Regularity of solutions to quantum master equations: A stochastic approach

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Abstract: We develop the connections between stochastic processes and operator theory to describe the evolution of open quantum systems. Using stochastic Schrödinger equations, we study Markovian quantum master equations (QMEs for short) whose coefficients involve unbounded operators. QMEs are operator evolution equations that govern the dynamics of the density operators, which are positive operators of trace 1. Let the initial density operator be regular in the sense that the expected values with respect to it of a large class of unbounded operators are well-defined. Then we prove, under general conditions, that the solution of the QME remains regular all the time. Our analysis is mainly based on probabilistic representations of solutions to QMEs and adjoint quantum master equations (AQMEs). As by-products we obtain probabilistic interpretations of regular solutions to QMEs and the uniqueness of the solution for the AQME.

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1. Introduction

This paper studies the evolution of the mean values of quantum observables represented by unbounded operators with the help of classical stochastic analysis. To establish the regularity of solutions of quantum master equations with unbounded coefficients we develop the relation between stochastic evolution equations and the operator equations describing the evolution of density operators in open quantum systems.

1.1. Context

In the usual set-up of open quantum systems, a small quantum system interacts weakly with a heat bath. The states of the small system are characterized by elements of a complex Hilbert space $(\mathfrak{h}, \langle \cdot, \cdot \rangle)$ and their statistical mixtures

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are modeled by positive operators in $\mathfrak{h}$ with unit trace, which are called density operators. In many physical situations, the density operators $\rho_t(\varrho)$ evolve according to the operator evolution equation

$$\begin{cases} \dfrac{d}{dt}\rho_t(\varrho) = G\rho_t(\varrho) + \rho_t(\varrho)G^* + \sum_{k=1}^{\infty} L_k\rho_t(\varrho)L_k^* \\ \rho_0(\varrho) = \varrho \end{cases}, \tag{1.1}$$

where $G, L_1, L_2, \ldots$ are given linear operators in $\mathfrak{h}$ such that formally

$$G = -iH - \frac{1}{2}\sum_{k=1}^{\infty} L_k^* L_k \tag{1.2}$$

with H self-adjoint operator.

We can derive the quantum master equation (1.1) by using the Born-Markov approximation (see, e.g., [12, 14, 28]) or by means of coupling limit methods (see, e.g., [1, 2, 3, 20, 43, 44]). The Hamiltonian H guides essentially the free evolution of the small quantum system and $L_1, L_2, \ldots$ govern the effect of the environment. The measurable physical quantities of the small quantum system are represented by self-adjoint operators in $\mathfrak{h}$, which are called observables. The mean value of the observable A at time t is given by $tr(\rho_t(\varrho)A)$, the trace of $\rho_t(\varrho)A$ (see, e.g., [15, 42] for deeper discussions of the postulates of quantum mechanics).

1.2. Primary aim

Our main goal is to make progress in the understanding of the evolution of $tr(\rho_t(\varrho)A)$ in cases, like boson systems, where the observables A are described by unbounded operators. Relevant examples of unbounded observables arise, for instance, from quantum oscillators (see, e.g., Remark 5.5) and systems formed by an indefinite number of particles (see, e.g., [10, 35]).

1.3. Specific objectives

From [17] and [21] we have that (1.1) has a unique solution (see Subsection 5.2 for details). Since we are interested in the well-posedness of the expected values of observables described by unbounded operators, this paper investigates when the solution of (1.1) is regular enough to guarantee the trace-class property of $\rho_t(\varrho)A$ for a large set of unbounded operators A.

In [22], Davies established the regularity of $\rho_t(\varrho)$ for a variable number of neutrons moving in a translation invariant external reservoir of unstable atoms. Chebotarev, García and Quezada [18] proved that $\rho_t(\varrho)$ is regular in a general framework (Remark 5.4 presents some details). Arnold and Sparber [4] got regularity results for a linear quantum master equation associated to a diffusion models with Hartree interaction. All of these works are based on methods from the operator theory.

This paper provides a general criterion under which the solutions of (1.1) keep the regularity of the initial data (see Definition 4.1 for the precise meaning of regular solution). To this end, we first bring together classes of regular density operators and random variables taking values in $\mathfrak{h}$. Indeed, we deduce that the density operator ϱ is regular if and only if there exists a regular $\mathfrak{h}$-valued random variable ξ such that $\varrho = \mathbb{E} \langle \xi, \cdot \rangle \xi$.

Second, we consider the stochastic evolution equation

$$X_t(\xi) = \xi + \int_0^t G X_s(\xi)\, ds + \sum_{k=1}^{\infty} \int_0^t L_k X_s(\xi)\, dW_s^k, \tag{1.3}$$

where $\left(W^k\right)_{k \in \mathbb{N}}$ is a sequence of real valued independent Wiener processes on a filtered complete probability space $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P}\right)$. We prove that

$$\rho_t(\varrho) := \mathbb{E} \langle X_t(\xi), \cdot \rangle X_t(\xi) \tag{1.4}$$

is the unique solution of (1.1) under a refined non-explosion condition of the type introduced by Chebotarev and Fagnola [17]. Hence, the regularity of $X_t(\xi)$ implies that of $\rho_t(\varrho)$.

By applying stochastic techniques we deduce that $\mathbb{E} \langle X_t(\xi), \cdot \rangle X_t(\xi)$ is a solution to (1.1) whenever ξ is regular. This leads to construct a semigroup $(\rho_t)_{t \geq 0}$ of trace-class operators satisfying (1.1). The adjoint semigroup of $(\rho_t)_{t \geq 0}$ solves the the operator evolution equation

$$\begin{cases} \dfrac{d}{dt}\mathcal{T}_t(A) = \mathcal{T}_t(A) G + G^* \mathcal{T}_t(A) + \sum_{k=1}^{\infty} L_k^* \mathcal{T}_t(A) L_k \\ \mathcal{T}_0(A) = A \end{cases}, \tag{1.5}$$

where $\mathcal{T}_t(A)$ is an unknown linear operator in $\mathfrak{h}$. To obtain the uniqueness of the solution for (1.1), using (1.3) we show the uniqueness of the solution to (1.5) whenever A is bounded. Thus, as a by-product, we establish a new criterion for the existence and uniqueness of the solution to (1.5) with bounded initial data (see, e.g., [16, 17, 18, 19, 25] for previous works on this problem). The adjoint quantum master equation (1.5) describes the dynamic of the observable A in the Heisenberg picture under, for instance, the Born-Markov approximation.

From (1.4) we obtain

$$\rho_t(\varrho) = \mathbb{E} \langle Y_t, \cdot \rangle Y_t, \tag{1.6}$$

where Y_t satisfies the non-linear stochastic Schrödinger equation on $\mathfrak{h}$ (driven by a standard cylindrical Brownian motion)

$$Y_t = Y_0 + \int_0^t G(Y_s)\, ds + \sum_{k=1}^{\infty} \int_0^t L_k(Y_s)\, dW_s^k. \tag{1.7}$$

Here for any $y \in \mathfrak{h}$, $L_k(y) = L_k y - \Re \langle y, L_k y \rangle y$, and

$$G(y) = Gy + \sum_{k=1}^{\infty} \left(\Re \langle y, L_k y \rangle L_k y - \frac{1}{2} \Re^2 \langle y, L_k\ y \rangle y \right).$$

The probabilistic representation (1.6) has been introduced in the physical literature by means of formal computations (see, e.g., [5, 12, 29]). Furthermore, Barchielli and Holevo [6] established essentially (1.4) and (1.6) in situations where $G, L_1, L_2, \ldots$ are bounded.

Relation (1.6) has given rise to efficient and accurate numerical schemes that compute the mean value of quantum observables at time t through the non-linear stochastic Schrödinger equations (see, e.g., [12, 34, 41] and references therein). On the other hand, (1.4) and (1.6) open a door to treat, for instance, the long time behavior of the evolution of open quantum systems by means of stochastic processes (see, e.g., [7, 9, 33]).

1.4. Outline

Section 2 presents notation, hypotheses, and preliminary results on the stochastic Schrödinger equations. Section 3 treats the existence and uniqueness of the solution for the adjoint quantum master equation, as well as its probabilistic representation. Section 4 addresses probabilistic interpretations of regular density operators. Section 5 states the main results. In particular, Subsection 5.1 constructs Schrödinger evolutions by means of stochastic Schrödinger equations. Subsection 5.2 focusses on (1.1) and Subsection 5.3 is concerned with a quantum oscillator. Section 6 is devoted to proofs.

2. Preliminary results

2.1. Notation

Throughout this paper, $(\mathfrak{h}, \langle \cdot, \cdot \rangle)$ is a separable complex Hilbert space whose scalar product $\langle \cdot, \cdot \rangle$ is linear in the second variable and anti-linear in the first one. We write $\mathfrak{B}(\mathfrak{h})$ for the Borel σ-algebra on $\mathfrak{h}$. Following Dirac notation, for any $x, y \in \mathfrak{h}$ the map $|x\rangle\langle y|$ is defined by $|x\rangle\langle y|(z) = \langle y, z \rangle x$ whenever $z \in \mathfrak{h}$.

Suppose that A is a linear operator in $\mathfrak{h}$. Then, $\mathcal{D}(A)$ stands for the domain of A and A^* denotes the adjoint of A. If A has a unique bounded extension to $\mathfrak{h}$, then we continue to write A for the closure of A. By I we mean the identity operator.

Let $\mathfrak{X}$, $\mathfrak{Z}$ be normed spaces. We write $\mathfrak{L}(\mathfrak{X}, \mathfrak{Z})$ for the set of all bounded operators from $\mathfrak{X}$ to $\mathfrak{Z}$ (together with norm $\|\cdot\|_{\mathfrak{L}(\mathfrak{X},\mathfrak{Z})}$). We abbreviate $\|\cdot\|_{\mathfrak{L}(\mathfrak{X},\mathfrak{Z})}$ to $\|\cdot\|$, if no misunderstanding is possible, and define $\mathfrak{L}(\mathfrak{X}) = \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$. Moreover, $\mathfrak{L}_1(\mathfrak{h})$ denotes the Banach space of trace-class operators on $\mathfrak{h}$ equipped with the trace norm $\|\cdot\|_1$. By $\mathfrak{L}_1^+(\mathfrak{h})$ we mean the subset of all non-negative trace-class operators on $\mathfrak{h}$.

Let C be a self-adjoint positive operator in $\mathfrak{h}$. Then, for any $x, y \in \mathcal{D}(C)$ we set $\langle x, y \rangle_C = \langle x, y \rangle + \langle Cx, Cy \rangle$ and $\|x\|_C = \sqrt{\langle x, x \rangle_C}$. As usual, $L^2(\mathbb{P}, \mathfrak{h})$ stands for the set of all square integrable random variables from $(\Omega, \mathfrak{F}, \mathbb{P})$ to $(\mathfrak{h}, \mathfrak{B}(\mathfrak{h}))$. We write $L^2_C(\mathbb{P}, \mathfrak{h})$ for the set of all $\xi \in L^2(\mathbb{P}, \mathfrak{h})$ satisfying $\xi \in \mathcal{D}(C)$ a.s. and

$\mathbb{E}\left\|\xi\right\|_C^2 < \infty$. The function $\pi_C : \mathfrak{h} \to \mathfrak{h}$ is defined by $\pi_C(x) = x$ if $x \in \mathcal{D}(C)$ and $\pi_C(x) = 0$ whenever $x \notin \mathcal{D}(C)$.

In the sequel, the letter K denotes generic constants. Notation 2.1 and 5.1 point out the meaning of the symbols $X(\xi)$ and ρ_t respectively.

2.2. Linear stochastic Schrödinger equation

We begin by specifying the notion of solution to (1.3).

Hypothesis 1. *Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$ such that $\mathcal{D}(C)$ is a subset of the domains of $G, L_1, L_2, \dots$ and the maps $G \circ \pi_C, L_1 \circ \pi_C, L_2 \circ \pi_C, \dots$ are measurable.*

Definition 2.1. *Let Hypothesis 1 hold. Assume that $\mathbb{T}$ is either $[0,\infty[$ or the interval $[0,T]$, with $T \in \mathbb{R}_+$. An adapted process $(X_t(\xi))_{t\in\mathbb{T}}$ taking values in $\mathfrak{h}$ with continuous sample paths is called strong C-solution of (1.3) on $\mathbb{T}$ with initial datum ξ if and only if:*

- *For any $t \in \mathbb{T}$, $\mathbb{E}\left\|X_t(\xi)\right\|^2 \le \mathbb{E}\left\|\xi\right\|^2$, $X_t(\xi) \in \mathcal{D}(C)$ a.s. and*
$$\sup_{s\in[0,t]} \mathbb{E}\left\|CX_s(\xi)\right\|^2 < \infty.$$
- *$\mathbb{P}$-a.s. for all $t \in \mathbb{T}$,*
$$X_t(\xi) = \xi + \int_0^t G\pi_C(X_s(\xi))\,ds + \sum_{k=1}^{\infty}\int_0^t L_k\pi_C(X_s(\xi))\,dW_s^k.$$

Notation 2.1. *The symbol $X(\xi)$ will be reserved for the strong C-solution of (1.3) with initial datum ξ.*

Let us introduce the basic assumptions of this paper.

Hypothesis 2. *Suppose that Hypothesis 1 holds. In addition, assume:*

(H2.1) The operator G belongs to $\mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$.
(H2.2) For all $x \in \mathcal{D}(C)$, $2\Re\langle x, Gx\rangle + \sum_{k=1}^{\infty}\left\|L_k x\right\|^2 = 0$.
(H2.3) Let $\xi \in L_C^2(\mathbb{P},\mathfrak{h})$ be $\mathfrak{F}_0$-measurable. Then for all $T > 0$, (1.3) has a unique strong C-solution on $[0,T]$ with initial datum ξ.

From [27] it follows that under Hypothesis 2, $\left\|X(\xi)\right\|^2$ is a martingale and $X(\xi)$ has the Markov property.

Theorem 2.1. *Let Hypothesis 2 hold and let $\xi \in L_C^2(\mathbb{P},\mathfrak{h})$. Then $\left(\left\|X_t(\xi)\right\|^2\right)_{t\ge0}$ is a martingale. Moreover, for any measurable bounded function $f : (\mathfrak{h}, \mathfrak{B}(\mathfrak{h})) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ we have*

$$\begin{aligned}\mathbb{E}\left(f(X_{s+t}(\xi)) \diagup \mathfrak{F}_s\right) &= \mathbb{E}\left(f(X_{s+t}(\xi)) \diagup X_s(\xi)\right) \qquad (2.1)\\ &= \int_{\mathfrak{h}} f(z)\,P_t(X_s(\xi), dz),\end{aligned}$$

where $P_t(x,\cdot) = \begin{cases} \mathbb{P}\circ (X_s(x))^{-1}, & \text{if } x \in \mathcal{D}(C) \\ \delta_x, & \text{if } x \notin \mathcal{D}(C) \end{cases}$.

