Error estimates for an LDG method applied to Signorini type problems

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Abstract

In this paper we propose and analyze a Local Discontinuous Galerkin method for an elliptic variational inequality of the first kind that corresponds to a Poisson equation with Signorini type condition on part of the boundary. The method uses piecewise polynomials of degree one for the field variable and of degree zero or one for the approximation of its gradient. We show optimal convergence for the method and illustrate it with some numerical experiments.

Key words. Local Discontinuous Galerkin, Signorini condition, variational inequality.
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1 Introduction

This paper is concerned with the numerical solution of a model problem that can be written in terms of a variational inequality of the first kind in a closed convex set. In differential form the problem is:

$$\left[\begin{array}{l} -\Delta u = f \qquad \text{in } \Omega, \\ u = g_0 \qquad \text{on } \Gamma_0, \\ u \geq 0, \quad \partial_{\boldsymbol{\nu}} u \geq g, \quad u\,(\partial_{\boldsymbol{\nu}} u - g) = 0 \qquad \text{on } \Gamma. \end{array}\right. \tag{1.1}$$

Here Ω is a polygonal domain in the plane whereas the polygonal curves Γ and Γ_0 form a subdivision of its boundary (see Figure 1 for a sketch of the geometry). This problem is a scalar version of the Signorini problem, that arises when we exchange the Laplace operator by the Navier–Lamé (linear elasticity) operator and all occurrences of the normal derivative by the normal stress. The Signorini problem is an important linear model for elastic behavior with friction conditions on the boundary (or on part of it). For more on the Signorini problem, we refer the reader to the already classical text [17]. Problem

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(1.1) includes a three conditions on the friction boundary Γ: two inequalities and a complementarity condition.

In variational form this problem corresponds to minimizing the functional

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f\, u - \int_\Gamma g\, u$$

over the set

$$K := \{u \in H^1(\Omega)\,:\, u = g_0 \quad \text{on } \Gamma_0, \qquad u \geq 0 \quad \text{on } \Gamma\}.$$

In its turn, this minimization problem can be written as a variational inequality of the first kind on the convex set K:

$$\left[\begin{array}{l} u \in K, \\ \displaystyle\int_\Omega \nabla u \cdot \nabla (v-u) \geq \int_\Omega f\,(v-u) + \int_\Gamma g\,(v-u) \qquad \forall v \in K. \end{array}\right.$$

Elementary results on this very simple type of variational inequalities can be found in [15] or [17]. Finite element approximation of this and more complicated variational inequalities has been the subject of intense research since the very dawn of this popular method. The above mentioned monographs include statements of results on this approximation. An introduction to this subject can be found in [4, Chapter 11], including results that already appeared in original papers as [7]. In addition to the usual inherent difficulties of variational inequalities associated to elliptic problems (such as regularity issues), finite element approximation adds the analytical difficulty of the appearance of a consistency error term due to the fact that the finite element approximation of the convex set K is not a subset of it. Apart from the interest of obtaining fine estimates for this approximation process, its theoretical study is a source of novel mathematical techniques in finite element analysis.

In this work we are going to propose the numerical solution of this problem by one of the many Discontinuous Galerkin (DG in the sequel) methods that appear in the literature. DG methods applied both to steady–state and dynamic problems have attracted the attention of theorists and practitioners (two seminal references are [1] and [16]). The number of different DG methods increased during the decade of the nineties. The need for organizing, classifying and finding common features (as well as differences) was fulfilled, in the case of DG methods applied to elliptic problems, in [3]. Many of the methods studied in that paper shared the possibility of being written both in primal and mixed form. In the primal form only the discretization of the original unknown (u in our case) is shown, whereas the mixed form emphasizes the fact that a numerical approximation of the flux (∇u in our case) is being computed at the same time. Among the advantages of DG methods, the possibility of handling very general partitions (with multiple hanging nodes) and varying the polynomial degree from element to element have been advertised as desirable characteristics for dealing with complicated geometries and using simple local adaptivity procedures.

Among the methods discussed in [3], there is the Local Discontinuous Galerkin (LDG) method, that appeared in its current formulation in [12]. The natural form for this method is the mixed one and the primal form appears here as just a trick for analysis.

The LDG method has grown since in new directions, by adding more possibilities of how numerical fluxes (the main ingredient of DG methods) depend on the different discrete variables, as in [10]. In particular, some of these methods admit hybridization, a very useful computational feature that appeared in [2] for some mixed and non–conforming methods, allowing to reduce computations to the skeleton of the discretization. LDG methods that can be hybridized cannot be however be written in primal form. Since the primal form is at the very heart of our proposal for the method, based on the discretization of the minimization problem, we are not able to deal with this promising feature at the present state of our research. Among some of the advantages of LDG is the fact that it is a naturally mixed method, offering simultaneous approximation of the primal variable and the flux, as well as the simplicity of the treatment of several kinds of non–linearities [8], [9].

In this paper we will present and analyze an LDG approximation of the minimization problem and will explain some practical issues related to implementation. To keep the analysis simple we will restrict ourselves to linear elements and will emphasize how DG methods that can be written in primal form can be applied to variational inequalities.

Notational foreword. To distinguish easily between scalar–valued and vector–valued quantities, we will always use boldface fonts for vectors. The symbols $\lesssim$ and $\approx$ will be used as accustomed in much of the finite element literature. When we write $a \lesssim b$ we mean that there exists a quantity $C > 0$ that does not depend on the discretization parameter h such that $a \le C\, b$. If $a \lesssim b \lesssim a$, we just write $a \approx b$.

2 The LDG gradient

Consider a plane domain Ω, whose boundary is composed of two closed polygonal lines Γ and Γ_0 (Figure 1). For given $f \in L^2(\Omega)$, $g_0 : \Gamma_0 \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$ (the regularity conditions on these two will be given in the sequel), we consider the problem

$$\left[\begin{array}{ll} -\Delta u = f & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma_0, \\ \partial_{\boldsymbol{\nu}} u = g & \text{on } \Gamma, \end{array}\right. \tag{2.1}$$

as well as an LDG discretization of it.

