

A Brézis–Browder principle on partially ordered spaces and related ordering theorems*

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Abstract

In this work, the existence of strong minimal points in quasi ordered spaces is studied. Through a simple extension of Brézis–Browder principle to partially ordered spaces, a very general strong minimal point existence theorem is proved. This theorem together with a generic quasi order and a new strong approximate solution notion are used to obtain two strong solution existence theorems and three general Ekeland variational principles in optimization problems where the objective space is quasi ordered. Then these results are applied to prove strong minimal point existence results and generalizations of Bishop–Phelps lemma in linear spaces, and Ekeland variational principles in set-valued optimization problems through a set solution criterion.

Keywords. Quasi order, existence of strong minimal points, ordering principle, Brézis–Browder principle, Bishop–Phelps lemma, Ekeland variational principle, vector optimization, set-valued optimization, set solution criterion, existence of strong efficient solutions.

Mathematics subject classification. 58E30, 46N10.

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1 Introduction

A classical question related to a vector optimization problem with single valued or set-valued objective function is to obtain conditions from which one can prove the existence of efficient solutions. Usually, these conditions are deduced through minimal point existence theorems of a set in ordered spaces by assuming some continuity assumption on the objective function (see [14, 16, 28, 32–34] and the references therein for a complete description of minimal point and efficient solution existence theorems in several spaces and optimization problems, respectively).

In the literature, sufficient conditions for the existence of minimal points in ordered linear spaces are based on cone-compactness or cone-completeness and boundedness assumptions compatible with the order relation via different kind of cones (Daniell, correct, closed and pointed, etc., see [16]). These conditions are proved by using various tools, like Zorn lemma (see, for instance, [6, 31]), Bishop–Phelps lemma (see [5]) or scalarization procedures (see, for instance, [27, 35]).

In the last years, some conditions for the existence of efficient solutions in vector optimization have been generalized to set-valued optimization with set solution criteria (see, for instance, [14, 23, 25, 30] and the references therein).

On the other hand, recently several authors have been interested to extend the well-known Ekeland variational principle (see [12]) to vector and set-valued optimization problems (see [9, 16, 20, 23, 24, 26] and the references therein), due to the important applications of this result in mathematical programming, control theory, convex analysis, geometry theory of Banach spaces, etc. (see [11, 13, 16]).

There exist a lot of relations between Brézis–Browder principle (see [2, 7]), Bishop–Phelps lemma and Ekeland variational principle (see, for instance, [28, pg. 160] and [4, 13, 23]). The main objectives of this work are, first, to derive Brézis–Browder principles on partially ordered spaces through weaker assumptions than the usual ones and second, to use them to prove more general strong minimal point/solution existence theorems and new versions of Bishop–Phelps lemma and Ekeland variational principle in different frameworks. The more general version of these new Brézis–Browder principles is based on the lower boundedness of certain maximal sets.

As a consequence we obtain a scalar version of the Brézis–Browder principle whose monotonicity assumption on the scalarization function is more general than the usual strict monotonicity and for which the standard boundedness assumption on the mentioned scalarization function is not necessary (Theorem 5.1).

Moreover, we prove two very general Ekeland variational principles that

work for functions whose image space is a magma (Theorem 6.2). The second one is based on a new approximate strong solution concept defined in a ordered magma that encompasses a lot of approximate efficiency notions introduced in vector optimization. Some properties of this new approximate strong solution concept are proved too.

As applications, we prove several sufficient conditions for the existence of efficient points in ordered linear spaces, from which one observes that the assumptions on the feasible set and the order cone are complementary in a certain sense, and new Ekeland variational principles that extend various similar recently published results. In particular we derive this kind of variational principles for scalar optimization problems where the perturbation function is a generalized Q-function, a new class of mappings introduced in this paper that generalizes properly the class of Q-functions (see [1]), and also for set-valued optimization problems with a set solution criterion.

The structure of this paper is as follows. In Section 2, the main notations are fixed and some basic definitions and results are recalled. In Section 3, a generic quasi order $\preceq_{g,C}$ is introduced, which collapses the partial orders usually considered in the objective space of vector and set-valued optimization problems, in Bishop–Phelps lemma and in several Ekeland variational principles too. In Section 4, a version of Brézis–Browder principle on partially ordered spaces is proved that encompasses two recent vectorial Brézis–Browder principles published in [10,36]. This extension is used in Section 5 to obtain a minimal point existence theorem, that is applied to derive strong solution existence theorems and Bishop–Phelps lemmas, and in Section 6, together with a new approximate strong solution concept, to prove Ekeland variational principles, all of them in optimization problems whose objective space is quasi ordered. Moreover, Section 5 and Section 6 contain applications to vector and set-valued optimization problems, respectively.

