ON NONLOCAL CONSERVATION LAWS MODELING SEDIMENTATION

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Abstract. The well-known kinematic sedimentation model by Kynch states that the settling velocity of small equal-sized particles in a viscous fluid is a function of the local solids volume fraction. This assumption converts the one-dimensional solids continuity equation into a scalar, nonlinear conservation law with a non-convex and local flux. The present work deals with a modification of this model, and is based on the assumption that either the solids phase velocity or the solid-fluid relative velocity at a given position and time depends on the concentration in a neighborhood via convolution with a symmetric kernel function with finite support. This assumption is justified by theoretical arguments arising from stochastic sedimentation models, and leads to a conservation law with a nonlocal flux. The alternatives of velocities for which the nonlocality assumption can be stated lead to different algebraic expressions for the factor that multiplies the nonlocal flux term. In all cases, solutions are in general discontinuous and need to be defined as entropy solutions. An entropy solution concept is introduced, jump conditions are derived and uniqueness of entropy solutions in shown. Existence of entropy solutions is established by proving convergence of a difference-quadrature scheme. It turns out that only for the assumption of nonlocality for the relative velocity it is ensured that solutions of the nonlocal equation assume physically relevant solution values between zero and one. Numerical examples illustrate the behaviour of entropy solutions of the nonlocal equation.

1. Introduction

1.1. **Scope.** We study a family of conservation laws with nonlocal flux defined by

$$u_t + \big(u(1-u)^\alpha V(K_a * u)\big)_x = 0, \quad x \in \mathbb{R}, \quad t \in (0,T], \tag{1.1}$$

together with the initial datum

$$u(0,x) = u_0(x), \quad 0 \le u_0(x) \le 1, \quad x \in \mathbb{R}. \tag{1.2}$$

Under idealizing assumptions, (1.1) represents a one-dimensional model for the sedimentation of small equal-sized spherical solid particles dispersed in a viscous fluid, where the local solids volume fraction $u = u(x,t)$ as a function of depth x and

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time t is sought. The parameter α satisfies either $\alpha = 0$ or $\alpha \geq 1$; for both choices there is justification from literature, and we study both in parallel. The function V is a hindered settling factor that can be chosen, for example, as

$$V(w) = (1-w)^n, \quad n \geq 1, \tag{1.3}$$

according to Richardson and Zaki [43], and which is herein supposed to depend on

$$(K_a * u)(x,t) = \int_{-2a}^{2a} K_a(y) u(x+y,t)\,\mathrm{d}y,$$

where K_a is a symmetric, non-negative piecewise smooth kernel function with support on $[-2a, 2a]$ for a parameter $a > 0$ and $\int_{\mathbb{R}} K_a(x)\,\mathrm{d}x = 1$. Usually, one defines a kernel $K = K(x)$ with support on $[-2,2]$ and sets $K_a(x) := a^{-1}K(a^{-1}x)$. Clearly, (1.1) can be considered as a nonlocal version of the kinematic sedimentation model due to Kynch [30], which gives rise to the local scalar conservation law

$$u_t + \big(uV(u)\big)_x = 0, \quad x \in \mathbb{R}, \quad t \in (0,T]. \tag{1.4}$$

In this paper we study the well-posedness of (1.1), (1.2). We establish uniqueness of solutions by an entropy solution concept, and existence by proving convergence of a difference-quadrature scheme based on the standard Lax-Friedrichs scheme. It turns out that for $\alpha = 0$, solutions are bounded by a constant that depends on the final time T, and are Lipschitz continuous if u_0 is Lipschitz continuous. In contrast, for $\alpha \geq 1$ solutions are in general discontinuous even if u_0 is smooth, but assume values within the interval $[0,1]$ for all times. Some numerical examples illustrate the solution behaviour, in particular the so-called effect of layering in sedimenting suspensions and the differences between the cases $\alpha = 0$ and $\alpha \geq 1$.

1.2. **Motivation of the nonlocal flux.** Kynch [30] carried out an analysis of sedimentation in which the suspension was approximated by a continuum. When diffusion is negligible, the one-dimensional continuity equation is [13]

$$u_t(x,t) + \big(u(x,t)v_{\mathrm{s}}(x,t)\big)_x = 0, \tag{1.5}$$

where $v_{\mathrm{s}}(x,t)$ is the solids phase velocity, or settling velocity, at position x at time t, and (1.4) corresponds to the assumption that v_{s} is an explicit function of u, $v_{\mathrm{s}} = v_{\mathrm{St}}V(u)$, where v_{St} is the Stokes velocity, i.e., the settling velocity of a single sphere in an unbounded fluid. If V is given by (1.3), that is, we employ

$$v_{\mathrm{s}}(x_0,t) = v_{\mathrm{St}}\big(1-u(x_0,t)\big)^n, \tag{1.6}$$

and assume that V depends on $K_a * u$ instead of u (detailed justification of this assumption will be provided in Section 2), then (1.5) takes the form

$$u_t + v_{\mathrm{St}}\big(u(1-K_a * u)^n\big)_x = 0. \tag{1.7}$$

A different approach consists in considering the solid and fluid mass conservation equations (1.5) and $-u_t + ((1-u)v_{\mathrm{f}})_x = 0$, where v_{f} is the fluid phase velocity. For batch settling we have the relation $v_{\mathrm{s}} = (1-u)v_{\mathrm{r}}$, where $v_{\mathrm{r}} := v_{\mathrm{s}} - v_{\mathrm{f}}$ is the solid-fluid relative velocity or slip velocity. This leads to the governing equation

$$u_t + \big(u(1-u)v_{\mathrm{r}}\big)_x = 0. \tag{1.8}$$

Assuming now that v_{r} (instead of v_{s}) has a nonlocal behaviour and requiring that the local versions based on constitutive assumptions for either v_{s} or v_{r} should coincide, we state the constitutive assumption for v_{r} as $v_{\mathrm{r}} = V(K_a * u)/(1-u)$. For instance, if we employ (1.3), then the exponent n should be reduced by one, so using the properly adapted Richardson-Zaki equation leads us to

$$v_{\mathrm{s}}(x_0,t)/v_{\mathrm{St}} = \bigl(1-u(x_0,t)\bigr)\bigl(1-(K_a * u)(x_0,t)\bigr)^{n-1},$$

from which we obtain the conservation law

$$u_t + v_{\mathrm{St}}\bigl(u(1-u)(1-K*u)^{n-1}\bigr)_x = 0. \tag{1.9}$$

Equations (1.7) and (1.9) represent the respective cases $\alpha = 0$ and $\alpha = 1$. It is relevant to write the exponent in (1.9) as "$n-1$" only if predictions made by the two versions are to be compared; since n can be chosen arbitrarily, for the mathematical analysis it is sufficient to consider the generic model (1.1)–(1.3). As we prove in this paper, the basic difference in solution behaviour between (1.8) and (1.9) is that solutions of (1.8) may assume values larger than one, while those of (1.9) are strictly limited to $[0,1]$. It seems to us that only (1.9) is suitable for the simulation of the complete sedimentation process from the dilute limit to the densely packed bed. Moreover, formulating a constitutive assumption for v_{r} rather than for v_{s} is consistent with one consequence of the principle of material objectivity (see e.g. [35]) stating that a constitutive relation should only be formulated for an objective quantity: not a single velocity (such as v_{s}), but only the difference between two velocities (such as v_{r}) is objective. In fact, already Richardson and Zaki [43] recognized that the functional relationship was between v_{r} and $1-u$ (V_{s} and ϵ in their notation). Equation (1.9) is the nonlocal approach that is analogous to theirs.

1.3. **Approximate dispersive local PDE and invariant region.** Insight into qualitative properties of the nonlocal PDE (1.1) can be gained by analyzing an approximate local PDE (the "effective" local PDE [52]) obtained by Taylor expansion of $K_a * u$. In a formal calculation, since K_a is even, we have $K_a * u = u + M_2 a^2 u_{xx} + \mathcal{O}(a^4)$, where $2M_2$ is the second moment of K_a, i.e.

$$2M_2 = \frac{1}{a^2}\int_{-2a}^{2a} K_a(x)x^2 \,\mathrm{d}x.$$

Thus, we can write

$$\begin{aligned} V(K_a * u) &= V\bigl(u + M_2a^2u_{xx} + \mathcal{O}(a^4)\bigr) \approx V(u) + a^2V'(u)\bigl(M_2u_{xx} + \mathcal{O}(a^2)\bigr) \\ &\approx V(u) + a^2M_2V'(u)u_{xx}. \end{aligned}$$

Assuming that the length scale of the solution is much larger than a, we replace $V(K_a * u)$ in (1.1) by $V(u) + a^2M_2V'(u)u_{xx}$ and obtain the approximate diffusive-dispersive local PDE

$$u_t + \bigl(u(1-u)^{\alpha}V(u)\bigr)_x = -a^2M_2\bigl(V'(u)u(1-u)^{\alpha}u_{xx}\bigr)_x. \tag{1.10}$$

Note that (1.10) depends on the choices of α and V independently; one cannot simply "absorb" $(1-u)^{\alpha}$ into the choice of V. Thus, for example, we expect qualitatively different solutions in the respective cases $\alpha = 0$ and $\alpha = 1$ with V given by (1.3) with exponents n and $n-1$, although both assumptions lead to the same PDE if V depends locally on u. Specifically, (1.10) reveals why we should expect bounded solutions for $\alpha \geq 1$. In fact, dispersive equations do, in general, not have invariant regions, i.e., one cannot guarantee that the solution takes values in a bounded u-interval for all times. However, for $\alpha \geq 1$ the term sitting inside the derivative on the right-hand side of (1.10) is multiplied by $u(1-u)$, regardless of the algebraic form of V, so for $u = 0$ and $u = 1$ (1.10) degenerates to the first-order conservation law $u_t + (u(1-u)^\alpha V(u))_x = 0$. The factor $u(1-u)$ has a "saturating" effect; it prevents solution values from leaving the interval $[0,1]$. Thus, we should expect that also the nonlocal PDE (1.1) satisfies an invariant region principle for $\alpha \geq 1$. This is indeed the case, as will be proved in Lemma 5.2 of Section 5.

1.4. **Related work.** Zumbrun [52] studied an equation equivalent to (1.1) in the case $\alpha = 0$ and $V(w) = v_{\mathrm{St}}(1-\beta w)$. This model for the sedimentation of a dilute particle in a viscous fluid was advanced by Rubinstein [44], and arises as the limiting case for $d \to 0$ from the more general equation

$$u_t + v_{\mathrm{st}}\big(u(1-\beta K_a * u)\big)_x = du_{xx}, \quad v_{\mathrm{St}}, \beta, d > 0, \tag{1.11}$$

derived from a kinetic theory by Rubinstein and Keller [45, 46] (see also our Section 2). For $\beta = 6.55$ [5], this model had been proposed earlier by Caflisch and Papanicolaou [14]. In coordinates $x' = x - v_{\mathrm{St}}t$ and for $\beta = 1$ (equivalent to rescaling u) and $d = 0$, (1.11) reduces to the equation actually studied in [52], namely

$$u_t + \big(uK_a * u\big)_x = 0, \tag{1.12}$$

where $K_a(x) := a^{-1}K(a^{-1}x)$ and K is the truncated parabola given by

$$K(x) = \frac{3}{8}\left(1 - \frac{x^2}{4}\right) \text{ for } |x| < 2; \quad K(x) = 0 \text{ otherwise.} \tag{1.13}$$

Zumbrun [52] showed global existence of weak solutions for the initial value problem (1.2), (1.12) in L^∞ and uniqueness in the class BV. Furthermore, he derived the effective local, dispersive, KdV-like PDE

$$u_t + (u^2)_x = -M_2 a^2 (uu_{xx})_x, \tag{1.14}$$

and showed by analyzing (1.14) that (1.12) supports travelling waves, but not viscous shocks. This result is based on the symmetry of K, which makes (1.12) completely dispersive. Moreover, an L^2 stability argument is invoked to conclude that smooth solutions of the Burgers-like first-order conservation law $u_t + (u^2)_x = 0$ arise from smooth solutions of (1.12) as $a \to 0$. Zumbrun [52] (see also [27]) also studied the effect of artificial diffusion added to (1.12), corresponding to $d > 0$, and showed that for the corresponding effective local PDE, i.e. (1.14) with du_{xx} added to the right-hand side, solutions of shock initial data converge to a stable, oscillatory travelling wave. He then discussed whether the resulting model is possibly sufficient to explain the phenomenon of layering in sedimentation. Much of his analysis is for a more general, but symmetric kernel K. Whatever the exact form of $K(x)$, it is clear that the interval over which it applies scales with the sphere radius a. We will compare our findings with those of Zumbrun in Section 5.4, see also Section 7.

Another spatially one-dimensional, nonlocal sedimentation model was studied by Sjögreen et al. [48]. Starting from a more involved model, they consider a hyperbolic-elliptic model problem given by (1.5) coupled with $-\eta(v_{\mathrm{s}})_{xx} + v_{\mathrm{s}} = u$, where $\eta > 0$ is a viscosity parameter. Clearly, at any fixed position x_0, $v_{\mathrm{s}}(x_0,t)$ will depend on $u(\cdot,t)$ as a whole; the nonlocal dependence is not limited to a neighborhood, as in [52] and herein. They prove that their model has a smooth solution, and present numerical solutions obtained by a high-order difference scheme.

The (local) kinematic model of sedimentation (1.4) is similar to the well-known Lighthill-Whitham-Richards (LWR) model of vehicular traffic. Sopasakis and Katsoulakis [49] extended the LWR model to a nonlocal version by a "look-ahead" rule, i.e. drivers choose their velocity taking account the density on a stretch of road ahead of them. Kurganov and Polizzi [29] showed that an extension of the well-known Nesshayu-Tadmor (NT) central nonoscillatory scheme [39] is suitable for the nonlocal model of [49], which can be written as (1.1) for $\alpha = 1$ and $V(w) = \exp(-w)$, and if we replace K_a by $K(y) = \mathcal{H}(y)\gamma^{-1}\varphi(y/\gamma)$, where $\mathcal{H}$ is the Heaviside function, $\gamma > 0$ is a constant proportional to the look-ahead distance, and either $\varphi = 1$ (according to [49]) or $\varphi(z) = 2 - 2z$ (as proposed in [29]) for $0 \le z \le 1$, and $\varphi = 0$ elsewhere. As pointed out in [29], the basic methods for conservation laws with local flux that should be adapted for (1.1) are central rather than upwind schemes for conservation laws, since the latter usually involve the (approximate) solution of Riemann problems, and no Riemann solver is available for (1.1). Though the second-order NT scheme produces better resolution, we herein rely on the Lax-Friedrichs scheme to be consistent with the entropy analysis.