Condition H2.2 is a weak version of (1.2). In [27] it is provided the following sufficient condition for the existence and uniqueness of strong C-solutions to (1.3).

Hypothesis 3. *Let C be a self-adjoint positive operator in $\mathfrak{h}$ with the properties:*

(H3.1) The operators $G, L_1, L_2, \ldots$ belong to $\mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$.

(H3.2) There exist non-negative real numbers α, β and a core $\mathfrak{D}_1$ of C^2 such that for all x in $\mathfrak{D}_1$,

$$2\Re\left\langle C^2x, Gx\right\rangle + \sum_{k=1}^{\infty}\|CL_kx\|^2 \leq \alpha\|x\|_C^2 + \beta.$$

(H3.3) There exist a core $\mathfrak{D}_2$ of C such that for any x in $\mathfrak{D}_2$,

$$2\Re\langle x, Gx\rangle + \sum_{k=1}^{\infty}\|L_kx\|^2 \leq 0.$$

Theorem 2.2. *Assume that Hypothesis 3 holds. Let ξ be a $\mathfrak{F}_0$-measurable random variable of $L_C^2(\mathbb{P},\mathfrak{h})$. Then (1.3) has a unique strong C-solution $(X_t(\xi))_{t\geq 0}$ with initial datum ξ. Moreover,*

$$\mathbb{E}\|CX_t(\xi)\|^2 \leq \exp(\alpha t)\left(\mathbb{E}\|C\xi\|^2 + \alpha t\mathbb{E}\|\xi\|^2 + \beta t\right).$$

Theorem 2.2 asserts that Condition H2.3 of Hypothesis 2 is general enough for many physical applications. Indeed, Conditions H3.1 and H3.2 of Hypothesis 3 together with Condition H2.2 of Hypothesis 2 are the underlying assumptions of this paper.

Remark 2.1. *Let A be a closable operator in $\mathfrak{h}$ such that $\mathcal{D}(C)\subset\mathcal{D}(A)$, where C is a self-adjoint positive operator in $\mathfrak{h}$. Applying the closed graph theorem we obtain $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$, which leads to a sufficient condition for H3.1, as well as for H2.1.*

Remark 2.2. *Let C be a self-adjoint positive operator in $\mathfrak{h}$ whose domain is contained in $\mathcal{D}(G)$. Assume that $2\Re\langle x, Gx\rangle + \sum_{k=1}^{\infty}\|L_kx\|^2 \leq 0$, for all $x \in \mathcal{D}(G)$. Then the numerical range of G is contained in the left half-plane of $\mathbb{C}$, and so G is closable. Therefore G lies in $\mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$ by Remark 2.1, and so Condition H3.1 holds.*

Remark 2.3. *Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$, together with $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$. Then $A\circ\pi_C : \mathfrak{h}\to\mathfrak{h}$ is measurable whenever $\mathfrak{h}$ is equipped with its Borel σ-algebra (see, e.g., [27] for details). Thus $G\circ\pi_C$ and $L_k\circ\pi_C$ are measurable under Condition H3.1.*

3. Adjoint quantum master equation

To deal with (1.1) we establish the uniqueness of the solution to (1.5) in case A is bounded operator.

Definition 3.1. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. Suppose that $A \in \mathfrak{L}(\mathfrak{h})$. A family of operators $(\mathcal{A}_t)_{t\geq 0}$ belonging to $\mathfrak{L}(\mathfrak{h})$ is a C-solution of (1.5) with initial datum A if and only if:*

(a) $\mathcal{A}_0 = A$.
(b) For all $t \geq 0$ and any x, y in $\mathcal{D}(C)$,

$$\frac{d}{dt}\langle x, \mathcal{A}_t y\rangle = \langle x, \mathcal{A}_t G y\rangle + \langle G x, \mathcal{A}_t y\rangle + \sum_{k=1}^{\infty}\langle L_k x, \mathcal{A}_t L_k y\rangle .$$

(c) For all $T \geq 0$, $\sup_{s\in[0,T]}\|\mathcal{A}_s\|_{\mathfrak{L}(\mathfrak{h})} < \infty$.
(d) For any $x, y \in \mathfrak{h}$, the application $t \longmapsto \langle x, \mathcal{A}_t y\rangle$ is continuous.

Theorem 3.1. *Suppose that Hypothesis 2 holds. Let A belong to $\mathfrak{L}(\mathfrak{h})$. Then, for every non-negative real number t there exists a unique $\mathcal{T}_t(A)$ in $\mathfrak{L}(\mathfrak{h})$ such that for any x, y in $\mathcal{D}(C)$,*

$$\langle x, \mathcal{T}_t(A)\, y\rangle = \mathbb{E}\langle X_t(x), A X_t(y)\rangle . \tag{3.1}$$

Moreover, $\|\mathcal{T}_t(A)\|_{\mathfrak{L}(\mathfrak{h})} \leq \|A\|_{\mathfrak{L}(\mathfrak{h})}$ for all $t \geq 0$, and any C-solution of (1.5) with initial datum A coincides with $(\mathcal{T}_t(A))_{t\geq 0}$.

Proof. The proofs fall naturally into Lemmata 6.1 and 6.2. □

The following theorem essentially states the existence of a solution to (1.5). From the physical point of view, Theorems 3.1 and 3.2 have interest by themselves, because (1.5) governs the the dynamic of the observable A in the Heisenberg picture.

Theorem 3.2. *Let Hypothesis 2 holds. Suppose that $A \in \mathfrak{L}(\mathfrak{h})$ and that $\mathcal{T}_t(A)$ is as in Theorem 3.1. Then $(\mathcal{T}_t(A))_{t\geq 0}$ is the unique C-solution of (1.5) with initial datum A.*

Proof. Lemmata 6.20 and 6.21 shows that $(\mathcal{T}_t(A))_{t\geq 0}$ is a C-solution of (1.5) with initial datum A. Theorem 3.1 now completes the proof. □

Remark 3.1. *[37] deals with the existence and uniqueness of solutions to (1.5) for A unbounded. Chebotarev and Fagnola [17] proved the Markov property of the family $(\mathcal{T}_t : \mathfrak{L}(\mathfrak{h}) \to \mathfrak{L}(\mathfrak{h}))_{t\geq 0}$ (i.e., $\mathcal{T}_t(I) = I$ for any $t \geq 0$) under a non-explosion criterion inherent to quantum physical systems. This result, which has been generalized in [18, 25], implies the uniqueness of the solution to (1.5) with A bounded. Hypothesis 3, the underlying assumption of Theorem 3.2, is a refined version of non-explosion criteria that guarantee the Markov property of the quantum dynamical semigroups.*

4. Probabilistic representations of regular density operators

The following notion of regular density operator was introduced by Chebotarev, García and Quezada [18] to investigate the identity preserving property of minimal quantum dynamical semigroups.

Definition 4.1. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. An operator ϱ belonging to $\mathfrak{L}_1^+(\mathfrak{h})$ is called C-regular if and only if*

$$\varrho = \sum_{n \in \mathfrak{I}} \lambda_n |u_n\rangle\langle u_n| \tag{4.1}$$

for some countable set $\mathfrak{I}$, summable non-negative real numbers $(\lambda_n)_{n\in\mathfrak{I}}$ and family $(u_n)_{n\in\mathfrak{I}}$ of elements of $\mathcal{D}(C)$, which together satisfy:

$$\sum_{n \in \mathfrak{I}} \lambda_n \|Cu_n\|^2 < \infty. \tag{4.2}$$

We write $\mathfrak{L}_{1,C}^+(\mathfrak{h})$ for the set of all C-regular density operators.

We next formulate the concept of C-regular operator in terms of random variables. This characterization of $\mathfrak{L}_{1,C}^+(\mathfrak{h})$ complements those given in [18] using operator theory (see also [19]).

Theorem 4.1. *Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$. Let ϱ be a linear operator in $\mathfrak{h}$. Then ϱ is C-regular iff $\varrho = \mathbb{E}|\xi\rangle\langle\xi|$ for some $\xi \in L_C^2(\mathbb{P},\mathfrak{h})$. Moreover, $\mathbb{E}|\xi\rangle\langle\xi|$ can be interpreted as a Bochner integral in both $\mathfrak{L}_1(\mathfrak{h})$ and $\mathfrak{L}(\mathfrak{h})$.*

Proof. The proof is divided into Lemmata 6.5 and 6.6. □

Using Theorem 4.3 below we can assert that the mean values of a large number of unbounded observables are well-posed when the density operators are C-regular. Theorem 4.3 also provides probabilistic interpretations of these expected values.

Theorem 4.2. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. Suppose that $\varrho = \mathbb{E}|\xi\rangle\langle\xi|$ with $\xi \in L_C^2(\mathbb{P},\mathfrak{h})$. Then the range of ϱ is contained in $\mathcal{D}(C)$.*

Proof. Deferred to Subsection 6.2. □

Theorem 4.3. *Adopt the assumptions of Theorem 4.2. In addition, suppose that A belongs to $\mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$ and B is a densely defined linear operator in $\mathfrak{h}$ such that $\mathcal{D}(C) \subset \mathcal{D}(B^*)$. Then $A\varrho B$ is densely defined and bounded. The unique bounded extension of $A\varrho B$ belongs to $\mathfrak{L}_1(\mathfrak{h})$ and is equal to $\mathbb{E}|A\xi\rangle\langle B^*\xi|$. Moreover, $tr(A\varrho B) = \mathbb{E}\langle B^*\xi, A\xi\rangle$ and $\mathbb{E}|A\xi\rangle\langle B^*\xi|$ is a well-defined Bochner integral in both $\mathfrak{L}_1(\mathfrak{h})$ and $\mathfrak{L}(\mathfrak{h})$.*

Proof. Deferred to Subsection 6.4. □

5. Quantum master equation

5.1. Modelling the evolution of density operators

We next introduce the density operators at time t by means of (1.3).

Theorem 5.1. *Suppose that Hypothesis 2 holds. Then, for every $t \geq 0$ there exists a unique operator ρ_t belonging to $\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))$ such that for each C-regular operator ϱ we have*

$$\rho_t(\varrho) = \mathbb{E}\,|X_t(\xi)\rangle\langle X_t(\xi)|\,, \tag{5.1}$$

where ξ is an arbitrary random variable in $L^2_C(\mathbb{P},\mathfrak{h})$ satisfying $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$ and $X(\xi)$ is the unique strong C-solution of (1.3) with initial datum ξ. Here we can interpret $\mathbb{E}\,|X_t(\xi)\rangle\langle X_t(\xi)|$ as a Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$ as well as in $\mathfrak{L}(\mathfrak{h})$. Moreover, $\|\rho_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))} \leq 1$ for all $t \geq 0$.

Proof. Deferred to Subsection 6.5. □

Notation 5.1. *From now on, ρ_t stands for the operator given by (5.1).*

In case the initial density operator ϱ is C-regular, Corollary 5.1 allows to define the density operator at time t as the average of all density operators associated to the pure states Y_t, where Y satisfies (1.7) and $\varrho = \mathbb{E}\,|Y_0\rangle\langle Y_0|$. This model has a sound physical basis (see, e.g., [5, 8, 30, 45]).

Definition 5.1. *Let C satisfy Hypothesis 1. Suppose that $\mathbb{T}$ is either $[0,+\infty[$ or $[0,T]$ provided $T \in [0,+\infty[$. We say that $\left(\mathbb{Q},(Y_t)_{t\in\mathbb{T}},(W_t)_{t\in\mathbb{T}}\right)$ is a C-solution of (1.7) with initial distribution θ on $\mathbb{T}$ if and only if:*

- *$W = \left(W^k\right)_{k\in\mathbb{N}}$ is a sequence of real valued independent Brownian motions on the filtered complete probability space $\left(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\in\mathbb{T}},\mathbb{Q}\right)$.*
- *$(Y_t)_{t\in\mathbb{T}}$ is an $\mathfrak{h}$-valued process with continuous sample paths such that the law of Y_0 coincides with θ and $\mathbb{Q}\left(\|Y_t\| = 1 \text{ for all } t \in \mathbb{T}\right) = 1$.*
- *For every $t \in \mathbb{T}$, $Y_t \in \mathcal{D}(C)$ $\mathbb{Q}$-a.s. and $\sup_{s\in[0,t]} \mathbb{E}_{\mathbb{Q}} \|CY_s\|^2 < \infty$.*
- *$\mathbb{Q}$-a.s., $Y_t = Y_0 + \int_0^t G(\pi_C(Y_s))\,ds + \sum_{k=1}^{\infty} \int_0^t L_k(\pi_C(Y_s))\,dW^k_s$ for all $t \in \mathbb{T}$.*

Corollary 5.1. *Let Hypothesis 2 hold. Then ρ_t is the unique element of $\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))$ such that $\rho_t(\varrho) = \mathbb{E}_{\mathbb{Q}}\,|Y_t\rangle\langle Y_t|$, for any C-regular operator ϱ. Here*

$$\left(\mathbb{Q},(Y_t)_{t\geq 0},(B_t)_{t\geq 0}\right)$$

is the C-solution of (1.7) with initial law θ satisfying $\int_{\mathfrak{h}} \|Cx\|^2\,\theta(dx) < \infty$, $\theta\left(\mathcal{D}(C) \cap \{x \in \mathfrak{h} : \|x\| = 1\}\right) = 1$, and $\varrho = \int |y\rangle\langle y|\,\theta(dy)$.

Proof. Deferred to Subsection 6.6. □

We proceed to state basic properties of ρ_t. Applying Theorems 4.1 and 5.3 we obtain that ρ_t leaves invariant the set of all C-regular density operators.

Theorem 5.2. *Assume that Hypothesis 2 holds. Let $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$. Then*

$$\mathbb{E}\rho_t(|\xi\rangle\langle\xi|) = \rho_t(\mathbb{E}|\xi\rangle\langle\xi|)$$

for all $t \geq 0$.

Proof. Deferred to Subsection 6.7. □

Theorem 5.3. *Adopt Hypothesis 2. Then $(\rho_t)_{t\geq 0}$ is a semigroup of contractions such that $\rho_t\left(\mathfrak{L}_1^+(\mathfrak{h})\right) \subset \mathfrak{L}_1^+(\mathfrak{h})$, $\rho_t\left(\mathfrak{L}_{1,C}^+(\mathfrak{h})\right) \subset \mathfrak{L}_{1,C}^+(\mathfrak{h})$, and*

$$\lim_{s\to t} tr\left|\rho_s(\varrho) - \rho_t(\varrho)\right| = 0 \tag{5.2}$$

provided that ϱ is C-regular.

