A proof of the Reynolds transport theorem in the space $W^{1,1}(Q)$. Extension to cylindrical coordinates

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Abstract

In this Note we give a proof of the Reynolds transport theorem for the case where the field belongs to the space $W^{1,1}(Q)$, Q being the trajectory under a motion of a reference configuration which is a bounded open subset of $\mathbb{R}^n$. We first consider the standard formula in Cartesian coordinates. Next, a particular version in three-dimensional cylindrical coordinates for an axisymmetric field and an axisymmetric motion keeping invariant the azimuthal coordinate is proved.

Résumé

Une démonstration du theorème du transport de Reynolds pour des champs dans $W^{1,1}(Q)$. Extension aux coordonnées cylindriques. Dans cette Note nous donnons une démonstration du théorème du transport de Reynolds pour le cas où le champ appartient à l'espace $W^{1,1}(Q)$, Q étant la "trajectoire" determinée par un mouvement d'une configuration de référence qui est un ouvert borné quelconque dans $\mathbb{R}^n$. On considére d'abord la version standard en coordonnées cartesiennes et ensuite une version particulière en coordonnées cylindriques en dimension trois pour un champ et un mouvement axisymétriques, celui-ci laissant invariant l'azimut.

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Version française abrégée

Les résultats principaux contenus dans cette Note sont les deux théorèmes suivants :

Theorem 0.1 *Soit $\widehat{\Omega}$ un ouvert borné dans $\mathbb{R}^n$ et $T > 0$. Soit $\boldsymbol{X} \in \mathcal{C}([0,T] \times \overline{\widehat{\Omega}}; \mathbb{R}^n)$ une application satisfaisant les hypothèses* **H1**–**H6** *ci-après, où $\boldsymbol{v}$ est défini par l'équation (1). On pose $\boldsymbol{\Psi}(t, \widehat{\boldsymbol{x}}) := (t, \boldsymbol{X}(t, \widehat{\boldsymbol{x}}))$ et $Q := \boldsymbol{\Psi}((0,T) \times \widehat{\Omega})$. Soit $\widehat{\sigma} \in \mathrm{L}^\infty(\widehat{\Omega})$ une fonction donnée et σ définie dans $\boldsymbol{\Psi}([0,T] \times \widehat{\Omega})$ par $\sigma(t, \boldsymbol{x}) := \widehat{\sigma}(\widehat{\boldsymbol{x}})$, où $\boldsymbol{x} := \boldsymbol{X}(t, \widehat{\boldsymbol{x}})$. Si $\phi \in \mathrm{W}^{1,1}(Q)$, $A \subset \widehat{\Omega}$ est un ensemble mesurable au sens de Lebesgue et $A_t := \boldsymbol{X}(t, A)$, alors on a l'égalité*

$$\frac{d}{dt} \int_{A_t} \sigma(t, \boldsymbol{x}) \phi(t, \boldsymbol{x})\, d\boldsymbol{x} = \int_{A_t} \sigma(t, \boldsymbol{x}) \left(\frac{\partial \phi}{\partial t}(t, \boldsymbol{x}) + (\mathbf{grad}_{\boldsymbol{x}}\, \phi)(t, \boldsymbol{x}) \cdot \boldsymbol{v}(t, \boldsymbol{x}) + \phi(t, \boldsymbol{x})(\mathrm{div}_{\boldsymbol{x}}\, \boldsymbol{v})(t, \boldsymbol{x}) \right) d\boldsymbol{x}$$

dans $\mathcal{D}'(0,T)$ et aussi p.p. sur $(0,T)$.

La preuve se fait par régularisation et passage à la limite, à partir d'une version du même résultat pour des fonctions regulières.

Le second théorème concerne le cas axisymmétrique. Por sa démonstration on utilise le theorème précédent. La seule difficulté vient du fait que la fonction $1/r$ est singulière sur l'axe $r = 0$.

Theorem 0.2 *Soit $\widehat{\Omega} \subset (0, \infty) \times \mathbb{R}$ un ouvert borné et $T > 0$. Soit $\boldsymbol{X} \in \mathcal{C}([0,T] \times \overline{\widehat{\Omega}}; [0, \infty) \times \mathbb{R})$ une application satisfaisant les hypothèses* **H1**–**H5** *et* **H7**–**H8**. *Alors, pour tout $\phi \in \mathrm{W}^{1,1}_r(Q)$ et tout ensemble $A \subset \widehat{\Omega}$ mesurable au sens de Lebesgue, on a*

$$\frac{d}{dt} \int_{A_t} \sigma \phi\, r\, dr\, dz = \int_{A_t} \sigma \frac{\partial \phi}{\partial t}\, r\, dr\, dz + \int_{A_t} \sigma \phi\, \mathrm{div}\, \boldsymbol{v}\, r\, dr\, dz + \int_{A_t} \sigma \boldsymbol{v} \cdot \mathbf{grad}\, \phi\, r\, dr\, dz$$

dans $\mathcal{D}'(0,T)$ et aussi p.p. sur $(0,T)$, où $\mathrm{div}\, \boldsymbol{v} := \frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z}$.