2 Notations and preliminaries

Let (\mathcal{V}, \leq) be a quasi ordered set, i.e., \leq is a binary relation on $\mathcal{V} \neq \emptyset$ that satisfies the reflexive and transitive properties, and let $(\mathcal{G}, \trianglelefteq)$ be a partially ordered set, i.e., $(\mathcal{G}, \trianglelefteq)$ is a quasi ordered set such that \trianglelefteq satisfies the anti-symmetric property too. Given $v_1, v_2 \in \mathcal{V}$ we write $v_1 < v_2$ if $v_1 \leq v_2$ and $v_1 \neq v_2$. Analogously, for $y_1, y_2 \in \mathcal{G}$ the notation $y_1 \triangleleft y_2$ means that $y_1 \trianglelefteq y_2$ and $y_1 \neq y_2$.

A sequence $(v_n) \subset \mathcal{V}$ is said to be nonincreasing for the order \leq (nonincreasing for short) if $v_n \leq v_m \forall n, m \in \mathbb{N}, n > m$. A mapping $\varphi : \mathcal{V} \rightarrow \mathcal{G}$ is said to be nondecreasing (resp. increasing) for the orders \leq and \trianglelefteq (nonde-

creasing or increasing for short) if $\varphi(v_1) \preceq \varphi(v_2)$ for all $v_1, v_2 \in \mathcal{V}$, $v_1 \leq v_2$ (resp. $\varphi(v_1) \triangleleft \varphi(v_2)$ for all $v_1, v_2 \in \mathcal{V}$, $v_1 < v_2$). The sets of nondecreasing and increasing mappings are denoted by $H(\mathcal{V}, \leq, \mathcal{G}, \preceq)$ and $H^s(\mathcal{V}, \leq, \mathcal{G}, \preceq)$, respectively (in order to shorten both notations, \leq or \preceq can be missing if any confusion is possible). Clearly, $H^s(\mathcal{V}, \mathcal{G}) \subset H(\mathcal{V}, \mathcal{G})$.

Let $(\mathcal{E}, +, \preceq)$ be a preordered magma with zero element such that

$$y, y_1, y_2 \in \mathcal{E}, \quad y_1 \preceq y_2 \Rightarrow y_1 + y \preceq y_2 + y, \quad (1)$$

i.e., (\mathcal{E}, \preceq) is a preordered set (\preceq is a transitive binary relation on $\mathcal{E} \neq \emptyset$), $+ : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ is a law of composition, there exists a zero element $0_{\mathcal{E}} \in \mathcal{E}$ ($y + 0_{\mathcal{E}} = 0_{\mathcal{E}} + y = y$ for all $y \in \mathcal{E}$) and property (1) is satisfied. The zero element $0_{\mathcal{E}}$ will be denoted by 0 if there is not confusion.

Let X be a nonempty set and consider a mapping $J : X \rightarrow \mathcal{V}$. We denote $\text{Im}(J) = J(X) := \bigcup_{x \in X} J(x)$, $\text{Graph}(J) = \{(x, J(x)) : x \in X\}$ and for each $v \in \mathcal{V}$, $S(J, v) := \{x \in X : J(x) \leq v\}$ and $S_0(J, v) = \{x \in X : J(x) < v\}$. In particular, if $X = \mathcal{V}$ and $J(v) = v$ for all $v \in \mathcal{V}$ we denote $S(\mathcal{V}, v) := S(J, v)$ and $S_0(\mathcal{V}, v) := S_0(J, v)$.

Given the following general optimization problem:

$$\leq \text{-Min } F(x) \text{ subject to } x \in X, \quad (2)$$

where $F : X \rightarrow \mathcal{V}$, we denote

$$\begin{aligned} \text{Min}(F, \leq) &:= \{x \in X : x \in S(F, F(z)), \forall z \in S(F, F(x))\}, \\ \text{Min}(F, <) &:= \{x \in X : S_0(F, F(x)) = \emptyset\}, \\ \text{SMin}(F, \leq) &:= \{x \in X : S(F, F(x)) = \{x\}\}. \end{aligned}$$

The elements of $\text{Min}(F, \leq)$, $\text{Min}(F, <)$ and $\text{SMin}(F, \leq)$ are called minimal solutions, nondominated solutions and strong minimal solutions (solutions, nondominated solutions and strong solutions for short) of problem (2), respectively. Strong minimal solutions are termed strict efficient solutions in [15] and references therein.