Proof. The proof is divided into Lemmata 6.12, 6.13 and 6.15. □

5.2. Regular solutions of quantum master equations

Using the linear stochastic Schrödinger equation we now show that $\rho_t(\varrho)$ satisfies (1.1) in both sense integral and $\mathfrak{L}_1(\mathfrak{h})$-weak, whenever ϱ is C-regular. Recall that throughout this paper $\rho_t(\varrho)$ denotes the operator introduced in Theorem 5.1. Previously, Davies [21] constructed the so-called minimal solution of (1.1) in case G is the infinitesimal generator of a strongly continuous contraction semigroup on $\mathfrak{h}$. That is, Davies obtained a contraction semigroup $(\mathcal{T}_{*t})_{t\geq 0}$ on $\mathfrak{L}_1(\mathfrak{h})$ whose infinitesimal generator is given formally by

$$\mathcal{L}_*(\varrho) = G\varrho + \varrho G^* + \sum_{k=1}^{\infty} L_k \varrho L_k^*.$$

Hypothesis 4. *The operators $G, L_1, L_2, \ldots$ are closable.*

Theorem 5.4. *Let Hypotheses 2 and 4 hold. Suppose that ϱ is C-regular. Then for all $t \geq 0$,*

$$\rho_t(\varrho) = \varrho + \int_0^t \left(G\rho_s(\varrho) + \rho_s(\varrho) G^* + \sum_{k=1}^{\infty} L_k \rho_s(\varrho) L_k^* \right) ds, \tag{5.3}$$

where we understand the above integral in the sense of Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$. Moreover, for any $A \in \mathfrak{L}(\mathfrak{h})$ and $t \geq 0$ we have

$$\frac{d}{dt} tr(A\rho_t(\varrho)) = tr\left(A \left(G\rho_t(\varrho) + \rho_t(\varrho) G^* + \sum_{k=1}^{\infty} L_k \rho_t(\varrho) L_k^* \right) \right). \tag{5.4}$$

Proof. Deferred to Subsection 6.9. □

Remark 5.1. *Let $G, L_1, L_2, \ldots$ be densely defined. Then Hypothesis 4 is equivalent to say that $G^*, L_1^*, L_2^*, \ldots$ are densely defined.*

We now establish that under Hypothesis 2, $\mathbb{E}\,|Y_t\rangle\langle Y_t|$ is the unique solution of a weak version of (5.4).

Theorem 5.5. *Let Hypothesis 2 hold. Then there exists a unique semigroup $(\widehat{\rho}_t)_{t\geq 0}$ of bounded operators on $\mathfrak{L}_1(\mathfrak{h})$ with the following properties:*

(i) For each non-negative real number T, $\sup_{t\in[0,T]} \|\widehat{\rho}_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))} < \infty$.
(ii) For any $x \in \mathcal{D}(C)$ and $A \in \mathfrak{L}(\mathfrak{h})$, the function $t \mapsto tr(\widehat{\rho}_t(|x\rangle\langle x|)A)$ is continuous.
(iii) For any $x \in \mathcal{D}(C)$ and $A \in \mathfrak{L}(\mathfrak{h})$,

$$\lim_{t\to 0+} \frac{1}{t}\left(tr\left(A\widehat{\rho}_t\left(|x\rangle\langle x|\right)\right) - tr\left(A\,|x\rangle\langle x|\right)\right)$$
$$= \langle x, AGx\rangle + \langle Gx, Ax\rangle + \sum_{k=1}^{\infty} \langle L_k x, AL_k x\rangle .$$

Moreover, $(\widehat{\rho}_t)_{t\geq 0}$ coincides with $(\rho_t)_{t\geq 0}$ and $\widehat{\rho}_t\left(\mathfrak{L}^+_{1,C}(\mathfrak{h})\right) \subset \mathfrak{L}^+_{1,C}(\mathfrak{h})$ for all $t \geq 0$.

Proof. Deferred to Subsection 6.10. □

Remark 5.2. *Adopt Hypothesis 2. By Theorem 5.5, the solutions of (1.1) remain in $\mathfrak{L}^+_{1,C}(\mathfrak{h})$ provided that the initial data are C-regular density operators. Theorem 4.3 leads to the mean values of the observables A with respect to these solutions are well-posed in case $\mathcal{D}(C) \subset \mathcal{D}(A)$. Thus we arrive at a sound mathematical description, which is in a good agreement with physical considerations, of the evolution of unbounded observables in the Schrödinger picture.*

Remark 5.3. *Suppose that $\mathcal{T}_{*t}(\varrho)$ is as in the first paragraph of this subsection and that $\mathcal{T}_t(I) = I$ for all $t \geq 0$, where $\mathcal{T}_t(I)$ is, roughly speaking, the solution of (1.5) with $A = I$. Let G be the infinitesimal generator of a strongly continuous contraction semigroup on $\mathfrak{h}$ and let Condition H2.2 hold for all $x \in \mathcal{D}(G)$. Then the linear span of $\{|x\rangle\langle y| : x, y \in \mathcal{D}(G)\}$ is a core for $\mathcal{L}_*$ (see, e.g., Proposition 3.32 of [25] for details). Hence $(\mathcal{T}_{*t})_{t\geq 0}$ is the unique strongly continuous semigroup on $\mathfrak{L}_1(\mathfrak{h})$ such that $(\mathcal{T}_{*t}(\varrho))_{t\geq 0}$ is a solution to (1.1) for $\varrho = |x\rangle\langle y|$, with $x, y \in \mathcal{D}(G)$.*

Remark 5.4. *Using methods from the operator theory, Chebotarev, García and Quezada [18] proved that $\mathcal{T}_{*t}(\mathfrak{L}_{1,C}(\mathfrak{h})) \subset \mathfrak{L}_{1,C}(\mathfrak{h})$ under assumptions that include the restrictions:*

- *G is the infinitesimal generator of strongly continuous semigroup of contractions and $\mathcal{D}\left(C^2\right) \subset \mathcal{D}(G) \subset \mathcal{D}(C)$.*
- *For any $\varrho \in \mathfrak{L}_{1,C^4}(\mathfrak{h})$, $tr(C\mathcal{L}_*(\varrho)C) \leq K tr(C\varrho C)$.*
- *The relation $\varrho \mapsto \sum_{k=1}^{\infty} L_k \varrho L_k^*$ defines a continuous function from*

$$\left(\mathfrak{L}_{1,C^4}(\mathfrak{h}), tr\left(\left|C^2 \cdot C^2\right|\right)\right)$$

to $(\mathfrak{L}_1(\mathfrak{h}), tr(|\cdot|))$.

- *$\mathcal{L}_*$ is continuous as an application from* $\left(\mathfrak{L}_{1,C^{2k}}\left(\mathfrak{h}\right), tr\left(\left|C^{2k}\cdot C^{2k}\right|\right)\right)$ *to* $\left(\mathfrak{L}_{1,C^{2k-2}}\left(\mathfrak{h}\right), tr\left(\left|C^{2k-2}\cdot C^{2k-2}\right|\right)\right)$ *for* $k=1,2$.

5.3. Example

This subsection briefly illustrates our main results by means of a one-dimensional quantum oscillator. A motivation comes from the fact that many quantum systems are described through quantum oscillators (see, e.g., Chapter V of [15] for discussion).

Example 1. *Consider* $\mathfrak{h}=l^2\left(\mathbb{Z}_+\right)$ *together with its canonical orthonormal basis* $\left(e_n\right)_{n\in\mathbb{Z}_+}$. *Let the closed operator* a *be given by* $ae_0=0$ *and* $ae_n=\sqrt{n}e_{n-1}$ *if* $n>0$. *Similarly,* $a^\dagger$ *is defined by* $a^\dagger e_n=\sqrt{n+1}e_{n+1}$ *for all* $n\in\mathbb{Z}_+$. *Set* $N=a^\dagger a$.

Choose $L_1=\alpha_1 a$, $L_2=\alpha_2 a^\dagger$, $L_3=\alpha_3 N$, $L_4=\alpha_4 a^2$, $L_5=\alpha_5\left(a^\dagger\right)^2$ *and* $L_6=\alpha_6 N^2$, *with* $\alpha_1,\ldots,\alpha_6$ *complex numbers. Take* $L_k=0$ *for all* $k\geq 7$ *and*

$$G=-iH-\frac{1}{2}\sum_{k=1}^{6}L_k^*L_k, \tag{5.5}$$

where $H=i\beta_1\left(a^\dagger-a\right)+\beta_2 N+\beta_3\left(a^\dagger\right)^2a^2$ *with* β_1, β_2, β_3 *real numbers.*

Remark 5.5. *In Example 1, the unbounded operator* N *gives the number of photons. The observables position and momentum operators are described by the unbounded operators* $i\left(a^\dagger-a\right)/\sqrt{2}$ *and* $i\left(a^\dagger+a\right)/\sqrt{2}$ *respectively.*

Remark 5.6. *Example 1 unifies some concrete physical systems like two-photon absorption and emission processes (see, e.g., [13, 26] and references therein) and ideal resonators interacting with two-level atoms (see, e.g., [24]). In this model,* $l^2\left(\mathbb{Z}_+\right)$ *represents, for instance, a single mode of a quantized electromagnetic field. The action of* $a^\dagger$ *and* a *on* e_n *make an energy quantum appear or disappear respectively.*

In the framework of Example 1, suppose that $|\alpha_4|\geq|\alpha_5|$ and that the initial density operator is regular enough. Then, according to the next theorem we have that the mean values of observables formed by a finite composition of the creation and annihilation operators (i.e., $a^\dagger$ and a) are well defined at any time.

Theorem 5.6. *Assume the setting of Example 1 and let* $\rho_t\left(\varrho\right)$ *be as in Theorem 5.1. Suppose that* $|\alpha_4|\geq|\alpha_5|$ *and that* p *is a natural number greater than or equal to* 4. *If* ϱ *lies in* $\mathfrak{L}^+_{1,N^p}\left(l^2\left(\mathbb{Z}_+\right)\right)$, *then* $\rho_t\left(\varrho\right)$ *is a* N^p*-regular operator that satisfies both (5.3) and (5.4). Moreover,* $\left(\rho_t\right)_{t\geq 0}$ *is the unique semigroup of bounded operators on* $\mathfrak{L}_1\left(l^2\left(\mathbb{Z}_+\right)\right)$ *for which Properties (i)-(iii) of Theorem 5.5 hold with* $C=N^p$.

Proof. Deferred to Section 6.11. □

6. Proofs

6.1. Proof of Theorem 3.1

Lemma 6.1. *Adopt the assumptions of Hypothesis 2 with the exception of Condition H2.2. Consider A in $\mathfrak{L}(\mathfrak{h})$. Then for every $t \geq 0$ there exists a unique $\mathcal{T}_t(A)$ belonging to $\mathfrak{L}(\mathfrak{h})$ for which (3.1) holds for all x, y in $\mathcal{D}(C)$. Moreover, $\|\mathcal{T}_t(A)\| \leq \|A\|$ for any $t \geq 0$.*

Proof. By Definition 2.1, $|\mathbb{E}\langle X_t(x), AX_t(y)\rangle| \leq \|A\| \|x\| \|y\|$, where $x, y \in \mathcal{D}(C)$. Hence the sesquilinear form over $\mathcal{D}(C) \times \mathcal{D}(C)$ given by

$$(x, y) \mapsto \mathbb{E}\langle X_t(x), AX_t(y)\rangle$$

can be extended uniquely to a sesquilinear form $[\cdot, \cdot]$ over $\mathfrak{h} \times \mathfrak{h}$ with the property that for any $x, y \in \mathfrak{h}$,

$$|[x, y]| \leq \|A\| \|x\| \|y\| . \tag{6.1}$$

There exists a unique bounded operator $\mathcal{T}_t(A)$ on $\mathfrak{h}$ such that

$$|[x, y]| = \langle x, \mathcal{T}_t(A) y\rangle$$

for all x, y in $\mathfrak{h}$, and so the lemma follows from (6.1). □

Lemma 6.2. *Let Hypothesis 2 hold. Assume that $(\mathcal{A}_t)_{t\geq 0}$ is a C-solution of (1.5) with initial datum $A \in \mathfrak{L}(\mathfrak{h})$. Then $\mathcal{A}_t = \mathcal{T}_t(A)$ for all $t \geq 0$, where $\mathcal{T}_t(A)$ is as in Therorem 3.1.*

Proof. Fix $x, y \in \mathcal{D}(C)$. We first prove that for any $t \geq 0$ and $n \in \mathbb{N}$,

$$\mathbb{E}\left\langle R_n X_t^{\tau_j}(x), \mathcal{A}_{t-t\wedge\tau_j} R_n X_t^{\tau_j}(y)\right\rangle \to_{j\to\infty} \mathbb{E}\langle R_n X_t(x), AR_n X_t(y)\rangle , \tag{6.2}$$

where $R_n = n(n + C)^{-1}$ and $\tau_j = \inf\{t \geq 0 : \|X_t(x)\| + \|X_t(y)\| > j\}$.

Combining Itô's formula with Condition H2.2 of Hypothesis 2 we obtain the martingale property of $\|X^{\tau_j}(z)\|^2$, with $z = x, y$. To this end, we can use an analysis similar to that in the proof of Theorem 2.1 (see, e.g., [27]). According to Theorem 2.1 we have $\mathbb{E}\|X_t(z)\|^2 = \|z\|^2$, and so

$$\mathbb{E}\left\|X_t^{\tau_j}(z)\right\|^2 = \mathbb{E}\|X_t(z)\|^2 = \|z\|^2 . \tag{6.3}$$

Choose $\epsilon_j = \langle R_n X_t(x), AR_n X_t(y)\rangle - \left\langle R_n X_t^{\tau_j}(x), \mathcal{A}_{t-t\wedge\tau_j} R_n X_t^{\tau_j}(y)\right\rangle$. Using Conditions (c) and (d) of Definition 3.1 gives $\lim_{j\to\infty} \epsilon_j = 0$. Since $\|R_n\| \leq 1$, applying Fatou's lemma we get

$$\begin{aligned} &2\mathbb{E}\left(\sup_{s\in[0,t]} \|\mathcal{A}_s\| \sum_{z=x,y} \|X_t(z)\|^2\right) \\ &\leq \liminf_{j\to\infty} \mathbb{E}\left(\sup_{s\in[0,t]} \|\mathcal{A}_s\| \sum_{z=x,y} \left(\|X_t(z)\|^2 + \left\|X_t^{\tau_j}(z)\right\|^2\right) - |\epsilon_j|\right). \end{aligned}$$

Therefore $\limsup_{j \to \infty} \mathbb{E} \left| \epsilon_j \right| \leq 0$ by (6.3). This implies (6.2).

