symmetry for the stresses, have been derived during the last decade using the finite element exterior calculus, a quite abstract framework involving several sophisticated mathematical tools (see, e.g. [4], [5], [6], [7]). In particular, the first elements using polynomial shape-functions that are known to be stable for the symmetric stress-displacement formulation in 2D are the ones provided in [7]. The corresponding lowest order element consists of piecewise cubic polynomials for the stress, with 24 degrees of freedom per triangle, and piecewise linear functions for the displacement. An analogue of this element in 3D, which considers piecewise quartic stresses with 162 degrees of freedom per tetrahedron, and piecewise linear displacements, was proposed in [1]. In turn, the stable elements with a weak symmetry condition for the stresses have been constructed in [4] and [6], and a new proof of some of the main results obtained there, which employs more elementary and classical techniques, was provided recently in [12]. In fact, the approaches in [12] for the 2D and 3D cases are based, respectively, on the use of stable Stokes elements and interpolation operators that keep the reduced symmetry. The resulting Arnold-Falk-Winther (AFW) element with the lowest polynomial degrees, which is referred to as of order 1, consists of piecewise linear approximations for the stress and piecewise constants functions for both the displacement and rotation unknowns.

Now, the main purpose of the present paper is to introduce and analyze a new finite element method for the 3D version of the interaction problem studied in [15]. To this end, and in order to simplify the approach from [15], we now incorporate the equilibrium of forces (see the first equation in (2.2) below) into the definition of the product space to which the stress $\boldsymbol{\sigma}$ of the solid and the pressure p of the fluid belong. In this way, we avoid the introduction of further unknowns (Lagrange multipliers) on the boundary of the solid, which otherwise would yield later on a more expensive Galerkin scheme. Moreover, the strategy involving a Lagrange multiplier on the transmission boundary would require the use of two finite element meshes satisfying a stability condition between their corresponding mesh sizes, which certainly constitutes a very cumbersome restriction in 3D computations. Hence, according to the availability of the new stable mixed finite elements for 3D linear elasticity with weak symmetry (which are described in the previous paragraph), we also propose here to approximate the unknowns of the solid and the fluid by the corresponding components of the AFW and Lagrange finite element subspaces of order 1, respectively. Thus, because of the coincidence between the polynomial shape-functions approximating $\boldsymbol{\sigma}\,\boldsymbol{\nu}$ and $-\,p\,\boldsymbol{\nu}$, we are able to generate a conforming finite element subspace for the pair $(\boldsymbol{\sigma}, p)$. In other words, the first equation in (2.2) is exactly satisfied at the discrete level, whence the matching described above gives rise to what we call a natural coupling of the Lagrange and AFW elements of lowest order with respect to that transmission condition. The rest of this work is organized as follows. In Sections 2 and 3 we describe the fluid-solid interaction problem and derive its continuous variational formulation. Then, in Section 4, we show that the resulting saddle point problem is well posed. Finally, the corresponding Galerkin scheme is analyzed in Section 5.

We end this section with some notations to be used below. Since in the sequel we deal with complex valued functions, we let $\mathbb{C}$ be the set of complex numbers, use the symbol $\imath$ for $\sqrt{-1}$, and denote by $\overline{z}$ and $|z|$ the conjugate and modulus, respectively, of each $z \in \mathbb{C}$. In addition, given any Hilbert space U, U^3 and $U^{3\times 3}$ denote, respectively, the space of vectors and tensors of order 3 with entries in U. In particular, $\mathbf{I}$ is the identity matrix of $\mathbb{C}^{3\times 3}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{C}^{3\times 3}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})$, the trace $\operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{3} \tau_{ii}$, the deviator tensor $\boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}$, the tensor product $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{3} \tau_{ij}\,\zeta_{ij}$, and the conjugate tensor $\overline{\boldsymbol{\tau}} := (\overline{\tau}_{ij})$. Finally, in what follows we utilize the standard terminology for Sobolev spaces and norms, employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The fluid-solid interaction problem

We are interested in the 3D version of the interaction problem studied in [15]. More precisely, we now consider an incident acoustic wave upon a bounded elastic body (obstacle) in $\mathbb{R}^3$ that is fully surrounded by a fluid, and aim to determine both the response of the body and the scattered wave. The boundary of the obstacle Ω_s is denoted by Σ. We assume that the incident wave and the volume force acting on the body exhibit a time-harmonic behaviour with frequency ω and amplitudes p_i and $\mathbf{f}$, respectively, so that p_i satisfies the Helmholtz equation in $\mathbb{R}^3\backslash\Omega_s$. Hence, we may consider that this interaction problem is posed in the frequency domain. In this way, and since, following [15], we plan to employ a mixed variational formulation in the solid, our main unknowns become the amplitude $\boldsymbol{\sigma} : \Omega_s \to \mathbb{C}^{3\times 3}$ of the Cauchy stress tensor, the amplitude $\mathbf{u} : \Omega_s \to \mathbb{C}^3$ of the displacement field, and the amplitude of the total (incident + scattered) pressure $p : \mathbb{R}^3\backslash\Omega_s \to \mathbb{C}$.

The fluid is assumed to be perfect, compressible, and homogeneous, with mass density ρ_f and wave number $\kappa_f := \dfrac{\omega}{v_0}$, where v_0 is the speed of sound in the linearized fluid. In addition, the solid is supposed to be isotropic and linearly elastic with mass density ρ_s and Lamé constants μ and λ, which means, in particular, that the corresponding constitutive equation is given by

$$\boldsymbol{\sigma} \,=\, \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in} \quad \Omega_s\,,$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\right)$ is the strain tensor of small deformations, ∇ is the gradient tensor, and $\mathcal{C}$ is the elasticity operator given by Hooke's law, that is

$$\mathcal{C}\,\boldsymbol{\zeta} \;:=\; \lambda\,\mathrm{tr}(\boldsymbol{\zeta})\,\mathbf{I} \,+\, 2\,\mu\,\boldsymbol{\zeta} \qquad \forall\,\boldsymbol{\zeta} \in [L^2(\Omega_s)]^{3\times 3}\,. \tag{2.1}$$

Consequently, under the hypotheses of small oscillations, both in the solid and the fluid, the unknowns $\boldsymbol{\sigma}$, $\mathbf{u}$, and p satisfy the elastodynamic and acoustic equations in time-harmonic regime, that is:

$$\begin{aligned} \mathbf{div}(\boldsymbol{\sigma}) \,+\, \kappa_s^2\,\mathbf{u} \quad &= \quad -\,\mathbf{f} \qquad \text{in} \quad \Omega_s\,,\\ \Delta p \,+\, \kappa_f^2\,p \quad &= \quad 0 \qquad\;\; \text{in} \quad \mathbb{R}^3\backslash\Omega_s\,, \end{aligned}$$

where the wave number κ_s of the solid is defined by $\sqrt{\rho_s}\,\omega$, together with the transmission conditions:

$$\begin{aligned} \boldsymbol{\sigma}\,\boldsymbol{\nu} \quad &= \quad -\,p\,\boldsymbol{\nu} \qquad \text{on} \quad \Sigma\,,\\ \rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu} \quad &= \quad \frac{\partial p}{\partial\boldsymbol{\nu}} \qquad \text{on} \quad \Sigma\,, \end{aligned} \tag{2.2}$$

and the behaviour at infinity given by

$$p - p_i \,=\, O(\mathtt{r}^{-1}) \tag{2.3}$$

and

$$\frac{\partial(p-p_i)}{\partial\mathtt{r}} \,-\, \imath\,\kappa_f\,(p-p_i) \,=\, o(\mathtt{r}^{-1})\,, \tag{2.4}$$

as $\mathtt{r} \,:=\, \|\mathbf{x}\| \,\to\, +\infty$, uniformly for all directions $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$. Hereafter, $\mathbf{div}$ stands for the usual divergence operator div acting on each row of the tensor, $\|\mathbf{x}\|$ is the euclidean norm of a vector $\mathbf{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \mathbb{R}^3$, and $\boldsymbol{\nu}$ denotes the unit outward normal on Σ, that is pointing toward Ω_f. The transmission conditions given in (2.2) constitute the equilibrium of forces and the equality of the normal displacements of the solid and fluid, whereas the equation (2.4) is known as the Sommerfeld radiation condition.

On the other hand, it is important to remark, as a consequence of (2.3) and (2.4), that the outgoing waves are absorbed by the far field. Motivated by this fact, and aiming to obtain a suitable simplification of our model problem, we now proceed as in [15] and introduce a sufficiently large polyhedral surface Γ approximating a sphere centered at the origin, define Ω_f as the annular domain bounded by Σ and Γ, and consider the Robin boundary condition:

$$\frac{\partial p}{\partial\boldsymbol{\nu}} \,-\, \imath\,\kappa_f\,p \,=\, g \,:=\, \frac{\partial p_i}{\partial\boldsymbol{\nu}} \,-\, \imath\,\kappa_f\,p_i \quad \text{on} \quad \Gamma\,, \tag{2.5}$$

where $\boldsymbol{\nu}$ denotes also the unit outward normal on Γ.

Therefore, throughout the rest of the paper we assume the Robin boundary condition (2.5), and, given $\mathbf{f} \in [L^2(\Omega_s)]^3$ and $g \in H^{-1/2}(\Gamma)$, consider the following fluid-solid interaction problem: Find $\boldsymbol{\sigma} \in H(\mathbf{div};\Omega_s)$, $\mathbf{u} \in [L^2(\Omega_s)]^3$, and $p \in H^1(\Omega_f)$, such that there holds in the distributional sense:

$$
\begin{array}{rcll}
\boldsymbol{\sigma} & = & \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) & \text{in} \quad \Omega_s\,, \\
\mathbf{div}(\boldsymbol{\sigma}) \,+\, \kappa_s^2\,\mathbf{u} & = & -\,\mathbf{f} & \text{in} \quad \Omega_s\,, \\
\Delta p \,+\, \kappa_f^2\,p & = & 0 & \text{in} \quad \Omega_f\,, \\
\boldsymbol{\sigma}\boldsymbol{\nu} & = & -\,p\boldsymbol{\nu} & \text{on} \quad \Sigma\,, \\
\rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu} & = & \dfrac{\partial p}{\partial \boldsymbol{\nu}} & \text{on} \quad \Sigma\,, \\
\dfrac{\partial p}{\partial \boldsymbol{\nu}} \,-\, \imath\,\kappa_f\,p & = & g & \text{on} \quad \Gamma\,.
\end{array} \tag{2.6}
$$

3 The continuous variational formulation

In this section we follow very closely [15] and employ primal and dual-mixed approaches in the fluid Ω_f and the solid Ω_s, respectively, to derive the full continuous variational formulation of (2.6). The main difference with the approach from [15] lies on the fact that, instead of weakly imposing the first trasmission condition on Σ, we proceed to incorporate the first equation of (2.2) (or fourth equation of (2.6)) into the definition of the space to which the unknowns $\boldsymbol{\sigma}$ and p belong. In turn, the second equation of (2.2) (or fifth equation of (2.6)) is handled as in [15], which means that it becomes a natural transmission condition. According to the above, we first multiply the acoustic equation by $q \in H^1(\Omega_f)$, integrate by parts, and use the Robin boundary condition, to obtain

$$
\int_{\Omega_f} \nabla p\cdot\nabla q \,-\, \kappa_f^2 \int_{\Omega_f} pq \,+\, \langle \frac{\partial p}{\partial \boldsymbol{\nu}}, q\rangle_\Sigma \,-\, \imath\,\kappa_f \int_\Gamma pq \,=\, \langle g, q\rangle_\Gamma\,, \tag{3.1}
$$

where, given $\mathcal{S} \in \{\Sigma, \Gamma\}$, $\langle \cdot,\cdot\rangle_{\mathcal{S}}$ stands for the duality pairing of $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$ with respect to the $L^2(\mathcal{S})$-inner product. Next, we replace $\dfrac{\partial p}{\partial \boldsymbol{\nu}}$ by $\rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu}$ on Σ and divide by $\rho_f\,\omega^2$, whence (3.1) becomes

$$
\frac{1}{\rho_f\,\omega^2}\int_{\Omega_f} \nabla p\cdot\nabla q \,-\, \frac{\kappa_f^2}{\rho_f\,\omega^2}\int_{\Omega_f} pq \,+\, \langle q\boldsymbol{\nu}, \mathbf{u}\rangle_\Sigma \,-\, \imath\,\frac{\kappa_f}{\rho_f\,\omega^2}\int_\Gamma pq \,=\, \frac{1}{\rho_f\,\omega^2}\langle g, q\rangle_\Gamma\,. \tag{3.2}
$$

On the other hand, in order to derive the mixed variational formulation in the solid Ω_s, we follow the usual procedure (see [2], [15] and [30]) and introduce the rotation

$$
\boldsymbol{\gamma} \,:=\, \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}) \,\in\, [L^2(\Omega_s)]^{3\times3}_{\mathtt{asym}}
$$

as a further unknown, where $[L^2(\Omega_s)]^{3\times3}_{\mathtt{asym}}$ denotes the space of asymmetric tensors with entries in $L^2(\Omega_s)$. In this way, the constitutive equation can be rewritten in the form

$$
\mathcal{C}^{-1}\,\boldsymbol{\sigma} \,=\, \boldsymbol{\varepsilon}(\mathbf{u}) \,=\, \nabla\mathbf{u} \,-\, \boldsymbol{\gamma}\,,
$$

which, multiplying by a function $\boldsymbol{\tau} \in H(\mathbf{div};\Omega_s)$ and integrating by parts, yields

$$
\int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\sigma} : \boldsymbol{\tau} \,+\, \int_{\Omega_s} \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,-\, \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}\rangle_\Sigma \,+\, \int_{\Omega_s} \boldsymbol{\tau} : \boldsymbol{\gamma} \,=\, 0\,. \tag{3.3}
$$

We recall here that

$$
H(\mathbf{div};\Omega_s) \,:=\, \Big\{ \boldsymbol{\tau} \in [L^2(\Omega_s)]^{3\times3} : \quad \mathbf{div}(\boldsymbol{\tau}) \,\in\, [L^2(\Omega_s)]^3 \Big\}
$$

endowed with the norm $\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\,\Omega_s)} \,:=\, \|\boldsymbol{\tau}\|^2_{[L^2(\Omega_s)]^{3\times 3}} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega_s)]^3}$ is a Hilbert space.