Related models with a nonlocal convective flux that have been analyzed within an entropy solution framework include the continuum model for the flow of pedestrians by Hughes [22], which gives rise to a multi-dimensional conservation law with a nonlocal flux; see also [16, 17]. However, in contrast to (1.1) the nonlocality in that model is not introduced by explicit convolution but via the solution of an eikonal equation. An entropy solution framework is employed in [18] to establish well-posedness for a hyperbolic-elliptic approximation of the original model of [22].

Another equation that can formally be expressed in the form (1.1), namely for $\alpha = 0$, $V(w) = w$ and with K_a replaced by the Cauchy kernel so that $K_a * u$ becomes the Hilbert transform Hu, is studied in [15]. This equation arises from several applications, including a one-dimensional model of the two-dimensional vortex sheet problem [3], and is analyzed in [15] with respect to existence of smooth solutions for smooth initial data. Equations that can formally be written as a first-order conservation law with nonlocal flux also arise from models of opinion formation [2].

1.5. **Outline of the paper.** The remainder of the paper is organized as follows. In Section 2 we motivate the assumption of nonlocal dependence of settling velocities and argue that it may describe the layering phenomenon in sedimentation. In Section 3 we describe the numerical scheme, which involves the approximate computation of $K_a * u$ by a quadrature formula. We state some assumptions on the functions u_0 and V and on the mesh for the numerical scheme, and derive some estimates on differences of the discrete convolution. In Section 4 we state the definition of entropy solutions of (1.1), (1.2), the jump conditions, and prove that entropy solutions are L^1 contractive with respect to initial data, and in particular unique. Section 5 is devoted to the proof of convergence of approximate solutions generated by the numerical scheme to entropy solutions, which is achieved by standard compactness bounds (Sect. 5.1), a cell entropy inequality, and Lax-Wendroff-type arguments (Sect. 5.2). In Section 5.3 we prove that for the case $\alpha = 0$, solutions are actually Lipschitz continuous provided that u_0 is Lipschitz continuous. In Section 6 we present numerical examples, paying particular attention to the layering phenomenon. Conclusions, limitations and possible extensions are addressed in Section 7.

2. Motivation of the nonlocal sedimentation model

2.1. **Nonlocal dependence of settling velocities.** The solution of the one-dimensional continuity equation (1.5) requires an initial condition, possibly boundary conditions, and an equation relating v_{s} to $u = u(x,t)$. Theoretical studies of this relationship in dilute, uniformly mixed suspensions of identical spheres are numerous. Those by Kermack et al. [26] and Batchelor [5] are especially notable. When $u(x,t)$ is not constant, the relationship between v_{s} and u is no longer obvious. The key assumption of Kynch's theory [30] is that v_{s} is determined by the "local solids concentration", which is the concentration $u(x,t)$ at a specified height and time. This relationship is expressed as $v_{\mathrm{s}} = v_{\mathrm{St}} V(u)$. This treatment is analyzed in detail by Bustos et al. [13]. In the three-dimensional reality approximated by the one-dimensional theory, $u(x,t)$ is the solids concentration at a horizontal plane [11, 41]. Though this is an excellent approximation (for the dependence of v_{s} on u), we may still improve it by a nonlocal dependence, as will be argued in this section.

The locality of the dependence in Kynch's theory contrasts sharply with the theoretical result that the velocity of each particle is determined by the size, position and orientation of all particles and the nature of the boundaries, if any [19]. (Of course, the orientation is irrelevant for spheres.) As a compromise between this result and the assumption by Kynch, Pickard and Tory [40] postulated that the settling velocity, $v_{\mathrm{s}}(x_0,t)$, of a test particle at x_0 is governed by a parameter

$$c(x_0,t) = \int_I w(x) u(x_0 + x, t)\,\mathrm{d}x, \qquad I \subseteq \mathbb{R}, \tag{2.1}$$

that is the convolution of local solids concentration with a weighting function. This parameter was introduced in the context of a stochastic model for which the smoothing effect was important [42] and later generalized to polydisperse suspensions [50]. The function $w(x)$ was specified to be positive in a neighborhood of zero, unimodal, and uniformly bounded with uniformly bounded mean, mode, and variance. When u is constant, we require that $c(x_0,t) = u(x_0,t)$. This implies that $\int_I w(x)\,\mathrm{d}x = 1$, see [20]. This means that the velocity of a sphere at x_0 is governed by the concentration in a contiguous region of finite width. In the limiting case, $w(x)$ is replaced by $\delta(x)$ and the sifting property of the Dirac delta function equates the parametric and local solids concentrations [41].

Beenakker and Mazur [9, 10] calculated the mean velocity of a test sphere in a dilute suspension of identical spheres settling toward an infinite horizontal flat plate. Assuming that all the spheres were placed according to a uniform distribution subject only to the condition that they do not overlap the test sphere or the solid boundary [51], they obtained an explicit expression for the mean velocity of a sphere at a given height. Neglecting terms of $\mathcal{O}(a/h)$, where a is the radius of the test sphere and h is its distance from the boundary, this can be written as [44]

$$v_{\mathrm{s}}(x_0,t) = v_{\mathrm{St}} \left(1 + \int_{-2a}^{2a} H_a(x) u(x_0 + x, t)\,\mathrm{d}x \right), \tag{2.2}$$

where

$$H_a(x) = \frac{15}{8a} \left(\frac{1}{4} \left(\frac{x}{a} \right)^2 - 1 \right). \tag{2.3}$$

Note that only spheres in the interval $[x_0 - 2a, x_0 + 2a]$ affect the mean velocity of the test sphere [9, 51]. This results from an exact cancellation, before taking the limit, of large terms in the regions above and below this interval [51]. Taking the limits first yields a divergent sedimentation velocity upon integration [9]. Equation (2.2) does not contradict the result that the velocity of the test sphere is affected by all the spheres in a suspension [8, 19, 36] because the variance of velocity is determined by the positions of all the spheres [51]. Velocity fluctuations in sedimentation, which account for hydrodynamic diffusion, are still being studied intensively [37].

When u is constant, insertion of (2.3) into (2.2) yields

$$v_{\mathrm{s}}(x_0,t) = v_{\mathrm{St}}\bigl(1-5u(x_0,t)\bigr), \quad \text{i.e., } V(u) = 1-5u. \tag{2.4}$$

This result also holds in a linear concentration gradient because the additional term in the integrand is an odd function (since $H_a(x)$ is even), and the integral is between symmetric limits. Apart from discontinuities, concentration normally varies smoothly over distances much greater than $4a$. Hence, this equation is a good approximation in nonlinear gradients, but only in very dilute suspensions. Higher-order two-sphere interactions can be added [9] to yield Batchelor's result for identical spheres [5], which is

$$v_{\mathrm{s}}(x_0,t) = v_{\mathrm{St}}\bigl(1-6.55u(x_0,t)\bigr), \quad \text{i.e., } V(u) = 1-6.55u. \tag{2.5}$$

This equation works well for colloidal dispersions in which Brownian motion maintains an essentially uniform distribution of sphere centers. However, experiments with non-Brownian spheres suggest that (2.4) is more accurate. Though the velocity of the spheres relative to the fluid is independent of the shape of the container [10], equation (2.3) applies only to dilute suspensions settling towards an infinite flat plate. Nevertheless, it seems likely, given the success of Kynch's theory, that a generalization of (2.3) should be a reasonable approximation at higher concentrations and for suspensions in finite containers.

Three-sphere and higher interactions are important at higher concentrations [6, 7, 23]. Special treatments involving intensive computation are necessary for concentrated suspensions [21, 31, 32, 33, 34, 38]. At higher concentrations, the dependence of v_{s} on u is nonlinear. The Richardson-Zaki [43] equation (1.6), corresponding to $V(u)$ given by (1.3), is widely used to predict the position of the interface and the propagation of concentration changes.

In the Pickard-Tory model, the dependence of the settling velocity, $v_{\mathrm{s}}(x_0,t)$, on $c(x_0,t)$ rather than $u(x_0,t)$ is similar to the dependence in [8], but not as specific. If we combine their model with the Richardson-Zaki equation, we obtain

$$\begin{aligned} \frac{v_{\mathrm{s}}(x_0,t)}{v_{\mathrm{St}}} &= \bigl(1-c(x_0,t)\bigr)^n \\ &= 1 - n\int_I w(x)u(x_0+x,t)\,\mathrm{d}x + \frac{n(n-1)}{2}\left[\int_I w(x)u(x_0+x,t)\,\mathrm{d}x\right]^2 - \dots. \end{aligned} \tag{2.6}$$

We can choose w to be an even function. Then (2.6) implies that $c(x_0,t) = u(x_0,t)$ in a linear concentration gradient. When u is small and constant or linear, we obtain the approximation $v_{\mathrm{s}}(x_0,t) \approx v_{\mathrm{St}}(1-nu(x_0,t))$, which agrees with (2.4) and (2.5).

Again, we choose w to be an even function and require that $\int_I w(x)\,\mathrm{d}x = 1$. Then (2.6) can be written as $v_{\mathrm{s}}(x_0,t) = v_{\mathrm{St}}(1-K*u)^n$, which yields (1.7), where $K*u$ is the convolution of K with u and $\int_I K(x)\,\mathrm{d}x = 1$.

2.2. **Layered sedimentation in suspensions.** Initially homogeneous suspensions of hydrophobic colloidal particles do not always sediment in smooth continuous fashion. Instead, layers of different concentrations are often observed after settling has proceeded for a time [47]. This phenomenon is accentuated when a very dilute suspension has an initial concentration gradient [47]. The upward propagation of a concentration gradient from the bottom of the container will eventually obliterate the layered form if we study this phenomenon in a closed vessel rather than just focussing on the zone slightly below the suspension-supernate interface.

The weighting functions $H_a(x), K(x), w(x)$, and $W(x)$ have an important influence near discontinuities in concentration. When the interval I overlaps the packed bed, these weighting functions introduce a concentration gradient [4]. Where Kynch's theory predicts a jump in u from u_0 to $u_{\max}$, which correponds to a so-called mode of sedimentation MS-1 [12, 13], weighting functions produce the same increase over a finite distance [4]. However, this gradient does not expand.

Our assumption of nonlocal dependence of the settling velocity provides an explanation of the layering phenomenon. In fact, when I overlaps the suspension-supernatant interface, the spheres near that interface settle faster than those below. For example, according to (2.2), a particle at the interface of a uniform suspension has an initial velocity of $v(x_0,t) = v_{\mathrm{St}}(1-2.5u(x_0,t))$ compared to that given by (2.4) for a sphere that is $2a$ or more below the interface. This causes an increase in concentration from u_0 to $u_0+\Delta u$ in a small region just below the interface. However, spheres near the bottom of this region settle faster than those in its middle because I includes a sub-region with $u = u_0$ as well as one with $u = u_0+\Delta u$. This concentration disturbance should propagate down the settling column. If equation (1.5) applies, the result would seem to be a gradual increase in concentration and perhaps some instability if the concentration near the top remains higher than that near the bottom. Since concentrated suspensions settle much more slowly than dilute ones, it would seem that layering would occur only in very dilute suspensions where slight increases in concentration cause only slight changes in settling velocity.

3. Preliminaries

3.1. **Assumptions and numerical scheme.**

We discretize (1.1) on a fixed grid given by $x_j = j\Delta x$ for $j \in \mathbb{Z}$ and $t_n = n\Delta t$ for $n \leq N := T/\Delta t$, where T is the finite final time. As usual, u_j^n approximates the cell average

$$u_j^n \approx \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(y,t_n)\,\mathrm{d}y, \tag{3.1}$$

and we define $U^n := (\dots, u_{j-1}^n, u_j^n, u_{j+1}^n, \dots)^{\mathrm{T}}$. The initial datum u_0 is discretized accordingly. We use the standard spatial difference operators $\Delta_+ u_j^n := u_{j+1}^n - u_j^n$, $\Delta_- u_j^n := u_j^n - u_{j-1}^n$, and $\Delta^2 u_j^n := \Delta_+\Delta_- u_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$. The obvious difficulty in defining a numerical scheme for (1.1) arises from the discretization of the integral. We approximate it by a quadrature formula given by

$$(K_a * u)_j^n \approx \tilde{u}_{a,j}^n := \sum_{i=-l}^{l} \gamma_i u_{j-i}^n, \quad \text{where } \gamma_i = \int_{x_{i-1/2}}^{x_{i+1/2}} K_a(y)\,\mathrm{d}y \text{ and } l = \left\lceil \frac{2a}{\Delta x}\right\rceil + 1,$$

i.e., l is the smallest integer larger or equal to $(2a/\Delta x)+1$.

Due to the properties of K (Eq. (1.13)), $\gamma_{-l} + \dots + \gamma_l = 1$. The computations and the numerical analysis are based on the Lax-Friedrichs scheme for a standard non-linear scalar conservation law. We summarize all assumptions on the initial datum u_0, the velocity function V and the mesh.

Assumption 3.1. *We assume that u_0 has compact support, $u_0(x) \geq 0$ for $x \in \mathbb{R}$ and $u_0 \in BV(\mathbb{R})$. The function $u \mapsto V(u)$ and its derivatives are locally Lipschitz continuous for $u \geq 0$ (which occurs, for example, if $V(\cdot)$ is a polynomial). When we send $\Delta x, \Delta t \downarrow 0$ then it is understood that $\lambda := \Delta t/\Delta x$ is kept constant.*

In addition to Assumption 3.1 for the case $\alpha \geq 1$ we suppose the following.

Assumption 3.2. *The inital datum satisfies $u_0(x) \leq 1$ for all $x \in \mathbb{R}$.*

Remark 3.1. *The same analysis remains valid for any smooth, positive and not necessarily compactly supported kernel with $\|K\|_1 = 1$ and $\|\partial_x K_a\|_1 < \infty$.*

From now on we let the function u^Δ be defined by

$$u^\Delta(x,t) = U_j^n \quad \text{for } (x,t) \in [j\Delta x, (j+1)\Delta x) \times [n\Delta t, (n+1)\Delta t).$$

We now prove two lemmas that will be used for the convergence analysis.

Although K_a (Eq. (1.13)) is just Lipschitz continuous on $\mathbb{R}$, on its support it is a smooth function. Having this in mind we can prove the following lemma.