Our next claim is that

$$\mathbb{E} \left\langle X_t \left(x \right), A X_t \left(x \right) \right\rangle = \left\langle x, \mathcal{A}_t y \right\rangle . \tag{6.4}$$

This establishes $\mathcal{A}_t = \mathcal{T}_t \left(A \right)$ by Lemma 6.1.

In order to prove (6.4), we consider the orthonormal basis $\left(e_n \right)_{n \in \mathbb{N}}$ of $\mathfrak{h}$. For any $u \in \mathfrak{h}$, we define $\bar{u}$ to be $\sum_{n \in \mathbb{N}} \overline{\langle e_n, u \rangle} e_n$. We set

$$F_n \left(s, u, v \right) = \left\langle R_n \bar{u}, \mathcal{A}_{t-s} R_n v \right\rangle$$

provided that $s \in [0, t]$ and $u, v \in \mathfrak{h}$. Since the range of R_n is contained in $\mathcal{D} \left(C \right)$, Condition (b) of Definition 3.1 shows

$$\frac{d}{ds} F_n \left(s, u, v \right) = - f_n \left(s, \bar{u}, v \right), \tag{6.5}$$

where $f_n \left(s, u, v \right)$ is equal to

$$\left\langle R_n u, \mathcal{A}_{t-s} G R_n v \right\rangle + \left\langle G R_n u, \mathcal{A}_{t-s} R_n v \right\rangle + \sum_{k=1}^{\infty} \left\langle L_k R_n u, \mathcal{A}_{t-s} L_k R_n v \right\rangle . \tag{6.6}$$

By $C R_n \in \mathfrak{L} \left(\mathfrak{h} \right)$, from Hypothesis 2 and Definition 3.1 we conclude that f_n is continuous. Now, we can apply Itô's formula and (6.5) to obtain

$$\begin{aligned} & \mathbb{E} F_n \left(t \wedge \tau_j, \overline{X_t^{\tau_j} \left(x \right)}, X_t^{\tau_j} \left(y \right) \right) - \mathbb{E} F_n \left(0, \overline{X_0^{\tau_j} \left(x \right)}, X_0^{\tau_j} \left(y \right) \right) \\ & = \mathbb{E} \int_0^{t \wedge \tau_j} \left(- f_n \left(s, X_s^{\tau_j} \left(x \right), X_s^{\tau_j} \left(y \right) \right) + g_n \left(s, X_s^{\tau_j} \left(x \right), X_s^{\tau_j} \left(y \right) \right) \right) ds, \end{aligned} \tag{6.7}$$

with $g_n \left(s, u, v \right)$ given by

$$\left\langle R_n u, \mathcal{A}_{t-s} R_n G v \right\rangle + \left\langle R_n G u, \mathcal{A}_{t-s} R_n v \right\rangle + \sum_{k=1}^{\infty} \left\langle R_n L_k u, \mathcal{A}_{t-s} R_n L_k v \right\rangle . \tag{6.8}$$

According to (6.2) we have

$$\mathbb{E} F_n \left(t \wedge \tau_j, \overline{X_t^{\tau_j} \left(x \right)}, X_t^{\tau_j} \left(y \right) \right) \longrightarrow_{j \to \infty} \mathbb{E} \left\langle R_n X_t \left(x \right), A R_n X_t \left(y \right) \right\rangle .$$

Using the dominated convergence theorem we tend $j \to \infty$ in (6.7) to get

$$\begin{aligned} & \mathbb{E} \left\langle R_n X_t \left(x \right), A R_n X_t \left(y \right) \right\rangle - \left\langle R_n x, \mathcal{A}_t R_n y \right\rangle \\ & = \mathbb{E} \int_0^t \left(- f_n \left(s, X_s \left(x \right), X_s \left(y \right) \right) + g_n \left(s, X_s \left(x \right), X_s \left(y \right) \right) \right) ds. \end{aligned} \tag{6.9}$$

Since $\left\| R_n \right\| \leq 1$ and R_n tends pointwise to I as $n \to \infty$, the dominated convergence theorem leads to

$$\lim_{n \to \infty} \mathbb{E} \int_0^t g_n \left(s, X_s \left(x \right), X_s \left(y \right) \right) ds = \mathbb{E} \int_0^t g \left(s, X_s \left(x \right), X_s \left(y \right) \right) ds$$

where $g(s,u,v)$ is described by (6.8) with all R_n deleted. Similarly, we define $f(s,u,v)$ by (6.6) with R_n replaced by I. Because $\|CR_n x\| \leq \|Cx\|$ and $\lim_{n\to\infty} CR_n x = Cx$ for all $x \in \mathcal{D}(C)$, using again the dominated convergence theorem we deduce that

$$\lim_{n\to\infty} \mathbb{E}\int_0^t f_n(s, X_s(x), X_s(y))\, ds = \mathbb{E}\int_0^t f(s, X_s(x), X_s(y))\, ds.$$

We now take the limit when $n \to \infty$ in (6.9) to obtain (6.4). □

6.2. Proof of Theorem 4.2

In order to prove Theorems 4.1, 4.2 and 4.3 we introduce the following two technical lemmata.

Lemma 6.3. *Suppose that ξ and χ belong to $L^2(\mathbb{P}, \mathfrak{h})$. Then $\mathbb{E}|\xi\rangle\langle\chi|$ defines an element of $\mathfrak{L}_1(\mathfrak{h})$, which moreover, is given by*

$$\langle x, \mathbb{E}|\xi\rangle\langle\chi| y\rangle = \mathbb{E}\langle x, \xi\rangle \langle \chi, y\rangle \tag{6.10}$$

for all $x, y \in \mathfrak{h}$. Here, $\mathbb{E}|\xi\rangle\langle\chi|$ is well-defined as a Bochner integral with values in both $\mathfrak{L}_1(\mathfrak{h})$ and $\mathfrak{L}(\mathfrak{h})$. In addition,

$$tr(\mathbb{E}|\xi\rangle\langle\chi|) = \mathbb{E}\langle\chi, \xi\rangle. \tag{6.11}$$

Proof. Since the image of $|\xi\rangle\langle\chi|$ lies in the set of all rank-one operators on $\mathfrak{h}$, $|\xi\rangle\langle\chi|$ takes values in $\mathfrak{L}_1(\mathfrak{h})$. Applying Parseval's equality yields

$$tr(A|\xi\rangle\langle\chi|) = \langle\chi, A\xi\rangle. \tag{6.12}$$

Hence $|\xi\rangle\langle\chi|$ is $\mathfrak{B}(\mathfrak{L}_1(\mathfrak{h}))$-measurable, because the dual of $\mathfrak{L}_1(\mathfrak{h})$ is formed by all maps $\varrho \mapsto tr(A\varrho)$ with $A \in \mathfrak{L}(\mathfrak{h})$.

Due to the absolute value of $|x\rangle\langle y|$ is $\|x\|\,|y\rangle\langle y|\,/\,\|y\|$ whenever $y \neq 0$, for any $x, y \in \mathfrak{h}$ we have

$$\||x\rangle\langle y|\|_1 = \|x\|\,\|y\|. \tag{6.13}$$

Combining $\xi, \chi \in L^2(\mathbb{P}, \mathfrak{h})$ with (6.13) shows that $\mathbb{E}\||\xi\rangle\langle\chi|\|_1 < \infty$, and so the Bochner integral $\mathbb{E}|\xi\rangle\langle\chi|$ is well-defined in the separable Banach space $\mathfrak{L}_1(\mathfrak{h})$. The application $(x,y) \mapsto |x\rangle\langle y|$ from $\mathfrak{h} \times \mathfrak{h}$ to $\mathfrak{L}(\mathfrak{h})$ is continuous, and in consequence the measurability of ξ and χ implies that $|\xi\rangle\langle\chi|$ is $\mathfrak{B}(\mathfrak{L}(\mathfrak{h}))$-measurable. Thus using $\|\cdot\|_{\mathfrak{L}(\mathfrak{h})} \leq \|\cdot\|_1$ we deduce that $|\xi\rangle\langle\chi|$ is Bochner $\mathbb{P}$-integrable in $\mathfrak{L}(\mathfrak{h})$ (see, e.g., [46] for a treatment of the Bochner integral in Banach spaces which in general are not separable). Since $\mathfrak{L}_1(\mathfrak{h})$ is continuously embedded in $\mathfrak{L}(\mathfrak{h})$, either of the interpretations of $\mathbb{E}|\xi\rangle\langle\chi|$ given above refers to the same operator.

For any x, y belonging to $\mathfrak{h}$, the linear function $A \mapsto \langle x, Ay\rangle$ is continuous as a map from $\mathfrak{L}(\mathfrak{h})$ to $\mathbb{C}$. This gives (6.10). Similarly, (6.12) leads to

$$tr(\mathbb{E}|\xi\rangle\langle\chi|) = \mathbb{E}tr(|\xi\rangle\langle\chi|) = \mathbb{E}\langle\chi, \xi\rangle,$$

because $tr(\cdot) \in \mathfrak{L}_1(\mathfrak{h})'$. □

Remark 6.1. *Under the assumptions of Lemma 6.3,* $\mathbb{E}\,|\xi\rangle\langle\chi|$ *can also be interpreted as a Bochner integral in the pointwise sense (see, e.g. [23] for details)*

Lemma 6.4. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. Suppose that $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$ and $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$. Then $A\xi$ belongs to $L^2(\mathbb{P},\mathfrak{h})$.*

Proof. By $A\xi = A\pi_C(\xi)$ $\mathbb{P}$-*a.s.*, from Remark 2.3 we deduce that $A\xi$ is strongly measurable. Since $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$ and $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$, $A\xi \in L^2(\mathbb{P},\mathfrak{h})$. □

Proof of Theorem 4.2. Let $x \in \mathcal{D}(C)$ and let $y \in \mathfrak{h}$. Using Lemma 6.3 yields

$$\langle Cx, \varrho y\rangle = \mathbb{E}\,\langle Cx, \xi\rangle\,\langle \xi, y\rangle = \mathbb{E}\,\langle x, C\xi\rangle\,\langle \xi, y\rangle\,.$$

In Lemma 6.4 we take $A = C$ to obtain $C\xi \in L^2(\mathbb{P},\mathfrak{h})$. Thus, Lemma 6.3 implies

$$\mathbb{E}\,\langle x, C\xi\rangle\,\langle \xi, y\rangle = \langle x, \mathbb{E}\,|C\xi\rangle\langle\xi|\,y\rangle\,,$$

and so $\langle Cx, \varrho y\rangle = \langle x, \mathbb{E}\,|C\xi\rangle\langle\xi|\,y\rangle$. Therefore $\varrho y \in \mathcal{D}(C^*) = \mathcal{D}(C)$ and $C\varrho y = \mathbb{E}\,|C\xi\rangle\langle\xi|\,y$. □

6.3. Proof of Theorem 4.1

Our proof starts with the easy construction of a random variable that represents a given C-regular operator. Then, we deduce the sufficient condition of Theorem 4.1 with the help of Theorem 4.2.

Lemma 6.5. *Suppose that ϱ belongs to $\mathfrak{L}^+_{1,C}(\mathfrak{h})$, where C is a self-adjoint positive operator in $\mathfrak{h}$. Then there exists ξ in $L^2_C(\mathbb{P},\mathfrak{h})$ such that $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$ and $\|\xi\|^2 = tr(\varrho)$ a.s.*

Proof. Assume that ϱ is positive. Write ϱ as in (4.1). Then, choose $\Omega = \mathfrak{I}$. Define $\mathbb{P}(\{n\}) = \lambda_n / tr(\varrho)$ and $\xi(n) = \sqrt{tr(\varrho)}u_n$ for any $n \in \mathfrak{I}$. According to (4.2) we have $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$. An elementary computation leads to $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$. On the other hand, take $\xi = 0$ in case $\varrho = 0$. □

Lemma 6.6. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. Suppose that $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$, with $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$. Then ϱ is C-regular.*

Proof. According to Lemma 6.3 we have $\varrho \in \mathfrak{L}^+_1(\mathfrak{h})$. Hence

$$\varrho = \sum_{n\in\mathfrak{I}} \lambda_n\,|u_n\rangle\langle u_n|\,,$$

where $\mathfrak{I}$ is a countable set, $(\lambda_n)_{n\in\mathfrak{I}}$ are summable positive real numbers and $(u_n)_{n\in\mathfrak{I}}$ is a orthonormal family of vectors of $\mathfrak{h}$. Using Theorem 4.2 yields $u_n \in \mathcal{D}(C)$ for all $n \in \mathfrak{I}$.

We extend $(u_n)_{n\in\mathfrak{I}}$ to an orthonormal basis $(e_n)_{n\in\mathfrak{I}'}$ of $\mathfrak{h}$ formed by elements of $\mathcal{D}(C)$. From Parseval's equality we obtain

$$\sum_{n\in\mathfrak{I}}\lambda_n\|Cu_n\|^2=\sum_{n\in\mathfrak{I}}\sum_{k\in\mathfrak{I}'}\lambda_n|\langle Cu_n,e_k\rangle|^2$$
$$=\sum_{k\in\mathfrak{I}'}\sum_{n\in\mathfrak{I}}\lambda_n\langle Ce_k,|u_n\rangle\langle u_n|\,Ce_k\rangle=\sum_{k\in\mathfrak{I}'}\langle Ce_k,\varrho Ce_k\rangle\,.$$

Combining Lemma 6.3 with Parseval's equality we now get

$$\sum_{n\in\mathfrak{I}}\lambda_n\|Cu_n\|^2=\sum_{k\in\mathfrak{I}'}\mathbb{E}\,|\langle\xi,Ce_k\rangle|^2=\mathbb{E}\sum_{k\in\mathfrak{I}'}|\langle C\xi,e_n\rangle|^2=\mathbb{E}\,\|C\xi\|^2\,.$$

This gives $\varrho\in\mathfrak{L}^+_{1,C}(\mathfrak{h})$. □

6.4. Proof of Theorem 4.3

In this subsection, we approximate $A\varrho B$ by $AR_n\varrho B$, where R_n is the Yosida approximation of $-C$. Then, combining Lemma 6.3 with a limit procedure we deduce our claim.

Proof of Theorem 4.3. Using Theorem 4.2 yields $\mathcal{D}(A\varrho B)=\mathcal{D}(B)$, and so $A\varrho B$ is densely defined.