Next, assuming that the test functions $\boldsymbol{\tau}$ and q satisfy $\boldsymbol{\tau}\boldsymbol{\nu} = -q\boldsymbol{\nu}$ on Σ and replacing back

$$\mathbf{u} \,=\, -\,\frac{1}{\kappa_s^2}\left(\mathbf{f} \,+\, \mathbf{div}(\boldsymbol{\sigma})\right),$$

into (3.3), gives

$$\int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} \,-\, \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \langle q\boldsymbol{\nu}, \mathbf{u}\rangle_\Sigma \,+\, \int_{\Omega_s} \boldsymbol{\tau} : \gamma \,=\, \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{f}\cdot\mathbf{div}(\boldsymbol{\tau})\,. \tag{3.4}$$

Finally, the symmetry of $\boldsymbol{\sigma}$ is imposed weakly through the relation

$$\int_{\Omega_s} \boldsymbol{\sigma} : \boldsymbol{\eta} \,=\, 0 \qquad \forall\,\boldsymbol{\eta} \,\in\, [L^2(\Omega_s)]^{3\times 3}_{\mathtt{asym}}.$$

In the sequel, for economy of notation, we represent duplets $(\boldsymbol{\sigma}, p)$ and $(\boldsymbol{\tau}, q)$ from $H(\mathbf{div};\,\Omega_s) \times H^1(\Omega_f)$ by $\widehat{\boldsymbol{\sigma}}$ and $\widehat{\boldsymbol{\tau}}$ respectively, and, as suggested by the above choice of $\boldsymbol{\tau}$ and q, we introduce the closed subspace

$$\mathbb{X} \,:=\, \Big\{\widehat{\boldsymbol{\tau}} \,=\, (\boldsymbol{\tau}, q) \in H(\mathbf{div};\,\Omega_s) \times H^1(\Omega_f) : \quad \boldsymbol{\tau}\,\boldsymbol{\nu} \,+\, q\,\boldsymbol{\nu} = 0 \quad \text{on } \Sigma\Big\}$$

of $H(\mathbf{div};\,\Omega_s) \times H^1(\Omega_f)$ endowed with the norm

$$\|\widehat{\boldsymbol{\tau}}\|^2_{\mathbb{X}} := \|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\,\Omega_s)} + \|q\|^2_{H^1(\Omega_f)}.$$

In addition, we also let from now on $\mathbb{Y} \,:=\, [L^2(\Omega_s)]^{3\times 3}_{\mathtt{asym}}$.

Hence, substracting (3.4) from (3.2), we arrive to the following variational formulation of (2.6): Find $(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma}) \,\in\, \mathbb{X} \times \mathbb{Y}$ such that

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \,+\, B(\widehat{\boldsymbol{\tau}}, \boldsymbol{\gamma}) \quad &= \quad F(\widehat{\boldsymbol{\tau}}) \qquad \forall\,\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}\,, \\ B(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\eta}) \quad &= \quad 0 \qquad\quad\; \forall\,\boldsymbol{\eta} \in \mathbb{Y}\,, \end{aligned} \tag{3.5}$$

where $F : \mathbb{X} \to \mathbb{C}$ is the linear functional

$$F(\widehat{\boldsymbol{\tau}}) \,:=\, -\frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{f}\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \frac{1}{\rho_f\,\omega^2}\,\langle g, q\rangle_\Gamma \qquad \forall\,\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}\,,$$

and $A : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$, and $B : \mathbb{X} \times \mathbb{Y} \to \mathbb{C}$ are the bilinear forms defined by

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \quad &:= \,-\int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} \,+\, \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \frac{1}{\rho_f\,\omega^2}\int_{\Omega_f} \nabla p\cdot\nabla q \\ &\quad -\, \frac{\kappa_f^2}{\rho_f\,\omega^2}\int_{\Omega_f} pq \,-\, \imath\,\frac{\kappa_f}{\rho_f\,\omega^2}\int_\Gamma pq \qquad \forall\,\widehat{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, p),\, \widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}\,, \end{aligned} \tag{3.6}$$

and

$$B(\widehat{\boldsymbol{\tau}}, \boldsymbol{\eta}) \,:=\, -\int_{\Omega_s} \boldsymbol{\tau} : \boldsymbol{\eta} \qquad \forall\,\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}, \quad \forall\,\boldsymbol{\eta} \in \mathbb{Y}\,, \tag{3.7}$$

It is easy to see that F, A, and B are all bounded with constants depending on ω, ρ_f, ρ_s, κ_f, and κ_s, in the case of F and A, and constants independent of the physical parameters for B. Concerning the form A, we also observe from (2.1) that the inverse operator $\mathcal{C}^{-1}$ reduces to

$$\mathcal{C}^{-1}\,\boldsymbol{\zeta} \,:=\, \frac{1}{2\,\mu}\,\boldsymbol{\zeta} \,-\, \frac{\lambda}{3\,\mu\,(3\lambda+2\mu)}\,\mathrm{tr}(\boldsymbol{\zeta})\,\mathbf{I} \qquad \forall\,\boldsymbol{\zeta} \,\in\, [L^2(\Omega_s)]^{3\times 3}\,,$$

which implies that

$$\int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\zeta} : \boldsymbol{\tau} \,=\, \frac{1}{2\,\mu}\int_{\Omega_s} \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \,+\, \frac{1}{3\,(3\lambda+2\mu)}\int_{\Omega_s} \mathrm{tr}(\boldsymbol{\zeta})\,\mathrm{tr}(\boldsymbol{\tau}) \quad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau} \,\in\, [L^2(\Omega_s)]^{3\times 3}\,,$$

and hence

$$\int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\zeta} : \overline{\boldsymbol{\zeta}} \,\geq\, \frac{1}{2\mu}\,\|\boldsymbol{\zeta}^{\mathtt{d}}\|^2_{[L^2(\Omega_s)]^{3\times 3}} \qquad \forall\,\boldsymbol{\zeta} \,\in\, [L^2(\Omega_s)]^{3\times 3}\,. \tag{3.8}$$

This estimate will be useful for our analysis below.

4 Analysis of the continuous variational formulation

In this section we proceed analogously as in [15] and employ a suitable decomposition of $\mathbb{X}$ to show that (3.5) becomes a compact perturbation of a well-posed problem. Firstly, we need to analyze an elasticity problem in Ω_s with Neumann boundary conditions. Then, this auxiliary problem yields the definition of an associated operator, which is employed to obtain the above mentioned decomposition.

4.1 An auxiliary Neumann problem

Let $\mathbb{RM}(\Omega_s)$ be the space of rigid body motions in Ω_s, that is

$$\mathbb{RM}(\Omega_s) := \Big\{ \mathbf{v} : \Omega_s \to \mathbb{C}^2 \;:\; \mathbf{v}(\mathbf{x}) = \mathbf{a} + \mathbf{b}\times\mathbf{x} \quad \forall\,\mathbf{x}\in\Omega_s\,,\; \mathbf{a},\,\mathbf{b}\,\in\mathbb{C}^3 \Big\}.$$

Then, given $\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}$, we consider the boundary value problem

$$\tilde{\boldsymbol{\sigma}} \,=\, \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})\,, \quad \mathbf{div}\,\tilde{\boldsymbol{\sigma}} \,=\, \mathbf{div}\,\boldsymbol{\tau} + \mathbf{r}(\widehat{\boldsymbol{\tau}}) \quad \text{in} \quad \Omega_s\,, \quad \tilde{\boldsymbol{\sigma}}\,\boldsymbol{\nu} \,=\, -q\boldsymbol{\nu} \quad \text{on} \quad \Sigma\,, \tag{4.1}$$

where $\mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ is defined according to (2.1) and $\mathbf{r}(\widehat{\boldsymbol{\tau}}) \in \mathbb{RM}(\Omega_s)$ is characterized by

$$\int_{\Omega_s} \mathbf{r}(\widehat{\boldsymbol{\tau}})\cdot\mathbf{w} \,=\, -\,\langle\, q\boldsymbol{\nu}, \mathbf{w}\,\rangle_\Sigma - \int_{\Omega_s} \mathbf{div}\,\boldsymbol{\tau}\cdot\mathbf{w} \qquad \forall\mathbf{w}\in\mathbb{RM}(\Omega_s).$$

Note that $\mathbf{r}(\widehat{\boldsymbol{\tau}})$ is just an auxiliary rigid motion that is needed to guarantee the usual compatibility condition required for the Neumann problem (4.1) (cf. [10, Theorem 9.2.30]).

Next, let us define the spaces

$$\tilde{\mathbf{H}} \,:=\, \Big\{\, \tilde{\boldsymbol{\tau}} \in H(\mathbf{div};\,\Omega_s) : \quad \tilde{\boldsymbol{\tau}}\,\boldsymbol{\nu} \,=\, \mathbf{0} \quad \text{on} \quad \Sigma \Big\}$$

and

$$\tilde{\mathbf{Q}} \,:=\, (\mathbf{I} - \mathbf{M})([L^2(\Omega_s)]^3) \times [L^2(\Omega_s)]^{3\times 3}_{\mathtt{asym}}\,,$$

where $\mathbf{M} : [L^2(\Omega_s)]^3 \to \mathbb{RM}(\Omega_s)$ is the $[L^2(\Omega_s)]^3$-orthogonal projector. Then, introducing the rotation $\tilde{\gamma} := \frac{1}{2}\left(\nabla\tilde{\mathbf{u}} - (\nabla\tilde{\mathbf{u}})^{\mathtt{t}}\right)$ as a further unknown, the dual-mixed variational formulation of (4.1) reduces to: Find $(\tilde{\boldsymbol{\sigma}}, (\tilde{\mathbf{u}}, \tilde{\gamma})) \in H(\mathbf{div};\Omega_s) \times \tilde{\mathbf{Q}}$ such that $\tilde{\boldsymbol{\sigma}}\boldsymbol{\nu} = -q\,\boldsymbol{\nu}$ on Σ and

$$\begin{aligned} \mathbf{a}(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\tau}}) + \mathbf{b}(\tilde{\boldsymbol{\tau}}, (\tilde{\mathbf{u}}, \tilde{\gamma})) &= \mathbf{0} \qquad \forall\, \tilde{\boldsymbol{\tau}} \in \tilde{\mathbf{H}}\,, \\ \mathbf{b}(\tilde{\boldsymbol{\sigma}}, (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}})) &= \int_{\Omega_s} (\mathbf{div}\,\boldsymbol{\tau} + \mathbf{r}(\widehat{\boldsymbol{\tau}}))\cdot\tilde{\mathbf{v}} \qquad \forall\,(\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}}) \in \tilde{\mathbf{Q}}\,, \end{aligned} \tag{4.2}$$

where $\mathbf{a} : H(\mathbf{div};\Omega_s) \times H(\mathbf{div};\Omega_s) \to \mathbb{C}$ and $\mathbf{b} : H(\mathbf{div};\Omega_s) \times \tilde{\mathbf{Q}} \to \mathbb{C}$ are the bilinear forms given by

$$\mathbf{a}(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\tau}}) := \int_{\Omega_s} \mathcal{C}^{-1}\tilde{\boldsymbol{\sigma}} : \tilde{\boldsymbol{\tau}} \qquad \forall\,(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\tau}}) \in H(\mathbf{div};\Omega_s) \times H(\mathbf{div};\Omega_s)\,, \tag{4.3}$$

and

$$\mathbf{b}(\tilde{\boldsymbol{\tau}}, (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}})) := \int_{\Omega_s} \tilde{\mathbf{v}}\cdot\mathbf{div}\,\tilde{\boldsymbol{\tau}} + \int_{\Omega_s} \tilde{\boldsymbol{\tau}} : \tilde{\boldsymbol{\eta}} \qquad \forall\,(\tilde{\boldsymbol{\tau}}, (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}})) \in H(\mathbf{div};\Omega_s) \times \tilde{\mathbf{Q}}\,.$$

The well-posedness of (4.2) is already well known (see, e.g. [5, Section 11.7, Theorem 11.7] or [16, Section 3, Theorem 3.1]). In addition, owing to the regularity result for the elasticity problem with Neumann boundary conditions (see, e.g. [18], [19]), we know that the solution $\tilde{\mathbf{u}}$ of (4.1) belongs to $[H^{1+\epsilon}(\Omega_s)]^3$, for some $\epsilon > 0$, and there holds

$$\|\tilde{\mathbf{u}}\|_{[H^{1+\epsilon}(\Omega_s)]^3} \le C\left\{\|\mathbf{div}\,\boldsymbol{\tau}\|_{[L^2(\Omega_s)]^3} + \|q\,\boldsymbol{\nu}\|_{[L^2(\Sigma)]^3}\right\} \le C\left\{\|\mathbf{div}\,\boldsymbol{\tau}\|_{[L^2(\Omega_s)]^3} + \|q\|_{H^1(\Omega_f)}\right\},$$

which, in turn, implies that the unique solution of (4.2) satisfies

$$(\tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}}, \tilde{\gamma}) \in [H^\epsilon(\Omega_s)]^{3\times3} \times [H^{1+\epsilon}(\Omega_s)]^3 \times [H^\epsilon(\Omega_s)]^{3\times3} \tag{4.4}$$

and

$$\|\tilde{\boldsymbol{\sigma}}\|_{[H^\epsilon(\Omega_s)]^{3\times3}} + \|\tilde{\mathbf{u}}\|_{[H^{1+\epsilon}(\Omega_s)]^3} + \|\tilde{\gamma}\|_{[H^\epsilon(\Omega_s)]^{3\times3}} \le C\left\{\|\mathbf{div}\,\boldsymbol{\tau}\|_{[L^2(\Omega_s)]^3} + \|q\|_{H^1(\Omega_f)}\right\}. \tag{4.5}$$

Note that the trace inequality in $H^1(\Omega_f)$ is used here to bound $\|q\,\boldsymbol{\nu}\|_{[L^2(\Sigma)]^3}$ by $C\,\|q\|_{H^1(\Omega_f)}$.