If $\emptyset \neq M \subset \mathcal{V}$ and we consider $X = M$ and $F(x) = x \forall x \in M$ then $\text{Min}(F, \leq)$, $\text{Min}(F, <)$ and $\text{SMin}(F, \leq)$ are the sets of minimal, nondominated and strong minimal points of M , respectively. We denote these sets by $\text{Min}(M, \leq)$, $\text{Min}(M, <)$ and $\text{SMin}(M, \leq)$.

In the sequel, \mathbb{R}_+^p denotes the nonnegative orthant of \mathbb{R}^p and $\mathbb{R}_+ := \mathbb{R}_+^1$. Moreover, if Y is a topological linear space, Y^* is the topological dual of Y and for a convex cone $D \subset Y$, we write the positive and strict positive polar cone of D by

$$\begin{aligned} D^+ &:= \{\xi \in Y^* : \xi(d) \geq 0, \forall d \in D\}, \\ D^{+s} &:= \{\xi \in Y^* : \xi(d) > 0, \forall d \in D \setminus \{0\}\}. \end{aligned}$$

It is obvious that $D^+ \subset H(Y, \leq_D, \mathbb{R})$ and $D^{+s} \subset H^s(Y, \leq_D, \mathbb{R})$, where

$$y, z \in Y, \quad y \leq_D z \iff z - y \in D.$$

This quasi order is used to model the decision maker's preferences in vector optimization problems.

Let us recall some particular kinds of order cones (see [16]). In the sequel, when Y is a topological space, $\text{int}(A)$ and $\text{cl}(A)$ denote the interior and the closure of a set $A \subset Y$.

Definition 2.1. Let $D \subset Y$ be a convex cone.

- (a) D is proper if $\{0\} \neq D \neq Y$.
- (b) D is solid if $\text{int}(D) \neq \emptyset$.
- (c) D is based if $D^{+s} \neq \emptyset$.
- (d) D is well-based if there exists a bounded convex set $B \subset Y$ such that $D = \mathbb{R}_+ B$ and $0 \notin \text{cl}(B)$.
- (e) Assume that Y is a normed space. D has the angle property if there exists $\xi_D \in Y^*$ and $\alpha > 0$ such that

$$D \subset \{y \in Y : \alpha \|y\| \leq \xi_D(y)\}.$$

- (f) D is normal if for all nets $(x_i), (y_i) \subset Y$ such that $0 \leq_D x_i \leq_D y_i \forall i$ one has $y_i \rightarrow 0 \Rightarrow x_i \rightarrow 0$.
- (g) Assume that (Y, τ) is a locally convex space and τ is defined by a family \mathcal{P} of seminorms. D is supernormal if for every $p \in \mathcal{P}$ there exists $\xi_p \in Y^*$ such that $p(d) \leq \xi_p(d)$ for all $d \in D$.

3 A general quasi order

Let \mathcal{V} be a nonempty set and consider two mappings $g : \mathcal{V} \rightarrow \mathcal{E}$ and $C : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{E}$.

Definition 3.1. We say that g and C define a domination structure in \mathcal{V} through the preordered space \mathcal{E} if the following relation is a quasi order in \mathcal{V} :

$$v_1, v_2 \in \mathcal{V}, \quad v_1 \preceq_{g,C} v_2 \iff g(v_1) + C(v_1, v_2) \preceq g(v_2).$$

The notation $\preceq_{g,C}$ says that the domination structure in \mathcal{V} is given by the relation \preceq of \mathcal{E} and the mappings g, C . For example, the domination structure $(\leq_D^l)_{g,C}$ is given by the set relation \leq_D^l , defined in Remark 3.1(c), and the mappings g, C .

To check if two mappings g and C define a domination structure we have the following trivial properties. Let us observe from Remark 3.1(b),(g) that these sufficient conditions are not necessary.

Lemma 3.1. Consider $g : \mathcal{V} \rightarrow \mathcal{E}$ and $C : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{E}$.

(a) If $C(v, v) \preceq 0 \forall v \in \mathcal{V}$, then $\preceq_{g,C}$ is reflexive.