1. Introduction

The Reynolds transport formula is an important analytical tool in continuum mechanics. The proof in the case of smooth motions and fields can be found in many books (see, for instance, [2] or [5]). However, to the best of the authors' knowledge, a rigorous and complete proof for non-smooth fields cannot be found in the literature. In the present Note we give a proof for the case where the field belongs to the Sobolev space $\mathrm{W}^{1,1}(Q)$, where Q is the (in general) non-cylindrical trajectory of an open bounded set under a smooth motion (see [2]). First, we prove the formula in the n-dimensional setting under mild regularity assumptions on the motion. These mild assumptions allow us to apply the resulting formula to prove a particular version of the Reynolds transport theorem for the case where the field and the motion are three-dimensional axisymmetric and the latter keeps invariant the azimuthal coordinate. We remark that Theorem 4.2 from [4, Chap. 8] is close to our n-dimensional result, but, there, $Q = \mathbb{R}^n$ and the field is assumed to be in $C(0,T; \mathrm{W}^{1,1}_{\mathrm{loc}}(R^n)) \cap \mathcal{C}^1(0,T; \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n))$.

2. A family of diffeomorphisms

In what follows we will make use of the following well-known facts. If Ω is a bounded open set in $\mathbb{R}^n$ and $\boldsymbol{f} \in \mathcal{C}(\overline{\Omega}; \mathbb{R}^n)$ is injective, then $\overline{\boldsymbol{f}(\Omega)} = \boldsymbol{f}(\overline{\Omega})$ and $\boldsymbol{f} : \overline{\Omega} \to \boldsymbol{f}(\overline{\Omega})$ is a homeomorphism. If, furthermore, $\boldsymbol{f} \in \mathcal{C}^1(\Omega; \mathbb{R}^n)$ and $D\boldsymbol{f}(\boldsymbol{x})$ is nonsingular at every $\boldsymbol{x} \in \Omega$, then $\boldsymbol{f}|_\Omega$ is an open mapping (hence $\boldsymbol{f}(\Omega)$ is open in $\mathbb{R}^n$) and, by virtue of the inverse mapping theorem, $\boldsymbol{f}^{-1} \in \mathcal{C}^1(\boldsymbol{f}(\Omega))$ and $(D(\boldsymbol{f}^{-1}))(\boldsymbol{f}(\boldsymbol{x})) = (D\boldsymbol{f}(\boldsymbol{x}))^{-1}$ for all $\boldsymbol{x} \in \Omega$.

Let $\widehat{\Omega}$ be a bounded open set in $\mathbb{R}^n$ and $T > 0$. Let $\boldsymbol{X} \in \mathcal{C}([0,T] \times \overline{\widehat{\Omega}}; \mathbb{R}^n)$ be a mapping satisfying the following assumptions:

H1: $\boldsymbol{X}(t,\cdot) : \overline{\widehat{\Omega}} \to \mathbb{R}^n$ is injective $\forall t \in [0,T]$;

H2: the partial derivative $D_{\hat{\boldsymbol{x}}}\boldsymbol{X}$ exists and is continuous and bounded in $[0,T] \times \widehat{\Omega}$;

H3: there exists a constant $\alpha > 0$ such that $\det(D_{\hat{\boldsymbol{x}}}\boldsymbol{X})(t,\widehat{\boldsymbol{x}}) \geq \alpha \quad \forall (t,\widehat{\boldsymbol{x}}) \in [0,T] \times \widehat{\Omega}$;

H4: the partial derivative $\frac{\partial \boldsymbol{X}}{\partial t}$ exists and is continuous in $[0,T] \times \overline{\widehat{\Omega}}$. At $t = 0$ ($t = T$) we consider here and in the sequel the right (resp. left) time derivative.

Let us define the mapping $\boldsymbol{\Psi} : [0,T] \times \overline{\widehat{\Omega}} \to \mathbb{R}^{n+1}$ by $\boldsymbol{\Psi}(t,\widehat{\boldsymbol{x}}) := (t, \boldsymbol{X}(t,\widehat{\boldsymbol{x}}))$ and the set $Q := \boldsymbol{\Psi}((0,T) \times \widehat{\Omega})$. Then, $\boldsymbol{\Psi}$ is injective, $\boldsymbol{\Psi} \in \mathcal{C}([0,T] \times \overline{\widehat{\Omega}}; \mathbb{R}^{n+1}) \cap \mathcal{C}^1((0,T) \times \widehat{\Omega}; \mathbb{R}^{n+1})$ and, from **H3**, the Jacobian matrix $D_{(t,\hat{\boldsymbol{x}})}\boldsymbol{\Psi}(t,\widehat{\boldsymbol{x}})$ is non singular for all $(t,\widehat{\boldsymbol{x}}) \in (0,T) \times \widehat{\Omega}$. Moreover, from the results recalled at the beginning of this section, we have that $\boldsymbol{\Psi} : [0,T] \times \overline{\widehat{\Omega}} \to \overline{Q}$ is a homeomorphism, $\boldsymbol{\Psi}|_{(0,T)\times\widehat{\Omega}}$ is an open mapping, Q is an open set in $\mathbb{R}^{n+1}$, $\boldsymbol{\Psi}^{-1} \in \mathcal{C}^1(Q)$ and $(D_{(t,\boldsymbol{x})}(\boldsymbol{\Psi}^{-1}))(t,\boldsymbol{x}) = (D_{(t,\hat{\boldsymbol{x}})}\boldsymbol{\Psi}(t,\widehat{\boldsymbol{x}}))^{-1}$, where $(t,\widehat{\boldsymbol{x}}) = \boldsymbol{\Psi}^{-1}(t,\boldsymbol{x})$ for all $(t,\boldsymbol{x}) \in Q$. This relation and **H2**–**H4** imply that $D_{(t,\boldsymbol{x})}(\boldsymbol{\Psi}^{-1})$ is bounded in Q.