4.2 The associated operator P

We now introduce the linear operators $P : \mathbb{X} \to H(\mathbf{div};\Omega_s)$ and $\mathbf{P} : \mathbb{X} \to \mathbb{X}$ defined by

$$P(\widehat{\boldsymbol{\tau}}) := \tilde{\boldsymbol{\sigma}} \quad \text{and} \quad \mathbf{P}(\widehat{\boldsymbol{\tau}}) := (P(\widehat{\boldsymbol{\tau}}), q) \qquad \forall\,\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}\,, \tag{4.6}$$

where $(\tilde{\boldsymbol{\sigma}}, (\tilde{\mathbf{u}}, \tilde{\gamma})) \in H(\mathbf{div};\Omega_s) \times \tilde{\mathbf{Q}}$ is the unique solution of (4.2). It is clear from (4.1) that

$$P(\widehat{\boldsymbol{\tau}})^{\mathtt{t}} = P(\widehat{\boldsymbol{\tau}}) \quad \text{in} \quad \Omega_s\,, \quad \mathbf{div}\,(P(\widehat{\boldsymbol{\tau}})) = \mathbf{div}\,\boldsymbol{\tau} + \mathbf{r}(\widehat{\boldsymbol{\tau}}) \quad \text{in} \quad \Omega_s \tag{4.7}$$

and

$$(P(\widehat{\boldsymbol{\tau}}))\,\boldsymbol{\nu} = -q\,\boldsymbol{\nu} \quad \text{on} \quad \Sigma\,. \tag{4.8}$$

Then, thanks to the continuous dependence result for (4.2), we find that

$$\|P(\widehat{\boldsymbol{\tau}})\|_{H(\mathbf{div};\Omega_s)} \le C\left\{\|\mathbf{div}\,\boldsymbol{\tau}\|_{[L^2(\Omega_s)]^3} + \|q\|_{H^1(\Omega_f)}\right\} \qquad \forall\,\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}\,,$$

which shows that $\mathbf{P}$ is bounded. Moreover, it is easy to see from (4.2), (4.6), (4.7), and (4.8) that $\mathbf{P}$ is actually a projector, and hence there holds

$$\mathbb{X} = \mathbf{P}(\mathbb{X}) \oplus (\mathbf{I} - \mathbf{P})(\mathbb{X})\,. \tag{4.9}$$

Finally, it is clear from (4.4) and (4.5) that $P(\widehat{\boldsymbol{\tau}}) \in [H^\epsilon(\Omega_s)]^{3\times3}$ and

$$\|P(\widehat{\boldsymbol{\tau}})\|_{[H^\epsilon(\Omega_s)]^{3\times3}} \le C\left\{\|\mathbf{div}\,\boldsymbol{\tau}\|_{[L^2(\Omega_s)]^3} + \|q\|_{H^1(\Omega_f)}\right\} \qquad \forall\,\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}\,. \tag{4.10}$$

4.3 Well-posedness of the continuous formulation

In order to show that our coupled problem (3.5) is well-posed, we now employ the stable decomposition (4.9) to reformulate (3.5) in a more suitable form. We begin by observing, according to (4.7), (4.8), the symmetry of $P(\widehat{\boldsymbol{\tau}})$, and the fact that $\nabla \mathbf{r} \in [L^2(\Omega_s)]^{3\times 3}_{\mathtt{asym}}$ $\forall\, \mathbf{r} \in \mathbb{RM}(\Omega_s)$, that for all $\widehat{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, p)$, $\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}$ there holds

$$\begin{aligned} \int_{\Omega_s} \Big\{ \mathbf{div}\,\boldsymbol{\sigma} - \mathbf{div}\, P(\widehat{\boldsymbol{\sigma}}) \Big\} \cdot \mathbf{div}\, P(\widehat{\boldsymbol{\tau}}) \;&=\; - \int_{\Omega_s} \mathbf{r}(\widehat{\boldsymbol{\sigma}}) \cdot \mathbf{div}\, P(\widehat{\boldsymbol{\tau}}) \\ = \int_{\Omega_s} \nabla \mathbf{r}(\widehat{\boldsymbol{\sigma}}) : P(\widehat{\boldsymbol{\tau}}) \;-\; \langle\, (P(\widehat{\boldsymbol{\tau}}))\,\boldsymbol{\nu}, \mathbf{r}(\widehat{\boldsymbol{\sigma}})\, \rangle_\Sigma \;&=\; \int_\Sigma (\mathbf{r}(\widehat{\boldsymbol{\sigma}}) \cdot \boldsymbol{\nu})\, q\,. \end{aligned} \tag{4.11}$$

Then, writing $\widehat{\boldsymbol{\sigma}} = \mathbf{P}(\widehat{\boldsymbol{\sigma}}) + (\mathbf{I} - \mathbf{P})(\widehat{\boldsymbol{\sigma}})$ and $\widehat{\boldsymbol{\tau}} = \mathbf{P}(\widehat{\boldsymbol{\tau}}) + (\mathbf{I} - \mathbf{P})(\widehat{\boldsymbol{\tau}})$ in (3.6), similarly as we did in [15], using the identity (4.11), and adding and substracting suitable terms, we find that A can be decomposed as

$$A(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \;=\; A_0(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \;+\; K(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \qquad \forall\, \widehat{\boldsymbol{\sigma}},\, \widehat{\boldsymbol{\tau}} \,\in\, \mathbb{X}\,, \tag{4.12}$$

where $A_0 : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ and $K : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ are the bounded and symmetric bilinear forms given by

$$A_0(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \;:=\; \mathcal{A}(\mathbf{P}(\widehat{\boldsymbol{\sigma}}), \mathbf{P}(\widehat{\boldsymbol{\tau}})) \;-\; \mathcal{A}((\mathbf{I} - \mathbf{P})(\widehat{\boldsymbol{\sigma}}), (\mathbf{I} - \mathbf{P})(\widehat{\boldsymbol{\tau}})) \tag{4.13}$$

with

$$\begin{aligned} \mathcal{A}(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \;:=\; \int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\sigma} : \boldsymbol{\tau} \;+\; \frac{1}{\kappa_s^2} \int_{\Omega_s} \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} \;+\; \frac{1}{\rho_f\,\omega^2} \int_{\Omega_f} \nabla p \cdot \nabla q \\ +\; \frac{\kappa_f^2}{\rho_f\,\omega^2} \int_{\Omega_f} p\, q \;-\; \imath\, \frac{\kappa_f}{\rho_f\,\omega^2} \int_{\Gamma} p\, q\,, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} K(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \;:=\; -\,2\,\mathcal{K}(\mathbf{P}(\widehat{\boldsymbol{\sigma}}), \mathbf{P}(\widehat{\boldsymbol{\tau}})) \;-\; \mathcal{K}((\mathbf{I} - \mathbf{P})(\widehat{\boldsymbol{\sigma}}), \mathbf{P}(\widehat{\boldsymbol{\tau}})) \;-\; \mathcal{K}(\mathbf{P}(\widehat{\boldsymbol{\sigma}}), (\mathbf{I} - \mathbf{P})(\widehat{\boldsymbol{\tau}})) \\ +\, \frac{2}{\kappa_s^2} \left\{ \int_{\Omega_s} \mathbf{r}(\widehat{\boldsymbol{\sigma}}) \cdot \mathbf{r}(\widehat{\boldsymbol{\tau}}) \;+\; \int_\Sigma \mathbf{r}(\widehat{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}\, p \;+\; \int_\Sigma \mathbf{r}(\widehat{\boldsymbol{\sigma}}) \cdot \boldsymbol{\nu}\, q \right\}, \end{aligned} \tag{4.15}$$

with

$$\mathcal{K}(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \;:=\; \int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} \;+\; \frac{\kappa_f^2}{\rho_f \omega^2} \int_{\Omega_f} p\, q \qquad \forall\, \widehat{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, p),\, \widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \,\in\, \mathbb{X}\,.$$

Next, we let $\mathbf{A}_0 : \mathbb{X} \to \mathbb{X}$, $\mathbf{K} : \mathbb{X} \to \mathbb{X}$, and $\mathbf{B} : \mathbb{X} \to \mathbb{Y}$ be the linear and bounded operators induced by the bilinear forms A_0, K, and B, respectively. In addition, we let $\mathbf{F} \in \mathbb{X}$ be the Riesz representant of F. Hence, using these notations and taking into account the decomposition (4.12), the variational formulation (3.5) can be rewritten as the following operator equation: Find $(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\gamma}) \in \mathbb{X} \times \mathbb{Y}$ such that

$$\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \boldsymbol{\gamma} \end{pmatrix} \;+\; \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \boldsymbol{\gamma} \end{pmatrix} \;=\; \begin{pmatrix} \mathbf{F} \\ \mathbf{0} \end{pmatrix}. \tag{4.16}$$

Throughout the rest of this section we prove that the matrix operators on the left hand side of (4.16) become invertible and compact, respectively.

Because of the saddle point structure of the matrix operator involving $\mathbf{A}_0$ and $\mathbf{B}$, and according to the well known Babuška-Brezzi theory, we begin the analysis with the continuous inf-sup condition for B, which, as we know, is equivalent to the surjectivity of $\mathbf{B}$. To this end, we observe from the definition of the bilinear form B (cf. (3.7)) that $\mathbf{B}(\widehat{\boldsymbol{\tau}}) \;:=\; -\,\frac{1}{2}\big(\boldsymbol{\tau} - \boldsymbol{\tau}^{\mathtt{t}}\big) \quad \forall\, \widehat{\boldsymbol{\tau}} \;:=\; (\boldsymbol{\tau}, q) \,\in\, \mathbb{X}$.

Lemma 4.1 *There exists $C_1 > 0$ such that*

$$\sup_{\substack{\widehat{\boldsymbol{\tau}}\in\mathbb{X}\\ \widehat{\boldsymbol{\tau}}\neq\mathbf{0}}} \frac{|\,B(\widehat{\boldsymbol{\tau}},\boldsymbol{\eta})\,|}{\|\widehat{\boldsymbol{\tau}}\|_{\mathbb{X}}} \,\geq\, C_1\,\|\boldsymbol{\eta}\|_{\mathbb{Y}} \qquad \forall\,\boldsymbol{\eta}\,\in\,\mathbb{Y}\,.$$

Proof. Given $\boldsymbol{\eta}\in\mathbb{Y}$ we let $\mathbf{z}\in[H^1(\Omega_s)]^3$ be the unique (up to a rigid motion) solution of the variational formulation

$$\int_{\Omega_s}\boldsymbol{\varepsilon}(\mathbf{z}):\boldsymbol{\varepsilon}(\mathbf{w}) \,=\, -\int_{\Omega_s}\mathbf{r}(\boldsymbol{\eta})\cdot\mathbf{w} \,-\,\int_{\Omega_s}\boldsymbol{\eta}:\nabla\mathbf{w} \qquad \forall\,\mathbf{w}\in[H^1(\Omega_s)]^3\,, \tag{4.17}$$

where $\mathbf{r}(\boldsymbol{\eta})\in\mathbb{RM}(\Omega_s)$ is characterized by

$$\int_{\Omega_s}\mathbf{r}(\boldsymbol{\eta})\cdot\mathbf{w} \,=\, -\int_{\Omega_s}\boldsymbol{\eta}:\nabla\mathbf{w} \qquad \forall\,\mathbf{w}\,\in\,\mathbb{RM}(\Omega_s)\,.$$

Then, defining $\boldsymbol{\zeta} := \boldsymbol{\varepsilon}(\mathbf{z})+\boldsymbol{\eta}$, we find from (4.17) that $\mathbf{div}\,\boldsymbol{\zeta} = \mathbf{r}(\boldsymbol{\eta})$ in Ω_s, whence $\boldsymbol{\zeta}\in H(\mathbf{div};\Omega_s)$, and thus $\boldsymbol{\zeta}\boldsymbol{\nu} = \mathbf{0}$ on Σ. It follows that $\widehat{\boldsymbol{\zeta}} := (-\boldsymbol{\zeta},0)\in\mathbb{X}$, and clearly $\mathbf{B}(\widehat{\boldsymbol{\zeta}}) = \boldsymbol{\eta}$, which proves the surjectivity of $\mathbf{B}$.

□

Our next goal is to prove that $\mathbf{A}_0$ is an isomorphism on the kernel of $\mathbf{B}$. For this purpose, we now introduce the decomposition

$$H(\mathbf{div};\Omega_s) \,=\, H_0(\mathbf{div};\Omega_s)\,\oplus\,\mathbb{C}\,\mathbf{I}\,,$$

where

$$H_0(\mathbf{div};\Omega_s) \,:=\, \left\{\boldsymbol{\tau}\in H(\mathbf{div};\Omega_s):\quad \int_{\Omega_s}\mathrm{tr}(\boldsymbol{\tau}) \,=\, 0\right\}.$$

This means that for any $\boldsymbol{\tau}\in H(\mathbf{div};\Omega_s)$ there exist unique $\boldsymbol{\tau}_0\in H_0(\mathbf{div};\Omega_s)$ and $d\in\mathbb{C}$ given by $d := \dfrac{1}{3\,|\Omega_s|}\displaystyle\int_{\Omega_s}\mathrm{tr}(\boldsymbol{\tau})$, where $|\Omega_s|$ denotes the measure of Ω_s, such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I}$.

Our subsequent analysis will strongly depend on the inequalities provided by the following three lemmata. In particular, note that Lemma 4.3 constitute an interesting generalization of [14, Lemma 2.2] (see also [15, Lemma 4.5]). In addition, we also remark that the coerciveness-type estimate provided by Lemma 4.4 is less restrictive than the analogue one given in [15, Lemma 4.6].

Lemma 4.2 *There exists $c_1 > 0$, depending only on Ω_s, such that*

$$c_1\,\|\boldsymbol{\tau}_0\|^2_{[L^2(\Omega_s)]^{3\times3}} \,\leq\, \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{[L^2(\Omega_s)]^{3\times3}} \,+\, \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{[L^2(\Omega_s)]^3} \qquad \forall\,\boldsymbol{\tau}\in H(\mathbf{div};\Omega_s)\,. \tag{4.18}$$

Proof. See [3, Lemma 3.1] or [11, Proposition 3.1, Chapter IV].