Lemma 3.1. *Suppose that $u^\Delta(\cdot, t_n) \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then*

$$\left|\Delta_+ \tilde{u}^n_{a,j}\right| \leq \|\partial_x K_a\|_\infty \left\|u^\Delta(\cdot,t_n)\right\|_1 \Delta x \quad \text{for } j \in \mathbb{Z}. \tag{3.2}$$

Proof. We compute

$$\Delta_- \tilde{u}^n_{a,j} = \sum_{i=-l}^{l} \gamma_i \Delta_+ u^n_{j-i} = \sum_{i=-l}^{l-1} u^n_{j-i}(\gamma_{i+1} - \gamma_i) + \gamma_l\big(u^n_{j+1+l} - u^n_{j-l}\big). \tag{3.3}$$

Since $K_a(2a) = 0$, we have

$$\gamma_l = \int_{2a-\Delta x/2}^{2a} K_a(x)\,\mathrm{d}x = \int_{2a-\Delta x/2}^{2a} \big|K_a(x) - K_a(2a)\big|\,\mathrm{d}x \leq \|\partial_x K_a\|_\infty \frac{\Delta x^2}{4}.$$

For $-l \leq i \leq l-1$ we find

$$\begin{aligned}\gamma_{i+1} - \gamma_i &= \int_{x_{i+1/2}}^{x_{i+3/2}} \big(K_a(x) - K_a(x-\Delta x)\big)\,\mathrm{d}x = \int_{x_{i+1/2}}^{x_{i+3/2}} \partial_x K_a(\xi_{i+1})\Delta x\,\mathrm{d}x \\ &\leq \|\partial_x K_a\|_\infty \Delta x^2,\end{aligned}$$

where $\xi_{i+1} \in [x_{i-1/2}, x_{i+3/2}]$. Applying the last two inequalities to the right-hand side of (3.3) and using that $u^\Delta(\cdot,t_n) \in L^1_{\mathrm{loc}}(\mathbb{R})$, we obtain

$$\left|\Delta_+ \tilde{u}^n_{a,j}\right| \leq \|\partial_x K_a\|_\infty \left(\sum_{i=-l}^{l-1} |u^n_{j-i}| + \frac{|u^n_{j+1+l}| + |u^n_{j-l}|}{4}\right)\Delta x^2,$$

which implies (3.2). □

In what follows, $\mathcal{C}_a$ always denotes a constant that is independent of $\Delta := (\Delta x, \Delta t)$, but depends on a, and that may change from one line to the next.

Lemma 3.2. *Suppose that $u^\Delta(\cdot,t_n) \in L^1_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then*

$$\left|\Delta^2 \tilde{u}^n_{a,j}\right| \leq \mathcal{C}_a \Delta x^2 \quad \text{for } j \in \mathbb{Z}. \tag{3.4}$$

Proof. We calculate

$$\begin{aligned}\Delta^2 \tilde{u}^n_{a,j} &= \sum_{i=-l}^{l} \big(\gamma_i u^n_{j+1-i} - 2\gamma_i u^n_{j-i} + \gamma_i u^n_{j-1-i}\big) \\ &= \sum_{i=-l+1}^{l-1} u^n_{j-i}\Delta^2\gamma_i + u^n_{j+l}\Delta_+\gamma_{-l} - u^n_{j-l}\Delta_-\gamma_l + \gamma_l\big(\Delta_+ u^n_{j+l} - \Delta_- u^n_{j-l}\big).\end{aligned} \tag{3.5}$$

Lemma 3.1 implies that there exists a constant $\mathcal{C}_a$ such that $\gamma_l \le \mathcal{C}_a \Delta x^2$, and therefore $\Delta_+ \gamma_{-l} \le \mathcal{C}_a \Delta x^2$ and $\Delta_- \gamma_l \le \mathcal{C}_a \Delta x^2$. Using the Taylor Theorem we get for $i \in \{-l+1, \dots, l-1\}$

$$\begin{aligned} \left|\Delta^2 \gamma_i\right| &= \left| \int_{x_{i-1/2}}^{x_{i+1/2}} \big(K_a(x+\Delta x) - 2K_a(x) + K_a(x-\Delta x)\big)\,\mathrm{d}x \right| \\ &= \left| \int_{x_{i-1/2}}^{x_{i+1/2}} \left(\partial_x^2 K_a(\xi_i^+) \frac{\Delta x^2}{2} + \partial_x^2 K_a(\xi_i^-) \frac{\Delta x^2}{2} \right) \mathrm{d}x \right| \le \|\partial_x^2 K_a\|_\infty \Delta x^3, \end{aligned}$$

where $\xi_i^+ \in [x_{i-1/2}, x_{i+3/2}]$ and $\xi_i^- \in [x_{i-3/2}, x_{i+1/2}]$. Consequently, using that $u^\Delta(\cdot, t_n) \in L^1_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we obtain from (3.5) the desired estimate (3.4). □

4. Definition and uniquenss of entropy solutions

4.1. **Definition of an entropy solution and jump conditions.** It is well known that solutions to a standard nonlinear conservation law like (1.4) are in general discontinuous even if the initial datum u_0 is smooth. The same will occur with the nonlocal equation (1.1), so we need to define solutions as weak solutions. Since weak solutions of conservation laws are, in general, not unique, a selection criterion must be imposed in order to single out the physically relevant solution. We select the solution through an entropy criterion, and the sought solutions are entropy solutions defined as follows. To facilitate notation we define $f(u) := u(1-u)^\alpha$.

Definition 4.1. *A measurable, non-negative function u is an* entropy solution *of the initial value problem* (1.1), (1.2) *if it satisfies the following conditions:*

(1) *We have $u \in L^\infty(\Pi_T) \cap L^1(\Pi_T) \cap BV(\Pi_T)$.*
(2) *The initial condition* (1.2) *is satisfied in the following sense:*

$$\lim_{t \downarrow 0} \int_{\mathbb{R}} \left|u(x,t) - u_0(x)\right| \mathrm{d}x = 0. \tag{4.1}$$

(3) *For all non-negative test functions $\varphi \in C_0^\infty(\Pi_T)$, the following entropy inequality is satisfied:*

$$\begin{aligned} \forall k \in \mathbb{R}: \quad \iint_{\Pi_T} \Big\{ &|u-k|\varphi_t + \operatorname{sgn}(u-k)\big(f(u) - f(k)\big) V(K_a * u)\varphi_x \\ &- \operatorname{sgn}(u-k) f(k) V'(K_a * u)(\partial_x K_a * u)\varphi \Big\}\,\mathrm{d}x\,\mathrm{d}t \ge 0. \end{aligned} \tag{4.2}$$

The Kružkov-type [28] entropy inequality (4.2) follows from a standard vanishing viscosity argument. It is also standard to deduce that an entropy solution is, in particular, a weak solution of (1.1), (1.2), which is defined by (1) and (2) of Definition 4.1, and the following equality, which must hold for all $\varphi \in C_0^\infty(\Pi_T)$:

$$\iint_{\Pi_T} \Big\{ u\,\varphi_t + f(u) V(K_a * u)\varphi_x - f(u) V'(K_a * u)(\partial_x K_a * u)\varphi \Big\}\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.3}$$

Assume that u is an entropy solution having a discontinuity at a point $(x_0, t_0) \in \Pi_T$ between the approximate limits u^+ and u^- of u taken with respect to $x > x_0$ and $x < x_0$, respectively. The propagation velocity s of the jump is given by the Rankine-Hugoniot condition, which is derived in a standard way from (4.3):

$$s = \sigma(u^+, u^-) V(K_a * u), \quad \sigma(u,v) := \frac{f(u) - f(v)}{u - v}, \tag{4.4}$$

where we utilize that $(K_a * u)(\cdot,t)$ is a Lipschitz continuous function of x. In addition, a discontinuity between two solution values needs to satisfy the following jump entropy condition, which is a consequence of (4.2):

$$\forall k \in \big(\min\{u^-,u^+\},\max\{u^-,u^+\}\big) : \sigma(u^+,k)V(K_a * u) \le s \le \sigma(u^-,k)V(K_a * u).$$

4.2. **Uniqueness of entropy solutions.** The uniqueness of entropy solutions is a consequence of a result proved in [24] regarding continuous dependence of entropy solutions with respect to the flux function. Precisely, we have the following theorem.

Theorem 4.1. *Assume that u and v are entropy solutions of* (1.1), (1.2) *with initial data u_0 and v_0, respectively. Then there exists a constant C_1 such that*

$$\big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^1(\mathbb{R})} \le C_1 \|u_0 - v_0\|_{L^1(\mathbb{R})} \quad \forall t \in (0,T].$$

In particular, an entropy solution of (1.1), (1.2) *is unique.*

Proof. Let u and v be entropy solutions of the respective initial value problems

$$\begin{aligned} u_t + \big(V(x,t)f(u)\big)_x &= 0, \quad V(x,t) := V\big((K_a * u)(x,t)\big); \quad u(x,0) = u_0(x), \\ v_t + \big(\tilde{V}(x,t)f(v)\big)_x &= 0, \quad \tilde{V}(x,t) := V\big((K_a * v)(x,t)\big); \quad v(x,0) = v_0(x). \end{aligned}$$

Following the proof of Theorem 1.3 of [24] and keeping in mind that u and v are of bounded variation, we obtain the following inequality, where $J := [0,\|u\|_\infty]$:

$$\begin{aligned} \big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^1(\mathbb{R})} &\le \|u_0 - v_0\|_{L^1(\mathbb{R})} \\ &\quad + \|f\|_{L^\infty(J)} \int_0^t \int_{\mathbb{R}} \big|V_x(x,s) - \tilde{V}_x(x,s)\big| \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \|f\|_{\mathrm{Lip}(J)} \int_0^t \int_{\mathbb{R}} \big|V(x,s) - \tilde{V}(x,s)\big|\big|v_x(x,t)\big| \,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{4.5}$$

where v_x must be understood in the sense of measures. Now we observe that

$$\begin{aligned} \big|V(x,s) - \tilde{V}(x,s)\big| &= \big|V\big((K_a * u)(x,s)\big) - V\big((K_a * v)(x,s)\big)\big| \\ &\le \|V'\|_\infty \big|\big(K_a * (u-v)\big)(x,s)\big| \\ &\le \|V'\|_\infty \|K_a\|_\infty \big\|u(\cdot,s)-v(\cdot,s)\big\|_{L^1(\mathbb{R})}, \\ \big|V_x(x,s) - \tilde{V}_x(x,s)\big| &= \big|V'\big(K_a * u(x,s)\big)(\partial_x K_a * u)(x,s) \\ &\qquad - V'\big(K_a * v(x,s)\big)(\partial_x K_a * v)(x,s)\big| \\ &\le \|V'\|_\infty \big|\big(\partial_x K_a * (u-v)\big)(x,s)\big| \\ &\quad + \|\partial_x K_a * v\|_\infty \|V''\|_\infty \big|\big(K_a * (u-v)\big)(x,s)\big|. \end{aligned}$$

Inserting the last expressions into the integrands in (4.5), using the properties of the kernel K_a and the fact that v has bounded variation we arrive at

$$\big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^1(\mathbb{R})} \le \|u_0 - v_0\|_{L^1(\mathbb{R})} + C_2 \int_0^t \big\|u(\cdot,s)-v(\cdot,s)\big\|_{L^1(\mathbb{R})} \,\mathrm{d}s.$$

Applying the integral form of the Gronwall inequality we finally obtain

$$\big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^1(\mathbb{R})} \le \|u_0 - v_0\|_{L^1(\mathbb{R})} \big(1 + C_2 t \exp(C_2 t)\big).$$

The second statement of the lemma follows by taking $u_0 = v_0$. □

5. Convergence analysis and existence of entropy solutions

5.1. **Compactness estimates.** We define $V_j^n := V(\tilde{u}_{a,j}^n)$. Then the marching formula for the approximation of solutions of (1.1), (1.2) reads

$$u_j^{n+1} = \frac{u_{j-1}^n + u_{j+1}^n}{2} - \frac{\lambda}{2} u_{j+1}^n \big(1 - u_{j+1}^n\big)^\alpha V_{j+1}^n + \frac{\lambda}{2} u_{j-1}^n \big(1 - u_{j-1}^n\big)^\alpha V_{j-1}^n. \tag{5.1}$$

We assume that $\lambda = \Delta t / \Delta x$ satisfies the following CFL condition:

$$\lambda \max_{u \leq u^*} \big|V(u)\big| < 1 \text{ for } \alpha = 0,\ u^* := \|K_a\|_\infty \|u_0\|_1; \tag{5.2}$$

$$\lambda \max_{0 \leq u \leq 1} \big|V(u)\big| < 1 \text{ for } \alpha \geq 1. \tag{5.3}$$

By the conservativity of the scheme (5.1) and the CFL condition, we immediately obtain the following lemma.

Lemma 5.1. *Under Assumption 3.1, the numerical approximation generated by* (5.1) *in the case* $\alpha = 0$ *satisfies*

$$\|U^n\|_1 \leq \|U_0\|_1 \quad \text{for } 0 \leq n \leq N.$$

The next step in the numerical analysis is to prove the L^∞ stability. Appealing to Lemma 3.1, we are in a position to prove the following lemma.

Lemma 5.2. *The numerical approximation generated by* (5.1) *satisfies*

$$0 \leq u_j^n \leq \begin{cases} C_3 & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha \geq 1, \end{cases} \quad \text{for } j \in \mathbb{Z} \text{ and } 0 \leq n \leq N, \tag{5.4}$$

where the constant C_3 is independent of Δ but depends on T.