Now, let $\Omega_t := \boldsymbol{X}(t,\widehat{\Omega})$. Using again the results recalled at the beginning of this section, we have that Ω_t is open in $\mathbb{R}^n$ and $\boldsymbol{X}(t,\cdot) : \overline{\widehat{\Omega}} \to \overline{\Omega}_t$ is a homeomorphism for all $t \in [0,T]$. Recall that $\boldsymbol{\Psi}^{-1} \in \mathcal{C}(\overline{Q})$. Moreover, it is of the form $\boldsymbol{\Psi}^{-1}(t,\boldsymbol{x}) = (t, \boldsymbol{P}(t,\boldsymbol{x})) \quad \forall (t,\boldsymbol{x}) \in \overline{Q}$, where $\boldsymbol{P}(t,\cdot) : \overline{\Omega}_t \to \overline{\widehat{\Omega}}$ is the inverse of $\boldsymbol{X}(t,\cdot)$ for all $t \in [0,T]$. Mapping $\boldsymbol{P}$ is called the *reference map* and $\boldsymbol{P}(t,\cdot) \in \mathcal{C}(\overline{\Omega}_t) \cap \mathcal{C}^1(\Omega_t)$ for all $t \in [0,T]$. For $0 < t < T$ this is clear, because $\boldsymbol{\Psi}^{-1} \in \mathcal{C}^1(Q)$; for $t = 0$ and $t = T$ we apply the inverse mapping theorem to $\boldsymbol{X}(t,\cdot)$. Notice that all the first-order derivatives of $\boldsymbol{P}$ are bounded in Q.

Since $\frac{\partial \boldsymbol{X}}{\partial t} \in \mathcal{C}([0,T] \times \overline{\widehat{\Omega}})$ and $\boldsymbol{\Psi}^{-1}(\overline{Q}) = [0,T] \times \overline{\widehat{\Omega}}$, we can define the velocity field $\boldsymbol{v} \in \mathcal{C}(\overline{Q})$ by

$$\boldsymbol{v}(t,\boldsymbol{x}) := \frac{\partial \boldsymbol{X}}{\partial t}(\boldsymbol{\Psi}^{-1}(t,\boldsymbol{x})) = \frac{\partial \boldsymbol{X}}{\partial t}(t, \boldsymbol{P}(t,\boldsymbol{x}))) \qquad \forall (t,\boldsymbol{x}) \in \overline{Q}. \tag{1}$$

The above equation can be rewritten as follows:

$$\frac{\partial \boldsymbol{X}}{\partial t}(t,\widehat{\boldsymbol{x}}) = \boldsymbol{v}(t, \boldsymbol{X}(t,\widehat{\boldsymbol{x}})) \qquad \forall (t,\widehat{\boldsymbol{x}}) \in [0,T] \times \overline{\widehat{\Omega}}. \tag{2}$$

3. The Liouville's formula

We introduce the following further assumption:

H5: the second order partial derivatives $\frac{\partial}{\partial t}(D_{\hat{\boldsymbol{x}}}\boldsymbol{X})$ and $D_{\hat{\boldsymbol{x}}}(\frac{\partial \boldsymbol{X}}{\partial t})$ exist and are continuous in $[0,T] \times \widehat{\Omega}$.

Notice that, from Schwarz theorem, they coincide in the interior of this set and then in $[0,T] \times \widehat{\Omega}$. From (1) and the chain rule, we have that $D_{\boldsymbol{x}}\boldsymbol{v}$ exists for all $(t,\boldsymbol{x}) \in Q \cup (\{0\} \times \Omega_0) \cup (\{T\} \times \Omega_T)$ and is continuous in Q. Differentiating (2) with respect to $\widehat{\boldsymbol{x}}$ and using the equality of the cross derivatives in $[0,T] \times \widehat{\Omega}$, we obtain

$$\left(\frac{\partial}{\partial t}(D_{\hat{\boldsymbol{x}}}\boldsymbol{X})\right)(t,\widehat{\boldsymbol{x}}) = (D_{\boldsymbol{x}}\boldsymbol{v})(t, \boldsymbol{X}(t,\widehat{\boldsymbol{x}}))D_{\hat{\boldsymbol{x}}}\boldsymbol{X}(t,\widehat{\boldsymbol{x}}) \qquad \forall (t,\widehat{\boldsymbol{x}}) \in [0,T] \times \widehat{\Omega},$$

which, by using Jacobi's formula for the derivative of a determinant (see, for instance, [2, Chap. II.3]), leads to the Liouville's formula:

$$\left(\frac{\partial}{\partial t}(\det D_{\hat{\boldsymbol{x}}}\boldsymbol{X})\right)(t,\widehat{\boldsymbol{x}}) = (\operatorname{div}_{\boldsymbol{x}} \boldsymbol{v})(t, \boldsymbol{X}(t,\widehat{\boldsymbol{x}})) \det D_{\hat{\boldsymbol{x}}}\boldsymbol{X}(t,\widehat{\boldsymbol{x}}) \qquad \forall (t,\widehat{\boldsymbol{x}}) \in [0,T] \times \widehat{\Omega}.$$

4. An (almost) classical version of the Reynolds theorem

The following additional assumption is enough to prove a Reynolds transport theorem:

H6: $\mathrm{div}_{\boldsymbol{x}}\, \boldsymbol{v}$ is bounded in Q.