□

Lemma 4.3 *There exists $c_2 > 0$ such that*

$$\mathrm{Re}\Big\{\mathcal{A}(\widehat{\boldsymbol{\tau}},\overline{\widehat{\boldsymbol{\tau}}})\Big\} \,\geq\, c_2\,\|\widehat{\boldsymbol{\tau}}\|^2_{\mathbb{X}} \qquad \forall\,\widehat{\boldsymbol{\tau}}\,\in\,\mathbb{X}\,. \tag{4.19}$$

Proof. Let $\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau},q)\in\mathbb{X}$ with $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I}$. We first notice, from the definition of $\mathcal{A}$ (cf. (4.14)) and the inequality (3.8), that

$$\mathrm{Re}\Big\{\mathcal{A}(\widehat{\boldsymbol{\tau}},\overline{\widehat{\boldsymbol{\tau}}})\Big\} \,\geq\, C_4\,\Big\{\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{[L^2(\Omega_s)]^{3\times3}} \,+\, \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{[L^2(\Omega_s)]^3} \,+\, \|q\|^2_{H^1(\Omega_f)}\Big\}\,. \tag{4.20}$$

On the other hand, since $\boldsymbol{\tau}\boldsymbol{\nu} = -q\boldsymbol{\nu}$ on Σ, we see that $-q\boldsymbol{\nu} = \boldsymbol{\tau}_0\boldsymbol{\nu} + d\boldsymbol{\nu}$ in $[H^{-1/2}(\Sigma)]^3$, from which, applying the trace theorem in $H(\mathbf{div};\Omega_s)$ together with the continuity of the canonical embedding $[L^2(\Sigma)]^3 \hookrightarrow [H^{-1/2}(\Sigma)]^3$ and the trace theorem in $H^1(\Omega_f)$, we deduce that

$$\begin{aligned} |d|\,\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Sigma)]^3} &\le \|\boldsymbol{\tau}_0\boldsymbol{\nu}\|_{[H^{-1/2}(\Sigma)]^3} + \|q\boldsymbol{\nu}\|_{[H^{-1/2}(\Sigma)]^3} \\ &\le C_1\left\{\|\boldsymbol{\tau}_0\|_{H(\mathbf{div};\Omega_s)} + \|q\|_{H^1(\Omega_f)}\right\}. \end{aligned}$$

It follows that

$$\begin{aligned} \|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega_s)} + \|q\|^2_{H^1(\Omega_f)} &= \|\boldsymbol{\tau}_0\|^2_{H(\mathbf{div};\Omega_s)} + 3\,d^2\,|\Omega_s| + \|q\|^2_{H^1(\Omega_f)} \\ &\le C_2\left\{\|\boldsymbol{\tau}_0\|^2_{H(\mathbf{div};\Omega_s)} + \|q\|^2_{H^1(\Omega_f)}\right\}, \end{aligned}$$

which, thanks to (4.18), yields

$$\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega_s)} + \|q\|^2_{H^1(\Omega_f)} \le C_3\left\{\|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{[L^2(\Omega_s)]^{3\times 3}} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{[L^2(\Omega_s)]^3} + \|q\|^2_{H^1(\Omega_f)}\right\}$$

for all $\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}$. The above estimate and (4.20) imply (4.19) and finish the proof.

□

In what follows we make frequent use of the linear and bounded operator $\Xi := (2\,\mathbf{P} - \mathbf{I}) : \mathbb{X} \to \mathbb{X}$.

Lemma 4.4 *There exists $C > 0$, depending on μ, c_1, c_2, κ_s, ρ_f, and ω^2, such that for each $\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X}$ there holds*

$$\mathrm{Re}\left\{A_0(\widehat{\boldsymbol{\tau}}, \Xi(\overline{\widehat{\boldsymbol{\tau}}}))\right\} \ge C\left\{\|\mathbf{P}(\widehat{\boldsymbol{\tau}})\|^2_{\mathbb{X}} + \|(\mathbf{I}-\mathbf{P})\widehat{\boldsymbol{\tau}}\|^2_{\mathbb{X}}\right\}. \tag{4.21}$$

Proof. Since $\mathbf{P}$ is a projector we easily observe that

$$\mathbf{P}\,\Xi(\widehat{\boldsymbol{\tau}}) = \mathbf{P}(\widehat{\boldsymbol{\tau}}) \quad \text{and} \quad (\mathbf{I}-\mathbf{P})\,\Xi(\widehat{\boldsymbol{\tau}}) = -(\mathbf{I}-\mathbf{P})(\widehat{\boldsymbol{\tau}}) \qquad \forall\,\widehat{\boldsymbol{\tau}} \in \mathbb{X},$$

which, according to the definition of A_0 (cf. (4.13)), gives

$$A_0(\widehat{\boldsymbol{\tau}}, \Xi(\overline{\widehat{\boldsymbol{\tau}}})) = \mathcal{A}(\mathbf{P}(\widehat{\boldsymbol{\tau}}), \mathbf{P}(\overline{\widehat{\boldsymbol{\tau}}})) + \mathcal{A}((\mathbf{I}-\mathbf{P})(\widehat{\boldsymbol{\tau}}), (\mathbf{I}-\mathbf{P})(\overline{\widehat{\boldsymbol{\tau}}})). \tag{4.22}$$

Hence, the inequality (4.21) follows directly from (4.22), Lemma 4.3, and the fact that both $\mathbf{P}(\widehat{\boldsymbol{\tau}})$ and $(\mathbf{I}-\mathbf{P})(\widehat{\boldsymbol{\tau}})$ belong to $\mathbb{X}$.

□

We now let $\mathbb{V}$ be the kernel of $\mathbf{B}$, that is $\mathbb{V} := \left\{\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X} : \quad \mathbf{B}(\widehat{\boldsymbol{\tau}}) = 0\right\}$, which, recalling that $\mathbf{B}(\widehat{\boldsymbol{\tau}}) := -\frac{1}{2}\left(\boldsymbol{\tau} - \boldsymbol{\tau}^{\mathtt{t}}\right)$ $\forall\,\widehat{\boldsymbol{\tau}} \in \mathbb{X}$, becomes $\mathbb{V} = \left\{\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X} : \quad \boldsymbol{\tau}^{\mathtt{t}} = \boldsymbol{\tau}\right\}$. Hence, we are now in a position to establish the weak coercivity of A_0 on $\mathbb{V}$.

Lemma 4.5 *There exists $C > 0$ such that*

$$\sup_{\substack{\widehat{\boldsymbol{\zeta}} \in \mathbb{V} \\ \widehat{\boldsymbol{\zeta}} \neq \mathbf{0}}} \frac{|\,A_0(\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\zeta}})\,|}{\|\widehat{\boldsymbol{\zeta}}\|_{\mathbb{X}}} \ge C\,\|\widehat{\boldsymbol{\tau}}\|_{\mathbb{X}} \qquad \forall\,\widehat{\boldsymbol{\tau}} \in \mathbb{V}. \tag{4.23}$$

In addition, there holds

$$\sup_{\widehat{\boldsymbol{\zeta}} \in \mathbb{V}} |\,A_0(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}})\,| > 0 \qquad \forall\,\widehat{\boldsymbol{\tau}} \in \mathbb{V},\ \widehat{\boldsymbol{\tau}} \neq \mathbf{0}. \tag{4.24}$$

Proof. Since $P(\widehat{\boldsymbol{\tau}})^{\mathrm{t}} = P(\widehat{\boldsymbol{\tau}})$ $\forall \widehat{\boldsymbol{\tau}} \in \mathbb{X}$ (cf. (4.7)), we find that $\mathbf{P}(\widehat{\boldsymbol{\tau}})$, and hence $\Xi(\widehat{\boldsymbol{\tau}})$, belong to $\mathbb{V}$ for each $\widehat{\boldsymbol{\tau}} \in \mathbb{V}$. In addition, it is easy to see, using for instance (4.21) (cf. Lemma 4.4), that for each $\widehat{\boldsymbol{\tau}} \in \mathbb{X}$, $\widehat{\boldsymbol{\tau}} \neq \mathbf{0}$, there holds $\Xi(\widehat{\boldsymbol{\tau}}) \neq \mathbf{0}$. According to the above, for each $\widehat{\boldsymbol{\tau}} \in \mathbb{V}$, $\widehat{\boldsymbol{\tau}} \neq \mathbf{0}$, we can write

$$\sup_{\substack{\widehat{\boldsymbol{\zeta}} \in \mathbb{V} \\ \widehat{\boldsymbol{\zeta}} \neq \mathbf{0}}} \frac{|\, A_0(\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\zeta}}) \,|}{\|\widehat{\boldsymbol{\zeta}}\|_{\mathbb{X}}} \,\geq\, \frac{|\, A_0(\widehat{\boldsymbol{\tau}}, \Xi(\overline{\widehat{\boldsymbol{\tau}}})) \,|}{\|\Xi(\overline{\widehat{\boldsymbol{\tau}}})\|_{\mathbb{X}}} \,\geq\, \frac{\mathrm{Re}\left\{ A_0(\widehat{\boldsymbol{\tau}}, \Xi(\overline{\widehat{\boldsymbol{\tau}}})) \right\}}{\|\Xi(\overline{\widehat{\boldsymbol{\tau}}})\|_{\mathbb{X}}}\,, \tag{4.25}$$

and applying (4.21), the stability of the decomposition (4.9), the fact that $\|\widehat{\boldsymbol{\tau}}\|_{\mathbb{X}} = \|\overline{\widehat{\boldsymbol{\tau}}}\|_{\mathbb{X}}$, and the boundedness of Ξ, we deduce that

$$\mathrm{Re}\left\{ A_0(\widehat{\boldsymbol{\tau}}, \Xi(\overline{\widehat{\boldsymbol{\tau}}})) \right\} \,\geq\, C\, \|\widehat{\boldsymbol{\tau}}\|_{\mathbb{X}}^2 \,\geq\, C\, \|(\Xi(\overline{\widehat{\boldsymbol{\tau}}}))\|_{\mathbb{X}}\, \|\widehat{\boldsymbol{\tau}}\|_{\mathbb{X}} \qquad \forall\, \widehat{\boldsymbol{\tau}} \in \mathbb{V}\,. \tag{4.26}$$

In this way, the inf-sup condition (4.23) follows straightforwardly after replacing (4.26) back into (4.25). Finally, the symmetry of A_0 and the estimate (4.26) yields (4.24) and complete the proof.

□

Lemma 4.6 *The operator* $\mathbf{K} : \mathbb{X} \to \mathbb{X}$ *is compact.*

Proof. We begin by recalling (cf. (4.10)) that there exists $\epsilon > 0$ such that $P(\widehat{\boldsymbol{\tau}}) \in [H^{\epsilon}(\Omega_s)]^{3\times 3}$ for all $\widehat{\boldsymbol{\tau}} \in \mathbb{X}$, which, according to the compact imbedding $H^{\epsilon}(\Omega_r) \stackrel{c}{\hookrightarrow} L^2(\Omega_r)$, for $r \in \{s, f\}$, yields the compacity of $\mathbf{P} : \mathbb{X} \to [L^2(\Omega_s)]^{3\times 3} \times L^2(\Omega_f)$. It follows that $\mathbf{P}^* : [L^2(\Omega_s)]^{3\times 3} \times L^2(\Omega_f) \to \mathbb{X}$, $\mathbf{P}^*\, \widetilde{\mathbf{K}}\, \mathbf{P}$, $(\mathbf{I} - \mathbf{P})^*\, \widetilde{\mathbf{K}}\, \mathbf{P}$, and $\mathbf{P}^*\, \widetilde{\mathbf{K}}\, (\mathbf{I} - \mathbf{P})$ are all compact, where $\widetilde{\mathbf{K}} : [L^2(\Omega_s)]^{3\times 3} \times L^2(\Omega_f) \to [L^2(\Omega_s)]^{3\times 3} \times L^2(\Omega_f)$ is the operator associated to the bilinear form $\mathcal{K}$. This shows that the operator induced by the first three terms defining K (cf. (4.15)) becomes compact, as well. Finally, it is clear that the remaining three terms on the right hand side of (4.15) constitute a finite rank operator. □

We are able now to establish the main result of this section.

Theorem 4.1 *Assume that the homogeneous problem associated to* (3.5) *has only the trivial solution. Then, given* $\mathbf{f} \in [L^2(\Omega_s)]^3$ *and* $g \in H^{-1/2}(\Gamma)$*, there exists a unique solution* $((\boldsymbol{\sigma}, p), \boldsymbol{\gamma}) \in \mathbb{X} \times \mathbb{Y}$ *to* (3.5) (*equivalently* (4.16))*. In addition, there exists* $C > 0$ *such that*

$$\|((\boldsymbol{\sigma}, p), \boldsymbol{\gamma})\|_{\mathbb{X}\times\mathbb{Y}} \,\leq\, C \left\{ \|\mathbf{f}\|_{[L^2(\Omega_s)]^2} \,+\, \|g\|_{H^{-1/2}(\Gamma)} \right\}.$$

Proof. It suffices to observe that the left hand side of (4.16) constitutes a Fredholm operator of index zero. In fact, Lemmata 4.1 and 4.5 imply that $\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ is an isomorphism, and Lemma 4.6 yields the compacity of $\begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$.

□

5 Analysis of the Galerkin scheme

In this section we introduce a Galerkin approximation of (3.5) and prove its well-posedness.

5.1 Preliminaries

We first let $\{\mathcal{T}_h\}_{h>0} := \{\mathcal{T}_{h_s}\}_{h_s>0} \cup \{\mathcal{T}_{h_f}\}_{h_f>0}$, where $\{\mathcal{T}_{h_s}\}_{h_s>0}$ and $\{\mathcal{T}_{h_f}\}_{h_f>0}$ are shape-regular families of triangulations of the polyhedral regions $\bar{\Omega}_s$ and $\bar{\Omega}_f$, respectively, by tetrahedrons T of diameter h_T with mesh sizes $h_s := \max\{ h_T : \quad T \in \mathcal{T}_{h_s} \}$, $h_f := \max\{ h_T : \quad T \in \mathcal{T}_{h_f} \}$, and $h := \max\{h_s, h_f\}$, and such that the vertices of $\{\mathcal{T}_{h_s}\}_{h_s>0}$ and $\{\mathcal{T}_{h_f}\}_{h_f>0}$ coincide on Σ. In what follows, given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^3$, $P_\ell(S)$ denotes the space of polynomials defined in S of total degree $\leq \ell$. Then, we define

$$\mathbf{H}_h := \Big\{ \boldsymbol{\tau}_h \in H(\mathbf{div}; \Omega_s) : \quad \boldsymbol{\tau}_h|_T \in [P_1(T)]^{3\times 3} \quad \forall\, T \in \mathcal{T}_{h_s} \Big\},$$

$$W_h := \Big\{ q_h \in C(\bar{\Omega}_f) : \quad q_h|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_{h_f} \Big\},$$

and introduce the finite element subspaces of $\mathbb{X}$ and $\mathbb{Y}$, given, respectively, by

$$\mathbb{X}_h := \Big\{ \widehat{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_h, q_h) \in \mathbf{H}_h \times W_h : \quad \boldsymbol{\tau}_h \boldsymbol{\nu} = -q_h \boldsymbol{\nu} \quad \text{on} \quad \Sigma \Big\}, \tag{5.1}$$

and

$$\mathbb{Y}_h := \Big\{ \boldsymbol{\eta}_h \in \mathbb{Y} : \quad \boldsymbol{\eta}_h|_T \in [P_0(T)]^{3\times 3} \quad \forall\, T \in \mathcal{T}_{h_s} \Big\}.$$

In addition, throughout the analysis below we will also need the spaces

$$\tilde{\mathbf{H}}_h := \Big\{ \tilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h : \quad \tilde{\boldsymbol{\tau}}_h \boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Sigma \Big\}$$

and

$$\mathbf{U}_h := \Big\{ \mathbf{v}_h \in [L^2(\Omega_s)]^3 : \quad \mathbf{v}_h|_T \in [P_0(T)]^3 \quad \forall\, T \in \mathcal{T}_{h_s} \Big\}.$$

Note that $\mathbf{H}_h \times \mathbf{U}_h \times \mathbb{Y}_h$ constitutes the lowest order mixed finite element approximation of the linear elasticity problem introduced recently by Arnold Falk and Winther (see [6], [5]). Moreover, the definition of $\mathbb{X}_h$ represents the announced coupling between the $H(\mathbf{div}; \Omega_s)$-component of the Arnold-Falk-Winther element (represented by $\mathbf{H}_h$) and the Lagrange finite elements (represented by W_h).