Suppose that $x\in\mathfrak{h}$ and $y\in\mathcal{D}(B)$. As in the proof of Lemma 6.2 we consider $R_n=n(n+C)^{-1}$, where $n\in\mathbb{N}$. Since $CR_nz\longrightarrow_{n\to\infty}Cz$ for any $x\in\mathcal{D}(C)$,

$$\langle x,A\varrho By\rangle=\lim_{n\to\infty}\langle x,AR_n\varrho By\rangle\,.$$

Lemma 6.3 now gives

$$\begin{aligned}\langle x,A\varrho By\rangle &= \lim_{n\to\infty}\mathbb{E}\left\langle(AR_n)^*x,\xi\right\rangle\langle\xi,By\rangle \qquad (6.14)\\ &= \lim_{n\to\infty}\mathbb{E}\,\langle x,AR_n\xi\rangle\langle\xi,By\rangle\,.\end{aligned}$$

By R_n commutes with C, $\|R_n\|\le 1$ leads to $\|AR_nz\|\le K\|z\|_C$. Hence the dominated convergence theorem shows that

$$\lim_{n\to\infty}\mathbb{E}\,\langle x,AR_n\xi\rangle\langle\xi,By\rangle=\mathbb{E}\,\langle x,A\xi\rangle\langle B^*\xi,y\rangle\,. \qquad (6.15)$$

Due to B is densely defined, B^* is closed. From Remark 2.1 we have $B^*\in\mathfrak{L}((\mathcal{D}(C),\|\cdot\|_C),\mathfrak{h})$. Applying Lemma 6.4 gives $A\xi,B^*\xi\in L^2(\mathbb{P},\mathfrak{h})$. By (6.14), (6.15) and Lemma 6.3,

$$\langle x,A\varrho By\rangle=\mathbb{E}\,\langle x,A\xi\rangle\langle B^*\xi,y\rangle=\langle x,\mathbb{E}\,|A\xi\rangle\langle B^*\xi|\,y\rangle\,.$$

The proof is completed by using Lemma 6.3. □

6.5. Proof of Theorem 5.1

We first establish, in our framework, the well know relation between Heisenberg and Schrödinger pictures.

Lemma 6.7. *Let Hypothesis 2 hold. Suppose that ξ belongs to $L^2_C(\mathbb{P},\mathfrak{h})$. Then*

$$tr\left(A\mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right|\right) = tr\left(\mathcal{T}_t(A)\,\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|\right) \tag{6.16}$$

for any $A\in\mathfrak{L}(\mathfrak{h})$. Here $\mathcal{T}_t(A)$ is the bounded operator described by Theorem 3.1.

Proof. Suppose that $A\in\mathfrak{L}(\mathfrak{h})$. For each $n\in\mathbb{N}$, we define $f_n:\mathfrak{h}\to\mathbb{C}$ to be

$$f_n(x) = \begin{cases} \langle x, Ax\rangle, & if\ \ \|x\|\le n \\ 0, & if\ \ \|x\|>n \end{cases}.$$

Moreover, set $P_t f_n(x) = \mathbb{E}\left(f_n(X_t(x))\right)$ for any $x\in\mathcal{D}(C)$. By Theorem 2.1,

$$\mathbb{E}\left(f_n(X_t(\xi))\right) = \mathbb{E}P_t f_n(\xi). \tag{6.17}$$

The dominated convergence theorem leads to

$$\lim_{n\to\infty}\mathbb{E}\left(f_n(X_t(\xi))\right) = \mathbb{E}\left\langle X_t(\xi), AX_t(\xi)\right\rangle. \tag{6.18}$$

Combining (6.18) with Theorem 3.1 yields $P_t f_n(x)\longrightarrow_{n\to\infty}\langle x,\mathcal{T}_t(A)\,x\rangle$. Since $\|P_t f_n(x)\|\le\|A\|\,\|x\|^2$, letting $n\to\infty$ in (6.17) we obtain

$$\mathbb{E}\left\langle X_t(\xi), AX_t(\xi)\right\rangle = \mathbb{E}\left\langle \xi, \mathcal{T}_t(A)\,\xi\right\rangle.$$

Theorem 4.3 now gives (6.16). □

The following lemma states that $\rho_t(\varrho)$ is well defined by (5.1) whenever ϱ is a C-regular operator.

Lemma 6.8. *Adopt Hypothesis 2. Suppose that ξ and φ are elements of $L^2_C(\mathbb{P},\mathfrak{h})$ satisfying $\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right| = \mathbb{E}\left|\varphi\right\rangle\left\langle\varphi\right|$. Then*

$$\mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right| = \mathbb{E}\left|X_t(\varphi)\right\rangle\left\langle X_t(\varphi)\right|.$$

Proof. Let $A\in\mathfrak{L}(\mathfrak{h})$. Using Lemma 6.7 yields

$$tr\left(A\mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right|\right) = tr\left(\mathcal{T}_t(A)\,\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|\right) = tr\left(A\mathbb{E}\left|X_t(\varphi)\right\rangle\left\langle X_t(\varphi)\right|\right).$$

Hence $\left\|\mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right| - \mathbb{E}\left|X_t(\varphi)\right\rangle\left\langle X_t(\varphi)\right|\right\|_{\mathfrak{L}_1(\mathfrak{h})} = 0$ (see, e.g., Proposition 9.12 of [38]). □

We now establish the contraction property of the restriction of ρ_t to $\mathfrak{L}^+_{1,C}(\mathfrak{h})$.

Lemma 6.9. *Let Hypothesis 2 hold. If $\varrho,\widetilde{\varrho}$ are C-regular, then*

$$tr\left|\rho_t(\varrho)-\rho_t(\widetilde{\varrho})\right| \le tr\left|\varrho-\widetilde{\varrho}\right|. \tag{6.19}$$

Proof. According to Lemma 6.7 we have

$$\begin{aligned} tr\left|\rho_t\left(\varrho\right)-\rho_t\left(\widetilde{\varrho}\right)\right| &= \sup_{A\in\mathfrak{L}(\mathfrak{h}),\|A\|=1}\left|tr\left(A\rho_t\left(\varrho\right)\right)-tr\left(A\rho_t\left(\widetilde{\varrho}\right)\right)\right| \\ &= \sup_{A\in\mathfrak{L}(\mathfrak{h}),\|A\|=1}\left|tr\left(\mathcal{T}_t\left(A\right)\varrho\right)-tr\left(\mathcal{T}_t\left(A\right)\widetilde{\varrho}\right)\right| \\ &\leq \sup_{A\in\mathfrak{L}(\mathfrak{h}),\|A\|=1}\left\|\mathcal{T}_t\left(A\right)\right\| tr\left|\varrho-\widetilde{\varrho}\right|. \end{aligned}$$

From Theorem 3.1 we obtain (6.19). □

Finally, we extend ρ_t to a bounded linear operator in $\mathfrak{L}_1(\mathfrak{h})$ by using density arguments.

Lemma 6.10. *Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$. Then $\mathfrak{L}_{1,C}^{+}(\mathfrak{h})$ is dense in $\mathfrak{L}_1^{+}(\mathfrak{h})$ with respect to the trace norm.*

Proof. Let $\varrho \in \mathfrak{L}_1^{+}(\mathfrak{h})$. Then there exist a sequence of orthonormal vectors $(u_j)_{j\in\mathbb{N}}$ for which $\varrho=\sum_{j\in\mathbb{N}}\lambda_j\,|u_j\rangle\langle u_j|$, with $\lambda_j\geq 0$ and $\sum_{j\in\mathbb{N}}\lambda_j<\infty$. Note that

$$tr\left|\varrho-\sum_{j=1}^{n}\lambda_j\,|u_j\rangle\langle u_j|\right|=\sum_{j=n+1}^{\infty}\lambda_j\longrightarrow_{n\to\infty}0. \tag{6.20}$$

On the other hand, for any $x,y\in\mathfrak{h}$ we have

$$\begin{aligned} tr\left||x\rangle\langle x|-|y\rangle\langle y|\right| &= \sup_{A\in\mathfrak{L}(\mathfrak{h}),\|A\|=1}\left|\langle x,Ax\rangle-\langle y,Ay\rangle\right| \\ &\leq \|x-y\|^2+2\|y\|\,\|x-y\|. \end{aligned}$$

Therefore $\{|x\rangle\langle x| : x\in\mathcal{D}(C)\}$ is a $\|\cdot\|_{\mathfrak{L}_1(\mathfrak{h})}$-dense subset of $\{|x\rangle\langle x| : x\in\mathfrak{h}\}$ by $\mathcal{D}(C)$ is dense in $\mathfrak{h}$. Combining this property with (6.20) we deduce the assertion of the lemma. □

Proof of Theorem 5.1. Combining Theorem 4.1 with Lemma 6.8 we obtain that (5.1) defines unambiguously a linear operator $\rho_t(\varrho)$ for any $\varrho\in\mathfrak{L}_{1,C}^{+}(\mathfrak{h})$ and $t\geq 0$.

Suppose that $\varrho\in\mathfrak{L}_1^{+}(\mathfrak{h})$. By Lemma 6.10, there exists a sequence $(\varrho_n)_{n\in\mathbb{N}}$ of C-regular operators for which $\lim_{n\to\infty}\|\varrho-\varrho_n\|_{\mathfrak{L}_1(\mathfrak{h})}\to 0$. According to Lemma 6.9 we have that the limit in $\mathfrak{L}_1(\mathfrak{h})$ of $\rho_t(\varrho_n)$ as $n\to\infty$ exists and does not depend on the choice of $(\varrho_n)_{n\in\mathbb{N}}$. Thus, we define

$$\rho_t\left(\varrho\right)=\lim_{n\to\infty}\rho_t\left(\varrho_n\right)\qquad in\ \mathfrak{L}_1\left(\mathfrak{h}\right).$$

Recall that every A belonging to $\mathfrak{L}(\mathfrak{h})$ has a unique decomposition of the form $A=\Re(A)+i\,\Im(A)$, where $\Re(A)$ and $\Im(A)$ and selfadjoint operators in $\mathfrak{h}$. For each $\varrho\in\mathfrak{L}_1(\mathfrak{h})$, we set

$$\rho_t\left(\varrho\right)=\rho_t\left(\Re\left(\varrho\right)_+\right)-\rho_t\left(\Re\left(\varrho\right)_-\right)+i\left(\rho_t\left(\Im\left(\varrho\right)_+\right)-\rho_t\left(\Im\left(\varrho\right)_-\right)\right).$$

Here A_+, A_- denotes respectively the positive and negative parts of the selfadjoint operator A (see, e.g., [11, 40] for details).

Assume that $\varrho = \varrho_1 - \varrho_2 + i\left(\varrho_3 - \varrho_4\right)$, with $\varrho_j \in \mathfrak{L}_{1,C}^{+}\left(\mathfrak{h}\right)$ for any $j = 1, \ldots, 4$. Since $\left\|\mathcal{T}_t\left(A\right)\right\| \leq \left\|A\right\|$, Lemma 6.7 yields

$$\begin{aligned} tr\left|\rho_t\left(\varrho\right)\right| &= \sup_{A\in\mathfrak{L}(\mathfrak{h}),\|A\|=1}\left|tr\left(A\rho_t\left(\varrho\right)\right)\right| \\ &= \sup_{A\in\mathfrak{L}(\mathfrak{h}),\|A\|=1}\left|tr\left(\mathcal{T}_t\left(A\right)\varrho\right)\right| \leq tr\left(\left|\varrho\right|\right). \end{aligned}$$

The construction of $\rho_t\left(\varrho\right)$ now implies $\left\|\rho_t\left(\varrho\right)\right\|_{\mathfrak{L}_1(\mathfrak{h})} \leq \left\|\varrho\right\|_{\mathfrak{L}_1(\mathfrak{h})}$ for all $\varrho \in \mathfrak{L}_1\left(\mathfrak{h}\right)$.

Let $\alpha \geq 0$. Consider two C-regular operators $\varrho, \widetilde{\varrho}$. By Definition 4.1, $\varrho + \alpha\widetilde{\varrho}$ belongs to $\mathfrak{L}_{1,C}^{+}\left(\mathfrak{h}\right)$. Applying Lemma 6.7 we obtain that for any $A \in \mathfrak{L}\left(\mathfrak{h}\right)$,

$$\begin{aligned} tr\left(\rho_t\left(\varrho + \alpha\widetilde{\varrho}\right)A\right) &= tr\left(\mathcal{T}_t\left(A\right)\varrho\right) + \alpha tr\left(\mathcal{T}_t\left(A\right)\widetilde{\varrho}\right) \\ &= tr\left(\left(\rho_t\left(\varrho\right) + \alpha\rho_t\left(\widetilde{\varrho}\right)\right)A\right). \end{aligned}$$

Therefore $\left\|\rho_t\left(\varrho + \alpha\widetilde{\varrho}\right) - \rho_t\left(\varrho\right) - \alpha\rho_t\left(\widetilde{\varrho}\right)\right\|_{\mathfrak{L}_1(\mathfrak{h})} = 0$, and so Lemma 6.10 leads to $\rho_t\left(\varrho + \alpha\widetilde{\varrho}\right) = \rho_t\left(\varrho\right) + \alpha\rho_t\left(\widetilde{\varrho}\right)$ for any $\varrho, \widetilde{\varrho} \in \mathfrak{L}_1^{+}\left(\mathfrak{h}\right)$. Careful algebraic manipulations now show the linearity of $\rho_t : \mathfrak{L}_1\left(\mathfrak{h}\right) \to \mathfrak{L}_1\left(\mathfrak{h}\right)$.

Finally, Lemma 6.10 guarantees the uniqueness of the operator belonging to $\mathfrak{L}\left(\mathfrak{L}_1\left(\mathfrak{h}\right)\right)$ for which (5.1) holds. □

6.6. Proof of Corollary 5.1

Proof. Let ξ be distributed according to θ. Set $\widetilde{\mathbb{Q}} = \left\|X_T\left(\xi\right)\right\|^2 \cdot \mathbb{P}$, where T is a real number greater that t. Moreover, we choose

$$\widetilde{Y}_t = \begin{cases} X_t\left(\xi\right) / \left\|X_t\left(\xi\right)\right\|, & \text{if } X_t\left(\xi\right) \neq 0 \\ 0, & \text{if } X_t\left(\xi\right) = 0 \end{cases}.$$

Let $B_t^k = W_t^k - \int_0^t \frac{1}{\left\|X_s(\xi)\right\|^2} d\left[W^k, X\left(\xi\right)\right]_s$ for any $k \in \mathbb{N}$. From [27] (see also [36]) we have that $\left(\mathbb{Q}, \left(Y_t\right)_{t\in[0,T]}, \left(B_t^k\right)_{t\in[0,T]}^{k\in\mathbb{N}}\right)$ is a C-solution of (1.7) with initial law θ. According to [27] (see also [36]) we have that (1.7) has a unique C-solution with initial distribution θ. Therefore the distribution of $\widetilde{Y}_t$ with respect to $\widetilde{\mathbb{Q}}$ coincides with the distribution of Y_t under $\mathbb{Q}$. By Theorem 2.1, $\left(\left\|X_t\right\|^2\right)_{t\in[0,T]}$ is a martingale. Hence for any $x \in \mathfrak{h}$,

$$\mathbb{E}_{\mathbb{Q}}\left|\left\langle x, Y_t\right\rangle\right|^2 = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left|\left\langle x, \widetilde{Y}_t\right\rangle\right|^2 = \mathbb{E}_{\mathbb{P}}\left(\left|\left\langle x, \widetilde{Y}_t\right\rangle\right|^2\left\|X_t\left(\xi\right)\right\|^2\right) = \mathbb{E}_{\mathbb{P}}\left|\left\langle x, X_t\left(\xi\right)\right\rangle\right|^2.$$

Applying (5.1) and the polarization identity gives $\rho_t\left(\varrho\right) = \mathbb{E}\left|Y_t\right\rangle\left\langle Y_t\right|$. □

6.7. Proof of Theorem 5.2

We begin by establishing the continuity of the map $\xi \mapsto \rho_t\left(\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|\right)$.