Theorem 4.1 *Let $\widehat{\Omega}$ be a bounded open set in $\mathbb{R}^n$ and $T > 0$. Let $\boldsymbol{X} \in \mathcal{C}([0,T] \times \overline{\widehat{\Omega}}; \mathbb{R}^n)$ be a mapping satisfying assumptions* **H1**–**H6**, *with $\boldsymbol{v}$ defined by equation (1). Let $\boldsymbol{\Psi}(t, \widehat{\boldsymbol{x}}) := (t, \boldsymbol{X}(t, \widehat{\boldsymbol{x}}))$ and $Q := \boldsymbol{\Psi}((0,T) \times \widehat{\Omega})$. Let $\widehat{\sigma} \in \mathrm{L}^\infty(\widehat{\Omega})$ and σ be defined in $\boldsymbol{\Psi}([0,T] \times \widehat{\Omega})$ by $\sigma(t, \boldsymbol{x}) := \widehat{\sigma}(\widehat{\boldsymbol{x}})$, where $\boldsymbol{x} := \boldsymbol{X}(t, \widehat{\boldsymbol{x}})$. If $\phi \in \mathcal{C}(\overline{Q}) \cap \mathcal{C}^1(Q)$ is such that $\frac{\partial \phi}{\partial x_i}$, $i = 1, \ldots n$, and $\frac{\partial \phi}{\partial t}$ are bounded in Q, then for any Lebesgue measurable set $A \subset \widehat{\Omega}$ and for all $t \in (0,T)$, we have*

$$\frac{d}{dt} \int_{A_t} \sigma(t, \boldsymbol{x}) \phi(t, \boldsymbol{x})\, d\boldsymbol{x} = \int_{A_t} \sigma(t, \boldsymbol{x}) \left(\frac{\partial \phi}{\partial t}(t, \boldsymbol{x}) + (\mathbf{grad}_{\boldsymbol{x}}\, \phi)(t, \boldsymbol{x}) \cdot \boldsymbol{v}(t, \boldsymbol{x}) + \phi(t, \boldsymbol{x}) (\mathrm{div}_{\boldsymbol{x}}\, \boldsymbol{v})(t, \boldsymbol{x}) \right) d\boldsymbol{x}, \tag{3}$$

where $A_t := \boldsymbol{X}(t, A)$.

Proof. For any Lebesgue measurable set $A \subset \widehat{\Omega}$ and $t \in [0,T]$, $A_t := \boldsymbol{X}(t, A)$ is a Lebesgue measurable subset of Ω_t. Moreover, σ is a measurable function and $\sigma(t, \cdot)$ is measurable for all $t \in [0,T]$ ([3, Th. 8.26]). The formula of change of variables ([3, Th. 8.26]) can be applied to obtain

$$\int_{A_t} \sigma(t, \boldsymbol{x}) \phi(t, \boldsymbol{x})\, d\boldsymbol{x} = \int_A \widehat{\sigma}(\widehat{\boldsymbol{x}}) \phi(t, \boldsymbol{X}(t, \widehat{\boldsymbol{x}})) \det D_{\hat{\boldsymbol{x}}} \boldsymbol{X}(t, \widehat{\boldsymbol{x}})\, d\widehat{\boldsymbol{x}} \qquad \forall t \in [0,T]. \tag{4}$$

Time derivation under the integral sign is justified by applying Lebesgue's dominated convergence theorem and using Liouville's formula, **H2**, **H4** and **H6**. This yields

$$\frac{d}{dt} \int_A \widehat{\sigma}(\widehat{\boldsymbol{x}}) \phi(t, \boldsymbol{X}(t, \widehat{\boldsymbol{x}})) \det D_{\hat{\boldsymbol{x}}} \boldsymbol{X}(t, \widehat{\boldsymbol{x}})\, d\widehat{\boldsymbol{x}} = \int_A \widehat{\sigma}(\widehat{\boldsymbol{x}}) \left(\frac{\partial \phi}{\partial t}(t, \boldsymbol{X}(t, \widehat{\boldsymbol{x}})) + (\mathbf{grad}_{\boldsymbol{x}}\, \phi)(t, \boldsymbol{X}(t, \widehat{\boldsymbol{x}})) \cdot \frac{\partial \boldsymbol{X}}{\partial t}(t, \widehat{\boldsymbol{x}}) \right.$$
$$\left. + \phi(t, \boldsymbol{X}(t, \widehat{\boldsymbol{x}})) (\mathrm{div}_{\boldsymbol{x}}\, \boldsymbol{v})(t, \boldsymbol{X}(t, \widehat{\boldsymbol{x}})) \right) \det D_{\hat{\boldsymbol{x}}} \boldsymbol{X}(t, \widehat{\boldsymbol{x}})\, d\widehat{\boldsymbol{x}}$$

for all $t \in (0,T)$. Since the last integral is equal to the right hand side of (3), we conclude the proof. □

Remark 1 The function $t \mapsto \int_{A_t} \sigma(t, \boldsymbol{x}) \phi(t, \boldsymbol{x})\, d\boldsymbol{x}$ is indeed in $\mathcal{C}([0,T]) \cap \mathcal{C}^1(0,T)$ and its first derivative is bounded. The continuity at $t = 0$ and $t = T$ arises from (4).