(b) If

$$C(v_1, v_3) \preceq C(v_1, v_2) + C(v_2, v_3) \quad \forall v_1, v_2, v_3 \in \mathcal{V}, \quad (3)$$

then $\preceq_{g,C}$ is transitive.

Remark 3.1. Let Y be a linear space, let $D \subset Y$ be a convex cone. The following domination structures are well-known:

(a) If $\mathcal{V} = Y$, $(\mathcal{E}, +, \preceq) = (Y, +, \leq_D)$, $g(v) = v$ and $C(v_1, v_2) = 0 \forall v, v_1, v_2 \in \mathcal{V}$ then $v_1 \preceq_{g,C} v_2$ if and only if $v_1 \leq_D v_2$.

(b) Consider $\mathcal{V} = Y$ and $(\mathcal{E}, +, \preceq) = (2^Y, +, \supset)$. Let $G : Y \rightarrow 2^Y$ be a set-valued mapping such that for each $y \in Y$, $0 \in G(y)$ and

$$d \in G(y), q \in G(y + d) \Rightarrow d + q \in G(y). \quad (4)$$

If $g(v) = \{v\}$ and $C(v_1, v_2) = G(v_1) \forall v, v_1, v_2 \in \mathcal{V}$ then $v_1 \preceq_{g,C} v_2$ if and only if $v_2 \in v_1 + G(v_1)$. This relation extends the previous one and is used for modeling variable preferences (see, for instance, [33, 35]). In particular, if $G(v) = D$ for all $v \in \mathcal{V}$ then $v_1 \preceq_{g,C} v_2$ if and only if $v_1 \leq_D v_2$. Let us observe that property (3) could not be satisfied. Indeed, consider $Y = \mathbb{R}^2$, $A = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$, $G(v) = A \forall v \in A$ and $G(v) = (\mathbb{R}^2 \setminus A) \cup \{(0, 0)\}$ if $v \notin A$. Then statement (4) is satisfied but $C((0, 0), v) + C(v, w) \not\subset C((0, 0), w)$ if $v = (x, y)$, $y < 0$, $\forall w \in \mathbb{R}^2$.

(c) As usual, we denote

$$A_1 + A_2 := \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\} \quad \forall A_1, A_2 \in 2^Y \setminus \{\emptyset\},$$

$A + \emptyset = \emptyset + A = \emptyset, \forall A \in 2^Y$. Throughout the paper, the following set relations due to Kuroiwa [29], for dealing with set-valued optimization problems, will be considered:

$$\begin{aligned} A_1, A_2 \in 2^Y, \quad A_1 \leq_D^l A_2 &\iff A_2 \subset A_1 + D, \\ A_1 \leq_D^u A_2 &\iff A_1 \subset A_2 - D. \end{aligned}$$

If $\mathcal{V} = 2^Y$, $(\mathcal{E}, +, \preceq) = (2^Y, +, \leq_D^l)$, $g(v) = v$ and $C(v_1, v_2) = \{0\}$ or $C(v_1, v_2) = D \forall v, v_1, v_2 \in \mathcal{V}$ then $v_1 \preceq_{g,C} v_2$ if and only if $v_1 \leq_D^l v_2$. If we consider \leq_D^u instead of \leq_D^l then $v_1 \preceq_{g,C} v_2$ if and only if $v_1 \leq_D^u v_2$.