Proof. We can rewrite (5.1) as

$$u_j^{n+1} = \frac{u_{j-1}^n}{2} \left(1 + \lambda \big(1 - u_{j-1}^n\big)^\alpha V_{j-1}^n\right) + \frac{u_{j+1}^n}{2} \left(1 - \lambda \big(1 - u_{j+1}^n\big)^\alpha V_{j+1}^n\right). \tag{5.5}$$

We consider first the case $\alpha = 0$. Using Assumption 3.1 we have

$$\tilde{u}_{a,j}^n = \sum_{i=-l}^{l} \left(\int_{x_{i-1/2}}^{x_{i+1/2}} K_a(y)\, \mathrm{d}y \right) u_{j-i}^n \leq \|K_a\|_\infty \sum_{i=-l}^{l} u_{j-i}^n \Delta x \leq \|K_a\|_\infty \|u_0\|_1,$$

and thanks to the local Lipschitz continuity of V we can bound $|V(\tilde{u}_{a,j}^n)|$ as a function of $\|K_a\|_\infty$ and $\|u_0\|_1$. Moreover, $|V'(\tilde{u}_{a,j}^n)|$ and $|V''(\tilde{u}_{a,j}^n)|$ can be bounded thanks to the assumptions on V and its derivatives. We can write

$$u_j^{n+1} = u_{j+1}^n \left(\frac{1}{2} - \frac{\lambda}{2} V_{j+1}^n \right) + u_{j-1}^n \left(\frac{1}{2} + \frac{\lambda}{2} V_{j+1}^n \right) - \frac{\lambda}{2} u_{j-1}^n \left(\Delta_+ V_j^n + \Delta_- V_j^n \right).$$

With Lemma 3.1 and the CFL condition we get

$$\begin{aligned} \big|u_j^{n+1}\big| &\leq \big|u_{j+1}^n\big| \left(\frac{1}{2} - \frac{\lambda}{2} V_{j+1}^n \right) + \big|u_{j-1}^n\big| \left(\frac{1}{2} + \frac{\lambda}{2} V_{j+1}^n \right) \\ &\quad + \lambda \big|u_{j-1}^n\big| \|V'\|_\infty \|\partial_x K_a\|_\infty \|u_0\|_1 \Delta x \\ &\leq \|U^n\|_\infty (1 + C_4 \Delta t), \end{aligned}$$

which means that

$$\big|u_j^{n+1}\big| \leq \|U^0\|_\infty (1 + C_4 \Delta t)^n = \|U^0\|_\infty \left(1 + C_4 \frac{T}{n} \right)^n \leq \|u_0\|_\infty \exp(C_4 T). \tag{5.6}$$

To handle the case $\alpha \geq 1$, we assume that $u_j^n \leq 1$ for all $j \in \mathbb{Z}$ (Assumption 3.2) and rewrite (5.1) as

$$\begin{aligned} u_j^{n+1} &= \frac{u_{j+1}^n}{2}\big(1 + \lambda u_{j+1}^n \big(1 - u_{j+1}^n\big)^{\alpha-1} V_{j+1}^n\big) - \frac{\lambda}{2} u_{j+1}^n \big(1 - u_{j+1}^n\big)^{\alpha-1} V_{j+1}^n \\ &\quad + \frac{u_{j-1}^n}{2}\,(1 - \lambda u_{j-1}^n (1 - u_{j-1}^n)^{\alpha-1} V_{j-1}^n) + \frac{\lambda}{2} u_{j-1}^n (1 - u_{j-1}^n)^{\alpha-1} V_{j-1}^n \\ &\leq \frac{u_{j+1}^n}{2}\big(1 + \lambda u_{j+1}^n \big(1 - u_{j+1}^n\big)^{\alpha-1} V_{j+1}^n\big) - \frac{\lambda}{2} \big(u_{j+1}^n\big)^2 \big(1 - u_{j+1}^n\big)^{\alpha-1} V_{j+1}^n \\ &\quad + \frac{u_{j-1}^n}{2}\,(1 - \lambda u_{j-1}^n (1 - u_{j-1}^n)^{\alpha-1} V_{j-1}^n) + \frac{\lambda}{2} u_{j-1}^n (1 - u_{j-1}^n)^{\alpha-1} V_{j-1}^n \\ &= \frac{u_{j+1}^n}{2} + u_{j-1}^n \left(\frac{1}{2} - \frac{\lambda}{2} u_{j-1}^n (1 - u_{j-1}^n)^{\alpha-1} V_{j-1}^n\right) + \frac{\lambda}{2} u_{j-1}^n (1 - u_{j-1}^n)^{\alpha-1} V_{j-1}^n. \end{aligned}$$

Because of the CFL condition, the last right-hand side is a convex combination of u_{j+1}^n, u_{j-1}^n and one. We therefore conclude that $u_j^{n+1} \leq 1$. The other inequality, $u_j^{n+1} \geq 0$ provided that $u_j^n \geq 0$ for all $j \in \mathbb{Z}$, follows in both cases $\alpha = 0$ and $\alpha \geq 1$ from the CFL condition. □

Remark 5.1. *Lemma 5.2 represents the most important estimate of this paper. Based on the discussion of the (local) effective PDE* (1.10) *we argued in Section 1.3 that one should expect an "invariant region" principle, namely that solutions assume values in* $[0, 1]$*, to hold for* (1.1), (1.2) *with* $\alpha \geq 1$*. The estimate* (5.4) *shows that this property indeed holds. This is an exceptional feature, since an invariant region principle does not hold for dispersive equations in general, and is not valid for* (1.1) *with* $\alpha = 0$*. In fact, from* (5.6) *we deduce that for* $\alpha = 0$*, one can guarantee that the model* (1.1), (1.2) *produces physically relevant results only if* $\|u_0\|_\infty$ *and the final time* T *are sufficiently small. The requirement of smallness for* $\|u_0\|_\infty$ *is consistent with the observation that the model development in Section 2.1 for* $\alpha = 0$ *is rigorously valid for dilute suspensions only.*

Since $u_j \geq 0$, we readily obtain the following corollary.

Corollary 5.1. *Under Assumption 3.1, the numerical solution generated by* (5.1) *in the case* $\alpha \geq 1$ *satisfies*

$$\|U^n\|_1 \leq \|U_0\|_1 \quad \text{for } 0 \leq n \leq N.$$

With the help of Lemma 3.2 we may prove the following uniform bound of total variation of the numerical approximation generated by (5.1).

Lemma 5.3. *The numerical approximation generated by* (5.1) *satisfies the following total variation bound, where* C_5 *does not depend on* Δ*:*

$$\sum_{j \in \mathbb{Z}} \left|u_j^n - u_{j-1}^n\right| \leq C_5 \quad \text{for } 0 \leq n \leq N.$$

Proof. Defining $w_{j-1/2}^n := u_j^n - u_{j-1}^n$ we get from the marching formula (5.1)

$$\begin{aligned} w_{j-1/2}^{n+1} = w_{j+1/2}^n &\left(\frac{1}{2} - \frac{\lambda}{2} f'\big(\xi_{j+1/2}^n\big) V_{j+1}^n\right) + w_{j-3/2}^n \left(\frac{1}{2} + \frac{\lambda}{2} f'\big(\xi_{j-3/2}^n\big) V_{j-1}^n\right) \\ &- \frac{\lambda}{2}\big(\Delta_+ V_j^n\big)\big(f'\big(\xi_{j-1/2}^n\big) w_{j-1/2}^n + f'\big(\xi_{j-3/2}^n\big) w_{j-3/2}^n\big) \\ &+ \frac{\lambda}{2} f\big(u_{j-2}^n\big)\big(-\Delta^2 V_j^n - \Delta^2 V_{j-1}^n\big), \end{aligned}$$

where $\xi^n_{j-1/2} \in [u^n_j, u^n_{j-1}]$. Using the Taylor theorem we obtain

$$\Delta^2 V^n_j = V'\big(\tilde u^n_{a,j}\big)\Delta^2 \tilde u^n_{a,j} + \frac{1}{2} V''\big(\alpha^n_{j+1/2}\big)\big(\Delta_+ \tilde u^n_{a,j}\big)^2 + \frac{1}{2} V''\big(\alpha^n_{j-1/2}\big)\left(\Delta_- \tilde u^n_{a,j}\right)^2,$$

where

$$\alpha^n_{j+1/2} \in [\tilde u^n_{a,j} \wedge \tilde u^n_{a,j+1}, \tilde u^n_{a,j} \vee \tilde u^n_{a,j+1}]$$

(where we define, as usual, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$). Thus, Lemmas 3.1 and 3.2 imply that

$$\Delta^2 V^n_j = \mathcal{O}(\Delta x^2).$$

Due to the CFL condition and using that $f(0) = 0$, we obtain that there exists a constant $\mathcal{C}_a$ such that

$$\begin{aligned}\big|w^{n+1}_{j-1/2}\big| &\le \big|w^n_{j+1/2}\big| \left(\frac{1}{2} - \frac{\lambda}{2} f'\big(\xi^n_{j+1/2}\big) V^n_{j+1}\right) + \big|w^n_{j-3/2}\big| \left(\frac{1}{2} + \frac{\lambda}{2} f'(\xi^n_{j-3/2}) V^n_{j-1}\right)\\ &\quad + \mathcal{C}_a \Delta t\big(\big|w^n_{j-1/2}\big| + \big|w^n_{j-3/2}\big| + \big|u^n_{j-2}\big|\Delta x\big).\end{aligned}$$

Summing over j and using Lemma 5.1 we find that there exist constants C_6 and C_7, which depend on a but not on Δ, such that

$$\mathrm{TV}(U^{n+1}) \le \mathrm{TV}(U^n)(1 + C_6\Delta t) + C_7\Delta t.$$

Finally, summing over n we obtain

$$\begin{aligned}\mathrm{TV}(U^{n+1}) &\le \mathrm{TV}(U^0)(1 + C_6\Delta t)^{n+1} + C_7\Delta t \sum_{p=0}^{n} (1 + C_6\Delta t)^p\\ &\le \mathrm{TV}(u_0)\exp(C_6 T)\left(1 + \frac{C_7}{C_6}\right).\end{aligned}$$

□

We also need that u^Δ satisfies the uniform L^1-Lipschitz continuity property with respect to time. This follows directly from the previous results.

Lemma 5.4. *The numerical approximation generated by* (5.1) *satisfies the following inequality, where C_8 depends on a, but not on Δ:*

$$\sum_{j\in\mathbb{Z}} \big|u^{n+1}_j - u^n_j\big| \le C_8\lambda \quad \text{for } 0 \le n < N.$$

Proof. Using the marching formula (5.1) we write

$$\begin{aligned}u^{n+1}_j - u^n_j &= \frac{1}{2}\Delta_+ u^n_j - \frac{1}{2}\Delta_- u^n_j - \frac{\lambda}{2}\left(f\big(u^n_{j+1}\big) - f\big(u^n_{j-1}\big)\right) V^n_{j+1}\\ &\quad - \frac{\lambda}{2} f\big(u^n_{j-1}\big)\left(V^n_{j+1} - V^n_{j-1}\right)\\ &= \frac{1}{2}\Delta_+ u^n_j - \frac{1}{2}\Delta_- u^n_j - \frac{\lambda}{2}\big(f'\big(\xi^n_{j+1/2}\big)(u^n_{j+1} - u^n_j)\\ &\quad + f'\big(\xi^n_{j-1/2}\big)\big(u^n_j - u^n_{j-1}\big)V^n_{j+1}\big) - f'(\xi) u^n_{j-1}\Delta_+ V^n_j.\end{aligned}$$

In the last expression we used that $f(0) = 0$. We conclude the proof by appealing to Lemma 5.3 and the fact that $\Delta t = \mathcal{O}(\Delta x)$. □

5.2. **Satisfaction of the entropy condition and existence result.** From Helly's theorem we have that u^Δ converges to a function $u \in L^\infty(\Pi_T) \cap L^1(\Pi_T) \cap BV(\Pi_T)$ as $\Delta \to 0$. It remains to prove that u satisfies the entropy inequality (4.2).

Theorem 5.1. *Assume that Assumptions 3.1 and 3.2 hold. Then the numerical solution generated by* (5.1) *converges to the unique entropy solution of* (1.1), (1.2).

Proof. We define the function

$$G_j^n(u, v, U^n) := \frac{1}{2}\big(u - \lambda f(u) V_{j+1}^n + v + \lambda f(v) V_{j-1}^n\big).$$

We can rewrite the scheme (5.1) as $u_j^{n+1} = G_j^n(u_{j+1}^n, u_{j-1}^n, U^n)$. Under the CFL condition, G_j^n is monotone its first two arguments for all $j \in \mathbb{Z}$, $0 \le n < N$. Using this property and omitting the third argument, which is always U^n, we obtain

$$\begin{aligned} \big|u_j^{n+1} - G_j^n(k,k)\big| &= \big|G_j^n\big(u_{j+1}^n, u_{j-1}^n\big) - G_j^n(k,k)\big| \\ &\le \big|G_j^n\big(u_{j+1}^n \vee k, u_{j-1}^n \vee k\big) - G_j^n\big(u_{j+1}^n \wedge k, u_{j-1}^n \wedge k\big)\big| \\ &= \big|u_j^n - k\big| - \big(\mathcal{G}_{j+}^n - \mathcal{G}_{j-}^n\big), \end{aligned} \tag{5.7}$$

where we define

$$\mathcal{G}_{j\pm}^n := \frac{\lambda}{2}\left[\big(f\big(u_{j\pm1}^n \vee k\big) - f\big(u_{j\pm1}^n \wedge k\big)\big) V_{j\pm1}^n - \frac{1}{\lambda}\Delta_\pm\big(\big|u_j^n - k\big|\big)\right].$$

On the other hand,

$$\begin{aligned} &\left|u_j^{n+1} - k + \frac{\lambda}{2} f(k)\big(V_{j+1}^n - V_{j-1}^n\big)\right| \\ &\ge \big|u_j^{n+1} - k\big| + \operatorname{sgn}\big(u_j^{n+1} - k\big)\frac{\lambda}{2} f(k)\big(V_{j+1}^n - V_{j-1}^n\big). \end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8) we arrive at the "cell entropy inequality"

$$\big|u_j^{n+1} - k\big| - \big|u_j^n - k\big| + \mathcal{G}_{j+}^n - \mathcal{G}_{j-}^n + \operatorname{sgn}\big(u_j^{n+1} - k\big)\frac{\lambda}{2} f(k)\big(V_{j+1}^n - V_{j-1}^n\big) \le 0. \tag{5.9}$$

We now establish convergence to a solution that satisfies (4.2) by a Lax-Wendroff-type argument. Multiplying the j-th inequality in (5.9) by $\int_{I_j} \varphi(x, t_n)\,\mathrm{d}x$, where φ is a non-negative test function, and summing the results over $j \in \mathbb{Z}$ and $0 \le n \le N-1$ we obtain the inequality $E_1 + E_2 + E_3 \le 0$, where we define

$$\begin{aligned} E_1 &:= \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} \big(\big|u_j^{n+1} - k\big| - \big|u_j^n - k\big|\big) \int_{I_j} \varphi(x, t_n)\,\mathrm{d}x, \\ E_2 &:= \frac{\lambda}{2} f(k) \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} \operatorname{sgn}(u_j^{n+1} - k)\big(V_{j+1}^n - V_{j-1}^n\big) \int_{I_j} \varphi(x, t_n)\,\mathrm{d}x, \\ E_3 &:= \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} \big(\mathcal{G}_{j+}^n - \mathcal{G}_{j-}^n\big) \int_{I_j} \varphi(x, t_n)\,\mathrm{d}x. \end{aligned}$$

By a standard summation by parts and using that φ has compact support, we get

$$E_1 = -\Delta t \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} \big|u_j^{n+1} - k\big| \int_{I_j} \frac{\varphi(x, t_{n+1}) - \varphi(x, t_n)}{\Delta t}\,\mathrm{d}x.$$

For E_2, we write $E_2 = E_2^a + E_2^b$ where

$$E_2^a := \frac{\lambda}{2} f(k) \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \bigl(\operatorname{sgn}\bigl(u_j^{n+1}-k\bigr) - \operatorname{sgn}\bigl(u_j^n-k\bigr)\bigr) \bigl(V_{j+1}^n - V_{j-1}^n\bigr) \int_{I_j} \varphi(x,t_n)\,\mathrm{d}x,$$

$$E_2^b := \frac{\lambda}{2} f(k) \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \operatorname{sgn}\bigl(u_j^n-k\bigr)\bigl(V_{j+1}^n - V_{j-1}^n\bigr) \int_{I_j} \varphi(x,t_n)\,\mathrm{d}x.$$