Remark 2 It can be proved that if $\phi \in \mathcal{C}(\overline{Q})$ has continuous and bounded first derivatives in $\boldsymbol{\Psi}([0,T] \times \widehat{\Omega})$, then (3) also holds for $t = 0$ and $t = T$.

Remark 3 Let $\mathcal{C}^1(\overline{\widehat{\Omega}}) := \{\phi \in \mathcal{C}(\overline{\widehat{\Omega}}) \cap \mathcal{C}^1(\widehat{\Omega}) : \ \frac{\partial \phi}{\partial x_i}$ have continuous extensions to all of $\overline{\widehat{\Omega}}$, $i = 1, \ldots, n\}$. If $\boldsymbol{X} \in \mathcal{C}^1([0,T]; \mathcal{C}^1(\overline{\widehat{\Omega}})^n)$ and $\det(D_{\hat{\boldsymbol{x}}} \boldsymbol{X})(t, \widehat{\boldsymbol{x}}) > 0 \ \ \forall (t, \widehat{\boldsymbol{x}}) \in [0,T] \times \overline{\widehat{\Omega}}$, then $\boldsymbol{X}$ satisfies **H2**–**H6**.

5. Reynolds theorem for functions in $\mathrm{W}^{1,1}(Q)$

Theorem 5.1 *Let $\widehat{\Omega}$, T, $\boldsymbol{X}$, $\boldsymbol{\Psi}$, Q, $\boldsymbol{v}$, $\widehat{\sigma}$ and σ be as in Theorem 4.1. If $\phi \in \mathrm{W}^{1,1}(Q)$, then, for any Lebesgue measurable set $A \subset \widehat{\Omega}$, function $t \mapsto \int_{A_t} \sigma(t, \boldsymbol{x}) \phi(t, \boldsymbol{x})\, d\boldsymbol{x}$ is in $\mathrm{W}^{1,1}(0,T)$ and equation (3) holds in the sense of distributions on $(0,T)$ and also a.e. in $(0,T)$.*

Proof. Since $\boldsymbol{\Psi} : (0,T) \times \widehat{\Omega} \to Q$ is a homeomorphism, $\boldsymbol{\Psi} \in \mathcal{C}^1((0,T) \times \widehat{\Omega})$, $\boldsymbol{\Psi}^{-1} \in \mathcal{C}^1(Q)$, $D_{(t,\hat{\boldsymbol{x}})} \boldsymbol{\Psi}$ is bounded in $(0,T) \times \widehat{\Omega}$ and $D_{(t,\boldsymbol{x})} \boldsymbol{\Psi}^{-1}$ is bounded in Q, we obtain from [1, Prop. IX.6] that $\widehat{\phi} := \phi \circ \boldsymbol{\Psi} \in \mathrm{W}^{1,1}((0,T) \times \widehat{\Omega})$. Now we construct an extension $\widetilde{\phi} \in \mathrm{W}^{1,1}((-T, 2T) \times \widehat{\Omega})$ of $\widehat{\phi}$ by reflection in time (see, for instance, [1, Lemma IX.2]). By virtue of Friedrich's theorem (see, for instance, [1, Th. IX.2]), there exists a sequence $\widetilde{\phi}_k \in \mathcal{D}(\mathbb{R}^{n+1})$ such that $\widetilde{\phi}_k \to \widetilde{\phi}$ in $\mathrm{L}^1((-T,2T)\times\widehat{\Omega})$ and in $\mathrm{W}^{1,1}_{\mathrm{loc}}((-T,2T)\times\widehat{\Omega})$. Let $\phi_k := \widetilde{\phi}_k \circ \boldsymbol{\Psi}^{-1}$; then $\phi_k \in \mathcal{C}(\overline{Q}) \cap \mathcal{C}^1(Q)$ and

$$\frac{\partial \phi_k}{\partial x_i}(t,\boldsymbol{x}) = \sum_{j=1}^{n} \frac{\partial \widetilde{\phi}_k}{\partial \widehat{x}_j}(t,\boldsymbol{P}(t,\boldsymbol{x}))\frac{\partial P_j}{\partial x_i}(t,\boldsymbol{x}), \qquad i=1,\dots n, \tag{5}$$

$$\frac{\partial \phi_k}{\partial t}(t,\boldsymbol{x}) = \frac{\partial \widetilde{\phi}_k}{\partial t}(t,\boldsymbol{P}(t,\boldsymbol{x})) + \sum_{j=1}^{n} \frac{\partial \widetilde{\phi}_k}{\partial \widehat{x}_j}(t,\boldsymbol{P}(t,\boldsymbol{x}))\frac{\partial P_j}{\partial t}(t,\boldsymbol{x}), \tag{6}$$

so that $\frac{\partial \phi_k}{\partial x_i}$ and $\frac{\partial \phi_k}{\partial t}$ are bounded in Q. By applying the previously obtained (almost) classical version of the Reynolds transport theorem to ϕ_k, we obtain for all $t \in (0,T)$,