We consider a sequence $\{\mathcal{T}_h\}$ of partitions of Ω, each formed by shape–regular triangles with possible hanging nodes. In principle, to avoid unnecessary difficulties, we consider triangulations that are refinements of conforming triangulations *à la Ciarlet* (see [5]). Bounded variation of the triangulation is also assumed, i.e., sizes of adjacent triangles have to be asymptotically comparable. We consider the broken Sobolev spaces

$$H^1(\mathcal{T}_h) := \prod_{T \in \mathcal{T}_h} H^1(T), \qquad \mathbf{H}^1(\mathcal{T}_h) := \prod_{T \in \mathcal{T}_h} \Big[H^1(T) \Big]^2.$$

Traces from these spaces on the boundaries Γ and Γ_0 are to be understood always element by element without further clarification. The interior skeleton $\mathcal{I}_h^\circ$ is defined to be the union

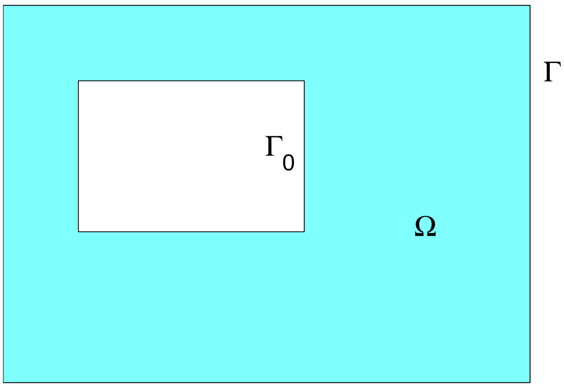

Figure 1: The very simplified model geometry. The Dirichlet boundary Γ_0 is set apart from the friction/Signorini boundary Γ to avoid unnecessary complications.

of all the edges of elements. The trace of functions in $H^1(\mathcal{T}_h)$ and $\mathbf{H}^1(\mathcal{T}_h)$ on the interior skeleton is a double valued function. We are going to use the standard notation of [3] for interelement averages and jumps: if $e = T_1 \cap T_2$ and $\boldsymbol{\nu}_{T_i}$ is the normal vector on e exterior to T_i, then we write

$$[\![u]\!]_e := u_{T_1}\boldsymbol{\nu}_{T_1} + u_{T_2}\boldsymbol{\nu}_{T_2}, \qquad \{u\}_e := \tfrac{1}{2}(u_{T_1} + u_{T_2}),$$
$$[\![\boldsymbol{\tau}]\!]_e := \boldsymbol{\tau}_{T_1}\cdot\boldsymbol{\nu}_{T_1} + \boldsymbol{\tau}_{T_2}\cdot\boldsymbol{\nu}_{T_2}, \qquad \{\boldsymbol{\tau}\}_e := \tfrac{1}{2}(\boldsymbol{\tau}_{T_1} + \boldsymbol{\tau}_{T_2}).$$

With these definitions, it is easy to prove the discrete divergence theorem

$$\int_\Omega \Big(\nabla_h v\cdot\boldsymbol{\tau} + v\,\mathrm{div}_h\boldsymbol{\tau}\Big) = \int_{\Gamma\cup\Gamma_0} v\,(\boldsymbol{\tau}\cdot\boldsymbol{\nu}) + \int_{\mathcal{I}_h^\circ}\Big([\![v]\!]\cdot\{\boldsymbol{\tau}\} + \{v\}[\![\boldsymbol{\tau}]\!]\Big), \tag{2.2}$$

where ∇_h and div_h denote respectively the gradient and divergence operators applied elementwise.

We next consider two bilinear forms depending on traces of functions of the broken Sobolev spaces, $A : H^1(\mathcal{T}_h)\times H^1(\mathcal{T}_h)\to\mathbb{R}$ and $S : H^1(\mathcal{T}_h)\times\mathbf{H}^1(\mathcal{T}_h)\to\mathbb{R}$, given by

$$A(u,v) := \int_{\mathcal{I}_h^\circ}\alpha[\![u]\!]\cdot[\![v]\!] + \int_{\Gamma_0}\alpha\,u\,v$$

and

$$S(u,\boldsymbol{\tau}) := \int_{\mathcal{I}_h^\circ}[\![u]\!]\cdot\big(\{\boldsymbol{\tau}\} - [\![\boldsymbol{\tau}]\!]\boldsymbol{\beta}\big) + \int_{\Gamma_0} u\,(\boldsymbol{\tau}\cdot\boldsymbol{\nu}).$$

They both constitute important elements in the definition of the LDG method. Here α and $\boldsymbol{\beta}$ are constant on each edge and we assume that

$$\alpha_e \approx h_e^{-1}, \qquad |\boldsymbol{\beta}_e| \lesssim 1 \qquad \forall e.$$

Finally we introduce the two discrete spaces for the LDG discretization

$$V_h := \prod_{T\in\mathcal{T}_h}\mathbb{P}_1(T), \qquad \boldsymbol{\Sigma}_h := \prod_{T\in\mathcal{T}_h}\Big[\mathbb{P}_r(T)\Big]^2, \qquad r\in\{0,1\},$$

$\mathbb{P}_r(T)$ being the space of bivariate polynomials of degree not greater than r. The global formulation of the LDG equations is given by

$$\left[\begin{array}{lll} \boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h, \quad u_h \in V_h, & & \\ \displaystyle\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}_h - \Big(\int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau}_h - S(u_h, \boldsymbol{\tau}_h) \Big) & = \displaystyle\int_{\Gamma_0} g_0 \, (\boldsymbol{\tau}_h \cdot \boldsymbol{\nu}) & \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h, \\ \Big(\displaystyle\int_\Omega \nabla_h v_h \cdot \boldsymbol{\sigma}_h - S(v_h, \boldsymbol{\sigma}_h) \Big) + A(u_h, v_h) & = \ell(v_h) + \displaystyle\int_{\Gamma_0} \alpha \, g_0 \, v_h & \forall v_h \in V_h, \end{array}\right. \tag{2.3}$$

with

$$\ell(v) := \int_\Omega f \, v + \int_\Gamma g \, v.$$

Derivation of these global equations from local equations, that can be easily interpreted in terms of local numerical fluxes, can be found in [3] or [8]. We will take them here as a given and proceed onwards.

We now define the operator $\mathbf{S}_h : H^1(\mathcal{T}_h) \to \boldsymbol{\Sigma}_h$ given by the relations

$$\int_\Omega \mathbf{S}_h v \cdot \boldsymbol{\tau}_h = S(v, \boldsymbol{\tau}_h) \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h,$$

the corrected gradient (the LDG gradient) operator

$$\nabla_h^* := \nabla_h - \mathbf{S}_h,$$

and the discrete function $\mathbf{g}_h^0 \in \boldsymbol{\Sigma}_h$ such that

$$\int_\Omega \mathbf{g}_h^0 \cdot \boldsymbol{\tau}_h = \int_{\Gamma_0} g_0 \, (\boldsymbol{\tau}_h \cdot \boldsymbol{\nu}) \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h.$$

Notice that in case it is needed, $\mathbf{g}_h^0$ can be computed on an element–by–element basis and that the support of $\mathbf{g}_h^0$ is limited to the set of triangles having an edge on Γ_0. Note also that for the exact solution of (2.1) we have

$$\nabla_h^* u = \nabla u - \mathbf{g}_h^0. \tag{2.4}$$

With these notations, the first equation in (2.3) can be written as

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}_h - \int_\Omega \nabla_h^* u_h \cdot \boldsymbol{\tau}_h = \int_\Omega \mathbf{g}_h^0 \cdot \boldsymbol{\tau}_h \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h.$$