Again summing by parts yields

$$\begin{aligned} E_2^a = &-\frac{\lambda}{2} f(k) \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \operatorname{sgn}\bigl(u_j^{n+1}-k\bigr) \int_{I_j} \varphi(x,t_n)\,\mathrm{d}x \\ &\qquad \times \bigl[V_{j+1}^{n+1} - V_{j-1}^{n+1} - \bigl\{V_{j+1}^n - V_{j-1}^n\bigr\}\bigr] \\ &-\frac{\lambda}{2} f(k) \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \operatorname{sgn}\bigl(u_j^{n+1}-k\bigr)\bigl(V_{j+1}^n - V_{j-1}^n\bigr) \\ &\qquad \times \int_{I_j} \bigl(\varphi(x,t_{n+1}) - \varphi(x,t_n)\bigr)\,\mathrm{d}x. \end{aligned}$$

Lemmas 5.4 and 3.1 and the fact that $\gamma_{i+1}-\gamma_i = \mathcal{O}(\Delta x^2)$, $\gamma_l = \mathcal{O}(\Delta x^2)$ yield

$$V_{j+1}^n - V_{j-1}^n = V'\bigl(\tilde{U}_{a,j}^n\bigr)\bigl(\tilde{u}_{a,j+1}^n - \tilde{u}_{a,j-1}^n\bigr) + \mathcal{O}(\Delta x^2) = \mathcal{O}(\Delta x),$$

and

$$\begin{aligned} &\tilde{u}_{a,j+1}^{n+1} - \tilde{u}_{a,j-1}^{n+1} - \bigl\{\tilde{u}_{a,j+1}^n - \tilde{u}_{a,j-1}^n\bigr\} \\ &= \sum_{i=-l}^{l-1} \bigl(u_{j-i}^{n+1} + u_{j-i-1}^{n+1}\bigr)(\gamma_{i+1}-\gamma_i) + \gamma_l\bigl(u_{j+1+l}^{n+1} + u_{j+l}^{n+1} - u_{j-l}^{n+1} - u_{j-l-1}^{n+1}\bigr) \\ &\quad - \left\{ \sum_{i=-l}^{l-1} \bigl(u_{j-i}^n + u_{j-i-1}^n\bigr)(\gamma_{i+1}-\gamma_i) + \gamma_l\bigl(u_{j+1+l}^n + u_{j+l}^n - u_{j-l}^n - u_{j-l-1}^n\bigr) \right\} \\ &= \sum_{i=-l}^{l-1} \bigl(u_{j-i}^{n+1} - u_{j-i}^n + u_{j-i-1}^{n+1} - u_{j-i-1}^n\bigr)(\gamma_{i+1}-\gamma_i) \\ &\quad + \gamma_l\bigl(u_{j+1+l}^{n+1} + u_{j+l}^{n+1} - u_{j-l}^{n+1} - u_{j-l-1}^{n+1} - \bigl[u_{j+1+l}^n + u_{j+l}^n - u_{j-l}^n - u_{j-l-1}^n\bigr]\bigr) \\ &= \mathcal{O}(\Delta x^2). \end{aligned}$$

Then, we can write

$$\begin{aligned} E_2^a = &-\frac{\lambda}{2} f(k) \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \operatorname{sgn}(u_j^{n+1}-k) \int_{I_j} \varphi(x,t_n)\,\mathrm{d}x \times \\ &\quad \times \Bigl[\bigl(V'\bigl(\tilde{u}_{a,j}^{n+1}\bigr) - V'\bigl(\tilde{u}_{a,j}^n\bigr)\bigr)\bigl(\tilde{u}_{a,j+1}^{n+1} - \tilde{u}_{a,j-1}^{n+1}\bigr) \\ &\qquad + V'\bigl(\tilde{u}_{a,j}^n\bigr)\bigl(\tilde{u}_{a,j+1}^{n+1} - \tilde{u}_{a,j-1}^{n+1} - \tilde{u}_{a,j+1}^n + \tilde{u}_{a,j-1}^n\bigr)\Bigr] + \mathcal{O}(\Delta x). \end{aligned}$$

Noting that $\gamma_i = \mathcal{O}(\Delta x)$ we have

$$\tilde{u}_{a,j}^{n+1} - \tilde{u}_{a,j}^n = \sum_{i=-l}^{l} \gamma_i\bigl(u_{j-i}^{n+1} - u_{j-i}^n\bigr) = \mathcal{O}(\Delta x),$$

and we conclude that $E_2^a = \mathcal{O}(\Delta x)$. Analogously, we obtain

$$E_2^b = \Delta t f(k) \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \operatorname{sgn}\bigl(u_j^n - k\bigr) V'\bigl(\tilde{u}_{a,j}^n\bigr) \frac{\tilde{u}_{a,j+1}^n - \tilde{u}_{a,j-1}^n}{2\Delta x} \int_{I_j} \varphi(x,t_n)\,\mathrm{d}x + \mathcal{O}(\Delta x).$$

It remains to analyze E_3. Another summation by parts gives us

$$\begin{aligned}
E_3 = &-\frac{\lambda}{2} \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \Bigl\{ \bigl(f\bigl(u_j^n \vee k\bigr) - f\bigl(u_j^n \wedge k\bigr)\bigr) V_j^n \\
&\quad \times \int_{I_j} \bigl(\varphi(x+\Delta x, t_n) - \varphi(x-\Delta x, t_n)\bigr)\,\mathrm{d}x \Bigr\} \\
&+ \frac{1}{2} \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \bigl|u_j^n - k\bigr| \int_{I_j} \bigl(\varphi(x+\Delta x,t_n) - 2\varphi(x,t_n) + \varphi(x-\Delta x,t_n)\bigr)\,\mathrm{d}x \\
= &-\Delta t \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \Bigl\{ \operatorname{sgn}\bigl(u_j^n - k\bigr)\bigl(f\bigl(u_j^n\bigr) - f(k)\bigr) V_j^n \\
&\quad \times \int_{I_j} \frac{\varphi(x+\Delta x,t_n) - \varphi(x-\Delta x,t_n)}{2\Delta x}\,\mathrm{d}x \Bigr\} + \mathcal{O}(\Delta x).
\end{aligned}$$

To conclude we must show that

$$\begin{aligned}
\mathcal{A} &:= \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{Z}} \left| \tilde{u}_{a,j}^n \Delta x - \int_{I_j} K_a * u(x,t_n)\,\mathrm{d}x \right| \to 0 \quad \text{as } \Delta \to 0, \\
\mathcal{B} &:= \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{Z}} \left| \tilde{u}_{a,j+1}^n - \tilde{u}_{a,j}^n - \int_{I_j} (\partial_x K_a * u)(x,t_n)\,\mathrm{d}x \right| \to 0 \quad \text{as } \Delta \to 0.
\end{aligned}$$

First, we proceed for $\mathcal{A}$. Using the definitions of $\tilde{u}_{a,j}^n$ and γ, we find that

$$\begin{aligned}
\mathcal{A} &= \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{Z}} \left| \sum_{i=-l}^{l} \int_{I_i} K_a(y) u_{j-i}^n \Delta x\,\mathrm{d}y - \int_{I_j} \sum_{i=-l}^{l} \int_{I_i} K_a(y) u(x-y,t_n)\,\mathrm{d}y\,\mathrm{d}x \right| \\
&= \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{Z}} \left| \sum_{i=-l}^{l} \int_{I_i} \int_{I_j} K_a(y) u_{j-i}^n\,\mathrm{d}x\,\mathrm{d}y - \sum_{i=-l}^{l} \int_{I_i} \int_{I_j} K_a(y) u(x-y,t_n)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&= \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{Z}} \left| \sum_{i=-l}^{l} \int_{I_i} \int_{I_j} K_a(y) \bigl(u_{j-i}^n - u(x-y,t_n)\bigr)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\le \Delta t \sum_{n=1}^{N-1} \sum_{i=-l}^{l} \int_{I_i} K_a(y) \sum_{j\in\mathbb{Z}} \int_{I_j} \bigl|u_{j-i}^n - u(x-y,t_n)\bigr|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Using the convergence of u^Δ and the bound of K_a we get the result. Now, we continue with $\mathcal{B}$. Proceeding as above, we find that

$$\begin{aligned}
\mathcal{B} = \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{Z}} \Biggl| &\sum_{i=-l}^{l} \int_{I_i} K_a(y) \bigl(u_{j+1-i}^n - u_{j-i}^n\bigr)\,\mathrm{d}y \\
&- \int_{I_j} \sum_{i=-l}^{l} \int_{I_i} \partial_y K_a(y) u(x-y)\,\mathrm{d}y\,\mathrm{d}x \Biggr|
\end{aligned}$$

$$
\begin{aligned}
&\leq \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{Z}} \Bigg| \sum_{i=-l+1}^{l-1} \int_{I_i} \big(K_a(y+\Delta x) - K_a(y)\big) u_{j-i}^n \,\mathrm{d}y \\
&\qquad\qquad - \int_{I_j} \sum_{i=-l+1}^{l-1} \int_{I_i} \partial_y K_a(y) u(x-y)\,\mathrm{d}y\,\mathrm{d}x \Bigg| \\
&\quad + \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{Z}} \left| \int_{I_{-l}} K_a(y) u_{j+l+1}^n \,\mathrm{d}y - \int_{I_j} \int_{I_{-l}} \partial_y K_a(y) u(x-y)\,\mathrm{d}y\,\mathrm{d}x \right| \\
&\quad + \Delta t \sum_{n=1}^{N-1} \sum_{j\in\mathbb{Z}} \left| - \int_{I_l} K_a(y) u_{j-l}^n \,\mathrm{d}y - \int_{I_j} \int_{I_l} \partial_y K_a(y) u(x-y)\,\mathrm{d}y\,\mathrm{d}x \right|.
\end{aligned}
$$

The last two terms of the last inequality are $\mathcal{O}(\Delta x)$ since $\partial_x K_a$ is bounded and $K_a(2a) = 0$. Finally, we use a Taylor expansion and the convergence of u^Δ to get the result for almost all $k \in \mathbb{R}$. Proceeding as in Lemmas 4.3 and 4.4 of [25] we may extend the analysis to all $k \in \mathbb{R}$. □

5.3. **An additional regularity result for $\alpha = 0$.**

Lemma 5.5. *Assume that $\alpha = 0$. Then the numerical solution generated by* (5.1) *converges to a Lipschitz continuous function u provided u_0 is also Lipschitz continuous.*

Proof. Defining $w_{j+1/2}^n := (u_{j+1}^n - u_j^n)/\Delta x$ we obtain from (5.1)

$$
\begin{aligned}
w_{j-1/2}^{n+1} &= w_{j+1/2}^n \left(\frac{1}{2} - \frac{\lambda}{2} V_{j+1}^n\right) + w_{j-3/2}^n \left(\frac{1}{2} + \frac{\lambda}{2} V_{j+1}^n\right) - w_{j-1/2}^n \frac{\lambda}{2} \Delta_+ V_j^n \\
&\quad - w_{j-3/2}^n \frac{\lambda}{2} \left(V_{j+1}^n - V_{j-2}^n\right) - u_{j-1}^n \frac{\lambda}{2\Delta x} \left(\Delta^2 V_j^n + \Delta^2 V_{j-1}^n\right).
\end{aligned}
$$

Using the CFL condition we have

$$
\begin{aligned}
\left|w_{j-1/2}^{n+1}\right| &\leq \left|w_{j+1/2}^n\right| \left(\frac{1}{2} - \frac{\lambda}{2} V_{j+1}^n\right) + \left|w_{j-3/2}^n\right| \left(\frac{1}{2} + \frac{\lambda}{2} V_{j+1}^n\right) + \frac{\lambda}{2} \left|w_{j-1/2}^n\right| \left|\Delta_+ V_j^n\right| \\
&\quad + \frac{\lambda}{2} \left|w_{j-3/2}^n\right| \left|V_{j+1}^n - V_{j-2}^n\right| + \left|u_{j-1}^n\right| \frac{\lambda}{2\Delta x} \left|\Delta^2 V_j^n + \Delta^2 V_{j-1}^n\right|.
\end{aligned}
$$

Lemmas 3.1, 3.2 and 5.2 imply that there exist constants C_9 and C_{10} such that

$$
\left|w_{j-1/2}^{n+1}\right| \leq \|W^n\|_\infty (1 + C_9 \Delta t) + C_{10} \Delta t.
$$

Following the same steps as in the proof of Lemma 5.3 we obtain

$$
\left|w_{j-1/2}^{n+1}\right| \leq \|W^0\|_\infty \exp(C_9 T) \left(1 + \frac{C_{10}}{C_9}\right).
$$

To conclude we notice that

$$
\begin{aligned}
w_{j+1/2}^0 &= \frac{u_{j+1}^0 - u_j^0}{\Delta x} = \frac{1}{\Delta x^2} \left(\int_{x_{j+1/2}}^{x_{j+3/2}} u_0(y)\,\mathrm{d}y - \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(y)\,\mathrm{d}y \right) \\
&= \frac{1}{\Delta x^2} \left(\int_{x_{j-1/2}}^{x_{j+1/2}} \big(u_0(y+\Delta x) - u_0(y)\big)\,\mathrm{d}y \right) \leq \|u_0\|_{\mathrm{Lip}}.
\end{aligned}
$$

The next step is to prove an analogous estimate for the discrete time derivative. Using (5.1) we can write

$$\begin{aligned} u_j^{n+1} - u_j^n = & \frac{u_{j+1}^n - u_j^n}{2} - \frac{u_j^n - u_{j-1}^n}{2} - \frac{\lambda}{2} V_{j+1}^n \left(u_{j+1}^n - u_{j-1}^n\right) \\ & - \frac{\lambda}{2} u_{j-1}^n \left(V_{j+1}^n - V_{j-1}^n\right). \end{aligned}$$

Multiplying this by Δt^{-1} and using that $\Delta t = \mathcal{O}(\Delta x)$ we find that there exists a constant C_{11}, which is independent of Δ, such that

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} &= \frac{u_{j+1}^n - u_j^n}{2\Delta t} - \frac{u_j^n - u_{j-1}^n}{2\Delta t} - \frac{V_{j+1}^n}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{u_{j-1}^n}{2\Delta x}\left(V_{j+1}^n - V_{j-1}^n\right) \\ &\leq C_{11}\left(\frac{\left|u_{j+1}^n - u_j^n\right|}{2\Delta x} + \frac{\left|u_j^n - u_{j-1}^n\right|}{2\Delta x}\right) - \frac{V_{j+1}^n}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \\ &\quad - \frac{u_{j-1}^n}{2\Delta x}\left(V_{j+1}^n - V_{j-1}^n\right) \end{aligned}$$