$$\frac{d}{dt}\int_{A_t} \sigma(t,\boldsymbol{x})\phi_k(t,\boldsymbol{x})\,d\boldsymbol{x} = \int_{A_t} \sigma(t,\boldsymbol{x})\left(\frac{\partial \phi_k}{\partial t}(t,\boldsymbol{x}) + (\mathbf{grad}_{\boldsymbol{x}}\,\phi_k)(t,\boldsymbol{x})\cdot\boldsymbol{v}(t,\boldsymbol{x}) + \phi_k(t,\boldsymbol{x})(\mathrm{div}_{\boldsymbol{x}}\,\boldsymbol{v})(t,\boldsymbol{x})\right)d\boldsymbol{x}. \tag{7}$$

The convergence of $\widetilde{\phi}_k$ to $\widehat{\phi}$ in $\mathrm{L}^1((0,T)\times\widehat{\Omega})$ implies the convergence of ϕ_k to ϕ in $\mathrm{L}^1(Q)$. Since $\phi = \widehat{\phi}\circ\boldsymbol{\Psi}^{-1}$, by applying [1, Prop. IX.6], we obtain formulas analogous to (5) and (6) with ϕ_k and $\widetilde{\phi}_k$ replaced respectively by ϕ and $\widetilde{\phi}$. Using these formulas as well as the convergence of $\widetilde{\phi}_k$ to $\widetilde{\phi}$ in $\mathrm{L}^1((-T,2T)\times\widehat{\Omega})\cap \mathrm{W}^{1,1}_{\mathrm{loc}}((-T,2T)\times\widehat{\Omega})$, equations (5) and (6), and the boundedness of the first partial derivatives of $\boldsymbol{P}$, we easily obtain that $\frac{\partial \phi_k}{\partial t} \to \frac{\partial \phi}{\partial t}$ in $\mathrm{L}^1(\boldsymbol{\Psi}((0,T)\times\widehat{\omega}))$ and $\frac{\partial \phi_k}{\partial x_i} \to \frac{\partial \phi}{\partial x_i}$ in $\mathrm{L}^1(\boldsymbol{\Psi}((0,T)\times\widehat{\omega}))$, $i=1,\dots,n$, for any open set $\widehat{\omega} \subset\subset \widehat{\Omega}$.

Let us assume first that $A \subset\subset \widehat{\Omega}$. Then, there exists an open set $\widehat{\omega}$ such that $\overline{A} \subset \widehat{\omega} \subset\subset \widehat{\Omega}$. The above convergences together with the boundedness of σ, $\boldsymbol{v}$ and $\mathrm{div}_{\boldsymbol{x}}\,\boldsymbol{v}$ allow us to pass to the limit in the integrals of (7) in the sense of $\mathrm{L}^1(0,T)$. Hence, function $t \mapsto \int_{A_t} \sigma(t,\boldsymbol{x})\phi(t,\boldsymbol{x})\,d\boldsymbol{x}$ is in $\mathrm{W}^{1,1}(0,T)$ and equation (3) holds in the sense of the distributions on $(0,T)$ and also a.e. in (0,T).

Now we consider the case of an arbitrary Lebesgue set $A \subset \widehat{\Omega}$. Let $\widehat{\Omega}_k$, $k \in \mathbb{N}$, be a sequence of open sets such that $\overline{\widehat{\Omega}}_k \subset \widehat{\Omega}_{k+1}$ and $\bigcup_{k=1}^{\infty} \widehat{\Omega}_k = \widehat{\Omega}$. Let $A^k := A \cap \widehat{\Omega}_k$. Since $A^k \subset\subset \widehat{\Omega}$, we have

$$\frac{d}{dt}\int_{A_t^k} \sigma(t,\boldsymbol{x})\phi(t,\boldsymbol{x})\,d\boldsymbol{x} = \int_{A_t^k} \sigma(t,\boldsymbol{x})\left(\frac{\partial \phi}{\partial t}(t,\boldsymbol{x}) + (\mathbf{grad}_{\boldsymbol{x}}\,\phi)(t,\boldsymbol{x})\cdot\boldsymbol{v}(t,\boldsymbol{x}) + \phi(t,\boldsymbol{x})(\mathrm{div}_{\boldsymbol{x}}\,\boldsymbol{v})(t,\boldsymbol{x})\right)d\boldsymbol{x} \tag{8}$$

in $\mathcal{D}'(0,T)$. Next, we pass to the limit as $k \to \infty$ in the integrals in the sense of $\mathrm{L}^1(0,T)$; for instance,