Hence, the finite element scheme associated to our coupled problem (3.5) is defined as: Find $\widehat{\boldsymbol{\sigma}}_h = (\boldsymbol{\sigma}_h, p_h) \in \mathbb{X}_h$ and $\boldsymbol{\gamma}_h \in \mathbb{Y}_h$ such that

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\tau}}_h) + B(\widehat{\boldsymbol{\tau}}_h, \boldsymbol{\gamma}_h) &= F(\widehat{\boldsymbol{\tau}}_h) \qquad \forall\, \widehat{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_h, q_h) \in \mathbb{X}_h, \\ B(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\eta}_h) &= 0 \qquad\qquad \forall \boldsymbol{\eta}_h \in \mathbb{Y}_h. \end{aligned} \tag{5.2}$$

The well-posedness of (5.2) will be proved below in Section 5.3. We previously collect in what remains of this section the approximation properties of the subspaces involved, and then in Sections 5.2 and 5.3 we analyze a Galerkin approximation of (4.2) with data in $\mathbb{X}_h$, which yields a mixed finite element approximation of the operator $\mathbf{P}|_{\mathbb{X}_h}$ (cf. (4.6)).

We begin with $\mathbf{H}_h$. Indeed, given $\delta \in (0, 1]$, we let $\mathcal{E}_h : [H^\delta(\Omega_s)]^{3\times 3} \cap H(\mathbf{div}; \Omega_s) \to \mathbf{H}_h$ be the usual BDM interpolation operator (see [11]), which is characterized by the identities

$$\int_F \mathcal{E}_h(\boldsymbol{\tau})\, \boldsymbol{\nu} \cdot \mathbf{p} = \int_F \boldsymbol{\tau}\, \boldsymbol{\nu} \cdot \mathbf{p} \quad \forall\, \mathbf{p} \in [P_1(F)]^3, \quad \forall \text{ face } F \text{ of } \mathcal{T}_{h_s}. \tag{5.3}$$

Moreover, the conmuting diagram property yields

$$\mathbf{div}(\mathcal{E}_h(\boldsymbol{\tau})) = \mathcal{P}_h(\mathbf{div}\, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in [H^\delta(\Omega_s)]^{3\times 3} \cap H(\mathbf{div}; \Omega_s), \tag{5.4}$$

where $\mathcal{P}_h : [L^2(\Omega_s)]^3 \to \mathbf{U}_h$ is the $[L^2(\Omega_s)]^3$-ortogonal projector. In addition, it is well known (see, e.g. [20, Theorem 3.16]) that there exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau} \in [H^\delta(\Omega_s)]^{3\times 3} \cap H(\mathbf{div};\Omega_s)$ there holds

$$\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|_{[L^2(T)]^{3\times 3}} \le C\, h_T^\delta \left\{ |\boldsymbol{\tau}|_{[H^\delta(T)]^{3\times 3}} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{[L^2(T)]^3} \right\} \quad \forall\, T \in \mathcal{T}_{h_s}\,. \tag{5.5}$$

We now let $\Pi_h : H^1(\Omega_f) \to W_h$ and $\mathcal{R}_h : [L^2(\Omega_s)]^{3\times 3} \to \mathbb{Y}_h$ be the corresponding orthogonal projectors with respect to the natural norms of each space. Then, we have (see [8], [11], [29]):

($\mathrm{AP}_h^{\boldsymbol{\sigma}}$) For each $\delta \in (0,1]$ and for each $\boldsymbol{\tau} \in [H^\delta(\Omega_s)]^{3\times 3}$, with $\mathbf{div}\,\boldsymbol{\tau} \in [H^\delta(\Omega_s)]^3$, there holds

$$\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|_{H(\mathbf{div};\Omega_s)} \le C\, h^\delta \left\{ \|\boldsymbol{\tau}\|_{[H^\delta(\Omega_s)]^{3\times 3}} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{[H^\delta(\Omega_s)]^3} \right\}.$$

(AP_h^p) For each $s \in (1,2]$ and for each $q \in H^s(\Omega_f)$, there holds

$$\|q - \Pi_h(q)\|_{H^1(\Omega_f)} \le C\, h^{s-1}\, \|q\|_{H^s(\Omega_f)}\,.$$

($\mathrm{AP}_h^{\boldsymbol{\gamma}}$) For each $s \in (0,1]$ and for each $\boldsymbol{\eta} \in [H^s(\Omega_s)]^{3\times 3} \cap [L^2(\Omega_s)]^{3\times 3}_{\mathtt{asym}}$, there holds

$$\|\boldsymbol{\eta} - \mathcal{R}_h(\boldsymbol{\eta})\|_{[L^2(\Omega_s)]^{3\times 3}} \le C\, h^s\, \|\boldsymbol{\eta}\|_{[H^s(\Omega_s)]^{3\times 3}}\,.$$

($\mathrm{AP}_h^{\mathbf{u}}$) For each $t \in (0,1]$ and for each $\mathbf{v} \in [H^t(\Omega_s)]^3$, there holds

$$\|\mathbf{v} - \mathcal{P}_h(\mathbf{v})\|_{[L^2(\Omega_s)]^3} \le C\, h^t\, \|\mathbf{v}\|_{[H^t(\Omega_s)]^3}\,.$$

Note here that ($\mathrm{AP}_h^{\boldsymbol{\sigma}}$) is actually a straightforward consequence of (5.4), (5.5), and ($\mathrm{AP}_h^{\mathbf{u}}$).

We end this section with an approximation property of our finite element subspace $\mathbb{X}_h$ (cf. (5.1)). For this purpose, we first proceed similarly as in [17, Section 5.2, Lemma 5.1] and assume from now on that $\{\mathcal{T}_{h_s}\}_{h_s>0}$ is quasi-uniform around Σ. This means that there exists an open neighborhood of Σ, say Ω_Σ, with Lipschitz boundary, and such that the elements of $\mathcal{T}_{h_s}$ intersecting that region are more or less of the same size. In other words, we define

$$\mathcal{T}_{\Sigma,h} := \left\{ T \in \mathcal{T}_{h_s} : \quad T \cap \Omega_\Sigma \neq \emptyset \right\},$$

and assume that there exists $c > 0$, independent of h, such that

$$\max_{T\in\mathcal{T}_{\Sigma,h}} h_T \le c \min_{T\in\mathcal{T}_{\Sigma,h}} h_T \qquad \forall\, h > 0\,. \tag{5.6}$$

Note that this assumption and the shape-regularity property of the meshes imply that Σ_h, the partition on Σ inherited from $\mathcal{T}_{h_s}$, is also quasi-uniform, which means that there exists $C > 0$, independent of h, such that

$$h_\Sigma := \max\left\{ \mathrm{diam}\,\{F\} : \quad F \text{ face of } \Sigma_h \right\} \le C \min\left\{ \mathrm{diam}\,\{F\} : \quad F \text{ face of } \Sigma_h \right\}.$$

In addition, the quasi-uniformity of Σ_h guarantees the inverse inequality on $\Phi_h(\Sigma)$, the subspace of $[L^2(\Sigma)]^3$ given by the piecewise polynomials of degree ≤ 1, that is, in particular,

$$\|\boldsymbol{\phi}_h\|_{[L^2(\Sigma)]^3} \le C\, h_\Sigma^{-1/2}\, \|\boldsymbol{\phi}_h\|_{[H^{-1/2}(\Sigma)]^3} \qquad \forall\, \boldsymbol{\phi}_h \in \Phi_h(\Sigma)\,. \tag{5.7}$$

Then, we are now in a position to establish the following lemma.

Lemma 5.1 *Given $\epsilon \in (0,1]$, define $\mathbb{X}_\epsilon := \Big\{ H(\mathbf{div};\Omega_s) \cap [H^\epsilon(\Omega_s)]^{3\times 3} \Big\} \times H^{1+\epsilon}(\Omega_f)$. Then, there exists a linear operator $\mathbb{I}_h : \mathbb{X}_\epsilon \longrightarrow \mathbb{X}_h$, such that for each $\widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \mathbb{X} \cap \mathbb{X}_\epsilon$ there holds*

$$\|\widehat{\boldsymbol{\tau}} - \mathbb{I}_h(\widehat{\boldsymbol{\tau}})\|_{\mathbb{X}} \,\le\, C \Big\{ \|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|_{H(\mathbf{div};\Omega_s)} \,+\, \|q - \Pi_h(q)\|_{H^1(\Omega_f)} \Big\}. \tag{5.8}$$

Proof. Given $\widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}, q) \in \mathbb{X} \cap \mathbb{X}_\epsilon$, we let $\boldsymbol{\varphi} : \Omega_s \mapsto \mathbb{R}^3$ be the unique solution (guaranteed by the Lax-Milgram Lemma) of the vectorial Laplace problem

$$\begin{aligned} \Delta \boldsymbol{\varphi} \,&=\, \frac{1}{|\Omega_s|} \int_\Sigma \Big\{ \mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu} \Big\} \quad \text{in} \quad \Omega_s \\ \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} \,&=\, \mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu} \quad \text{on} \quad \Sigma\,, \qquad \int_{\Omega_s} \boldsymbol{\varphi} \,=\, \mathbf{0}\,, \end{aligned} \tag{5.9}$$

whose corresponding continuous dependence result states that

$$\|\boldsymbol{\varphi}\|_{[H^1(\Omega_s)]^3} \,\le\, C\, \|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|_{[H^{-1/2}(\Sigma)]^3}\,. \tag{5.10}$$

Actually, since the Neumann data $\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}$, being a piecewise polynomial of degree ≤ 1, belongs to $[H^\delta(\Sigma)]^3$ for any $\delta \in [0, 1/2)$, we deduce that we have at least $[H^{3/2}(\Omega_s)]^3$-regularity for the solution $\boldsymbol{\varphi}$ and

$$\|\boldsymbol{\varphi}\|_{[H^{3/2}(\Omega_s)]^3} \,\le\, C\, \|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|_{[L^2(\Sigma)]^3}\,. \tag{5.11}$$

Moreover, since $\Omega_s^{\mathtt{int}} := \Omega_s \backslash \Omega_\Sigma$ is an interior region of Ω_s, the interior elliptic regularity estimate (see, e.g. [26, Theorem 4.16]) says that

$$\|\boldsymbol{\varphi}\|_{[H^2(\Omega_s^{\mathtt{int}})]^3} \,\le\, C\, \|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|_{[H^{-1/2}(\Sigma)]^3}\,. \tag{5.12}$$

Next, we define $\boldsymbol{\zeta} := \nabla \boldsymbol{\varphi}$ in Ω_s, whence $\boldsymbol{\zeta} \in [H^{1/2+\delta}(\Omega_s)]^{3\times 3}$, and observe from (5.9) that

$$\mathbf{div}\, \boldsymbol{\zeta} \,=\, \frac{1}{|\Omega_s|} \int_\Sigma \Big\{ \mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu} \Big\} \quad \text{in} \quad \Omega_s\,, \quad \text{and} \quad \boldsymbol{\zeta}\,\boldsymbol{\nu} \,=\, \mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu} \quad \text{on} \quad \Sigma\,, \tag{5.13}$$

which, in particular, implies that $\boldsymbol{\zeta} \in H(\mathbf{div};\Omega_s)$. Hence, we now set

$$\mathbb{I}_h(\widehat{\boldsymbol{\tau}}) \,:=\, \big(\mathcal{E}_h(\boldsymbol{\tau} - \boldsymbol{\zeta}), \Pi_h(q) \big) \,\in\, \mathbf{H}_h \times W_h\,,$$

and show that $\mathbb{I}_h$ is well defined, that is $\mathbb{I}_h(\widehat{\boldsymbol{\tau}}) \in \mathbb{X}_h$. In fact, employing the characterization (5.3) and the second identity in (5.13), we find that for each face $F \subseteq \Sigma$ and for each $\mathbf{p} \in [P_1(F)]^3$, there holds

$$\int_F \mathcal{E}_h(\boldsymbol{\zeta})\,\boldsymbol{\nu} \cdot \mathbf{p} \,=\, \int_F \boldsymbol{\zeta}\,\boldsymbol{\nu} \cdot \mathbf{p} \,=\, \int_F \Big\{ \mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu} \Big\} \cdot \mathbf{p}$$

which, noting that $\Big\{ \mathcal{E}_h(\boldsymbol{\tau} - \boldsymbol{\zeta})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu} \Big\}\Big|_F \in [P_1(F)]^3$, yields $\mathcal{E}_h(\boldsymbol{\tau} - \boldsymbol{\zeta})\,\boldsymbol{\nu} \,=\, -\,\Pi_h(q)\,\boldsymbol{\nu}$ on Σ.