Since $\nabla_h : V_h \to \boldsymbol{\Sigma}_h$, this equation is equivalent to the identity

$$\boldsymbol{\sigma}_h = \nabla_h^* u_h + \mathbf{g}_h^0.$$

Substituting in the second equation of (2.3), we obtain

$$a_h(u_h, v_h) = \ell_h(v_h) \qquad \forall v_h \in V_h, \tag{2.5}$$

where

$$a_h(u,v) := \int_\Omega \nabla_h^* u \cdot \nabla_h^* v + A(u,v)$$

and

$$\ell_h(v) := \ell(v) + \ell_h^0(v), \qquad \ell_h^0(v) := \int_{\Gamma_0} \alpha\, g_0\, v - \int_\Omega \mathbf{g}_h^0 \cdot \nabla_h^* v. \tag{2.6}$$

Notice that (2.5) is equivalent to the minimization problem

$$\tfrac{1}{2} a_h(u_h,u_h) - \ell_h(u_h) = \min!, \qquad u_h \in V_h. \tag{2.7}$$

Here and in the sequel we will use Zeidler's shorthand writing for minimization problems (see [20] for example): problem (2.7) is to be read as the *minimization* (in the sense of finding the argument where the minimum takes place) *of the functional* $\frac{1}{2}a(u_h,u_h) - \ell_h(u_h)$ *subject to the condition* $u_h \in V_h$.

Some analytical properties concerning elements that have appeared here will be given in Section 4

3 Variational problem and its LDG discretization

Consider the set

$$K := \Big\{ u \in H^1(\Omega) \;:\; u = g_0 \quad \text{on } \Gamma_0, \qquad u \geq 0 \quad \text{on } \Gamma \Big\}$$

and the bilinear form

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v.$$

The minimization problem

$$\tfrac{1}{2} a(u,u) - \ell(u) = \min!, \qquad u \in K \tag{3.1}$$

is equivalent (see [4] or [11]) to the variational inequality

$$\left[\begin{array}{l} u \in K, \\ a(u, v-u) \geq \ell(v-u), \qquad \forall v \in K. \end{array}\right. \tag{3.2}$$

By elementary results on quadratic minimization on convex sets of Hilbert spaces these problems are uniquely solvable. Note that they are weak forms of (1.1). Consider now the discrete set

$$K_h := \{ u_h \in V_h \;:\; u_h \geq 0 \quad \text{on } \Gamma \}$$

and note that we have dropped the Dirichlet condition on Γ_0 in the definition. Motivated by the LDG discretization of the boundary value problem (2.7), we consider the discrete minimization problem

$$\tfrac{1}{2} a_h(u_h,u_h) - \ell_h(u_h) = \min!, \qquad u_h \in K_h. \tag{3.3}$$

This problem is equivalent to

$$\left[\begin{array}{l} u_h \in K_h, \\ a_h(u_h, v_h - u_h) \geq \ell_h(v_h - u_h) \qquad \forall v_h \in K_h, \end{array}\right. \tag{3.4}$$

and to (K_h is a closed conic set with vertex on the origin)

$$\left[\begin{array}{l} u_h \in K_h, \\ a_h(u_h, u_h) = \ell_h(u_h), \\ a_h(u_h, v_h) \geq \ell_h(v_h) \qquad \forall v_h \in K_h. \end{array}\right. \tag{3.5}$$

Before carrying out the analysis of the method let us detail how the functional and its gradient can be effectively computed. Everything reduces to the ability to evaluate $a_h(v_h, w_h)$ and $\ell_h^0(v_h)$ for arbitrary $v_h, w_h \in V_h$. In fact, in ℓ_h most terms can be handled in a simple way, and we only have to take care of how to compute

$$\int_\Omega \mathbf{g}_h^0 \cdot \nabla_h^* v_h.$$

Let us choose a basis $\{\varphi_i\}$ for V_h and another one $\{\boldsymbol{\sigma}_i\}$ of $\boldsymbol{\Sigma}_h$, constructed by joining bases for the spaces on each element. Consider then the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ whose elements are

$$a_{ij} := \int_\Omega \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_i, \qquad b_{ij} := \int_\Omega \nabla_h \varphi_j \cdot \boldsymbol{\sigma}_i - S(\varphi_j, \boldsymbol{\sigma}_i), \qquad c_{ij} := A(\varphi_j, \varphi_i)$$

and the vector $\mathbf{d}$ with elements

$$d_i := \int_{\Gamma_0} g_0(\boldsymbol{\sigma}_i \cdot \boldsymbol{\nu}).$$

Note that $\mathbf{A}$ and $\mathbf{C}$ are square matrices of different size and $\mathbf{A}$ is block–diagonal (with a block for each triangle).

Proposition 1 *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be the respective vectors of coefficients of* v_h *and* w_h

$$v_h = \sum_i v_i \varphi_i, \qquad w_h = \sum_i w_i \varphi_i.$$

Then

$$a_h(v_h, w_h) = \mathbf{w}^\top \Big(\mathbf{C} + \mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B}\Big) \mathbf{v}, \qquad \int_\Omega \mathbf{g}_h^0 \cdot \nabla_h^* v_h = \mathbf{v}^\top \mathbf{B}^\top \mathbf{A}^{-1} \mathbf{d}.$$

Proof. Given $v_h \in V_h$, the corrected gradient $\nabla_h^* v_h$ is characterized by the equations

$$\left[\begin{array}{l} \nabla_h^* v_h \in \boldsymbol{\Sigma}_h, \\ \displaystyle\int_\Omega \nabla_h^* v_h \cdot \boldsymbol{\tau}_h = \int_\Omega \nabla_h v_h \cdot \boldsymbol{\tau}_h - S(v_h, \boldsymbol{\tau}_h) \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h. \end{array}\right.$$

If we write

$$\nabla_h^* v_h = \sum_j s_j \boldsymbol{\sigma}_j,$$

then the vector of coefficients, say $\mathbf{s}$, is the solution to the equations $\mathbf{A}\mathbf{s} = \mathbf{B}\mathbf{v}$. Hence

$$\int_\Omega \nabla_h^* v_h \cdot \nabla_h^* w_h = \Big(\mathbf{A}^{-1}\mathbf{B}\mathbf{w}\Big)^\top \mathbf{A}\Big(\mathbf{A}^{-1}\mathbf{B}\mathbf{v}\Big)$$

and the first identity follows readily. The second one is a straightforward consequence of the formula we have given for $\mathbf{s}$. □

Note that this result is just a matrix formalization of how the LDG equations are set into primal form by using the Schur complement of

$$\begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{bmatrix},$$

which is just the global matrix corresponding to the system (2.3).