$$\begin{aligned}
&\int_0^T \left| \int_{A_t^k} \sigma(t,\boldsymbol{x})\,\mathbf{grad}_{\boldsymbol{x}}\,\phi(t,\boldsymbol{x})\cdot\boldsymbol{v}(t,\boldsymbol{x})\,d\boldsymbol{x} - \int_{A_t} \sigma(t,\boldsymbol{x})\,\mathbf{grad}_{\boldsymbol{x}}\,\phi(t,\boldsymbol{x})\cdot\boldsymbol{v}(t,\boldsymbol{x})\,d\boldsymbol{x}\right| dt \\
&\qquad \le \int_0^T \int_{A_t\setminus A_t^k} |\sigma(t,\boldsymbol{x})|\,|(\mathbf{grad}_{\boldsymbol{x}}\,\phi(t,\boldsymbol{x}))\cdot\boldsymbol{v}(t,\boldsymbol{x})|\;d\boldsymbol{x}\,dt \\
&\qquad = \int_{\boldsymbol{\Psi}((0,T)\times(A\setminus A^k))} |\sigma(t,\boldsymbol{x})|\,|(\mathbf{grad}_{\boldsymbol{x}}\,\phi)(t,\boldsymbol{x}))\cdot\boldsymbol{v}(t,\boldsymbol{x})|\;d\boldsymbol{x}\,dt
\end{aligned}$$

and this integral tends to zero because of the Lebesgue's dominated convergence theorem. By proceeding analogously with the remaining integrals in (8), we conclude that (3) holds in the sense of distributions on $(0,T)$ and also a.e. in $(0,T)$. Thus we conclude the proof. □

6. The case of cylindrical coordinates: Reynolds theorem for functions in $\mathrm{W}_r^{1,1}(Q)$

Consider cylindrical coordinates (r, θ, z) in $\mathbb{R}^3$, with corresponding unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z$. Let $\widetilde{\Omega} \subset \mathbb{R}^3$ be an open axisymmetric bounded set and $\widehat{\Omega} := \{(\widehat{r}, \widehat{z}) \in (0, \infty) \times \mathbb{R} : \ (\widehat{r}, 0, \widehat{z}) \in \widetilde{\Omega}\}$ the interior of its meridian section. Let $\widetilde{\boldsymbol{X}} : [0, T] \times \overline{\widetilde{\Omega}} \to \mathbb{R}^3$ be an axisymmetric motion which keeps invariant the azimuthal coordinate θ. Then $\widetilde{\boldsymbol{X}}$ is determined by the mapping $\boldsymbol{X} : [0, T] \times \overline{\widehat{\Omega}} \to [0, \infty) \times \mathbb{R}$ which gives the cylindrical coordinates (r, z) of the point $\widetilde{\boldsymbol{X}}(t, \widetilde{x})$ in terms of t and the cylindrical coordinates $(\widehat{r}, \widehat{z})$ of $\widetilde{x} \in \overline{\widetilde{\Omega}}$. In this case, the velocity field $\widetilde{\boldsymbol{v}}$ is also axisymmetric and meridian; namely, $\widetilde{\boldsymbol{v}} = v_r \boldsymbol{e}_r + v_z \boldsymbol{e}_z$. Instead of working with $\widetilde{\boldsymbol{X}}$ and $\widetilde{\boldsymbol{v}}$, we deal with their corresponding descriptions in $\widehat{\Omega}$: $\boldsymbol{X}$ and $\boldsymbol{v} := (v_r, v_z)$. We denote $\operatorname{div} \boldsymbol{v} := \frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z}$, which actually corresponds to $\operatorname{div} \widetilde{\boldsymbol{v}}$ written in polar coordinates.

We use the notation (t, r, z) for a generic point of $\mathbb{R} \times [0, \infty) \times \mathbb{R}$. If G is an open set in $\mathbb{R}^3$ included in $\mathbb{R} \times [0, \infty) \times \mathbb{R}$, we denote $\mathrm{L}^1_r(G) := \left\{\phi : G \to \mathbb{R} \text{ measurable} : \int_G |\phi| \, r \, dr \, dz \, dt < \infty\right\}$ and $\mathrm{W}^{1,1}_r(G) := \{\phi \in \mathrm{L}^1_r(G) : \ \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial z} \in \mathrm{L}^1_r(G)\}$. Moreover, we denote $\mathbf{grad}\, \phi := \frac{\partial \phi}{\partial r} \boldsymbol{e}_r + \frac{\partial \phi}{\partial z} \boldsymbol{e}_z$.

Theorem 6.1 *Let $\widehat{\Omega} \subset (0, \infty) \times \mathbb{R}$ be a bounded open set and $T > 0$. Let $\boldsymbol{X} \in \mathcal{C}([0, T] \times \overline{\widehat{\Omega}}; [0, \infty) \times \mathbb{R})$ be a mapping satisfying assumptions* **H1**–**H5** *and, furthermore,*

H7: *$\forall (t, \widehat{r}, \widehat{z}) \in [0, T] \times \overline{\widehat{\Omega}}$, $\boldsymbol{X}(t, \widehat{r}, \widehat{z})$ lies on the axis $r = 0$ if and only if $(\widehat{r}, \widehat{z})$ lies on the axis $\widehat{r} = 0$.*

Let $\boldsymbol{\Psi}$, Q, $\widehat{\sigma}$ and σ be as in Theorem 4.1. Let $\boldsymbol{v}$ be defined by equation (1) and assume that

H8: *$\operatorname{div} \boldsymbol{v} = \frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z}$ is bounded in Q.*

If $\phi \in \mathrm{W}^{1,1}_r(Q)$, then, for any Lebesgue measurable set $A \subset \widehat{\Omega}$, the equation

$$\frac{d}{dt} \int_{A_t} \sigma \phi \, r \, dr \, dz = \int_{A_t} \sigma \frac{\partial \phi}{\partial t} r \, dr \, dz + \int_{A_t} \sigma \boldsymbol{v} \cdot \mathbf{grad}\, \phi \, r \, dr \, dz + \int_{A_t} \sigma \phi \operatorname{div} \boldsymbol{v} \, r \, dr \, dz \tag{9}$$

holds in the sense of distributions on $(0, T)$ and also a.e. in $(0, T)$.