Lemma 6.11. *Adopt Hypothesis 2. Let ξ and ξ_n, with $n \in \mathbb{N}$, be random variables in $L^2_C\left(\mathbb{P}, \mathfrak{h}\right)$ satisfying $\mathbb{E}\left\|\xi - \xi_n\right\|^2 \longrightarrow_{n\to\infty} 0$. Then $\rho_t\left(\mathbb{E}\left|\xi_n\right\rangle\left\langle\xi_n\right|\right)$ converges to $\rho_t\left(\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|\right)$ in $\mathfrak{L}\left(\mathfrak{h}\right)$ as $n \to \infty$.*

Proof. Let $x \in \mathfrak{h}$. Combining (5.1) with the linearity of (1.3) we get

$$\begin{aligned}&\left\|\rho_t\left(\mathbb{E}\left|\xi_n\right\rangle\left\langle\xi_n\right|\right)x - \rho_t\left(\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|\right)x\right\|\\ &\leq \mathbb{E}\left|\left\langle X_t\left(\xi_n\right), x\right\rangle\right|\left\|X_t\left(\xi_n - \xi\right)\right\| + \mathbb{E}\left|\left\langle X_t\left(\xi - \xi_n\right), x\right\rangle\right|\left\|X_t\left(\xi\right)\right\|\end{aligned}$$

Since $\mathbb{E}\left\|X_t\left(\eta\right)\right\|^2 \leq \mathbb{E}\left\|\eta\right\|^2$ for any η belonging to $L^2_C\left(\mathbb{P}, \mathfrak{h}\right)$,

$$\begin{aligned}&\left\|\rho_t\left(\mathbb{E}\left|\xi_n\right\rangle\left\langle\xi_n\right|\right)x - \rho_t\left(\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|\right)x\right\|\\ &\leq \left\|x\right\|\left(\mathbb{E}\left\|\xi - \xi_n\right\|^2 + 2\sqrt{\mathbb{E}\left\|\xi - \xi_n\right\|^2}\sqrt{\mathbb{E}\left\|\xi\right\|^2}\right).\end{aligned}$$

$\square$

Proof of Theorem 5.2. There exits a sequence $\left(\xi_n\right)_n$ of $\left(\mathcal{D}\left(C\right), \left\|\cdot\right\|\right)$-valued random variables with finite ranges such that $\left\|\xi_n - \xi\right\|$ converges monotonically to 0 (see, e.g., Lemma 1.1 of [23]). Since ρ_t is linear, an easy computation shows that $\mathbb{E}\rho_t\left(\left|\xi_n\right\rangle\left\langle\xi_n\right|\right) = \rho_t\left(\mathbb{E}\left|\xi_n\right\rangle\left\langle\xi_n\right|\right)$. By Lemma 6.11, $\rho_t\left(\mathbb{E}\left|\xi_n\right\rangle\left\langle\xi_n\right|\right)$ converges to $\rho_t\left(\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|\right)$ in $\mathfrak{L}\left(\mathfrak{h}\right)$. Hence

$$\mathbb{E}\rho_t\left(\left|\xi_n\right\rangle\left\langle\xi_n\right|\right) \longrightarrow_{n\to\infty} \rho_t\left(\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|\right) \qquad \textit{in } \mathfrak{L}\left(\mathfrak{h}\right). \tag{6.21}$$

On the other hand, Lemma 6.11 implies

$$\left\|\rho_t\left(\left|\xi_n\right\rangle\left\langle\xi_n\right|\right) - \rho_t\left(\left|\xi\right\rangle\left\langle\xi\right|\right)\right\|_{\mathfrak{L}\left(\mathfrak{h}\right)} \longrightarrow_{n\to\infty} 0.$$

For any $x, y \in \mathfrak{h}$ we have $\left\|\left|x\right\rangle\left\langle y\right|\right\| = \left\|x\right\|\left\|y\right\|$, and so

$$\left\|\rho_t\left(\left|\xi_n\right\rangle\left\langle\xi_n\right|\right)\right\| \leq \left\|\xi_n\right\|^2 \leq 2\left(\left\|\xi_1 - \xi\right\|^2 + \left\|\xi\right\|^2\right).$$

Therefore the dominated convergence theorem leads to

$$\mathbb{E}\left\|\rho_t\left(\left|\xi_n\right\rangle\left\langle\xi_n\right|\right) - \rho_t\left(\left|\xi\right\rangle\left\langle\xi\right|\right)\right\|_{\mathfrak{L}\left(\mathfrak{h}\right)} \longrightarrow_{n\to\infty} 0.$$

Consequently, $\mathbb{E}\rho_t\left(\left|\xi_n\right\rangle\left\langle\xi_n\right|\right)$ converges to $\mathbb{E}\rho_t\left(\left|\xi\right\rangle\left\langle\xi\right|\right)$ in $\mathfrak{L}\left(\mathfrak{h}\right)$. The theorem now follows from (6.21). $\square$

6.8. Proof of Theorem 5.3

We first obtain the semigroup property of $(\rho_t)_{t\geq 0}$.

Lemma 6.12. *Let Hypothesis 2 hold. Suppose that ϱ is C-regular. Then $\rho_t(\varrho)$ belongs to $\mathfrak{L}_{1,C}^{+}(\mathfrak{h})$, with $t\geq 0$, and $\rho_{t+s}(\varrho)=\rho_t\circ\rho_s(\varrho)$ for all $s,t\geq 0$.*

Proof. Since $X_t(\xi)\in L_C^2(\mathbb{P},\mathfrak{h})$, combining Theorem 4.1 with (5.1) gives

$$\rho_t\left(\mathfrak{L}_{1,C}^{+}(\mathfrak{h})\right)\subset\mathfrak{L}_{1,C}^{+}(\mathfrak{h}). \tag{6.22}$$

Let x,y belong to $\mathfrak{h}$. Consider $\xi\in L_C^2(\mathbb{P},\mathfrak{h})$ satisfying $\rho=\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$. Define

$$p_n(z)=\begin{cases}\langle z,x\rangle\langle y,z\rangle, & if\ \ |\langle z,x\rangle\langle y,z\rangle|\leq n\\ n\langle z,x\rangle\langle y,z\rangle/|\langle z,x\rangle\langle y,z\rangle|, & if\ \ |\langle z,x\rangle\langle y,z\rangle|>n\end{cases},$$

where $z\in\mathfrak{h}$ and $n\in\mathbb{N}$. According to Theorem 2.1, we have

$$\mathbb{E}\left(p_n\left(X_{t+s}(\xi)\right)\right)=\mathbb{E}P_t\left(p_n\right)\left(X_s(\xi)\right), \tag{6.23}$$

with $P_t(p_n)(z)=\mathbb{E}\left(p_n\left(X_t(z)\right)\right)$ for all $z\in\mathcal{D}(C)$.

Applying the dominated convergence theorem gives

$$\lim_{n\to\infty}\mathbb{E}\left(p_n\left(X_r(z)\right)\right)=\mathbb{E}\left\langle X_r(z),x\right\rangle\left\langle y,X_r(z)\right\rangle=\left\langle y,\rho_r\left(|z\rangle\langle z|\right)x\right\rangle$$

for any $z\in\mathcal{D}(C)$ and $r\geq 0$. Thus $\lim_{n\to\infty}P_t(p_n)(z)=\langle y,\rho_t(|z\rangle\langle z|)x\rangle$ and $\lim_{n\to\infty}\mathbb{E}\left(p_n\left(X_{t+s}(\xi)\right)\right)=\langle y,\rho_{t+s}(|z\rangle\langle z|)x\rangle$. By the dominated convergence theorem, letting $n\to\infty$ in (6.23) gives

$$\langle y,\rho_{t+s}(\varrho)x\rangle=\mathbb{E}\left\langle y,\rho_t\left(|X_s(\xi)\rangle\langle X_s(\xi)|\right)x\right\rangle.$$

Theorem 5.2 now shows that $\langle y,\rho_{t+s}(\varrho)x\rangle=\langle y,\rho_t\left(\mathbb{E}\left|X_s(\xi)\right\rangle\left\langle X_s(\xi)\right|\right)x\rangle$. □

Lemma 6.13. *Under Hypothesis 2, $(\rho_t)_{t\geq 0}$ is a semigroup of contractions which leaves $\mathfrak{L}_1^{+}(\mathfrak{h})$ invariant.*

Proof. Recall that Theorem 5.1 asserts that $\|\rho_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))}\leq 1$. Since $\rho_t(\varrho)$ is positive whenever ϱ is C-regular, using Lemma 6.10 yields $\langle x,\rho_t(\varrho)x\rangle\geq 0$ for any $\varrho\in\mathfrak{L}_1^{+}(\mathfrak{h})$ and $x\in\mathfrak{h}$.

Suppose that $\varrho=\varrho_1-\varrho_2+i(\varrho_3-\varrho_4)$, where $\varrho_1,\ldots,\varrho_4$ are C-regular operators. Applying (5.1) gives $\rho_0(\varrho)=\varrho$. From Lemma 6.12 we obtain $\rho_{t+s}(\varrho)=\rho_t\circ\rho_s(\varrho)$ for any $s,t\geq 0$. Then, combining Lemma 6.10 with density arguments we deduce that $(\rho_t)_{t\geq 0}$ is a semigroup. □

In order to prove the continuity of the map $t\mapsto\rho_t(\varrho)$ when ϱ is C-regular, we next verifies the continuity of the function $t\mapsto X_t(\xi)$ in the mean square sense.

Lemma 6.14. *Let Hypothesis 2 hold and let $\xi\in L_C^2(\mathbb{P},\mathfrak{h})$. Then for any $T\geq 0$,*

$$\lim_{s\to T}\mathbb{E}\left\|X_s(\xi)-X_t(\xi)\right\|^2=0. \tag{6.24}$$

Proof. Suppose that $t \geq s \geq 0$. Then $\mathbb{E}\left\| X_s(\xi) - X_t(\xi) \right\|^2$ is less than or equal to $2\left(\mathbb{E}\left\| \int_s^t GX_r(\xi)\, dr \right\|^2 + \mathbb{E}\left\| \sum_{k=1}^{\infty} \int_s^t L_k X_r(\xi)\, dW_r^k \right\|^2 \right)$. By Condition H2.2 of Hypothesis 2, $\mathbb{E}\left\| X_s(\xi) - X_t(\xi) \right\|^2$ is less than or equal to

$$2\left((t-s) \int_s^t \mathbb{E}\left\| GX_r(\xi) \right\|^2 dr - \int_s^t 2\mathbb{E}\Re \left\langle X_r(\xi), GX_r(\xi) \right\rangle dr \right).$$

Using $\sup_{s\in[0,T+1]} \mathbb{E}\left\| CX_s(\xi) \right\|^2 < \infty$ and $G \in \mathfrak{L}\left(\left(\mathcal{D}(C), \left\| \cdot \right\|_C \right), \mathfrak{h} \right)$ gives (6.24). □

Lemma 6.15. *Adopt Hypothesis 2, together with $\varrho \in \mathfrak{L}_{1,C}^{+}(\mathfrak{h})$. Then the map $t \mapsto \rho_t(\varrho)$ from $[0,\infty[$ to $\mathfrak{L}_1(\mathfrak{h})$ is continuous.*

Proof. Consider $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$ such that $\varrho = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$. Theorem 4.3 yields

$$\mathbb{E}\left\| X_t(\xi) \right\|^2 \leq \mathbb{E}\left\| \xi \right\|^2 = tr(\varrho)$$

for any $t \geq 0$. By Theorem 4.3 and the Cauchy-Schwarz inequality,

$$\begin{aligned} tr\left| \rho_t(\varrho) - \rho_s(\varrho) \right| &= \sup_{A\in\mathfrak{L}(\mathfrak{h}), \|A\|=1} \left| \mathbb{E}\left\langle X_t(\xi), AX_t(\xi) \right\rangle - \left\langle X_s(\xi), AX_s(\xi) \right\rangle \right| \\ &\leq 2\left(tr(\varrho) \right)^{1/2} \left(\mathbb{E}\left\| X_t(\xi) - X_s(\xi) \right\|^2 \right)^{1/2}. \end{aligned}$$

Lemma 6.14 now implies (5.2). □

6.9. Proof of Theorem 5.4

First, we establish the weak continuity of the map $t \mapsto AX_t(\xi)$ when A is relatively bounded by C. Hence a probabilistic version of the right-hand side of (5.4) is continuous as a function from $[0, +\infty[$ to $\mathbb{C}$.

Lemma 6.16. *Assume that Hypothesis 2 holds. If ξ belongs to $L_C^2(\mathbb{P}, \mathfrak{h})$ and if A lies in $\mathfrak{L}\left(\left(\mathcal{D}(C), \left\| \cdot \right\|_C \right), \mathfrak{h} \right)$, then*

$$\lim_{s\to t} \mathbb{E}\left\langle \psi, AX_s(\xi) \right\rangle = \mathbb{E}\left\langle \psi, AX_t(\xi) \right\rangle \tag{6.25}$$

for all $\psi \in L^2(\mathbb{P}, \mathfrak{h})$ and $t \geq 0$.