We now aim to prove (5.8). We first observe, applying the triangle inequality, that

$$\|\widehat{\boldsymbol{\tau}} - \mathbb{I}_h(\widehat{\boldsymbol{\tau}})\|_{\mathbb{X}}^2 \,\le\, 2\, \|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|_{H(\mathbf{div};\Omega_s)}^2 \,+\, 2\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{H(\mathbf{div};\Omega_s)}^2 \,+\, \|q - \Pi_h(q)\|_{H^1(\Omega_f)}^2\,. \tag{5.14}$$

Then, using the first identity in (5.13), which says that $\mathbf{div}(\boldsymbol{\zeta}) \in \mathbf{U}_h$, and (5.4), we deduce that

$$\begin{aligned} \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{H(\mathbf{div};\Omega_s)}^2 \,&=\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{[L^2(\Omega_s)]^{3\times 3}}^2 \,+\, \|\mathbf{div}\,\boldsymbol{\zeta}\|_{[L^2(\Omega_s)]^3}^2 \\ &\le\, C \Big\{ \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{[L^2(\Omega_s)]^{3\times 3}}^2 \,+\, \|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|_{[H^{-1/2}(\Sigma)]^3}^2 \Big\}\,. \end{aligned} \tag{5.15}$$

Now, adding and substracting $\boldsymbol{\tau}\,\boldsymbol{\nu} \,=\, -\,q\,\boldsymbol{\nu}$ on Σ, and applying the trace theorems in $H(\mathbf{div};\Omega_s)$ and $H^1(\Omega_f)$, we find that

$$
\begin{aligned}
\|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|_{[H^{-1/2}(\Sigma)]^3} \,&\le\, \|\big(\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\big)\,\boldsymbol{\nu}\|_{[H^{-1/2}(\Sigma)]^3} \,+\, \|\big(q \,-\, \Pi_h(q)\big)\,\boldsymbol{\nu}\|_{H^{-1/2}(\Sigma)} \\
&\le\, C\,\Big\{\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|_{H(\mathbf{div};\,\Omega_s)} \,+\, \|q \,-\, \Pi_h(q)\|_{L^2(\Sigma)}\Big\} \\
&\le\, C\,\Big\{\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|_{H(\mathbf{div};\,\Omega_s)} \,+\, \|q - \Pi_h(q)\|_{H^1(\Omega_f)}\Big\}\,.
\end{aligned}
\tag{5.16}
$$

It remains to estimate $\|\mathcal{E}_h(\boldsymbol{\zeta})\|_{[L^2(\Omega_s)]^{3\times3}}$. In fact, defining the sets

$$
\Omega_{\Sigma,h} \,:=\, \cup\Big\{\,T: \quad T\in\mathcal{T}_{\Sigma,h}\Big\} \quad \text{and} \quad \Omega_{s,h}^{\texttt{int}} \,:=\, \Omega_s\backslash\Omega_{\Sigma,h} \,\subseteq\, \Omega_s^{\texttt{int}}\,,
$$

and using the stability of $\mathcal{E}_h$ when applied to $[H^1(\Omega_{s,h}^{\texttt{int}})]^3$, and the estimate (5.12), we find that

$$
\begin{aligned}
\|\mathcal{E}_h(\boldsymbol{\zeta})\|_{[L^2(\Omega_s)]^{3\times3}} \,&\le\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{[L^2(\Omega_{s,h}^{\texttt{int}})]^{3\times3}} \,+\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{[L^2(\Omega_{\Sigma,h})]^{3\times3}} \\
&\le\quad C\,\|\boldsymbol{\varphi}\|_{[H^2(\Omega_s^{\texttt{int}})]^3} \,+\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{[L^2(\Omega_{\Sigma,h})]^{3\times3}} \\
&\le\quad C\,\|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|_{[H^{-1/2}(\Sigma)]^3} \,+\, \|\mathcal{E}_h(\boldsymbol{\zeta})\|_{[L^2(\Omega_{\Sigma,h})]^{3\times3}}\,.
\end{aligned}
\tag{5.17}
$$

In turn, adding and substracting $\boldsymbol{\zeta} \,=\, \nabla\boldsymbol{\varphi}$, and utilizing the upper bound (5.10), the estimates (5.5) (with $\delta = 1/2$) and (5.11), the first identity in (5.13), the quasi-uniformity bound (5.6), and the inverse inequality (5.7), we arrive at

$$
\begin{aligned}
&\|\mathcal{E}_h(\boldsymbol{\zeta})\|^2_{[L^2(\Omega_{\Sigma,h})]^{3\times3}} \,\le\, C\,\Big\{\|\boldsymbol{\zeta} - \mathcal{E}_h(\boldsymbol{\zeta})\|^2_{[L^2(\Omega_{\Sigma,h})]^{3\times3}} \,+\, \|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|^2_{[H^{-1/2}(\Sigma)]^3}\Big\} \\
&\le\, C\sum_{T\in\mathcal{T}_{\Sigma,h}} h_T\,\|\boldsymbol{\varphi}\|^2_{[H^{3/2}(T)]^3} \,+\, C\,\|\mathbf{div}\,\boldsymbol{\zeta}\|^2_{[L^2(\Omega_s)]^3} \,+\, C\,\|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|^2_{[H^{-1/2}(\Sigma)]^3} \\
&\le\, C\,h_\Sigma\,\|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|^2_{[L^2(\Sigma)]^3} \,+\, C\,\|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|^2_{[H^{-1/2}(\Sigma)]^3} \\
&\le\, C\,\|\mathcal{E}_h(\boldsymbol{\tau})\,\boldsymbol{\nu} \,+\, \Pi_h(q)\,\boldsymbol{\nu}\|^2_{[H^{-1/2}(\Sigma)]^3}\,.
\end{aligned}
\tag{5.18}
$$

Finally, (5.14), (5.15), (5.16), (5.17) and (5.18) finish the proof.

□

5.2 Numerical analysis of the auxiliary Neumann problem

Given $\widehat{\boldsymbol{\tau}}_h \,=\, (\boldsymbol{\tau}_h, q_h) \,\in\, \mathbb{X}_h \,\subseteq\, \mathbb{X}$, we recall from (4.2) and (4.6) that $\mathbf{P}(\widehat{\boldsymbol{\tau}}_h) \,=\, (P(\widehat{\boldsymbol{\tau}}_h), q_h) \,:=\, (\tilde{\boldsymbol{\sigma}}, q_h)$, where $(\tilde{\boldsymbol{\sigma}}, (\tilde{\mathbf{u}}, \tilde{\gamma}))$ is the unique element in $H(\mathbf{div};\,\Omega_s)\times\tilde{\mathbf{Q}}$ such that $\tilde{\boldsymbol{\sigma}}\boldsymbol{\nu} \,=\, -\,q_h\,\boldsymbol{\nu}$ on Σ and

$$
\begin{aligned}
\mathbf{a}(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\tau}}) \,+\, \mathbf{b}(\tilde{\boldsymbol{\tau}}, (\tilde{\mathbf{u}}, \tilde{\gamma})) \quad &= \quad \mathbf{0} \qquad \forall\,\tilde{\boldsymbol{\tau}}\in\tilde{\mathbf{H}}\,, \\
\mathbf{b}(\tilde{\boldsymbol{\sigma}}, (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}})) \quad &= \quad \int_{\Omega_s} (\mathbf{div}\,\boldsymbol{\tau}_h + \mathbf{r}(\widehat{\boldsymbol{\tau}}_h))\cdot\tilde{\mathbf{v}} \qquad \forall\,(\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}})\in\tilde{\mathbf{Q}}\,.
\end{aligned}
\tag{5.19}
$$

In this case, the regularity estimate for (5.19) (cf. (4.5)) becomes

$$
\|\tilde{\boldsymbol{\sigma}}\|_{[H^\epsilon(\Omega_s)]^{3\times3}} + \|\tilde{\mathbf{u}}\|_{[H^{1+\epsilon}(\Omega_s)]^3} + \|\tilde{\gamma}\|_{[H^\epsilon(\Omega_s)]^{3\times3}} \,\le\, C\Big\{\,\|\mathbf{div}\,\boldsymbol{\tau}_h\|_{[L^2(\Omega_s)]^3} \,+\, \|q_h\|_{H^1(\Omega_f)}\,\Big\}\,. \tag{5.20}
$$

Hence, we now consider the following Galerkin approximation of (5.19): Find $(\tilde{\boldsymbol{\sigma}}_h, (\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\gamma}}_h)) \in \mathbf{H}_h \times \tilde{\mathbf{Q}}_h$ such that $\tilde{\boldsymbol{\sigma}}_h \boldsymbol{\nu} = -q_h \boldsymbol{\nu}$ on Σ and

$$
\begin{aligned}
\mathbf{a}(\tilde{\boldsymbol{\sigma}}_h, \tilde{\boldsymbol{\tau}}_h) + \mathbf{b}(\tilde{\boldsymbol{\tau}}_h, (\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\gamma}}_h)) &= \mathbf{0} \qquad \forall\, \tilde{\boldsymbol{\tau}}_h \in \tilde{\mathbf{H}}_h\,, \\
\mathbf{b}(\tilde{\boldsymbol{\sigma}}_h, (\tilde{\mathbf{v}}_h, \tilde{\boldsymbol{\eta}}_h)) &= \int_{\Omega_s} (\mathbf{div}\, \boldsymbol{\tau}_h + \mathbf{r}(\widehat{\boldsymbol{\tau}}_h)) \cdot \tilde{\mathbf{v}}_h \qquad \forall\, (\tilde{\mathbf{v}}_h, \tilde{\boldsymbol{\eta}}_h) \in \tilde{\mathbf{Q}}_h\,,
\end{aligned}
\tag{5.21}
$$

where $\tilde{\mathbf{Q}}_h := (\mathbf{I} - \mathbf{M})(\mathbf{U}_h) \times \mathbb{Y}_h$ is a finite element subspace of $\tilde{\mathbf{Q}}$.

Note that the solution of (5.19) can be defined, equivalently, as $(\tilde{\boldsymbol{\sigma}}, (\tilde{\mathbf{u}}, \tilde{\boldsymbol{\gamma}})) := (\tilde{\boldsymbol{\sigma}}_0 + \boldsymbol{\tau}_h, (\tilde{\mathbf{u}}, \tilde{\boldsymbol{\gamma}}))$, where $(\tilde{\boldsymbol{\sigma}}_0, (\tilde{\mathbf{u}}, \tilde{\boldsymbol{\gamma}}))$ is the unique element in $\tilde{\mathbf{H}} \times \tilde{\mathbf{Q}}$ such that

$$
\begin{aligned}
\mathbf{a}(\tilde{\boldsymbol{\sigma}}_0, \tilde{\boldsymbol{\tau}}) + \mathbf{b}(\tilde{\boldsymbol{\tau}}, (\tilde{\mathbf{u}}, \tilde{\boldsymbol{\gamma}})) &= -\,\mathbf{a}(\boldsymbol{\tau}_h, \tilde{\boldsymbol{\tau}}) \qquad \forall\, \tilde{\boldsymbol{\tau}} \in \tilde{\mathbf{H}}\,, \\
\mathbf{b}(\tilde{\boldsymbol{\sigma}}_0, (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}})) &= \int_{\Omega_s} (\mathbf{div}\, \boldsymbol{\tau}_h + \mathbf{r}(\widehat{\boldsymbol{\tau}}_h)) \cdot \tilde{\mathbf{v}} - \mathbf{b}(\boldsymbol{\tau}_h, (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}})) \qquad \forall\, (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}}) \in \tilde{\mathbf{Q}}\,.
\end{aligned}
\tag{5.22}
$$

Similarly, we may look for the solution of (5.21) in the form $(\tilde{\boldsymbol{\sigma}}_h, (\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\gamma}}_h)) := (\tilde{\boldsymbol{\sigma}}_{0,h} + \boldsymbol{\tau}_h, (\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\gamma}}_h))$, where $(\tilde{\boldsymbol{\sigma}}_{0,h}, (\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\gamma}}_h)) \in \tilde{\mathbf{H}}_h \times \tilde{\mathbf{Q}}_h$ is such that

$$
\begin{aligned}
\mathbf{a}(\tilde{\boldsymbol{\sigma}}_{0,h}, \tilde{\boldsymbol{\tau}}_h) + \mathbf{b}(\tilde{\boldsymbol{\tau}}_h, (\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\gamma}}_h)) &= -\,\mathbf{a}(\boldsymbol{\tau}_h, \tilde{\boldsymbol{\tau}}_h) \quad \forall\, \tilde{\boldsymbol{\tau}}_h \in \tilde{\mathbf{H}}_h, \\
\mathbf{b}(\tilde{\boldsymbol{\sigma}}_{0,h}, (\tilde{\mathbf{v}}_h, \tilde{\boldsymbol{\eta}}_h)) &= \int_{\Omega_s} (\mathbf{div}\, \boldsymbol{\tau}_h + \mathbf{r}(\widehat{\boldsymbol{\tau}}_h)) \cdot \tilde{\mathbf{v}}_h - \mathbf{b}(\boldsymbol{\tau}_h, (\tilde{\mathbf{v}}_h, \tilde{\boldsymbol{\eta}}_h)) \quad \forall\, (\tilde{\mathbf{v}}_h, \tilde{\boldsymbol{\eta}}_h) \in \tilde{\mathbf{Q}}_h.
\end{aligned}
\tag{5.23}
$$

It is clear that (5.23) constitutes a conforming Galerkin approximation of (5.22).

In what follows we apply the discrete Babuška-Brezzi theory to show the unique solvability, stability, and convergence of (5.21) (equivalently (5.23)). We provide first the discrete inf-sup condition for $\mathbf{b}$ on $\tilde{\mathbf{H}}_h \times \tilde{\mathbf{Q}}_h$, which has already been established in [5].

Lemma 5.2 *There exist $\tilde{\beta} > 0$, independent of h, such that for each $(\tilde{\mathbf{v}}_h, \tilde{\boldsymbol{\eta}}_h) \in \tilde{\mathbf{Q}}_h$, there hold*

$$
\sup_{\substack{\tilde{\boldsymbol{\tau}}_h \in \tilde{\mathbf{H}}_h \\ \tilde{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\, \mathbf{b}(\tilde{\boldsymbol{\tau}}_h, (\tilde{\mathbf{v}}_h, \tilde{\boldsymbol{\eta}}_h))\,|}{\|\tilde{\boldsymbol{\tau}}_h\|_{H(\mathbf{div};\,\Omega_s)}} \;\geq\; \tilde{\beta}_1\, \|(\tilde{\mathbf{v}}_h, \tilde{\boldsymbol{\eta}}_h)\|_{[L^2(\Omega_s)]^3 \times [L^2(\Omega_s)]^{3\times 3}}\,. \tag{5.24}
$$

Proof. See [5, Section 11.7, Theorem 11.9].