4 Some technical properties

With respect to the discrete norm

$$|\!|\!| u |\!|\!|_h := \Big[\int_\Omega |\nabla_h u|^2 + A(u,u) \Big]^{1/2}, \tag{4.1}$$

the LDG–corrected gradient is uniformly bounded

$$\|\nabla_h^* u\|_{0,\Omega} \lesssim |\!|\!| u |\!|\!|_h \qquad \forall u \in H^1(\mathcal{T}_h). \tag{4.2}$$

The proof of this can be found in [14], [13] or [8] or, with a somewhat different language, in [3]. Hence the bilinear form a_h is $h-$uniformly bounded (this is just part of [13, Lemma 3.2] or [18, Proposition 3.1])

$$|a_h(u,v)| \lesssim |\!|\!| u |\!|\!|_h |\!|\!| v |\!|\!|_h \qquad \forall u, v \in H^1(\mathcal{T}_h). \tag{4.3}$$

It is also $h-$uniformly elliptic

$$|\!|\!| u |\!|\!|_h^2 \lesssim a_h(u,u) \qquad \forall u \in H^1(\mathcal{T}_h), \tag{4.4}$$

as follows from arguments that are well known in the literature of DG methods (see the same references as for (4.2)). This fact proves unique solvability of (2.5) and of (3.4).

We next recall two useful bounds that appear often in the DG and FEM literature. The first one is the local version of the trace theorem that can be obtained by a scaling argument from the reference element (see [1] for instance):

$$h^{1/2}\|v\|_{0,e} \lesssim \|v\|_{0,T} + h|v|_{1,T}, \qquad e \in \mathcal{E}(T), \qquad v \in H^1(T). \tag{4.5}$$

In this inequality h is any of h_T or h_e, which are assumed to be asymptotically equivalent and $\mathcal{E}(T)$ is the set of edges of T (which might be consisting of more than three edges if there are hanging nodes). The second property concerns a bound of norms and appears in several different formulations in different parts of the literature. We will state it here as a Proposition and show how it can be proven by adapting a result from [19].

Proposition 2

$$\|v_h\|_{0,\Omega} \lesssim |||v_h|||_h \qquad \forall\, v_h \in V_h. \tag{4.6}$$

Proof. We will show how to prove this result for the more general space

$$W_h := \prod_{T\in\mathcal{T}_h} \mathbb{P}_k(T),$$

with any $k \geq 0$ fixed. First of all we construct a new shape regular triangulation $\mathcal{T}_{\underline{h}}$, which is a refinement of $\mathcal{T}_h$, has no hanging nodes and is such that sizes of edges of the new triangulation that proceed from subdividing an edge of the old one are comparable. This can be done because of the requirements done on our initial triangulation at the beginning of Section 3.

On the skeleton of $\mathcal{T}_{\underline{h}}$ we consider a new piecewise constant positive function $\alpha_{\underline{h}}$ (with the same behavior as α) and with that we define the norm $|||\cdot|||_{\underline{h}}$. Note that $W_h \subset W_{\underline{h}}$ and that

$$|||v_h|||_{\underline{h}} \approx |||v_h|||_h \qquad \forall v_h \in W_h, \tag{4.7}$$

since there are no jumps of v_h in newly created edges and $\alpha_{\underline{h}} \approx \alpha$ on $\mathcal{I}_h^\circ \cup \Gamma_0$. We now consider the space $\mathbf{V}_{\underline{h}}$ of Raviart–Thomas elements of order k on the triangulation $\mathcal{T}_{\underline{h}}$ with homogeneous boundary conditions on Γ, i.e., functions $\mathbf{v}_{\underline{h}} \in \mathbf{H}(\mathrm{div},\Omega)$ such that

$$\mathbf{v}_{\underline{h}}|_T \in [\mathbb{P}_k(T)]^2 + \mathbb{P}_k(T)\, m, \qquad \forall T \in \mathcal{T}_{\underline{h}}, \qquad m(\mathbf{x}) := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and that $\mathbf{v}_{\underline{h}} \cdot \boldsymbol{\nu} = 0$ on Γ. Note that

$$\|v_{\underline{h}}\|_{0,\Omega} \lesssim \sup_{\mathbf{0}\neq\mathbf{v}_{\underline{h}}\in\mathbf{V}_{\underline{h}}} \frac{\left|\displaystyle\int_\Omega v_{\underline{h}}\,(\mathrm{div}\,\mathbf{v}_{\underline{h}})\right|}{\|\mathbf{v}_{\underline{h}}\|_{\mathrm{div},\Omega}} \qquad \forall v_{\underline{h}} \in W_{\underline{h}}, \tag{4.8}$$

by a well–known result on Raviart–Thomas elements (see [6] for instance).

With these elements in hand we can go to the statement and proof of [19, Theorem 3.1] and notice that we can do the following modifications (we refer to notations of [19]): (a) the factor h^{-1} in front of each integral on edges can be substituted by the function α inside the integral; (b) the triangulation is not needed to be quasiuniform anymore (this is a requirement at the beginning of Section 2 in [19]) since we use the local mesh–sizes; (c) the definition of the discrete operator B^* can be adapted for the space $\mathbf{V}_{\underline{h}}$ that includes homogeneous boundary conditions and therefore edges of the triangulation lying on Γ can be ignored in the definition of the bilinear form $\mathcal{A}$. These simple adaptations allow us to use [19, Theorem 3.1] and (4.8) to prove that

$$\|v_{\underline{h}}\|_{0,\Omega} \lesssim |||v_{\underline{h}}|||_{\underline{h}} \qquad \forall v_{\underline{h}} \in W_{\underline{h}},$$

which coupled with (4.7) proves the result. Note again that the adaptations are necessary since the original result is stated for quasi–uniform grids without hanging nodes and including all edges in the discrete norm (we do not have the edges on Γ in it). $\square$

Proposition 3

$$|\ell_h^0(v_h)| \lesssim h^{-1/2} \|g_0\|_{0,\Gamma_0} |\!|\!| v_h |\!|\!|_h \qquad \forall\, v_h \in V_h.$$

Proof. Since $\alpha_e \approx h_e^{-1}$, it follows readily that

$$\Big| \int_{\Gamma_0} \alpha\, g_0\, v_h \Big| \lesssim \Big(\int_{\Gamma_0} \alpha g_0^2 \Big)^{1/2} |\!|\!| v_h |\!|\!|_h \lesssim h^{-1/2} |\!|\!| v_h |\!|\!|_h \|g_0\|_{0,\Gamma_0}.$$