Proof. Let $(s_n)_n$ be a sequence of non-negative real numbers converging to t. Since $\left(\left(X_{s_n}(\xi), AX_{s_n}(\xi), CX_{s_n}(\xi) \right) \right)_n$ is a bounded sequence in $L^2\left(\mathbb{P}, \mathfrak{h}^3 \right)$ with $\mathfrak{h}^3 = \mathfrak{h} \times \mathfrak{h} \times \mathfrak{h}$, there exists a subsequence $\left(s_{n(k)} \right)_k$ for which

$$\begin{aligned} &\left(X_{s_{n(k)}}(\xi), AX_{s_{n(k)}}(\xi), CX_{s_{n(k)}}(\xi) \right) \\ &\longrightarrow_{k\to\infty} (Y, U, V) \quad \text{weakly in } L^2\left(\mathbb{P}, \mathfrak{h}^3 \right). \end{aligned} \tag{6.26}$$

Set

$$\mathfrak{M} = \left\{ (\eta, A\eta, C\eta) : \eta \in L_C^2(\mathbb{P}, \mathfrak{h}) \right\}.$$

Then $\mathfrak{M}$ is a linear manifold of $L^2\left(\mathbb{P},\mathfrak{h}^3\right)$ closed with respect to the strong topology. In fact, suppose that $\left(\left(\eta_n, A\eta_n, C\eta_n\right)\right)_n$ is a sequence of elements of $\mathfrak{M}$ that converges to $\left(\eta_1,\eta_2,\eta_3\right)$ in $L^2\left(\mathbb{P},\mathfrak{h}^3\right)$. Hence there exists a subsequence $\left(\left(\eta_{n(j)}, A\eta_{n(j)}, C\eta_{n(j)}\right)\right)_{j\in\mathbb{N}}$ converging almost surely to $\left(\eta_1,\eta_2,\eta_3\right)$. Therefore $\eta_1\in\mathcal{D}(C)$ and $\eta_3 = C\eta_1$ by C is closed. Using $A\in\mathfrak{L}\left(\left(\mathcal{D}(C),\left\|\cdot\right\|_C\right),\mathfrak{h}\right)$ gives $\eta_2 = A\eta_1$.

For any $k\in\mathbb{N}$, $\left(X_{s_{n(k)}}(\xi), AX_{s_{n(k)}}(\xi), CX_{s_{n(k)}}(\xi)\right)$ belongs to $\mathfrak{M}$. Since $\mathfrak{M}$ is a closed linear manifold of $L^2\left(\mathbb{P},\mathfrak{h}^3\right)$, (6.26) implies $(Y,U,V)\in\mathfrak{M}$ (see, e.g., Subsection III.1.6 of [32]). We conclude from Lemma 6.14 that $Y = X_t(\xi)$, hence that $U = AX_t(\xi)$, and finally that $AX_{s_{n(k)}}(\xi)\longrightarrow_{k\to\infty} AX_t(\xi)$ weakly in $L^2(\mathbb{P},\mathfrak{h})$. □

Lemma 6.17. *Let Hypothesis 2 hold. Fix* $\xi\in L^2_C(\mathbb{P},\mathfrak{h})$ *and* $A\in\mathfrak{L}(\mathfrak{h})$. *Then*

$$
\begin{aligned}
t\mapsto\quad & \mathbb{E}\left\langle GX_t(\xi), AX_t(\xi)\right\rangle + \mathbb{E}\left\langle X_t(\xi), AGX_t(\xi)\right\rangle \\
& + \sum_{k=1}^{\infty}\mathbb{E}\left\langle L_kX_t(\xi), AL_kX_t(\xi)\right\rangle
\end{aligned}
$$

is continuous as a function from $[0,+\infty[$ *to* $\mathbb{C}$.

Proof. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that t_n converges to t. Applying Lemma 6.14 gives $AX_{t_n}(\xi)\longrightarrow_{n\to\infty} AX_t(\xi)$ in $L^2(\mathbb{P},\mathfrak{h})$, and so

$$
\lim_{n\to\infty}\mathbb{E}\left\langle GX_{t_n}(\xi), AX_{t_n}(\xi)\right\rangle = \mathbb{E}\left\langle GX_t(\xi), AX_t(\xi)\right\rangle \tag{6.27}
$$

by Lemma 6.16 (see, e.g., Subsection III.1.7 of [32]). From (6.27) we obtain that the map $t\mapsto\mathbb{E}\left\langle A^*X_t(\xi), GX_t(\xi)\right\rangle$ is continuous. Moreover, according to (6.27) we have $\mathbb{E}\Re\left\langle X_{t_n}(\xi), GX_{t_n}(\xi)\right\rangle\to_{n\to\infty}\mathbb{E}\Re\left\langle X_t(\xi), GX_t(\xi)\right\rangle$. Thus Condition H2.2 of Hypothesis 2 leads to

$$
\sum_{k=1}^{\infty}\mathbb{E}\left\|L_kX_{t_n}(\xi)\right\|^2\longrightarrow_{n\to\infty}\sum_{k=1}^{\infty}\mathbb{E}\left\|L_kX_t(\xi)\right\|^2. \tag{6.28}
$$

Applying Lemma 6.16 we get that $L_kX_{t_n}(\xi)$ converges weakly in $L^2(\mathbb{P},\mathfrak{h})$ to $L_kX_t(\xi)$ as n tends to ∞, where $k\in\mathbb{N}$. Then (6.28) implies that $L_kX_{t_n}(\xi)$ converges strongly in $L^2(\mathbb{P},\mathfrak{h})$ to $L_kX_t(\xi)$ as $n\to\infty$. Conversely, suppose that

$$
\limsup_{n\to\infty}\mathbb{E}\left\|L_jX_{t_n}(\xi)\right\|^2 > \mathbb{E}\left\|L_jX_t(\xi)\right\|^2 \tag{6.29}
$$

for a given $j\in\mathbb{N}$. Since $\mathbb{E}\left\|L_kX_t(\xi)\right\|^2\le\liminf_{n\to\infty}\mathbb{E}\left\|L_kX_{t_n}(\xi)\right\|^2$, Fatou's lemma shows

$$
\sum_{k\neq j}\mathbb{E}\left\|L_kX_t(\xi)\right\|^2\le\liminf_{n\to\infty}\sum_{k\neq j}\mathbb{E}\left\|L_kX_{t_n}(\xi)\right\|^2. \tag{6.30}
$$

According to (6.28) and (6.29) we have

$$\begin{aligned}\liminf_{n\to\infty}\sum_{k\neq j}\mathbb{E}\left\|L_kX_{t_n}(\xi)\right\|^2 &= \sum_{k=1}^{\infty}\mathbb{E}\left\|L_kX_t(\xi)\right\|^2-\limsup_{n\to\infty}\mathbb{E}\left\|L_jX_{t_n}(\xi)\right\|^2\\ &< \sum_{k\neq j}\mathbb{E}\left\|L_kX_t(\xi)\right\|^2,\end{aligned}$$

contrary to (6.30). Therefore $\limsup_{n\to\infty}\mathbb{E}\left\|L_jX_{t_n}(\xi)\right\|^2\leq\mathbb{E}\left\|L_jX_t(\xi)\right\|^2$, and so $L_kX_{t_n}(\xi)\longrightarrow_{n\to\infty}L_kX_t(\xi)$ in $L^2(\mathbb{P},\mathfrak{h})$. This gives

$$\mathbb{E}\left\langle L_kX_{t_n}(\xi),AL_kX_{t_n}(\xi)\right\rangle\longrightarrow_{n\to\infty}\mathbb{E}\left\langle L_kX_t(\xi),AL_kX_t(\xi)\right\rangle.$$

From Condition H2.2 it follows that $\sum_{k=1}^{n}\mathbb{E}\left\langle L_kX_t(\xi),AL_kX_t(\xi)\right\rangle$ converges to $\sum_{k=1}^{\infty}\mathbb{E}\left\langle L_kX_t(\xi),AL_kX_t(\xi)\right\rangle$ as $n\to\infty$ uniformly on any finite interval. Then $t\mapsto\sum_{k=1}^{\infty}\mathbb{E}\left\langle L_kX_t(\xi),AL_kX_t(\xi)\right\rangle$ is continuous. □

Second, combining the regularity of $X(\xi)$ with Itô's formula we obtain that $\mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right|$ satisfies an integral version of (1.1).

Lemma 6.18. *Suppose that Hypothesis 2 hold and that $\xi\in L^2_C(\mathbb{P},\mathfrak{h})$. We define $\mathcal{L}_*(\xi,t)$ to be*

$$\mathbb{E}\left|GX_t(\xi)\right\rangle\left\langle X_t(\xi)\right|+\mathbb{E}\left|X_t(\xi)\right\rangle\left\langle GX_t(\xi)\right|+\sum_{k=1}^{\infty}\mathbb{E}\left|L_kX_t(\xi)\right\rangle\left\langle L_kX_t(\xi)\right|.$$

Then $\mathcal{L}_*(\xi,t)$ is a trace-class operator on $\mathfrak{h}$ whose trace-norm is uniformly bounded with respect to t on bounded time intervals. The series involved in the definition of $\mathcal{L}_*$ converges in $\mathfrak{L}_1(\mathfrak{h})$. Moreover, the application $t\mapsto tr(A\mathcal{L}_*(\xi,t))$ is continuous as a function from $[0,\infty[$ to $\mathbb{C}$ for any $A\in\mathfrak{L}(\mathfrak{h})$.

Proof. By Condition H2.2 of Hypothesis 2, using (6.13) and Lemma 6.3 we get

$$\begin{aligned}&\left\|\mathbb{E}\left|GX_s(\xi)\right\rangle\left\langle X_s(\xi)\right|\right\|_1+\left\|\mathbb{E}\left|X_s(\xi)\right\rangle\left\langle GX_s(\xi)\right|\right\|_1\\&+\sum_{k=1}^{\infty}\left\|\mathbb{E}\left|L_kX_s(\xi)\right\rangle\left\langle L_kX_s(\xi)\right|\right\|_1\leq 3\mathbb{E}\left(\left\|X_t(\xi)\right\|\left\|GX_t(\xi)\right\|\right).\end{aligned}$$

From $G\in\mathfrak{L}\left(\left(\mathcal{D}(C),\left\|\cdot\right\|_C\right),\mathfrak{h}\right)$ we obtain

$$\mathbb{E}\left(\left\|X_t(\xi)\right\|\left\|GX_t(\xi)\right\|\right)\leq K\sqrt{\mathbb{E}\left\|\xi\right\|^2}\sqrt{\mathbb{E}\left\|X_t(\xi)\right\|_C^2},$$

and the first two assertions of the lemma follow.

Using Lemma 6.3 we deduce that $tr(A\mathcal{L}_*(\xi,t))$ is equal to

$$\mathbb{E}\left\langle X_t(\xi),AGX_t(\xi)\right\rangle+\mathbb{E}\left\langle GX_t(\xi),AX_t(\xi)\right\rangle+\sum_{k=1}^{\infty}\mathbb{E}\left\langle L_kX_t(\xi),AL_kX_t(\xi)\right\rangle,$$

whenever $A\in\mathfrak{L}(\mathfrak{h})$. Thus Lemmata 6.17 implies the continuity of the function $t\mapsto tr(A\mathcal{L}_*(\xi,t))$. □

Lemma 6.19. *Adopt Hypothesis 2 together with* $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$. *Then for all* $t \geq 0$

$$\rho_t(\varrho) = \varrho + \int_0^t \mathcal{L}_*(\xi,s)\,ds, \tag{6.31}$$

where $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$, $\mathcal{L}_*(\xi,s)$ *is as in Lemma 6.18, and we understand the above integral in the sense of Bochner integral in* $\mathfrak{L}_1(\mathfrak{h})$.

Proof. Since the dual of $\mathfrak{L}_1(\mathfrak{h})$ consists in all linear maps $\varrho \mapsto tr(A\varrho)$ with $A \in \mathfrak{L}(\mathfrak{h})$, the last assertion of Lemma 6.18 implies that $t \mapsto \mathcal{L}_*(\xi,t)$ is measurable as a function from $[0,\infty[$ to $\mathfrak{L}_1(\mathfrak{h})$. Using Lemma 6.18 we deduce that $t \mapsto \mathcal{L}_*(\xi,t)$ is a Bochner integrable $\mathfrak{L}_1(\mathfrak{h})$-valued function on bounded intervals.

Let $x \in \mathfrak{h}$. For each $n \in \mathbb{N}$, we choose $\tau_n = \inf\{s \geq 0 : \|X_s(\xi)\| > n\}$. Then, using Condition H2.2 of Hypothesis 2 yields

$$\begin{aligned}&\mathbb{E}\sum_{k=1}^{\infty}\int_0^{t\wedge\tau_n} \left\|\langle X_s(\xi),x\rangle L_k X_s(\xi) + \langle L_k X_s(\xi),x\rangle X_s(\xi)\right\|^2 ds \\ &\leq -4n^3\|x\|^2\,\mathbb{E}\int_0^{t\wedge\tau_n}\|GX_s\|\,ds.\end{aligned}$$

Therefore $\mathbb{E}\sum_{k=1}^{\infty}\int_0^{t\wedge\tau_n}\left(\langle X_s(\xi),x\rangle L_k X_s(\xi) + \langle L_k X_s(\xi),x\rangle X_s(\xi)\right)dW_s^k = 0$ by G belongs to $\mathfrak{L}\left(\left(\mathcal{D}(C),\|\cdot\|_C\right),\mathfrak{h}\right)$. The complex Itô formula now leads to

$$\mathbb{E}\left\langle X_{t\wedge\tau_n}(\xi),x\right\rangle X_{t\wedge\tau_n}(\xi) = \mathbb{E}\langle\xi,x\rangle\xi + \mathbb{E}\int_0^{t\wedge\tau_n} L_x(X_s(\xi))\,ds, \tag{6.32}$$

where for any $z \in \mathcal{D}(C)$ we write

$$L_x(z) = \langle z,x\rangle Gz + \langle Gz,x\rangle z + \sum_{k=1}^{\infty}\langle L_k z,x\rangle L_k z.$$

Since $X(\xi)$ has continuous sample paths, $\tau_n \nearrow_{n\to\infty} \infty$. By Conditions H2.1 and H2.2 of Hypothesis 2, applying the dominated convergence we obtain

$$\lim_{n\to\infty}\mathbb{E}\int_0^{t\wedge\tau_n} L_x(X_s(\xi))\,ds = \mathbb{E}\int_0^t L_x(X_s(\xi))\,ds. \tag{6.33}$$

Set $\epsilon_n = \langle X_t(\xi),x\rangle X_t(\xi) - \left\langle X_{t\wedge\tau_n}(\xi),x\right\rangle X_{t\wedge\tau_n}(\xi)$. From Theorem 2.1 we have $\mathbb{E}\|X_t(\xi)\|^2 = \mathbb{E}\|\xi\|^2$, and so Fatou's lemma implies

$$2\|x\|\,\mathbb{E}\|\xi\|^2 \leq \liminf_{n\to\infty}\mathbb{E}\left(\|x\|\,\|X_t(\xi)\|^2 + \|x\|\,\|X_{t\wedge\tau_n}(\xi)\|^2 - |\epsilon_n|\right). \tag{6.34}$$