□

Next, we prove that $\mathbf{a}$ is strongly coercive on the discrete kernel of $\mathbf{b}$, which is given by

$$
\tilde{S}_h := \left\{ \tilde{\boldsymbol{\tau}}_h \in \tilde{\mathbf{H}}_h : \quad \int_{\Omega_s} \tilde{\mathbf{v}}_h \cdot \mathbf{div}\, \tilde{\boldsymbol{\tau}}_h + \int_{\Omega_s} \tilde{\boldsymbol{\tau}}_h : \tilde{\boldsymbol{\eta}}_h = 0 \quad \forall\, (\tilde{\mathbf{v}}_h, \tilde{\boldsymbol{\eta}}_h) \in \tilde{\mathbf{Q}}_h \right\}.
$$

However, given $\tilde{\boldsymbol{\tau}}_h \in \tilde{S}_h \subseteq \tilde{\mathbf{H}}_h$ and $\mathbf{v}_h \in \mathbf{U}_h$, we find, defining $\tilde{\mathbf{v}}_h := (\mathbf{I} - \mathbf{M})(\mathbf{v}_h) \in (\mathbf{I} - \mathbf{M})(\mathbf{U}_h)$, integrating by parts, and noting that $\nabla \mathbf{M}(\mathbf{v}_h) \in \mathbb{Y}_h$, that

$$
\begin{aligned}
&\int_{\Omega_s} \mathbf{v}_h \cdot \mathbf{div}\, \tilde{\boldsymbol{\tau}}_h = \int_{\Omega_s} (\tilde{\mathbf{v}}_h + \mathbf{M}(\mathbf{v}_h)) \cdot \mathbf{div}\, \tilde{\boldsymbol{\tau}}_h \\
&= \int_{\Omega_s} \tilde{\mathbf{v}}_h \cdot \mathbf{div}\, \tilde{\boldsymbol{\tau}}_h - \int_{\Omega_s} \nabla \mathbf{M}(\mathbf{v}_h) : \tilde{\boldsymbol{\tau}}_h + \langle \tilde{\boldsymbol{\tau}}_h \boldsymbol{\nu}, \mathbf{M}(\mathbf{v}_h) \rangle_\Sigma = 0\,,
\end{aligned}
$$

whence $\tilde{S}_h$ actually becomes

$$\tilde{S}_h := \left\{ \tilde{\boldsymbol{\tau}}_h \in \tilde{\mathbf{H}}_h : \quad \int_{\Omega_s} \mathbf{v}_h \cdot \mathbf{div}\, \tilde{\boldsymbol{\tau}}_h \,+\, \int_{\Omega_s} \tilde{\boldsymbol{\tau}}_h : \tilde{\boldsymbol{\eta}}_h = 0 \quad \forall\, (\mathbf{v}_h, \tilde{\boldsymbol{\eta}}_h) \in \mathbf{U}_h \times \mathbb{Y}_h \right\}.$$

In this way, since $\mathbf{div}\, \tilde{\boldsymbol{\tau}}_h \in \mathbf{U}_h$ for all $\tilde{\boldsymbol{\tau}}_h \in \tilde{\mathbf{H}}_h$, we deduce that

$$\tilde{S}_h := \left\{ \tilde{\boldsymbol{\tau}}_h \in \tilde{\mathbf{H}}_h : \quad \mathbf{div}\, \tilde{\boldsymbol{\tau}}_h = \mathbf{0} \quad \text{in} \quad \Omega_s \quad \text{and} \quad \int_{\Omega_s} \tilde{\boldsymbol{\tau}}_h : \tilde{\boldsymbol{\eta}}_h = 0 \qquad \forall\, \tilde{\boldsymbol{\eta}}_h \in \mathbb{Y}_h \right\}.$$

The strong coerciveness of $\mathbf{a}$ on a space containing $\tilde{S}_h$ is established next.

Lemma 5.3 *There exists $\alpha > 0$, independent of h, such that*

$$\mathbf{a}(\tilde{\boldsymbol{\tau}}_h, \overline{\tilde{\boldsymbol{\tau}}}_h) \,\geq\, \alpha\, \|\tilde{\boldsymbol{\tau}}_h\|^2_{H(\mathbf{div};\,\Omega_s)} \quad \forall\, \tilde{\boldsymbol{\tau}}_h \in \tilde{\mathbf{H}}_h \quad \textit{such that} \quad \mathbf{div}\, \tilde{\boldsymbol{\tau}}_h = \mathbf{0} \quad \textit{in} \quad \Omega_s\,.$$

Proof. Let $\tilde{\boldsymbol{\tau}}_h = \tilde{\boldsymbol{\tau}}_{h,0} + d_h\, \mathbf{I}$ as indicated, with $\tilde{\boldsymbol{\tau}}_{h,0} \in H_0(\mathbf{div};\, \Omega_s)$ and $d_h \in \mathbb{C}$. Since $\mathbf{0} = \mathbf{div}\, \tilde{\boldsymbol{\tau}}_h = \mathbf{div}\, \tilde{\boldsymbol{\tau}}_{h,0}$ in Ω_s and $\tilde{\boldsymbol{\tau}}_h\, \boldsymbol{\nu} = \mathbf{0}$ on Σ, it follows from (3.8), (4.3), Lemma 4.2, and [15, Lemma 4.5] (see also Lemma 4.3 or [14, Lemma 2.2]), that

$$\mathbf{a}(\tilde{\boldsymbol{\tau}}_h, \overline{\tilde{\boldsymbol{\tau}}}_h) \,\geq\, \frac{1}{2\,\mu}\, \|\tilde{\boldsymbol{\tau}}^{\mathtt{d}}_h\|_{[L^2(\Omega_s)]^{3\times 3}} \,\geq\, \frac{c_1}{2\,\mu}\, \|\tilde{\boldsymbol{\tau}}_{h,0}\|^2_{[L^2(\Omega_s)]^{3\times 3}} \,=\, \frac{c_1}{2\,\mu}\, \|\tilde{\boldsymbol{\tau}}_{h,0}\|^2_{H(\mathbf{div};\,\Omega_s)} \,\geq\, \alpha\, \|\tilde{\boldsymbol{\tau}}_h\|^2_{H(\mathbf{div};\,\Omega_s)}\,,$$

which finishes the proof.

□

We are now in a position to state the unique solvability, stability, and convergence of (5.21).

Theorem 5.1 *Given $\widehat{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_h, q_h) \in \mathbb{X}_h$, there exists a unique $(\tilde{\boldsymbol{\sigma}}_h, (\tilde{\mathbf{u}}_h, \tilde{\gamma}_h)) \in \mathbf{H}_h \times \tilde{\mathbf{Q}}_h$ solution of* (5.21)*. Moreover, there exist C, $\tilde{C} > 0$, independent of h, such that*

$$\|\tilde{\boldsymbol{\sigma}}_h\|_{H(\mathbf{div};\,\Omega_s)} \,+\, \|\tilde{\mathbf{u}}_h\|_{[L^2(\Omega_s)]^3} \,+\, \|\tilde{\gamma}_h\|_{[L^2(\Omega_s)]^{3\times 3}} \,\leq\, C \left\{ \|\boldsymbol{\tau}_h\|_{[H(\mathbf{div};\,\Omega_s)} \,+\, \|q_h\|_{H^1(\Omega_f)} \right\} \tag{5.25}$$

and

$$\begin{aligned} &\|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{H(\mathbf{div};\,\Omega_s)} \,+\, \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{[L^2(\Omega_s)]^3} \,+\, \|\tilde{\gamma} - \tilde{\gamma}_h\|_{[L^2(\Omega_s)]^{3\times 3}} \\ &\leq\, \tilde{C} \left\{ \|(\mathbf{I} - \mathcal{E}_h)(\tilde{\boldsymbol{\sigma}})\|_{H(\mathbf{div};\,\Omega_s)} \,+\, \|(\mathbf{I} - \mathcal{P}_h)(\tilde{\mathbf{u}})\|_{[L^2(\Omega_s)]^3} \,+\, \|(\mathbf{I} - \mathcal{R}_h)(\tilde{\gamma})\|_{[L^2(\Omega_s)]^{3\times 3}} \right\}, \end{aligned} \tag{5.26}$$

where $(\tilde{\boldsymbol{\sigma}}, (\tilde{\mathbf{u}}, \tilde{\gamma})) \in H(\mathbf{div};\, \Omega_s) \times \tilde{\mathbf{Q}}$ is the unique solution of (5.19).

Proof. It follows straightforwardly from Lemma 5.2, Lemma 5.3, and the Babuška-Brezzi theory (see [11], [29]) applied to the continuous and discrete formulations (5.19) and (5.21) (equivalently (5.22) and (5.23)).

□

5.3 A mixed finite element approximation of $\mathbf{P}|_{\mathbb{X}_h}$

As suggested by the previous analysis, we now introduce the linear operators $P_h : \mathbb{X}_h \to \mathbf{H}_h$ and $\mathbf{P}_h : \mathbb{X}_h \to \mathbb{X}_h$ defined by

$$P_h(\widehat{\boldsymbol{\tau}}_h) \,=\, \tilde{\boldsymbol{\sigma}}_h \quad \text{and} \quad \mathbf{P}_h(\widehat{\boldsymbol{\tau}}_h) \,:=\, (P_h(\widehat{\boldsymbol{\tau}}_h), q_h) \qquad \forall\, \widehat{\boldsymbol{\tau}}_h \,=\, (\boldsymbol{\tau}_h, q_h) \,\in\, \mathbb{X}_h\,,$$

where $(\tilde{\boldsymbol{\sigma}}_h, (\tilde{\mathbf{u}}_h, \tilde{\boldsymbol{\gamma}}_h)) \in \mathbf{H}_h \times \tilde{\mathbf{Q}}_h$ is the unique solution of (5.21). The operator $\mathbf{P}_h$ constitutes what we call the mixed finite element approximation of $\mathbf{P}|_{\mathbb{X}_h}$. It follows from Theorem 5.1 (cf. (5.25)) that $\mathbf{P}_h$ is a linear and bounded operator. In addition, it is clear from (5.21) that

$$P_h(\widehat{\boldsymbol{\tau}}_h)\,\boldsymbol{\nu} \,=\, -\,q_h\,\boldsymbol{\nu} \quad \text{on} \quad \Sigma \qquad \text{and} \qquad \int_{\Omega_s} P_h(\widehat{\boldsymbol{\tau}}_h) : \tilde{\boldsymbol{\eta}}_h \,=\, 0 \quad \forall\, \tilde{\boldsymbol{\eta}}_h \,\in\, \mathbb{Y}_h\,. \tag{5.27}$$

Our next goal is to estimate $\|P(\widehat{\boldsymbol{\tau}}_h) - P_h(\widehat{\boldsymbol{\tau}}_h)\|_{H(\mathbf{div};\Omega_s)} \,=\, \|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{H(\mathbf{div};\Omega_s)}$ for each $\widehat{\boldsymbol{\tau}}_h \in \mathbb{X}_h$. More precisely, we have the following result.

Lemma 5.4 *Let $\epsilon > 0$ be the parameter defining the regularity of the solution of* (5.19). *Then, there exists $C > 0$, independent of h, such that*

$$\|P(\widehat{\boldsymbol{\tau}}_h) \,-\, P_h(\widehat{\boldsymbol{\tau}}_h)\|_{H(\mathbf{div};\Omega_s)} \,\leq\, C\,h^{\epsilon}\left\{\|\mathbf{div}\,\boldsymbol{\tau}_h\|_{[L^2(\Omega_s)]^3} \,+\, \|q_h\|_{H^1(\Omega_f)}\right\} \qquad \forall\,\widehat{\boldsymbol{\tau}}_h \,\in\, \mathbb{X}_h\,. \tag{5.28}$$

Proof. It suffices to show that the right hand side of (5.26) is bounded by the right hand side of (5.28). Indeed, using ($\mathrm{AP}_h^{\mathbf{u}}$), ($\mathrm{AP}_h^{\boldsymbol{\gamma}}$), and the regularity estimate (5.20), we easily find that

$$\begin{aligned} &\|(\mathbf{I} - \mathcal{P}_h)(\tilde{\mathbf{u}})\|_{[L^2(\Omega_s)]^2} \,+\, \|(\mathbf{I} - \mathcal{R}_h)(\tilde{\boldsymbol{\gamma}})\|_{[L^2(\Omega_s)]^{2\times 2}} \\ &\leq C\,h^{\epsilon}\left\{\|\tilde{\mathbf{u}}\|_{[H^{1+\epsilon}(\Omega_s)]^2} \,+\, \|\tilde{\boldsymbol{\gamma}}\|_{[H^{\epsilon}(\Omega_s)]^{2\times 2}}\right\} \,\leq\, C\,h^{\epsilon}\left\{\|\mathbf{div}\,\boldsymbol{\tau}_h\|_{[L^2(\Omega_s)]^3} \,+\, \|q_h\|_{H^1(\Omega_f)}\right\}. \end{aligned} \tag{5.29}$$

Now, in order to bound $\|(\mathbf{I} - \mathcal{E}_h)(\tilde{\boldsymbol{\sigma}})\|_{H(\mathbf{div};\Omega_s)}$ we proceed exactly as in [15, Lemma 5.4]. In fact, using that

$$\mathbf{div}\,\tilde{\boldsymbol{\sigma}} \,=\, \mathbf{div}\,\boldsymbol{\tau}_h \,+\, \mathbf{r}(\widehat{\boldsymbol{\tau}}_h) \quad \text{in} \quad \Omega_s\,, \tag{5.30}$$

and then applying the approximation property (5.5), the regularity estimate (5.20), and the boundedness of $\mathbf{r}$, we deduce that

$$\begin{aligned} \|(\mathbf{I} - \mathcal{E}_h)(\tilde{\boldsymbol{\sigma}})\|_{[L^2(\Omega_s)]^{3\times 3}} \,&\leq\, C\,h^{\epsilon}\left\{|\tilde{\boldsymbol{\sigma}}|_{[H^{\epsilon}(\Omega_s)]^{3\times 3}} + \|\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\|_{[L^2(\Omega_s)]^3}\right\} \\ &\leq \quad C\,h^{\epsilon}\left\{\|\mathbf{div}\,\boldsymbol{\tau}_h\|_{[L^2(\Omega_s)]^3} \,+\, \|q_h\|_{H^1(\Omega_f)}\right\}. \end{aligned} \tag{5.31}$$

Furthermore, it follows from (5.4) and (5.30) that

$$\|\mathbf{div}\,\tilde{\boldsymbol{\sigma}} \,-\, \mathbf{div}(\mathcal{E}_h(\tilde{\boldsymbol{\sigma}}))\|_{[L^2(\Omega_s)]^3} \,=\, \|(\mathbf{I} - \mathcal{P}_h)(\mathbf{div}(\tilde{\boldsymbol{\sigma}}))\|_{[L^2(\Omega_s)]^3} \,=\, \|(\mathbf{I} - \mathcal{P}_h)(\mathbf{r}(\widehat{\boldsymbol{\tau}}_h))\|_{[L^2(\Omega_s)]^3}\,,$$

whence, ($\mathrm{AP}_h^{\mathbf{u}}$), the fact that all the norms in $\mathbb{RM}(\Omega_s)$ are equivalent (with constants certainly independent of h), and the boundedness of $\mathbf{r}$, imply that

$$\begin{aligned} \|\mathbf{div}\,\tilde{\boldsymbol{\sigma}} \,-\, \mathbf{div}(\mathcal{E}_h(\tilde{\boldsymbol{\sigma}}))\|_{[L^2(\Omega_s)]^3} \,&\leq\, C\,h\,\|\mathbf{r}(\widehat{\boldsymbol{\tau}}_h)\|_{[H^1(\Omega_s)]^3} \,\leq\, C\,h\,\|\mathbf{r}(\widehat{\boldsymbol{\tau}}_h)\|_{[L^2(\Omega_s)]^3} \\ &\leq \quad C\,h\left\{\|\mathbf{div}\,\boldsymbol{\tau}_h\|_{[L^2(\Omega_s)]^3} \,+\, \|q_h\|_{H^1(\Omega_f)}\right\}. \end{aligned} \tag{5.32}$$

In this way, (5.31) and (5.32) give the required estimate for $\|(\mathbf{I} - \mathcal{E}_h)(\tilde{\boldsymbol{\sigma}})\|_{H(\mathbf{div};\Omega_s)}$, which, together with (5.29) and (5.26), yields (5.28) and finishes the proof.