On the other hand, using a scaling argument on the triangles that have edges on Γ_0, it is simple to see that

$$\|\mathbf{u}_h\|_{0,\Gamma_0} \lesssim h^{-1/2} \|\mathbf{u}_h\|_{0,\Omega} \qquad \forall \mathbf{u}_h \in \mathbf{\Sigma}_h. \tag{4.9}$$

Using the definition of $\mathbf{g}_h^0 \in \mathbf{\Sigma}_h$, it follows that

$$\|\mathbf{g}_h^0\|_{0,\Omega}^2 = \int_{\Gamma_0} g_0\, (\mathbf{g}_h^0 \cdot \boldsymbol{\nu}) \lesssim \|g_0\|_{0,\Gamma_0} \|\mathbf{g}_h^0\|_{0,\Gamma_0},$$

which together with (4.9) proves that

$$\|\mathbf{g}_h^0\|_{0,\Omega} \lesssim h^{-1/2} \|g_0\|_{0,\Gamma_0}.$$

We now go back to the remaining term of the definition of ℓ_h^0

$$\Big| \int_{\Omega} \mathbf{g}_h^0 \cdot \nabla_h^* v_h \Big| \lesssim \|\mathbf{g}_h^0\|_{0,\Omega} \|\nabla_h^* v_h\|_{0,\Omega} \lesssim h^{-1/2} \|g_0\|_{0,\Gamma_0} |\!|\!| v_h |\!|\!|_h,$$

where in the last inequality we have used (4.6). This finishes the proof. $\square$

Proposition 4

$$|\ell(v_h)| \lesssim h^{-1/2} |\!|\!| v_h |\!|\!|_h \Big(\|f\|_{0,\Omega} + \|g\|_{0,\Gamma} \Big) \qquad \forall v_h \in V_h.$$

Proof. The term involving f is easily bounded by $\|f\|_{0,\Omega} |\!|\!| v_h |\!|\!|_h$ using (4.6). On the other hand, applying (4.5) on each edge e on the boundary Γ, it is simple to prove that

$$\|v_h\|_{0,\Gamma} \lesssim h^{-1} \|v_h\|_{0,\Omega}^2 + h \|\nabla_h v_h\|_{0,\Omega}^2 \lesssim h^{-1} |\!|\!| v_h |\!|\!|_h,$$

where we have used (4.6) again. Hence

$$\Big| \int_{\Gamma} g\, v_h \Big| \le \|g\|_{0,\Gamma} \|v_h\|_{0,\Gamma} \lesssim h^{-1/2} \|g\|_{0,\Gamma} |\!|\!| v_h |\!|\!|_h.$$

Note that we can also bound using first (4.5) and then (4.6)

$$\int_{\Gamma} |v_h| \le \Big(\sum_{e \subset \Gamma} h_e \int_e |v_h|^2 \Big)^{1/2} \lesssim |\!|\!| v_h |\!|\!|_h, \tag{4.10}$$

from which we can obtain the bound

$$\Big| \int_{\Gamma} g\, v_h \Big| \lesssim \|g\|_{L^\infty(\Gamma)} |\!|\!| v_h |\!|\!|_h,$$

assuming more regularity for g. The inequality given in the statement will be enough for our purposes. $\square$

Lemma 5 *Let $\eta_h \in \prod_{e\in\mathcal{E}_h^\Gamma} \mathbb{P}_1(e)$ be such that $\eta_h \geq 0$ (here $\mathcal{E}_h^\Gamma$ is the set of edges of the triangulation that are contained in Γ). Then there exists $\rho \in H^1(\Gamma)$ such that $\rho \geq 0$ and*

$$\int_\Gamma |\eta_h - \rho| \leq 2h^3 \int_\Gamma \eta_h.$$

Proof. Beginning in one node of the partition of Γ and using the arc parameterization from that point, we can identify functions on Γ with functions on $[0, L]$, where $L :=$ length (Γ). Let t_i be the points of this interval corresponding to the nodes of the partition. We decompose

$$\eta_h = \sum_i \eta_i$$

where $\operatorname{supp} \eta_i \subseteq [t_{i-1}, t_{i+1}]$, $\eta_i(t_j^\pm) = 0$ for all $j \neq i$ and $\eta_i(t_i^\pm) = \eta_h(t_i^\pm)$. Assume that $\eta_i(t_i^+) > \eta_i(t_i^-)$ as in Figure 2 (in the other case the construction is similar but moving to the left) and take the point $\tau_i := t_i + h_i^4$, where $h_i := t_{i+1} - t_i$ (we assume $h_i < 1$ henceforth).

Now we define a function ρ_i such that $\rho_i \equiv 0$ outside $[t_{i-1}, t_{i+1}]$, $\rho_i \equiv \eta_i$ in $[t_{i-1}, t_i] \cup [\tau_i, t_{i+1}]$ and in the remaining interval $[t_i, \tau_i]$, ρ_i is linear. Figure 2 shows the construction of this function. It is clear that $0 \leq \rho_i \leq \eta_i$. Notice that

$$\int_{t_{i-1}}^{t_{i+1}} (\eta_i - \rho_i) = \int_{t_i}^{\tau_i} (\eta_i - \rho_i) \leq h_i^4 \eta_i(t_i^+) \leq 2h_i^3 \int_{t_i}^{t_{i+1}} \eta_i \leq 2h^3 \int_0^L \eta_i,$$

where the first inequality is simply seen in Figure 2: we have overestimated the small triangle separating both graphs by a rectangle with base of length h_i^4 and height $\eta_i(t_i^+)$. We then define (see Figure 3 for the final aspect of this function) $\rho := \sum_i \rho_i$, which is a Lipschitz continuous function and satisfies $0 \leq \rho \leq \eta_h$. Moreover

$$0 \leq \int_0^L |\eta_h - \rho| = \int_0^L (\eta_h - \rho) = \sum_i \int_{t_{i-1}}^{t_{i+1}} (\eta_i - \rho_i) \leq 2h^3 \sum_i \int_0^L \eta_i = 2h^3 \int_0^L \eta_h$$

and the proof is thus finished. □

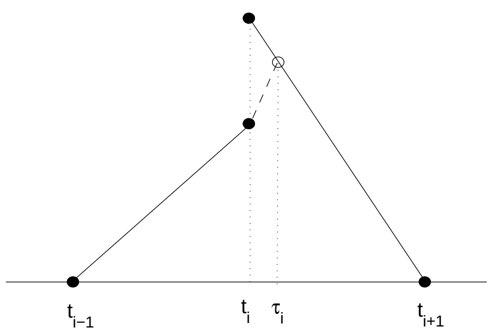

Figure 2: The function η_i in solid line and the construction of ρ_i (dashed line) by joining the lower limit of η_i on t_i with a point in the upwards direction.