Proof. First we consider the case where the intersection of $\partial \widehat{\Omega}$ with the axis $\widehat{r} = 0$ is empty. By virtue of **H7** and the compactness of $\overline{Q}$, the r coordinate is bounded from below in this set by a strictly positive constant. Hence $\mathrm{W}^{1,1}_r(Q) = \mathrm{W}^{1,1}(Q)$ and assumption **H8** coincides with **H6**. Thus, by applying Theorem 5.1 for Cartesian coordinates (r, z) to the function $r\phi \in \mathrm{W}^{1,1}(Q)$, we obtain (9).

Next we deal with the case in which $\partial \widehat{\Omega}$ has a nonempty intersection with the axis $\widehat{r} = 0$. Let $\phi \in \mathrm{W}^{1,1}_r(Q)$ and A be a Lebesgue measurable subset of $\widehat{\Omega}$. For all $\epsilon > 0$, let $\widehat{\Omega}_\epsilon := \widehat{\Omega} \cap \{(\widehat{r}, \widehat{z}) : \ \widehat{r} > \epsilon)\}$, $A^\epsilon := A \cap \widehat{\Omega}_\epsilon$ and $Q_\epsilon := \boldsymbol{\Psi}((0, T) \times \widehat{\Omega}_\epsilon)$. We apply the setting of the previous case with the sets $\widehat{\Omega}$, Q and A replaced with $\widehat{\Omega}_\epsilon$, Q_ϵ and A^ϵ, respectively. This leads to an equation analogous to (9) with A_t replaced by A^ϵ_t. Now, by using the dominated convergence theorem, we can pass to the limit as $\epsilon \to 0$ in all the integrals in such equation, in the sense of $\mathrm{L}^1(0, T)$, which leads to (9) in $\mathcal{D}'(0, T)$ and a.e. in $(0, T)$. Thus we conclude the proof. $\square$

Remark 4 If $\boldsymbol{X}$ is smooth enough, for instance $\boldsymbol{X} \in \mathcal{C}^1([0, T]; \mathcal{C}^1(\overline{\Omega})^2)$, and $\det(D_{\widehat{\boldsymbol{x}}} \boldsymbol{X})(t, \widehat{\boldsymbol{x}}) > 0 \quad \forall (t, \widehat{\boldsymbol{x}}) \in [0, T] \times \overline{\widehat{\Omega}}$, then $\boldsymbol{X}$ satisfies **H2**–**H5**. However, this is not necessarily the case with **H8**. In fact, $\operatorname{div} \boldsymbol{v}$ is the sum of $\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z}\right)$ and $\frac{v_r}{r}$; the first term is clearly bounded in Q, but not the latter. A bound of this can be found, for instance, if we further assume:

H9: $\forall a > 0$ small enough, $\widehat{\Omega} \cap \{(\widehat{r}, \widehat{z}) : \ 0 < \widehat{r} < a\}$ is a finite disjoint union of trapezoids.

Assumption **H9** is close to the one assumed in [6] and can be further relaxed. In what follows, we give a brief sketch of the proof of this claim. Taking into account **H7** and the relationship $\frac{v_r(t, r, z)}{r} = \frac{\partial_t X_1(t, \hat{r}, \hat{z})}{X_1(t, \hat{r}, \hat{z})}$ (where X_1 is the first component of $\boldsymbol{X}$), it is enough to bound the last quotient for $\widehat{r} > 0$ small enough. With this aim, first note that assumption **H7** implies that $X_1(t, 0, \widehat{z}) = 0$ and, hence, $\partial_t X_1(t, 0, \widehat{z}) = 0$

for $(t,0,z) \in [0,T] \times \overline{\widehat{\Omega}}$. From this, the mean value theorem and some laborious computations allow us to conclude **H8**. These computations make use of assumption **H9** and the facts that $\frac{\partial X_1}{\partial \hat{z}} = 0$ on the axis $\widehat{r} = 0$ and $\frac{\partial X_1}{\partial \hat{r}}$ is bounded below away from zero for $\widehat{r} > 0$ sufficiently small, which in its turn follows from **H3**, **H7** and the assumed smoothness of $\boldsymbol{X}$.

Remark 5 Assumption **H1** and **H7** are natural when $\boldsymbol{X}$ is the description in a meridian plane of a three-dimensional axisymmetric motion $\widetilde{\boldsymbol{X}}$ keeping invariant the azimuthal coordinate. Moreover, in such a case, assumptions **H2**−**H5** and **H8** are satisfied provided $\widetilde{\boldsymbol{X}}$ is sufficiently smooth, for instance, when $\widetilde{\boldsymbol{X}} \in \mathcal{C}^1([0,T];\mathcal{C}^1(\overline{\widetilde{\Omega}})^3))$ and $\det(D_{\tilde{x}}\widetilde{\boldsymbol{X}})(t,\widetilde{x}) > 0 \ \ \forall (t,\widetilde{x}) \in [0,T] \times \overline{\widetilde{\Omega}}$.

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