Then, using that u^Δ is Lipschitz continuous respect to the space variable and Lemma 3.1, we get that the solution generated by the numerical method converges to a Lipschitz continuous function. □

Remark 5.2. *Lemma 5.5 is not a surprise since in the simplest case, V constant, the conservation law becomes a linear advection equation, whose solution has a regularity that is the same as that of the initial data. Moreover, the limit function u will be a Lipschitz continuous weak solution of* (1.1), (1.2) *will automatically be an entropy solution, and stability and uniqueness are immediate from Theorem 4.1.*

5.4. **Comparison with the analysis by Zumbrun** [52]**.** The equation studied by Zumbrun, (1.12), is equivalent (up to a coordinate transformation) to (1.1) with $\alpha = 0$ and $V(w) = w$. The local existence of a bounded solution u with bounded spatial derivative u_x (provided that u_0 has corresponding properties) is proved in [52] by a fixed-point argument applied to the transport equation $u_t + (uK_a * v)_x = 0$ with given v. In general, global solutions inherit their regularity from u_0; in particular, if $\mathrm{TV}(u_0)$ is bounded, then (1.12) will have a BV solution u, which is unique following an L^1 argument with a discussion of entropy production terms at isolated discontinuities. In the present work, existence of a solution of (1.1), (1.2) is shown by the convergence of a difference scheme, covering a wider range of cases of α and V. Moreover, our Lemma 5.5 is a rough equivalent of Zumbrun's result concerning the regularity of u in terms of that of u_0. Both the analysis of [52] and ours rely on estimates on u or u^Δ that blow up when $a \to 0$. This holds, in particular, for the L^∞-stability estimates of [52, Sect. 2]. However, as is shown in [52, Sect. 4], an L^2-stability argument can be invoked to prove that smooth solutions of (1.12) converge in L^∞ at an $\mathcal{O}(a^2)$ rate to smooth solutions of $u_t + (u^2)_x = 0$. (Of course, this result holds for smooth u_0 and a sufficiently small final time T.) The proof of this result in [52] depends on the linearity of V, and does not carry over to more general functions V or to $\alpha = 1$.

A detailed discussion is devoted in [52] to the existence of travelling wave solutions to (1.12) and (1.14), that is, of solutions of the form $u(x,t) = \varphi(x - st)$ with $\varphi(\xi) \to \varphi(\pm\infty)$ as $\xi \to \infty$ with either $\varphi(\infty) \neq \varphi(-\infty)$, as for a "viscous shock", or $\varphi(\infty) = \varphi(-\infty)$ corresponding to a solitary wave solution. (The "viscous shock"-type solution is of particular interest in the context of the sedimentation model, since it corresponds to the evolution of the suspension-supernate interface.) Roughly speaking, the result of [52] is that neither (1.12) nor (1.14) admit viscous shocks, but that both equations do admit solitary wave solutions. However, as mentioned in Section 1.4, solutions of (1.14) with an additional diffusion term du_{xx}, $d > 0$ with Riemann-like initial data do converge to a stable oscillatory travelling wave of "viscous shock" type. The analysis of travelling waves for (1.1) is outside the scope of this paper, but the numerical results presented in Section 6 suggest that (1.1) for all values of α with a nonlinear function V equally supports oscillatory travelling waves of "viscous shock" type.

6. Numerical Examples

The numerical examples illustrate the qualitative behaviour of the solutions of (1.1), (1.2), with $\alpha = 0$ and $\alpha \geq 1$, and demostrate the convergence properties of the numerical scheme. For the first purpose, we select a relatively fine discretization and present the corresponding numerical solution as profiles at selected times, while the convergence properties of the scheme are illustrated by partly including error histories in some examples.

6.1. **Example 1.** We calculate the numerical solution of (1.1), (1.2) with $\alpha = 0$ for the hindered settling factor (1.3) with $n = 5$, and the kernel K given by (1.13) with $a = 0.2$. We are especially interested in phenomena produced at the suspension-supernate interface of a sedimenting suspension, and therefore employ the following Riemann initial data corresponding to the initial state of this interface for a concentrated and a dilute suspension, respectively:

$$u_0(x) = \begin{cases} 0.0 & \text{for } x \leq 0.2, \\ 0.6 & \text{for } x > 0.2, \end{cases} \quad \text{and} \quad u_0(x) = \begin{cases} 0.0 & \text{for } x \leq 0.2, \\ 0.01 & \text{for } x > 0.2. \end{cases} \tag{6.1}$$

In both cases we use $\Delta x = 0.0005$ and $\lambda = 0.2$. The results are shown in Figures 1 and 2 for the respective cases of an initially concentrated and dilute suspension. As predicted in Section 2.2, we obtain the formation of layers of mass due to the non-constancy of the initial data. We also plot the corresponding solution for the local equation (1.4), which we call the "Kynch solution."

We can conjecture from these simulations, that even though u_0 is not smooth, the presence of the kernel has a regularizating effect since we do not observe the formation of discontinuities. Moreover, we see that the numerical solution is not in $[0, 1]$ for the concentrated suspension accordingly with Lemma 5.2 even though u_0 assumes values from that interval. In Table 1 we show the error at $t_1 = 1$ and $t_2 = 3$ in the L^1 norm for u (denoted by $e^{t_i}_{\mathrm{c/d}}$, $i = 1, 2$) where we take as a reference the solution calculated with $\Delta x = 0.0005$. As expected for the Lax-Friedrichs method, we obtain an experimental order of convergence one. In addition to Table 1 we show in Figure 3 the "graphical" approximation.

6.2. **Example 2.** We study now the behaviour of the numerical solution to (1.1), (1.2) with $\alpha = 1$. We use V as given by (1.3) with $n = 4$ and K given by (1.13) with $a = 0.2$. We again utilize the initial datum (6.1) with $\Delta x = 0.0005$ and $\lambda = 0.2$. The results are plotted in Figures 4 and 5.

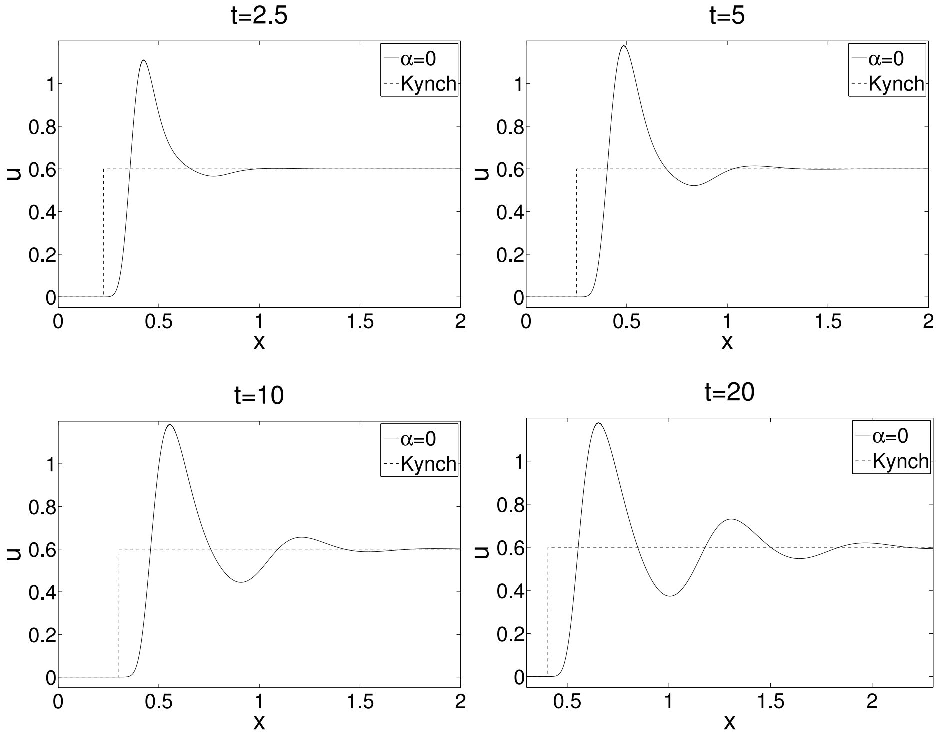

FIGURE 1. Example 1: Numerical solution of (1.1), (1.2) with $\alpha = 0$ and $a = 0.2$ for the hindered settling factor (1.3) with $n = 5$, for an initially concentrated suspension at $t = 2.5$, 5, 10 and 20.

Δx	$e_{\mathrm{c}}^{t_1}$	conv. rate	$e_{\mathrm{c}}^{t_2}$	conv. rate	$e_{\mathrm{d}}^{t_1}$	conv. rate	$e_{\mathrm{d}}^{t_2}$	conv. rate
1.00E-2	8.62E-3	-	1.33E-2	-	1.29E-4	-	2.25E-4	-
5.00E-3	6.24E-3	0.47	1.07E-2	0.31	8.34E-5	0.63	1.53E-4	0.56
4.00E-3	5.45E-3	0.60	9.07E-3	0.46	7.10E-5	0.72	1.33E-4	0.61
2.00E-3	3.26E-3	0.74	6.37E-3	0.61	3.91E-5	0.86	7.89E-5	0.75
1.25E-3	1.99E-4	1.05	4.11E-3	0.93	2.27E-5	1.16	4.73E-5	1.09

TABLE 1. Example 1: Numerical error for u at $t_1 = 1$ and $t_2 = 3$.

We observe the presence of layers but of smaller amplitude than those observed in Example 6.1. We explain this by the different flux function. We also observe more pronounced gradients in the solution, which is in agreement with results proved in Section 5. In Table 2 we show the error at $t_1 = 1$ and $t_2 = 3$ in the L^1 norm for u where we take as a reference the solution calculated with $\Delta x = 0.0005$ as in Example 6.1. We again get an experimental order of convergence one. Figure 6 shows the graphical approximation.

6.3. **Example 3.** We now examine how changes in the parameter a affect qualitatively the numerical solution of (1.1), (1.2) for $\alpha = 0$ and $\alpha = 1$. We use (1.3)

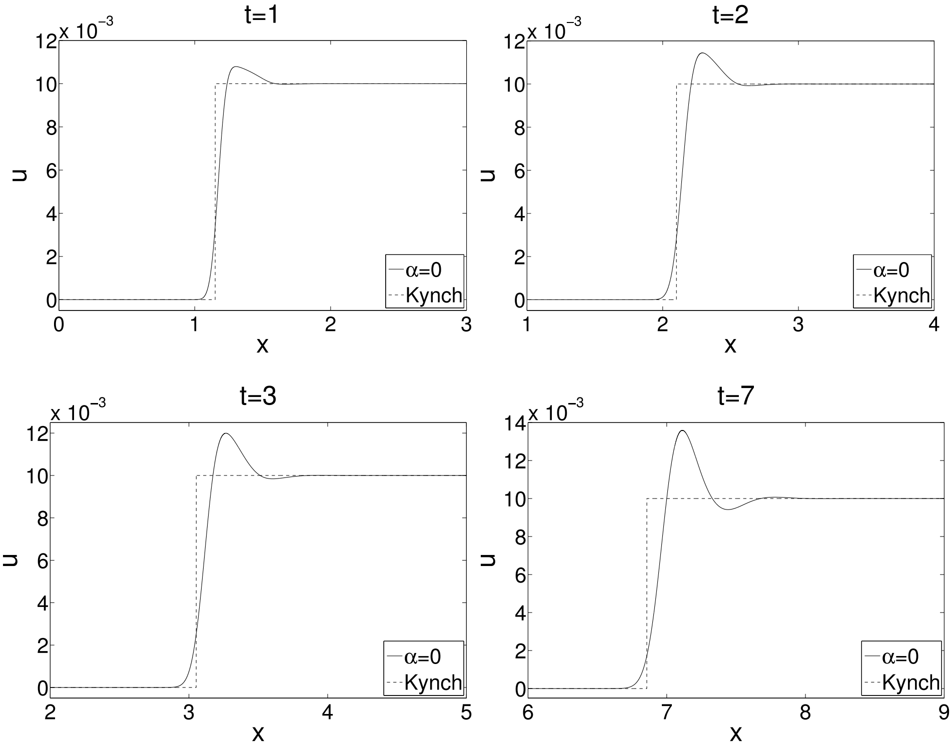

FIGURE 2. Example 1: Numerical solution of (1.1), (1.2) with $\alpha = 0$ and $a = 0.2$ for the hindered settling factor (1.3) with $n = 5$, for an initially dilute suspension at $t = 1$, 2, 3 and 7.