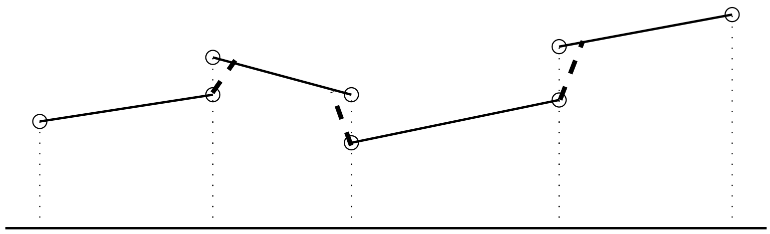

Figure 3: The result of the construction in Lemma 5: a continuous polygonal with roughly the same area. The difference is a very small multiple of the area.

Proposition 6 *Let u_h be the solution to* (3.4). *Then*

$$\inf_{v\in K}\int_\Gamma |u_h - v| \lesssim h^{5/2}\Big(\|f\|_{0,\Omega} + \|g_0\|_{0,\Gamma_0} + \|g\|_{0,\Gamma}\Big).$$

Proof. Take $\rho : \Gamma \to \mathbb{R}$ as in Lemma 5 and construct $v \in H^1(\Omega)$ such that $v = 0$ on Γ_0 and $v = \rho$ on Γ. Then, by (4.10)

$$\int_\Gamma |u_h - v| = \int_\Gamma |u_h - \rho| \le 2h^3 \int_\Gamma u_h \lesssim h^3 \|\!|u_h|\!\|_h.$$

We now can use the ellipticity property (4.4) as well as the characterization (3.5) for u_h to obtain

$$\|\!|u_h|\!\|_h^2 \lesssim a_h(u_h, u_h) = \ell_h(u_h) = \ell(u_h) - \ell_h^0(u_h).$$

Using Propositions 3 and 4 and the above bounds the result follows readily. □

5 Analysis

The residual of the discrete variational inequality when applied to the exact solution and tested with a general $w \in H^1(\mathcal{T}_h)$ is

$$R_h(w) := a_h(u, w) - \ell_h(w).$$

Proposition 7 *For arbitrary $v_h \in K_h$ and $v \in K$*

$$\|\!|u - u_h|\!\|_h^2 \lesssim \|\!|u - u_h|\!\|_h \|\!|u - v_h|\!\|_h + R_h(v_h - u_h) + R_h(v - u).$$

Proof. First we use ellipticity (4.4) to obtain

$$\|\!|u - u_h|\!\|_h^2 \lesssim a_h(u - u_h, u - u_h) = a_h(u - u_h, u - v_h) + a_h(u - u_h, v_h - u_h). \qquad (5.1)$$

For all $w \in H^1(\Omega)$ such that $w|_{\Gamma_0} = 0$ it follows that $\nabla_h^* w = \nabla w$ and therefore

$$\ell_h^0(w) = -\int_\Omega \mathbf{g}_h^0 \cdot \nabla w.$$

Because of (2.4), if w is as before, then

$$a_h(u, w) = \int_\Omega (\nabla u - \mathbf{g}_h^*) \cdot \nabla w = a(u, w) + \ell_h^0(w). \tag{5.2}$$

Applying this to $w = v - u$ with $v \in K$, we prove that the solution to (3.2) satisfies

$$a_h(u, v-u) \geq \ell(v-u) + \ell_h^0(v-u) = \ell_h(v-u) \qquad \forall v \in K.$$

This inequality together with the definition of u_h as the solution to (3.4) prove that for arbitrary $v_h \in K_h$ and $v \in K$,

$$\begin{aligned} a_h(u - u_h, v_h - u_h) &= a_h(u, v_h - u + v - u_h) - a_h(u, v-u) - a_h(u_h, v_h - u_h) \\ &\leq a_h(u, v_h - u + v - u_h) - \ell_h(v-u) - \ell_h(v_h - u_h) \\ &= R_h(v_h - u_h) + R_h(v - u). \end{aligned}$$

The result follows now from this inequality, (5.1) and the uniform boundedness of the discrete bilinear form (4.3). □

Proposition 8 *For all $w \in H^1(\Omega)$ such that $w|_{\Gamma_0} = 0$,*

$$R_h(w) = \int_\Gamma (\partial_{\boldsymbol{\nu}} u - g)\, w.$$

Proof. Because of (5.2) it follows that $R_h(w) = a(u, w) - \ell(w)$ and therefore the result is a simple application of Green's formula and the fact that $-\Delta u = f$. □

To shorten some forthcoming expressions, we will write $\boldsymbol{\xi}_h \in \boldsymbol{\Sigma}_h$ for the best $L^2(\Omega)$ approximation of ∇u in $\boldsymbol{\Sigma}_h$, i.e.,

$$\left[\begin{array}{l} \boldsymbol{\xi}_h \in \boldsymbol{\Sigma}_h, \\ \displaystyle\int_\Omega \boldsymbol{\xi}_h \cdot \boldsymbol{\tau}_h = \int_\Omega \nabla u \cdot \boldsymbol{\tau}_h, \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h. \end{array}\right.$$

Proposition 9 *If $u \in H^{3/2+\varepsilon}(\Omega)$, for some $\varepsilon > 0$, then*

$$R_h(w_h) = S(w_h, \nabla u - \boldsymbol{\xi}_h) + \int_\Gamma (\partial_{\boldsymbol{\nu}} u - g)\, w_h \qquad \forall w_h \in V_h.$$

Proof. Using (2.4), it is simple to see that

$$\begin{aligned} a_h(u, w_h) &= \int_\Omega (\nabla u - \mathbf{g}_h^0) \cdot \nabla_h^* w_h + \int_{\Gamma_0} \alpha\, g_0\, w_h \\ &= \int_\Omega \nabla u \cdot \nabla_h w_h + \ell_h^0(w_h) - \int_\Omega \nabla u \cdot \mathbf{S}_h w_h, \end{aligned}$$

and therefore, using the definition of $\boldsymbol{\xi}_h$ and of the discrete operator $\mathbf{S}_h$, we obtain

$$R_h(w_h) = \int_\Omega \nabla u \cdot \nabla_h w_h - \ell(w_h) - S(w_h, \boldsymbol{\xi}_h). \tag{5.3}$$

We then apply the discrete divergence theorem (identity (2.2) with $\boldsymbol{\tau} = \nabla u$ and $v = w_h$) to obtain

$$\int_\Omega \nabla u \cdot \nabla_h w_h - \int_\Omega f\, w_h = \int_{\mathcal{I}_h^\circ} \nabla u \cdot [\![w_h]\!] + \int_{\Gamma\cup\Gamma_0} (\partial_{\boldsymbol{\nu}} u)\, w_h = S(w_h, \nabla u) + \int_\Gamma (\partial_{\boldsymbol{\nu}} u) w_h,$$

which substituted into (5.3) proves the statement. □

For our convergence estimate, that proves that the method has optimal order of convergence (order one, since we are dealing with $H^1(\Omega)$ norms and $\mathbb{P}_1$ elements), we assume additional regularity requirements. These ones are sufficient to get full order of convergence, although they will not be satisfied by most solutions for which there will be a reduced order of convergence. Note that the freedom given by the triangulation to introduce hanging nodes makes the method able to deal with simple adaptive strategies. This will be the aim of future work.