In the proof of Theorem 2.1 we deduced that $\mathbb{E}\|X_{t\wedge\tau_n}(\xi)\|^2 = \mathbb{E}\|\xi\|^2$, by means of Itô's formula and Condition H2.2. Thus

$$2\|x\|\,\mathbb{E}\|\xi\|^2 \leq 2\|x\|\,\mathbb{E}\|\xi\|^2 - \limsup_{n\to\infty}\mathbb{E}|\epsilon_n|.$$

Consequently $\lim_{n\to\infty} \mathbb{E}\left|\epsilon_n\right| = 0$, and so

$$\lim_{n\to\infty} \mathbb{E}\left\langle X_{t\wedge\tau_n}\left(\xi\right), x\right\rangle X_{t\wedge\tau_n}\left(\xi\right) = \mathbb{E}\left\langle X_t\left(\xi\right), x\right\rangle X_t\left(\xi\right). \tag{6.35}$$

Due to (6.33) and (6.35), letting first $n \to \infty$ in (6.32) and then using Fubini's theorem we get

$$\mathbb{E}\left\langle X_t\left(\xi\right), x\right\rangle X_t\left(\xi\right) = \mathbb{E}\left\langle \xi, x\right\rangle \xi + \int_0^t \mathbb{E}L_x\left(X_s\left(\xi\right)\right). \tag{6.36}$$

Combining the dominated convergence theorem with Condition H2.2 yields

$$\mathbb{E}\sum_{k=1}^{\infty}\left\langle L_k X_s\left(\xi\right), x\right\rangle L_k X_s\left(\xi\right) = \sum_{k=1}^{\infty}\mathbb{E}\left\langle L_k X_s\left(\xi\right), x\right\rangle L_k X_s\left(\xi\right).$$

Therefore $\int_0^t \mathbb{E}L_x\left(X_s\left(\xi\right)\right) = \int_0^t \mathcal{L}_*\left(\xi, s\right) x ds$ by Lemmata 6.3 and 6.18. Combining Theorem 5.1 with Lemma 6.18 leads to (6.31). □

We proceed to show (5.3) and (5.4) with the help of Hypothesis 4.

Proof of Theorem 5.4. By Theorem 4.1 there exists $\xi \in L^2_C\left(\mathbb{P}, \mathfrak{h}\right)$ such that $\varrho = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$. Theorem 4.3 gives $AG\rho_t\left(\varrho\right) = \mathbb{E}\left|AGX_t\left(\xi\right)\right\rangle\left\langle X_t\left(\xi\right)\right|$. Applying Hypothesis 4 we get that $G^*, L_1^*, L_2^*, \ldots$ are densely defined and $G^{**}, L_1^{**}, \ldots$ coincide with the closures of $G, L_1, \ldots$ respectively (see, e.g., Theorem III.5.29 of [32]). According to Theorem 4.3 we have $A\rho_t\left(\varrho\right) G^* = \mathbb{E}\left|AX_t\left(\xi\right)\right\rangle\left\langle GX_t\left(\xi\right)\right|$ and $AL_k\rho_t\left(\varrho\right) L_k^* = \mathbb{E}\left|AL_kX_t\left(\xi\right)\right\rangle\left\langle L_kX_t\left(\xi\right)\right|$. Hence $\mathcal{L}_*\left(\xi, t\right) = G\rho_t\left(\varrho\right) + \rho_t\left(\varrho\right) G^* + \sum_{k=1}^{\infty} L_k\rho_t\left(\varrho\right) L_k^*$. Lemma 6.19 now yields (5.3), and so

$$tr\left(A\rho_t\left(\varrho\right)\right) = tr\left(A\varrho\right) + \int_0^t tr\left(A\mathcal{L}_*\left(\xi, s\right)\right) ds$$

for all $t \geq 0$. Using the continuity of $\mathcal{L}_*\left(\xi, \cdot\right)$ we obtain (5.4). □

6.10. Proof of Theorem 5.5

Lemma 6.20. *Let Hypothesis 2 hold. Then $\left(\rho_t\right)_{t\geq 0}$ is a semigroup of bounded operators on $\mathfrak{L}_1\left(\mathfrak{h}\right)$ that satisfies Properties (i)-(iii) of Theorem 5.5.*

Proof. According to Theorem 5.3 we have that $\left(\rho_t\right)_{t\geq 0}$ is a semigroup of bounded operators on $\mathfrak{L}_1\left(\mathfrak{h}\right)$ satisfying Property (i) of Theorem 5.5.

Fix $\varrho = \left|x\right\rangle\left\langle x\right|$, with $x \in \mathcal{D}\left(C\right)$. Thus ϱ is a C-regular operator, and so (5.2) leads to Property (ii). By Lemma 6.18, using Lemma 6.19 we obtain

$$\lim_{t\to 0+}\frac{1}{t}\left(tr\left(A\widehat{\rho}_t\left(\varrho\right)\right) - tr\left(A\varrho\right)\right) = tr\left(A\mathcal{L}_*\left(x, 0\right)\right).$$

Lemma 6.3 yields $tr\left(A\mathcal{L}_*\left(x, 0\right)\right) = \left\langle x, AGx\right\rangle + \left\langle Gx, Ax\right\rangle + \sum_{k=1}^{\infty}\left\langle L_kx, AL_kx\right\rangle$. □

Lemma below makes it legitimate to use in our context the duality relation between quantum master equations and adjoint quantum master equations.

Lemma 6.21. *Assume that Hypothesis 2 holds, together with $A \in \mathfrak{L}(\mathfrak{h})$. Let $(\widehat{\rho}_t)_{t\geq 0}$ be a semigroup of bounded operators on $\mathfrak{L}_1(\mathfrak{h})$ satisfying Properties (i)-(iii) of Theorem 5.5. Then $(\widehat{\rho}_t^*(A))_{t\geq 0}$ is a C-solution of (1.5) with initial datum A. Here $(\widehat{\rho}_t^*)_{t\geq 0}$ is the adjoint semigroup of $(\widehat{\rho}_t)_{t\geq 0}$ (see, e.g., Section 1.10 of [39] for details), that is, $(\widehat{\rho}_t^*)_{t\geq 0}$ is the unique semigroup of bounded operators on $\mathfrak{L}(\mathfrak{h})$ such that*

$$tr\left(\widehat{\rho}_t(\varrho) B\right) = tr\left(\widehat{\rho}_t^*(B)\varrho\right) \tag{6.37}$$

for all $B \in \mathfrak{L}(\mathfrak{h})$ and $\varrho \in \mathfrak{L}_1(\mathfrak{h})$.

Proof. Using (6.37) we deduce that for any vectors $x, y \in \mathfrak{h}$ whose norm is 1,

$$\begin{aligned} \left|\langle y, \widehat{\rho}_t^*(A) x\rangle\right| &= \left|tr\left(\widehat{\rho}_t^*(A)|x\rangle\langle y|\right)\right| \\ &\leq tr\left(\left|\widehat{\rho}_t(|x\rangle\langle y|)A\right|\right) \leq \|A\| \|\widehat{\rho}_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))} tr\left(\left||x\rangle\langle y|\right|\right). \end{aligned}$$

We conclude from (6.13) that $tr\left(\left||x\rangle\langle y|\right|\right) = 1$, hence that $\left|\langle y, \widehat{\rho}_t^*(A) x\rangle\right| \leq \|A\| \|\widehat{\rho}_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))}$, and finally that

$$\|\widehat{\rho}_t^*(A)\|_{\mathfrak{L}(\mathfrak{h})} \leq \|A\| \|\widehat{\rho}_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))}. \tag{6.38}$$

Applying Property (i) of Theorem 5.5 gives Property (c) of Definition 3.1.

Let $x \in \mathfrak{h}$. As in the proofs of Lemma 6.2 and Theorem 4.3, we define R_n to be $n(n+C)^{-1}$, $n \in \mathbb{N}$. According to (6.37) we have

$$\langle R_n x, \widehat{\rho}_t^*(A) R_n x\rangle = tr\left(\widehat{\rho}_t^*(A)|R_n x\rangle\langle R_n x|\right) = tr\left(\widehat{\rho}_t(|R_n x\rangle\langle R_n x|)A\right).$$

Since $R_n x \in \mathcal{D}(C)$, Property (ii) of Theorem 5.5 implies the continuity of the function $t \mapsto \langle R_n x, \widehat{\rho}_t^*(A) R_n x\rangle$. By (6.38),

$$\begin{aligned} & \left|\langle x, \widehat{\rho}_t^*(A) x\rangle - \langle x, \widehat{\rho}_s^*(A) x\rangle\right| \\ \leq\ & 2\|A\| \left(\|\widehat{\rho}_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))} + \|\widehat{\rho}_s\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))}\right) \|x\| \|x - R_n x\| \\ & + \left|\langle R_n x, \widehat{\rho}_t^*(A) R_n x\rangle - \langle R_n x, \widehat{\rho}_s^*(A) R_n x\rangle\right|. \end{aligned}$$

Using $R_n x \longrightarrow_{n\to\infty} x$ we deduce that the map $t \mapsto \langle x, \widehat{\rho}_t^*(A) x\rangle$ is continuous. The polarization identity leads to Property (d) of Definition 3.1.

Assume that $x \in \mathcal{D}(C)$. By (6.37), combining $\widehat{\rho}_{t+s}^*(A) = \widehat{\rho}_s^*(\widehat{\rho}_t^*(A))$ with Property (iii) of Theorem 5.5 yields

$$\begin{aligned} & \lim_{s\to 0+} \frac{1}{s}\left(\langle x, \widehat{\rho}_{t+s}^*(A) x\rangle - \langle x, \widehat{\rho}_t^*(A) x\rangle\right) \\ &= \lim_{s\to 0+} \frac{1}{s}\left(tr\left(\widehat{\rho}_s(|x\rangle\langle x|)\widehat{\rho}_t^*(A)\right) - tr\left(|x\rangle\langle x|\widehat{\rho}_t^*(A)\right)\right) = \mathcal{L}\left(\widehat{\rho}_t^*(A), x\right). \end{aligned}$$

with $\mathcal{L}\left(\widehat{\rho}_t^*(A), x\right) = \langle x, \widehat{\rho}_t^*(A) Gx\rangle + \langle Gx, \widehat{\rho}_t^*(A) x\rangle + \sum_{k=1}^{\infty} \langle L_k x, \widehat{\rho}_t^*(A) L_k x\rangle$. Thus

$$\frac{d}{dt}^{+} \langle x, \widehat{\rho}_t^*(A) x\rangle = \mathcal{L}\left(\widehat{\rho}_t^*(A), x\right) \tag{6.39}$$

From (6.38) and Condition H2.2 we get that $\sum_{k=1}^{\infty} \langle L_k x, \widehat{\rho}_t^* (A) L_k x\rangle$ is uniformly convergent on bounded intervals, and so $t \mapsto \sum_{k=1}^{\infty} \langle L_k x, \widehat{\rho}_t^* (A) L_k x\rangle$ is continuous. Hence the application $t \mapsto \frac{d}{dt}^+ \langle x, \widehat{\rho}_t^* (A) x\rangle$ is continuous. Therefore $\langle x, \widehat{\rho}_t^* (A) x\rangle$ is continuously differentiable (see, e.g., Section 2.1 of [39]). Property (b) of Definition 3.1 now follows from (6.39). □

Proof of Theorem 5.5. Let $(\widehat{\rho}_t)_{t\geq 0}$ be a semigroup of bounded operators on $\mathfrak{L}_1(\mathfrak{h})$ satisfying Properties (i)-(iii) of Theorem 5.5. Let $(\widehat{\rho}_t^*)_{t\geq 0}$ be the adjoint semigroup of $(\widehat{\rho}_t)_{t\geq 0}$. Combining Lemma 6.21 with Theorem 3.1 we obtain $\widehat{\rho}_t^* (A) = \mathcal{T}_t (A)$, where $t \geq 0$, $A \in \mathfrak{L}(\mathfrak{h})$ and $(\mathcal{T}_t (A))_{t\geq 0}$ is given by Theorem 3.1.

Let $\varrho \in \mathfrak{L}_{1,C}^+(\mathfrak{h})$ and $A \in \mathfrak{L}(\mathfrak{h})$. Appying Lemma 6.7 and (6.37) yields

$$tr\left(\rho_t(\varrho) A\right) = tr\left(\mathcal{T}_t(A)\varrho\right) = tr\left(\widehat{\rho}_t^*(A)\varrho\right) = tr\left(\widehat{\rho}_t(\varrho) A\right).$$

Therefore $\rho_t(\varrho) = \widehat{\rho}_t(\varrho)$. Lemma 6.10 now implies that $\rho_t(\varrho) = \widehat{\rho}_t(\varrho)$ for all ϱ belonging to $\mathfrak{L}_1^+(\mathfrak{h})$, and so $\rho_t = \widehat{\rho}_t$. Finally, Lemma 6.20 and Theorem 5.3 complete the proof. □

6.11. Proof of Theorem 5.6

Proof of Theorem 5.6. According to (5.5) we have $\mathcal{D}(G) = \mathcal{D}(N^4)$. By (5.5), Remark 2.2 shows that G is a closable operator with the property

$$G \in \mathfrak{L}\left(\left(\mathcal{D}(N^p), \|\cdot\|_{N^p}\right), \mathfrak{h}\right).$$

Let x be a vector of $\mathfrak{h}$ whose coordinates $x_n := \langle e_n, x\rangle$ are equal to 0 for all $n \in \mathbb{Z}_+$ except a finite number. Then, an easy computation leads to

$$\begin{aligned} & 2\Re\langle N^{2p}x, Gx\rangle + \sum_{k=1}^{\infty} \|N^p L_k x\|^2 \\ = \quad & 2\beta_1 \sum_{n=1}^{\infty} \sqrt{n+1}\left((n+1)^{2p} - n^{2p}\right)\Re\left(x_n \overline{x_{n+1}}\right) \\ & +4p\left(|\alpha_5|^2 - |\alpha_4|^2\right)\sum_{n=0}^{\infty} n^{2p+1}|x_n|^2 + \sum_{n=0}^{\infty} f(n)|x_n|^2, \end{aligned}$$

where f is a $2p$-degree polynomial whose coefficients depend of $|\alpha_k|^2$ with $k = 1, 2, 4, 5$. Hence N^p satisfies Hypothesis 3, and so applying Theorem 2.2 we obtain that N^p fulfills Condition H2.3 of Hypothesis 2. Now, the proof is completed by using Theorems 5.4 and 5.5. □

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