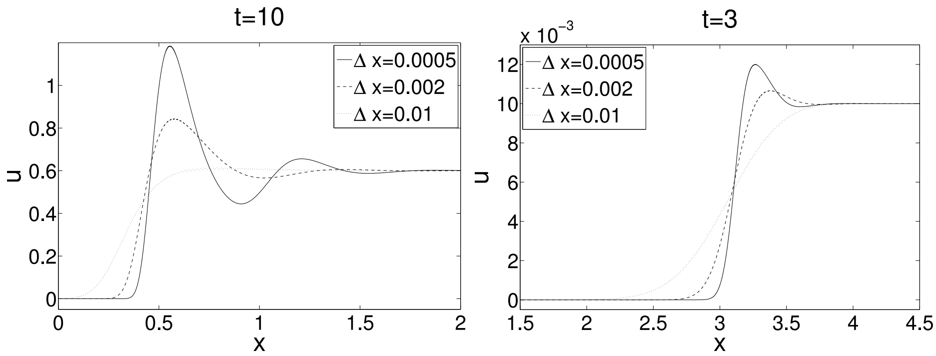

FIGURE 3. Example 1: Numerical solution of (1.1), (1.2) with $\alpha = 0$ and $a = 0.2$ for the hindered settling factor (1.3) with $n = 5$ for $\Delta x = 0.01$, $\Delta x = 0.002$ and $\Delta x = 0.0005$.

with $n = 5$ for $\alpha = 0$ and correspondingly, (1.3) with $n = 4$ for $\alpha = 1$. In both cases, K is given by (1.13) with the parameter $a = 0.4$, 0.2, 0.1 and 0.01. The initial datum is (6.1) for the two cases of a concentrated and a dilute suspension

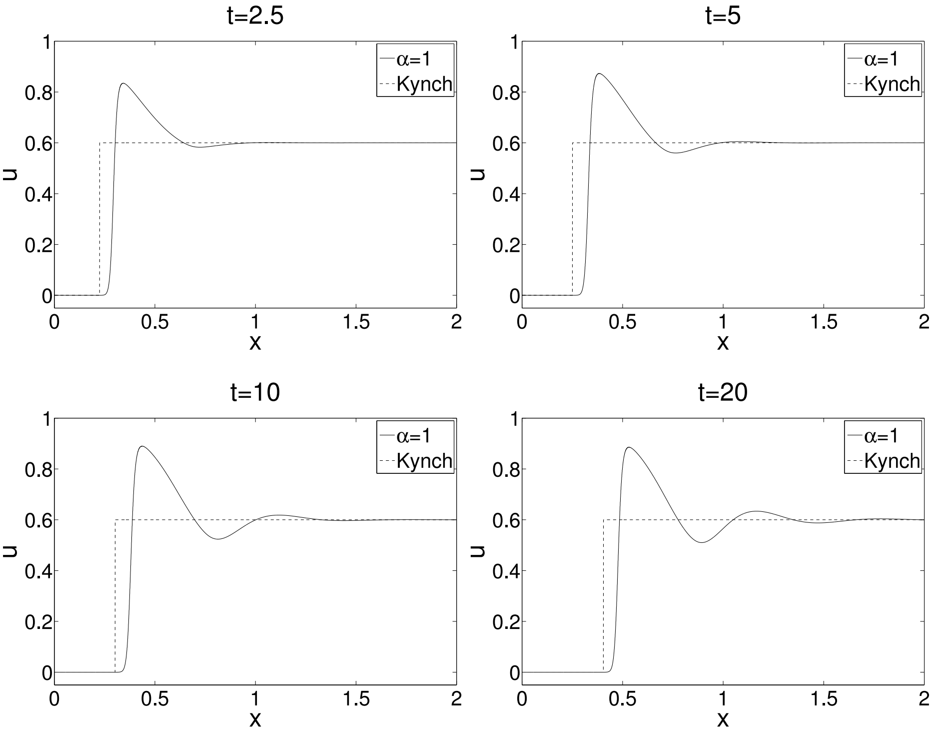

FIGURE 4. Example 2: Numerical solution of (1.1), (1.2) with $\alpha = 1$ and $a = 0.2$ for the hindered settling factor (1.3) with $n = 4$ for an initially concentrated suspension at $t = 2.5$, 5, 10 and 20.

Δx	$e_c^{t_1}$	conv. rate	$e_c^{t_2}$	conv. rate	$e_d^{t_1}$	conv. rate	$e_d^{t_2}$	conv. rate
1.00E-2	7.06E-3	-	9.66E-3	-	1.28E-4	-	2.22E-4	-
5.00E-3	4.67E-3	0.59	6.93E-3	0.48	8.18E-5	0.64	1.46E-4	0.60
4.00E-3	3.95E-3	0.76	5.97E-3	0.67	6.96E-5	0.73	1.27E-4	0.63
2.00E-3	2.08E-3	0.92	3.30E-3	0.86	3.83E-5	0.86	7.43E-5	0.77
1.25E-3	1.15E-3	1.26	1.84E-4	1.24	2.22E-5	1.16	4.43E-5	1.10

TABLE 2. Example 2: Numerical error for u at $t_1 = 1$ and $t_2 = 3$.

with $\Delta x = 0.0005$ and $\lambda = 0.2$. Figure 7 shows the results at $t = 10$ and $t = 7$ in the concentrated and dilute case, respectively.

The case $a = 0.01$ was calculated with $\Delta x = 0.0002$ since if we consider the parameter a "close" to Δx we get the Kynch result because the stencil of the convolution includes just a few points, and the numerical scheme can be viewed as a mollification scheme [1]. We observe a more strongly oscillatory behaviour with $a = 0.2$ and $a = 0.1$, and that the period of the oscillation is proportional to the value of a for both cases. The peak in the case $\alpha = 0$ occurs for $a = 0.4$ and in the case $\alpha = 1$ there is no difference between the peak with $a = 0.2$ and $a = 0.4$. We explain this by the dispersive behaviour of the formulation.

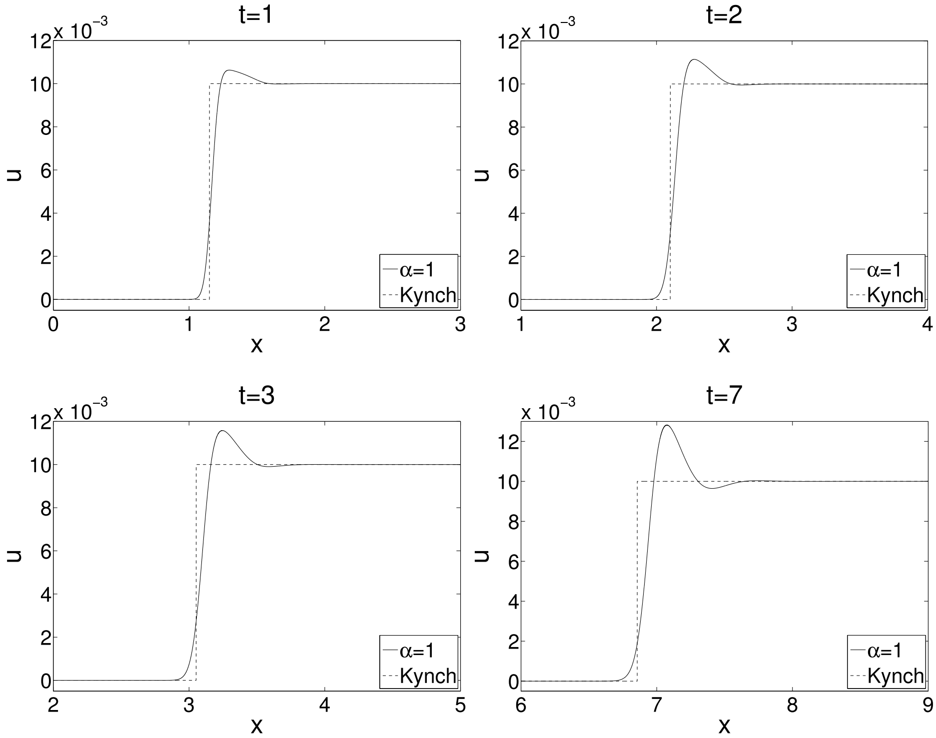

FIGURE 5. Example 2: Numerical solution of (1.1), (1.2) with $\alpha = 1$ and $a = 0.2$ for the hindered settling factor (1.3) with $n = 4$ for an initially dilute suspension at $t = 1$, 2, 3 and 7.

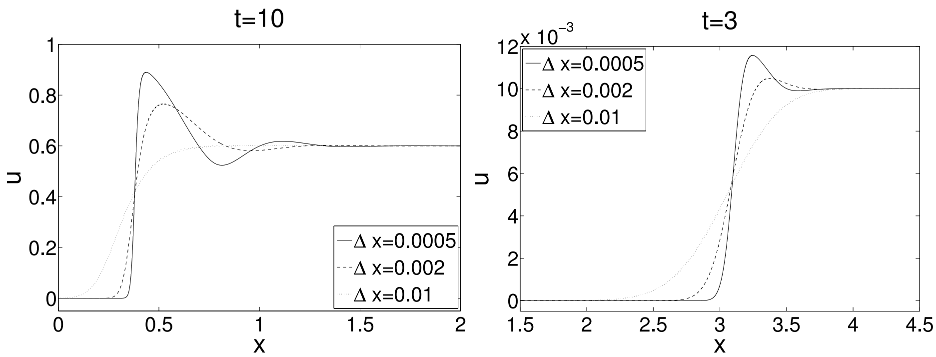

FIGURE 6. Example 2: Numerical solution of (1.1), (1.2) with $\alpha = 1$ and $a = 0.2$ for the hindered settling factor (1.3) with $n = 4$ for an initially concentrated (left) and dilute (right) suspension for $\Delta x = 0.01$, $\Delta x = 0.002$ and $\Delta x = 0.0005$.

6.4. **Example 4.** The idea of the present example is try to reproduce the layered sedimentation observed by Siano [47] in a batch process. The obvious difficulty appears when we are “close” to the boundary since in a batch process we have

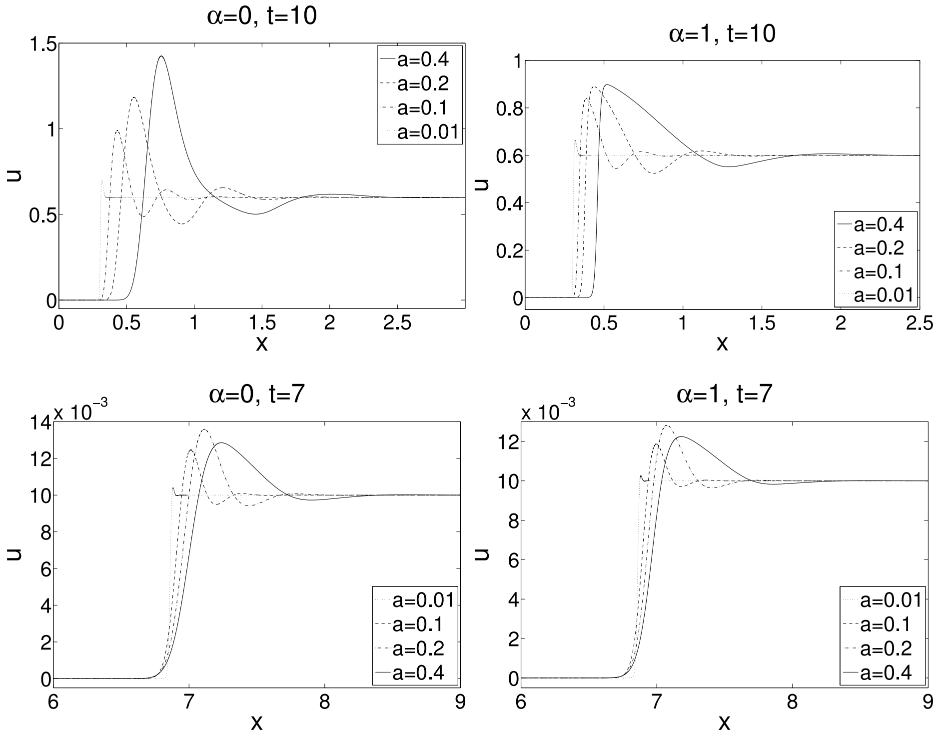

FIGURE 7. Example 3: Numerical solution of (1.1), (1.2) for the indicated values of α with $a = 0.4$, 0.2, 0.1 and 0.01 (top) for an initially concentrated suspension, at $t = 10$ and (bottom) for an initially dilute suspension, at $t = 7$.

a zero flux condition and for the numerical computations we have to extrapolate values in order to compute the numerical fluxes. To solve this problem, we assume that outside the volume control we have initial concentration values, 0 to the left and 1 to the right. In Figures 8 and 9 we show the numerical results for $\alpha = 1$, with $V(u) = (1-u)^4$, K as in (1.13), $a = 0.025$, $\Delta x = 0.00025$, $\lambda = 0.5$ and the respective initial datum for concentrated and dilute suspensions given by

$$u_0(x) = \begin{cases} 0 & \text{for } x < 0, \\ 0.5 & \text{for } 0 \le x < 1, \\ 1 & \text{for } x \ge 1 \end{cases} \quad \text{and} \quad u_0(x) = \begin{cases} 0 & \text{for } x < 0, \\ 0.05 & \text{for } 0 \le x < 1, \\ 1 & \text{for } x \ge 1. \end{cases} \tag{6.2}$$

In each figure we also plot the solution obtained by the local model (Kynch solution). We observe that the layers smooth after a while.

6.5. **Example 5.** In Figures 10–12 we plot the solution for u^Δ for $\alpha = 1$, with $V(u) = (1-u)^4$, K as in (1.13), $a = 0.025$ and $a = 0.5$ and we consider two different initial data. For the first one we take u_0 as in Example 4 and u_0 given by

$$u_0(x) = \begin{cases} 0 & \text{for } x < 0, \\ 0.4 + 0.2x & \text{for } 0 \le x < 1, \\ 1 & \text{for } x \ge 1 \end{cases} \tag{6.3}$$

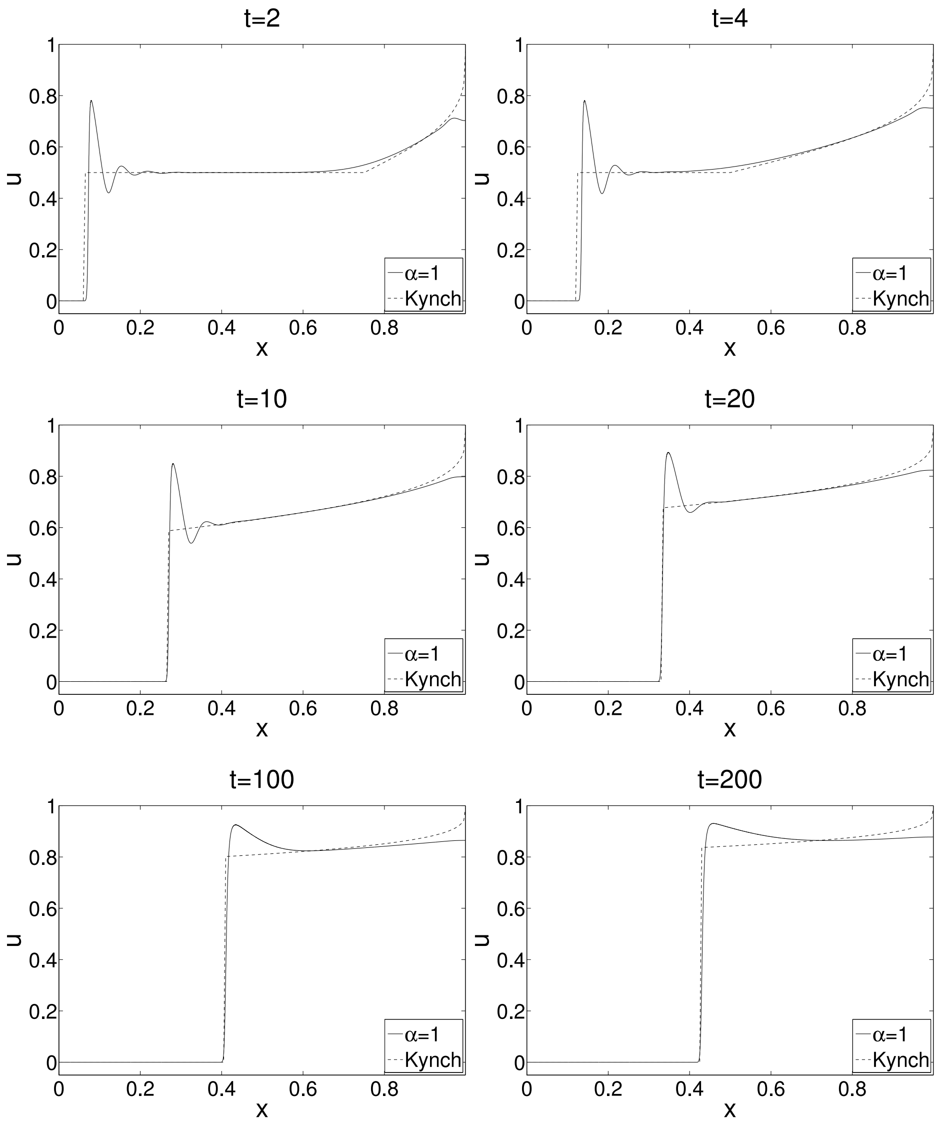

FIGURE 8. Example 4: Numerical solution of (1.1), (1.2) with $\alpha = 1$ for the hindered settling factor (1.3) with $n = 4$ and $a = 0.025$ for an initially concentrated suspension.