Theorem 10 *Assume that $u \in H^{3/2+\varepsilon}(\Omega) \cap H^2(\mathcal{T}_h)$, for some $\varepsilon > 0$, and that $\partial_{\boldsymbol{\nu}} u - g \in L^\infty(\Gamma)$. Then*

$$|\!|\!| u - u_h |\!|\!|_h \lesssim h\Big(|u|_{2,\mathcal{T}_h} + \|f\|_{0,\Omega} + \|g_0\|_{0,\Gamma_0} + \|g\|_{0,\Gamma} + \|\partial_{\boldsymbol{\nu}} u - g\|_{L^\infty(\Gamma)} \Big).$$

Proof. Using Propositions 7, 8 and 9, we prove that

$$\begin{aligned} |\!|\!| u - u_h |\!|\!|_h^2 \; &\lesssim \; |\!|\!| u - u_h |\!|\!|_h |\!|\!| u - v_h |\!|\!|_h + S(v_h - u_h, \nabla u - \boldsymbol{\xi}_h) \\ &\quad + \int_\Gamma (\partial_{\boldsymbol{\nu}} u - g)(v_h - u) + \int_\Gamma (\partial_{\boldsymbol{\nu}} u - g)(v - u_h) \qquad \forall v_h \in K_h, \quad \forall v \in K. \end{aligned}$$

Using some standard bounds on L^2 projections (see [14, Section3] for a proof in this same language), it is possible to prove that

$$|S(v_h - u_h, \nabla u - \boldsymbol{\xi}_h)| \lesssim h |u|_{2,\mathcal{T}_h} |\!|\!| v_h - u_h |\!|\!|_h \le h |u|_{2,\mathcal{T}_h} \Big(|\!|\!| u - v_h |\!|\!|_h + |\!|\!| u - u_h |\!|\!|_h \Big).$$

Taking v as in the proof of Proposition 6, we easily obtain

$$\Big| \int_\Gamma (\partial_{\boldsymbol{\nu}} u - g)\,(v - u_h) \Big| \lesssim h^{5/2} \|\partial_{\boldsymbol{\nu}} u - g\|_{L^\infty(\Gamma)} D_1,$$

with $D_1 := \|f\|_{0,\Omega} + \|g_0\|_{0,\Gamma_0} + \|g\|_{0,\Gamma}$. Hence, using a weighted Cauchy–Schwarz inequality we can eliminate all occurrences of $|\!|\!| u - u_h |\!|\!|_h$ in the right–hand side and obtain (with $D_2 := \|\partial_{\boldsymbol{\nu}} u - g\|_{L^\infty(\Gamma)}$)

$$|\!|\!| u - u_h |\!|\!|_h^2 \lesssim |\!|\!| u - v_h |\!|\!|_h^2 + h^2 |u|_{2,\mathcal{T}_h}^2 + D_2 \left(D_1 h^{5/2} + \int_\Gamma |u - v_h| \right), \qquad \forall v_h \in K_h.$$

The result is proved once we show that

$$\inf_{v_h \in K_h} \left(h \|\!| u - v_h \|\!|_h + \int_\Gamma |u - v_h| \right) \lesssim h^2 |u|_{2,\mathcal{T}_h},$$

regrouping terms afterwards to get the desired inequality. This can be done using an element–by–element interpolant. In this way the requirement of $v_h \geq 0$ on Γ is naturally satisfied. Terms involving jumps in the discrete norm (there will be jumps where hanging nodes occur) and the $L^1(\Gamma)$ norm are moved to interior estimates using (4.5). Finally, we apply well–known results on approximation by piecewise polynomials to obtain the result. $\square$

6 Numerical examples

In this section we show the performance of the method, considering the $\mathbb{P}_1 - [\mathbb{P}_0]^2$ and $\mathbb{P}_1 - [\mathbb{P}_1]^2$ approximation. The code has been written in Matlab, making use of the `quadprog` routine for quadratic minimization with linear constraints. In the three examples, we will take $\Omega := \Omega_1 \setminus \overline{\Omega}_0$, where $\Omega_1 := (0,2) \times (0,1)$ and $\Omega_0 := (1/2, 3/2) \times (1/4, 3/4)$. We also take $\Gamma_0 := \partial\Omega_0$, while $\Gamma := \partial\Omega \backslash \Gamma_0 = \partial\Omega_1$ is divided into four parts:

$$\begin{aligned}
\Gamma_1 &:= \{(x,0) \,:\, 0 \leq x \leq 2\}\,, \\
\Gamma_2 &:= \{(2,y) \,:\, 0 \leq y \leq 1\}\,, \\
\Gamma_3 &:= \{(x,1) \,:\, 0 \leq x \leq 2\}\,, \\
\Gamma_4 &:= \{(0,y) \,:\, 0 \leq y \leq 1\}.
\end{aligned}$$

We measure the errors

$$\boldsymbol{e}_h(u) := \|\!| u - u_h \|\!|_h, \quad \boldsymbol{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega},$$

$$\boldsymbol{e}_0(u) := \|u - u_h\|_{0,\Omega}, \quad \boldsymbol{e}_T := \left(\boldsymbol{e}_h(u)^2 + \boldsymbol{e}_0(\boldsymbol{\sigma})^2 \right)^{1/2}.$$

Finally $r_h(u)$, $r_0(\boldsymbol{\sigma})$, $r_0(u)$ and r_T denote the corresponding experimental rates of convergence for the given error measures.