□

5.4 Well-posedness of the discrete formulation

In this section we prove the well-posedness of our mixed finite element scheme (5.2). To this end, as established by a classical result on projection methods for Fredholm operators of index zero (see, e.g. Theorem 13.7 in [25]), it suffices to show that the Galerkin scheme associated to the isomorphism $\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ is well-posed. Therefore, in what follows we prove that A_0 and B (cf. (4.13), (3.7)) satisfy the corresponding inf-sup conditions on the finite element subspace $\mathbb{X}_h \times \mathbb{Y}_h$, thus providing the discrete analogues of Lemmata 4.1 and 4.5.

We begin with the discrete inf-sup condition for B.

Lemma 5.5 *There exists $\beta > 0$, independent of h, such that for each $\boldsymbol{\eta}_h \in \mathbb{Y}_h$, there holds*

$$\sup_{\substack{\widehat{\boldsymbol{\tau}}_h \in \mathbb{X}_h \\ \widehat{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\, B(\widehat{\boldsymbol{\tau}}_h, \boldsymbol{\eta}_h)\,|}{\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}} \;\geq\; \beta\, \|\boldsymbol{\eta}_h\|_{[L^2(\Omega_s)]^{3\times 3}}\,.$$

Proof. It suffices to notice that

$$\sup_{\substack{\widehat{\boldsymbol{\tau}}_h \in \mathbb{X}_h \\ \widehat{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\, B(\widehat{\boldsymbol{\tau}}_h, \boldsymbol{\eta}_h)\,|}{\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}}} \;\geq\; \sup_{\substack{(\boldsymbol{\tau}_h, 0) \in \mathbb{X}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{|\, B((\boldsymbol{\tau}_h, 0), \boldsymbol{\eta}_h)\,|}{\|(\boldsymbol{\tau}_h, 0)\|_{\mathbb{X}}} \;=\; \sup_{\substack{\boldsymbol{\tau}_h \in \tilde{\mathbf{H}}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\left|\displaystyle\int_{\Omega_s} \boldsymbol{\tau}_h : \boldsymbol{\eta}_h\right|}{\|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\,\Omega_s)}}$$

and then to take $\tilde{\mathbf{v}}_h \,=\, \mathbf{0}$ in (5.24). □

We now let $\mathbb{V}_h$ be the discrete kernel of $\mathbf{B}$, that is

$$\begin{aligned} \mathbb{V}_h \,&:=\, \{\, \widehat{\boldsymbol{\tau}} = (\boldsymbol{\tau}_h, q_h) \,\in\, \mathbb{X}_h : \quad B(\widehat{\boldsymbol{\tau}}_h, \boldsymbol{\eta}_h) \,=\, 0 \quad \forall\, \boldsymbol{\eta}_h \,\in\, \mathbb{Y}_h \,\} \\ &=\, \left\{ \widehat{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_h, q_h) \,\in\, \mathbb{X}_h : \quad \int_{\Omega_s} \boldsymbol{\tau}_h : \boldsymbol{\eta}_h \,=\, 0 \quad \forall\, \boldsymbol{\eta}_h \in \mathbb{Y}_h \right\}. \end{aligned}$$

Then, the discrete weak coercivity of A_0 is established as follows.

Lemma 5.6 *There exist C, $h_1 \,>\, 0$, independent of h such that for each $h \,\leq\, h_1$ there holds*

$$\sup_{\substack{\widehat{\boldsymbol{\zeta}}_h \in \mathbb{V}_h \\ \widehat{\boldsymbol{\zeta}}_h \neq \mathbf{0}}} \frac{|\, A_0(\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\zeta}}_h)\,|}{\|\widehat{\boldsymbol{\zeta}}_h\|_{\mathbb{X}}} \;\geq\; C\, \|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}} \qquad \forall\, \widehat{\boldsymbol{\tau}}_h \,\in\, \mathbb{V}_h\,. \tag{5.33}$$

In addition, for each $h \,\leq\, h_1$ there holds

$$\sup_{\widehat{\boldsymbol{\zeta}}_h \in \mathbb{V}_h} |\, A_0(\widehat{\boldsymbol{\zeta}}_h, \widehat{\boldsymbol{\tau}}_h)\,| \;>\; 0 \qquad \forall\, \widehat{\boldsymbol{\tau}}_h \,\in\, \mathbb{V}_h\,,\, \widehat{\boldsymbol{\tau}}_h \,\neq\, \mathbf{0}\,. \tag{5.34}$$

Proof. Let us introduce the linear and bounded operator $\Xi_h \,:=\, (2\,\mathbf{P}_h \,-\, \mathbf{I}) \,:\, \mathbb{X}_h \,\rightarrow\, \mathbb{X}_h$, which constitutes a discrete approximation of the operator $\Xi \,:=\, (2\,\mathbf{P} \,-\, \mathbf{I}) \,:\, \mathbb{X} \rightarrow \mathbb{X}$ (cf. Section 4.3). It follows from Lemma 5.4 that

$$\|\Xi(\widehat{\boldsymbol{\tau}}_h) \,-\, \Xi_h(\widehat{\boldsymbol{\tau}}_h)\|_{H(\mathbf{div};\,\Omega_s)} \;\leq\; C\, h^{\epsilon} \left\{ \|\mathbf{div}\,\boldsymbol{\tau}_h\|_{[L^2(\Omega_s)]^3} \,+\, \|q_h\|_{H^1(\Omega_f)} \right\} \;\leq\; \tilde{C}\, h^{\epsilon}\, \|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}} \quad \forall\, \widehat{\boldsymbol{\tau}}_h \,\in\, \mathbb{X}_h\,,$$

and hence, using the boundedness of A_0, the inequality (4.21) (cf. Lemma 4.4), and the stability of the decomposition (4.9), we find that for each $\widehat{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_h, q_h) \in \mathbb{X}_h$ there holds

$$\begin{aligned}\Big|\,\mathrm{Re}\Big\{ A_0(\widehat{\boldsymbol{\tau}}_h, \Xi_h \overline{\widehat{\boldsymbol{\tau}}}_h) \Big\}\,\Big| \;&\geq\; \Big|\,\mathrm{Re}\Big\{ A_0(\widehat{\boldsymbol{\tau}}_h, \Xi(\overline{\widehat{\boldsymbol{\tau}}}_h) \Big\}\,\Big| \;-\; \tilde{C}\, h^{\epsilon}\, \|\widehat{\boldsymbol{\tau}}_h\|^2_{\mathbb{X}} \\ &\geq\; C\Big\{ \|\mathbf{P}(\widehat{\boldsymbol{\tau}}_h)\|^2_{\mathbb{X}} \,+\, \|(\mathbf{I}-\mathbf{P})(\widehat{\boldsymbol{\tau}}_h)\|^2_{\mathbb{X}} \Big\} \;-\; \tilde{C}\, h^{\epsilon}\, \|\widehat{\boldsymbol{\tau}}_h\|^2_{\mathbb{X}} \\ &\geq\; \Big\{ C \,-\, \tilde{C}\, h^{\epsilon} \Big\}\, \|\widehat{\boldsymbol{\tau}}_h\|^2_{\mathbb{X}}\,.\end{aligned}$$

Thus, from this estimate we deduce the existence of $h_1 > 0$ such that for each $h \leq h_1$ there holds

$$\Big|\,\mathrm{Re}\Big\{ A_0(\widehat{\boldsymbol{\tau}}_h, \Xi_h(\overline{\widehat{\boldsymbol{\tau}}}_h)) \Big\}\,\Big| \;\geq\; C\, \|\widehat{\boldsymbol{\tau}}_h\|^2_{\mathbb{X}} \;\geq\; C\, \|(\Xi_h(\overline{\widehat{\boldsymbol{\tau}}}_h))\|_{\mathbb{X}}\, \|\widehat{\boldsymbol{\tau}}_h\|_{\mathbb{X}} \qquad \forall\, \widehat{\boldsymbol{\tau}} \in \mathbb{X}_h\,, \tag{5.35}$$

where the boundedness of Ξ_h has also been used in the last inequality.

In this way, since (5.27) implies that $\Xi_h(\widehat{\boldsymbol{\tau}}_h) \in \mathbb{V}_h$ for each $\widehat{\boldsymbol{\tau}}_h \in \mathbb{V}_h$, we realize that the discrete inf-sup condition (5.33) follows straightforwardly from (5.35), noting also from this inequality that $\Xi_h(\widehat{\boldsymbol{\tau}}_h) \neq \mathbf{0}$ for each $\widehat{\boldsymbol{\tau}}_h \neq \mathbf{0}$. Finally, similarly as in the continuous case (cf. Lemma 4.5), the symmetry of A_0 and the estimate (5.35) yield the discrete inf-sup condition (5.34).

□

The well-posedness and convergence of the discrete scheme (5.2) can now be established.

Theorem 5.2 *Assume that the homogeneous problem associated to* (3.5) *has only the trivial solution. Let $h_1 > 0$ be the constant provided by Lemma 5.6. Then, for each $h \leq h_1$, the mixed finite element scheme* (5.2) *has a unique solution $(\widehat{\boldsymbol{\sigma}}_h = (\boldsymbol{\sigma}_h, p_h), \boldsymbol{\gamma}_h) \in \mathbb{X}_h \times \mathbb{Y}_h$. In addition, there exist C_1, $C_2 > 0$, independent of h, such that*

$$\|((\boldsymbol{\sigma}_h, p_h), \boldsymbol{\gamma}_h)\|_{\mathbb{X}\times\mathbb{Y}} \;\leq\; C_1 \sup_{\substack{(\boldsymbol{\tau}_h, q_h)\in\mathbb{X}_h \\ (\boldsymbol{\tau}_h, q_h)\neq\mathbf{0}}} \frac{|F(\boldsymbol{\tau}_h, q_h)|}{\|(\boldsymbol{\tau}_h, q_h)\|_{\mathbb{X}}} \;\leq\; C_1 \Big\{ \|\mathbf{f}\|_{[L^2(\Omega_s)]^3} \,+\, \|g\|_{H^{-1/2}(\Gamma)} \Big\}$$

and

$$\begin{aligned}&\|((\boldsymbol{\sigma}, p), \boldsymbol{\gamma}) \,-\, ((\boldsymbol{\sigma}_h, p_h), \boldsymbol{\gamma}_h)\|_{\mathbb{X}\times\mathbb{Y}} \\ &\quad\leq\; C_2 \inf_{((\boldsymbol{\tau}_h, q_h), \boldsymbol{\eta}_h)\in\mathbb{X}_h\times\mathbb{Y}_h} \|((\boldsymbol{\sigma}, p), \boldsymbol{\gamma}) \,-\, ((\boldsymbol{\tau}_h, q_h), \boldsymbol{\eta}_h)\|_{\mathbb{X}\times\mathbb{Y}}\,.\end{aligned} \tag{5.36}$$

Furthermore, if there exists $\delta \in (0,1]$ such that $\boldsymbol{\sigma} \in [H^{\delta}(\Omega_s)]^{3\times 3}$, $\mathbf{div}\,\boldsymbol{\sigma} \in [H^{\delta}(\Omega_s)]^3$, $p \in H^{1+\delta}(\Omega_f)$, and $\boldsymbol{\gamma} \in [H^{\delta}(\Omega_s)]^{3\times 3}$, then there holds

$$\begin{aligned}&\|((\boldsymbol{\sigma}, p), \boldsymbol{\gamma}) \,-\, ((\boldsymbol{\sigma}_h, p_h), \boldsymbol{\gamma}_h)\|_{\mathbb{X}\times\mathbb{Y}} \\ &\leq\; C_3\, h^{\delta}\, \Big\{ \|\boldsymbol{\sigma}\|_{[H^{\delta}(\Omega_s)]^{3\times 3}} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{[H^{\delta}(\Omega_s)]^3} \,+\, \|p\|_{H^{1+\delta}(\Omega_f)} \,+\, \|\boldsymbol{\gamma}\|_{[H^{\delta}(\Omega_s)]^{3\times 3}} \Big\}\,,\end{aligned} \tag{5.37}$$

with a constant $C_3 > 0$, independent of h.

Proof. Thanks to Lemmata 5.5 and 5.6, the proof of the first part is a direct application of Theorem 13.7 in [25], whereas the rate of convergence (5.37) follows directly from the Cea estimate (5.36), and the approximation properties ($\mathrm{AP}_h^{\boldsymbol{\sigma}}$), ($\mathrm{AP}_h^{p}$), ($\mathrm{AP}_h^{\boldsymbol{\gamma}}$), and the special one for $\mathbb{X}_h$ given by Lemma 5.1 (cf. (5.8)).

□

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