for the concentrated case and

$$u_0(x) = \begin{cases} 0 & \text{for } x < 0, \\ 0.04 + 0.02x & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x \geq 1, \end{cases} \tag{6.4}$$

for the dilute case. We also use a nonlinear scale in color in order to highlight the layering phenomenon, which is supposed to appear in the range of concentrations close to the initial concentration. We observe the presence of layers in the case

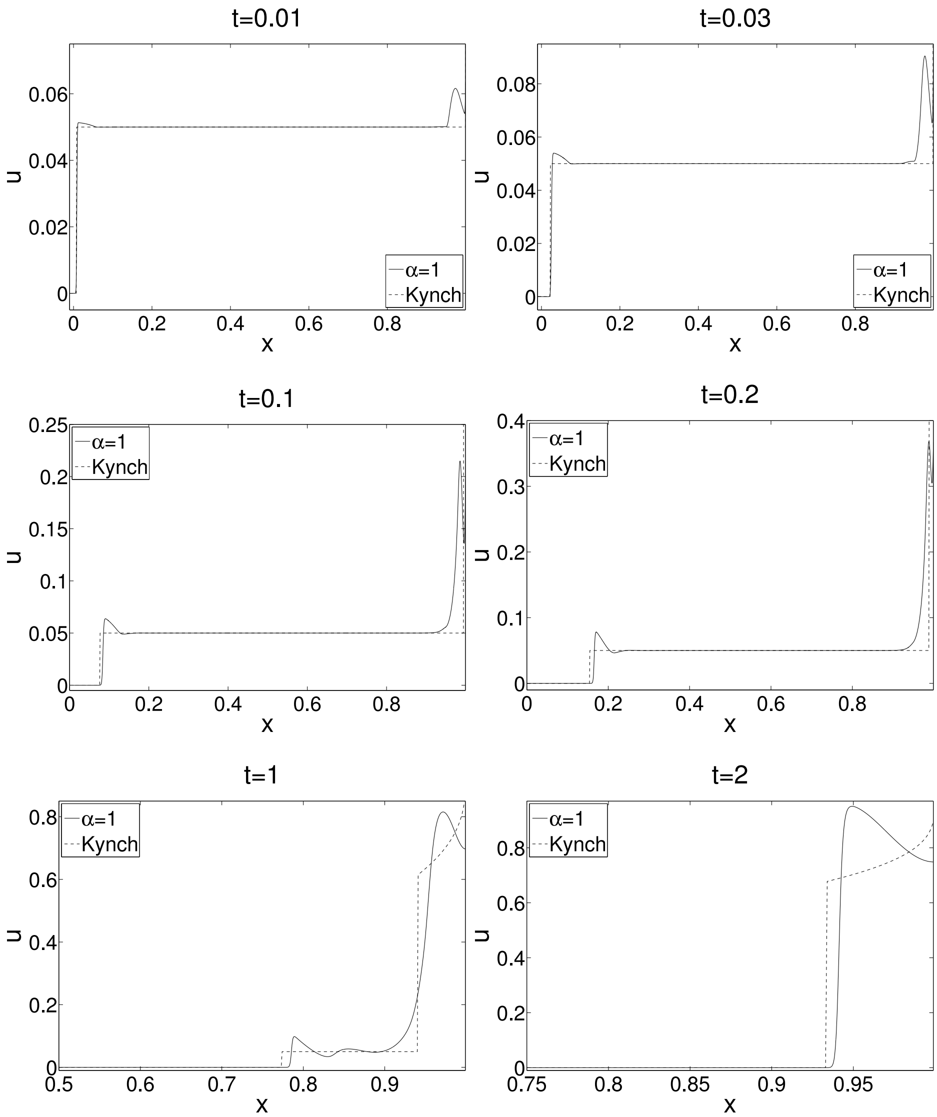

FIGURE 9. Example 4: Numerical solution of (1.1), (1.2) with $\alpha = 1$ for the hindered settling factor (1.3) with $n = 4$ and $a = 0.025$ for an initially dilute suspension.

with u_0 given by the Riemann data (6.2) in a more pronounced form than for the linear initial data (6.3) and (6.4). As we explain in Section 2.2, the presence of layers occurs only if the initial concentration exhibits strong variation, e.g. a jump between zero and a positive constant. We also see, comparing Figures 10 and 12, that the "width" of the layer is proportional to the parameter a.

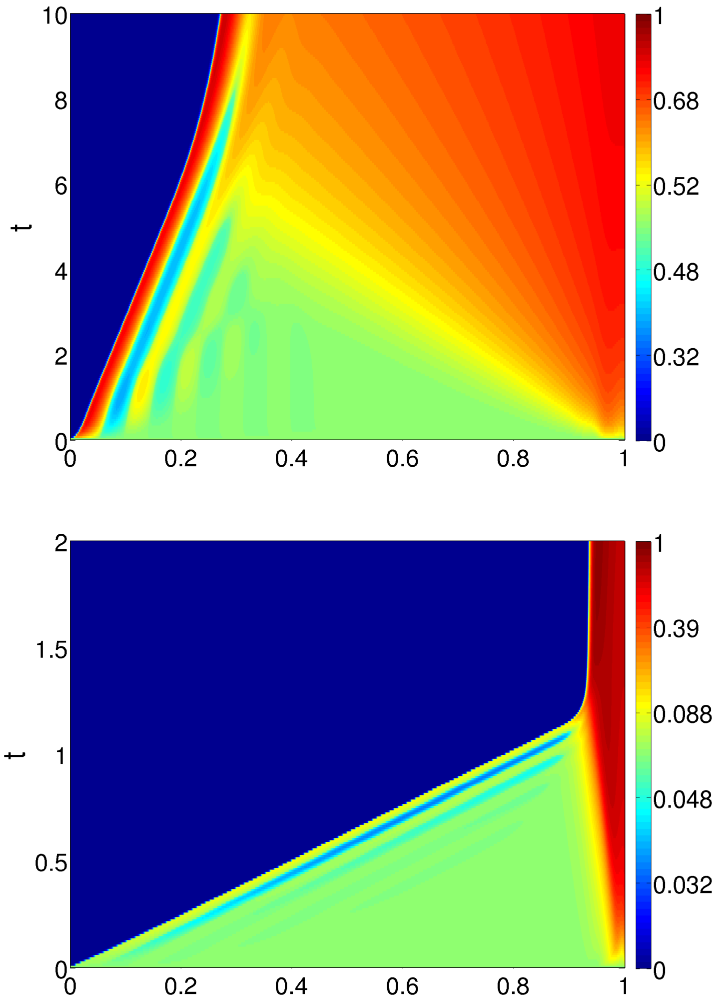

FIGURE 10. Example 5: Numerical solution of (1.1), (1.2) with $\alpha = 1$ for the hindered settling factor (1.3) with $n = 4$ and $a = 0.025$ for an initially concentrated (above) and dilute (below) suspension with u_0 constant.

7. CONCLUSIONS

We study a greater variety of models than the one proposed in [52], which corresponds to $\alpha = 0$ and a linear function V. The model corresponding to $\alpha = 1$ is consistent with (2.4) and (2.5) in the dilute limit $\phi \to 0$, but assumes values in $[0, 1]$ only and therefore can be applied to the whole range of concentrations. The treatment of the boundary conditions can possibly be improved. Our analysis

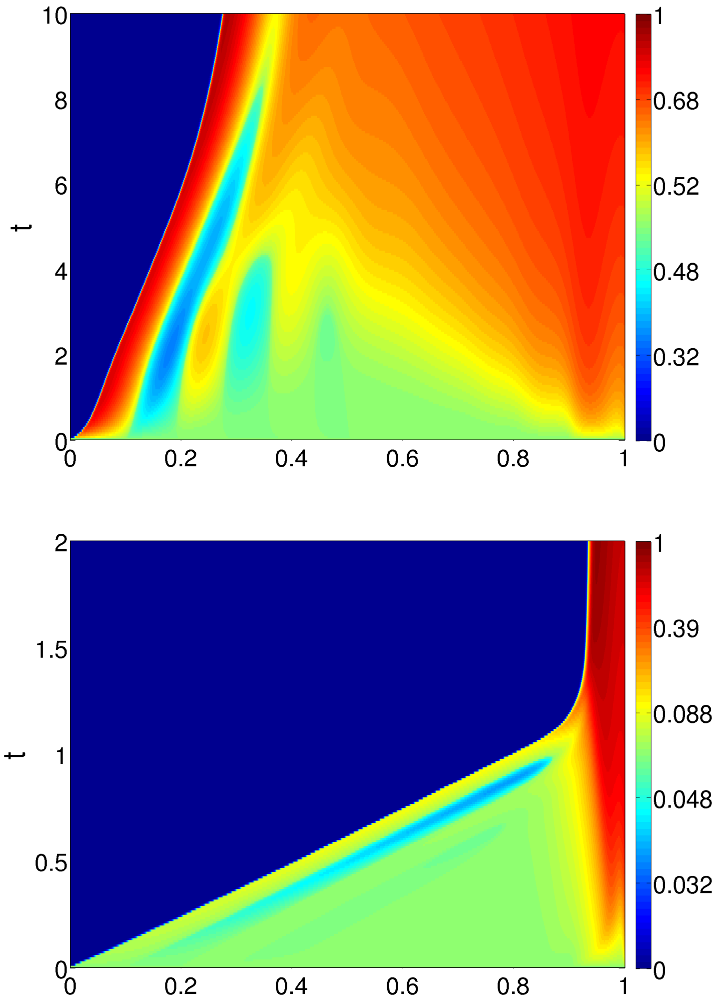

FIGURE 11. Example 5: Numerical solution of (1.1), (1.2) with $\alpha = 1$ for the hindered settling factor (1.3) with $n = 4$ and $a = 0.05$ for an initially concentrated (above) and dilute (below) suspension with u_0 constant.

shows that a reasonably simple difference-quadrature schemes converges to the entropy solution. However, since it is based on the Lax-Friedrichs scheme, high-order versions should be used for practical computations.

We have conducted numerical experiments aiming at assessing whether (1.1) can possibly explain the phenomenon of layering in sedimentation. The numerical experiments, and especially the plots of Figures 10–12, illustrate that (1.1) indeed produces patterns that are similar to layering, namely vertical fluctuations of concentration of $\mathcal{O}(a)$ with beneath the suspension-supernate interface. These oscillatory travelling waves of "viscous shock" type disappear when they start to

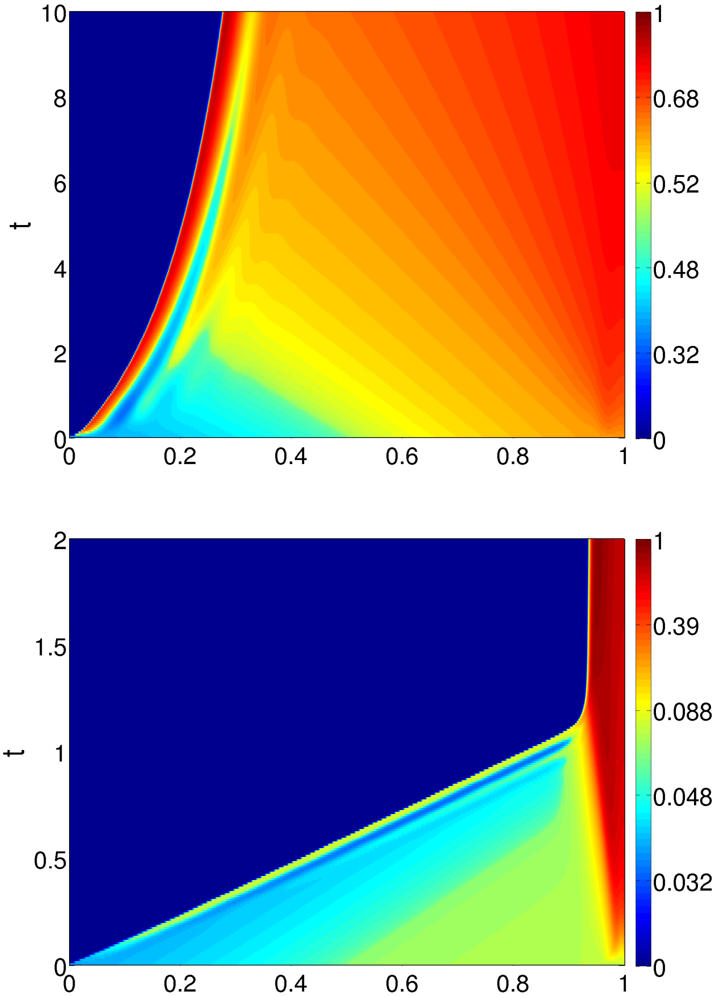

FIGURE 12. Example 5: Numerical solution of (1.1), (1.2) with $\alpha = 1$ for the hindered settling factor (1.3) with $n = 4$ and $a = 0.025$ for an initially concentrated (above) and dilute (below) suspension with a linear initial concentration u_0.

interfere with solution information propagating upwards (in the direction of decreasing x). One should mention, however, that this phenomenon differs from "layering" as observed by Siano [47] in that the solution exhibits oscillations rather than staircasing. As mentioned in [52] it would be interesting to explore further whether (1.1) produces solutions more similar to the staircasing phenomenon if this equation were equipped with additional standard or nonstandard diffusion terms.

Finally, a systematic travelling wave analysis of (1.1), which would extend the results of [52], is still lacking. Such an analysis could explain whether new phenomena, e.g. nonclassical shocks, should be expected when one considers the formal limit $a \to 0$ of entropy solutions of (1.1), especially in the case $\alpha \geq 1$. Unfortunately, most of the constants appearing in the compactness estimates of Section 5.1 are not uniform with respect to a, i.e. they blow up when $a \to 0$. It is therefore not clear at the moment whether a sequence of entropy solutions converges to a meaningful limit as $a \to 0$.

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