Example #1. We take $u(x,y) = xy$, and consider

$$g(x,y) := \begin{cases} -20 & \text{on } \Gamma_1\,, \\ y & \text{on } \Gamma_2\,, \\ x & \text{on } \Gamma_3\,, \\ -20 & \text{on } \Gamma_4\,. \end{cases}$$

Note that $\partial_{\boldsymbol{\nu}} u = g$ on $\Gamma_2 \cup \Gamma_3$ and $u = 0$ on $\Gamma_1 \cup \Gamma_4$, so the transition points between the two complementary conditions are corners of the domain that are always nodes of the grid. The numerical results are described in Table 1 and Table 2. As predicted by our theory, the method displays order one of convergence. Moreover, in $L^2(\Omega)$, we observe superconvergence for the primal variable u.

dof	$\boldsymbol{e}_h(u)$	$r_h(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_T$	r_T
80	0.8542	——	0.2688	——	0.0942	——	0.8955	——
320	0.3923	1.1226	0.1332	1.0136	0.0198	2.2506	0.4143	1.1121
1280	0.1442	1.4435	0.0665	1.0022	0.0035	2.4929	0.1588	1.3832
5120	0.0566	1.3494	0.0333	0.9982	0.0007	2.3914	0.0657	1.2742
20480	0.0238	1.2496	0.0167	0.9983	0.0001	2.2766	0.0291	1.1762
81920	0.0106	1.1608	0.0083	0.9989	3.058E-5	2.1777	0.0135	1.1035

Table 1: Numerical results for example 1 ($\mathbb{P}_1 - [\mathbb{P}_0]^2$ approximation)

dof	$\boldsymbol{e}_h(u)$	$r_h(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_T$	r_T
144	0.4003	——	0.1776	——	0.0213	——	0.4379	——
576	0.1628	1.2978	0.0873	1.0248	0.0047	2.1942	0.1847	1.2452
2304	0.0801	1.0233	0.0408	1.0970	0.0011	2.0637	0.0899	1.0391
9216	0.0396	1.0150	0.0196	1.0594	0.0003	2.0349	0.0442	1.0239
36864	0.0197	1.0081	0.0096	1.0335	0.0001	2.0110	0.0219	1.0130

Table 2: Numerical results for example 1 ($\mathbb{P}_1 - [\mathbb{P}_1]^2$ approximation)

Example #2. Here we first introduce a truncation function

$$\psi(z) := \begin{cases} 0 & z < 1/2\,, \\ 16(z - \frac{1}{2})^2(7 - 8z) & 1/2 \leq z \leq 3/4\,, \\ 1 & 3/4 \leq z\,. \end{cases}$$

Then we consider the data of the problem (except for g) such that the exact solution is $u(x, y) = \psi(x^2 + y^2)$. Since $u(x, u) = 0$ for $x^2 + y^2 \leq 1/2$, it is convenient to decompose the sides Γ_1 and Γ_4 in two pieces each:

$$\Gamma_{1a} := \{(x, 0) \,:\, 0 \leq x \leq 1/\sqrt{2}\}\,, \quad \Gamma_{1b} := \Gamma_1 \backslash \Gamma_{1a}\,,$$

and similarly $\Gamma_4 = \Gamma_{4a} \cup \Gamma_{4b}$. We then define

$$g := \begin{cases} \partial_{\boldsymbol{\nu}} u & \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_{1b} \cup \Gamma_{4b}\,, \\ -24 & \text{otherwise}\,. \end{cases}$$

The constant -24 is chosen to be a strict lower bound of the normal derivative, so that the contact points are $(0, 1/\sqrt{2})$ and $(1/\sqrt{2}, 0)$. These are never going to be nodes of the triangulations.

The numerical results are described in Tables 3 and 4. Convergence order is again one (the solution is smooth) and the superconvergence in $L^2(\Omega)$ is still appreciated, although the order of superconvergence seems to be reduced.

Example #3. We consider the data of the problem (except for g) such that the exact solution is $u(x, y) = \psi(x^2 + y^2) - x(2 - x)y(1 - y)$, with ψ being the same truncation

dof	$\boldsymbol{e}_h(u)$	$r_h(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_T$	r_T
80	4.8240	——	3.1745	——	0.9674	——	5.7748	——
320	7.4959	——	2.2388	0.5038	0.4655	1.0553	7.8231	——
1280	4.9192	0.6077	1.6315	0.4565	0.1499	1.6343	5.1827	0.5940
5120	2.9431	0.7411	1.0092	0.6930	0.0613	1.2893	3.1113	0.7362
20480	1.8366	0.6803	0.5884	0.7784	0.0239	1.3606	1.9285	0.6900
81920	0.9250	0.9895	0.3111	0.9194	0.0069	1.7977	0.9759	0.9827

Table 3: Numerical results for example 2 ($\mathbb{P}_1 - [\mathbb{P}_0]^2$ approximation)

dof	$\boldsymbol{e}_h(u)$	$r_h(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_T$	r_T
144	2.6932	——	2.5803	——	0.2846	——	3.7298	——
576	2.5133	0.0998	2.2982	0.1670	0.1905	0.5790	3.4056	0.1312
2304	1.8149	0.4697	1.5134	0.6027	0.0916	1.0567	2.3631	0.5272
9216	1.1761	0.6258	0.9245	0.7111	0.0423	1.1157	1.4960	0.6596
36864	0.6491	0.8576	0.3770	1.2939	0.0144	1.5575	0.7506	0.9949
147456	0.3248	0.9987	0.1553	1.2801	0.0042	1.7750	0.3600	1.0601

Table 4: Numerical results for example 2 ($\mathbb{P}_1 - [\mathbb{P}_1]^2$ approximation)

function introduced in the second example. Since the normal derivative of the bubble is positive, we can set g as in the previous example, too. The numerical results are described in Tables 5 and 6.

dof	$\boldsymbol{e}_h(u)$	$r_h(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_T$	r_T
80	4.7223	——	3.1314	——	0.9365	——	5.6662	——
320	7.6019	——	2.2430	0.4814	0.4757	0.9771	7.9259	——
1280	4.9491	0.6192	1.6329	0.4580	0.1513	1.6530	5.2115	0.6049
5120	2.9538	0.7446	1.0101	0.6929	0.0619	1.2882	3.1218	0.7393
20480	1.8409	0.6821	0.5891	0.7780	0.0240	1.3656	1.9329	0.6916
81920	0.9264	0.9908	0.3115	0.9192	0.0075	1.6735	0.9774	0.9838

Table 5: Numerical results for example 3 ($\mathbb{P}_1 - [\mathbb{P}_0]^2$ approximation)

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dof	$\boldsymbol{e}_h(u)$	$r_h(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_T$	r_T
144	2.7085	——	2.6046	——	0.2819	——	3.7577	——
576	2.5407	0.0923	2.3285	0.1617	0.1941	0.5384	3.4464	0.1248
2304	1.8186	0.4824	1.5161	0.6190	0.0916	1.0839	2.3677	0.5416
9216	1.1774	0.6272	0.9251	0.7128	0.0423	1.1146	1.4973	0.6611
36864	0.6501	0.8569	0.3780	1.2911	0.0144	1.5523	0.7520	0.9936
147456	0.3253	0.9988	0.1553	1.2830	0.0042	1.7719	0.3605	1.0608

Table 6: Numerical results for example 3 ($\mathbb{P}_1 - [\mathbb{P}_1]^2$ approximation)

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