- (d) Consider a metric space (X, p) . If $\mathcal{V} = X \times \mathbb{R}$, $(\mathcal{E}, +, \preceq) = (\mathbb{R}, +, \leq)$, $g(x, r) = r$ and $C((x_1, r_1), (x_2, r_2)) = p(x_1, x_2) \forall (x, r), (x_1, r_1), (x_2, r_2) \in \mathcal{V}$ then $(x_1, r_1) \preceq_{g,C} (x_2, r_2)$ if and only if $r_1 + p(x_1, x_2) \leq r_2$. This relation was defined by Bishop–Phelps [5] to prove a maximal point existence lemma.
- (e) Consider the uniform space (X, \mathcal{U}) , where the uniform topology \mathcal{U} is generated by the quasimetrics $(q_\lambda)_{\lambda \in \Lambda}$ (see [22, Definition 2]), and $\mathcal{V} = X \times Y$, $(\mathcal{E}, +, \preceq) = (2^Y, +, \leq_D^u)$. If $g(x, y) = \{y\}$, $k \in D \setminus \{0\}$ and $C((x_1, y_1), (x_2, y_2)) = \cup_{\lambda \in \Lambda} \{q_\lambda(x_1, x_2)k\} \forall (x, y), (x_1, y_1), (x_2, y_2) \in \mathcal{V}$ then $(x_1, y_1) \preceq_{g,C} (x_2, y_2)$ if and only if $y_1 + q_\lambda(x_1, x_2)k \leq_D y_2$ for all $\lambda \in \Lambda$. This relation has been used in [22] to obtain Ekeland variational principles in uniform spaces for functions with values in a linear space.
- (f) Let (X, p) be a metric space. If $(Y, \|\cdot\|)$ is normed, $k \in D \setminus \{0\}$, $\mathcal{V} = X \times Y$, $(\mathcal{E}, +, \preceq) = (Y, +, \leq_D)$, $g(x, y) = y$ and $C((x_1, y_1), (x_2, y_2)) = (p(x_1, x_2) + \|y_1 - y_2\|)k \forall (x, y), (x_1, y_1), (x_2, y_2) \in \mathcal{V}$ then $(x_1, y_1) \preceq_{g,C} (x_2, y_2)$ if and only if $y_1 + (p(x_1, x_2) + \|y_1 - y_2\|)k \leq_D y_2$. This relation was used in [24] to prove a vector-valued Ekeland variational principle in a vector optimization problem.
- (g) Consider a metric space (X, p) , $k \in D$ and $\xi \in D^+$ such that $\xi(k) = 1$. In [17], the authors use the following relation in $X \times Y$ to obtain minimal point theorems:

$$(x_1, y_1) \preceq_{k,\xi} (x_2, y_2) \\ \iff (x_1, y_1) = (x_2, y_2) \text{ or } \begin{cases} y_1 + p(x_1, x_2)k \leq_D y_2, \\ \xi(y_1) < \xi(y_2). \end{cases}$$

It is easy to check that $\preceq_{k,\xi}$ is the domination structure given in $\mathcal{V} = X \times Y$ by $(\mathcal{E}, +, \preceq) = (2^Y, +, \supset)$, $g(x, y) = \{y\}$ and

$$C((x_1, y_1), (x_2, y_2)) = \begin{cases} \{0\} & \text{if } (x_1, y_1) = (x_2, y_2), \\ \{d \in D : \xi(d) > 0\} & \text{if } x_1 = x_2, y_1 \neq y_2, \\ p(x_1, x_2)k + D & \text{if } x_1 \neq x_2. \end{cases}$$

Let us observe that the set-valued mapping C does not satisfy property (3). Indeed, if $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $D = \mathbb{R}_+^2$, $k = (1, 1)$ and $\xi(d_1, d_2) = (1/2)(d_1 + d_2)$ then $C((0, (1, 1)), (0, (1, 1))) = \{(0, 0)\}$,

$$C((0, (1, 1)), (1, (1, 1))) = C((1, (1, 1)), (0, (1, 1))) = (1, 1) + \mathbb{R}_+^2$$

and

$$\begin{aligned} & C((0, (1, 1)), (1, (1, 1))) + C((1, (1, 1)), (0, (1, 1))) \\ & = (2, 2) + \mathbb{R}_+^2 \not\subset C((0, (1, 1)), (0, (1, 1))). \end{aligned}$$

- (h) Consider a nonempty set A and let $K : A \times A \rightrightarrows 2^D$ be a set-valued D -metric (see [20]). If $\mathcal{V} = A \times Y$, $(\mathcal{E}, +, \preceq) = (2^Y, +, \leq_D^l)$, $g(a, y) = \{y\}$ and $C((a_1, y_1), (a_2, y_2)) = K(a_1, a_2) \forall (a, y), (a_1, y_1), (a_2, y_2) \in \mathcal{V}$ then $(a_1, y_1) \preceq_{g,C} (a_2, y_2)$ if and only if $y_2 \in y_1 + K(a_1, a_2) + D$. This relation was introduced by Gutiérrez et al. in [20] to obtain a set-valued version of the Ekeland variational principle in vector optimization problems with multivalued objectives.

4 A generalized Brézis–Browder principle

In this section, a vector-valued version of Brézis–Browder principle on partially ordered spaces is obtained that extends and unifies several similar results of [10, 36]. Such a vectorial version will show, in the next section (Theorem 5.1), the standard boundedness assumption on the scalar function in the original Brézis–Browder principle is superfluous. First we recall some concepts.

Let $\mathcal{F} \subset 2^{\mathcal{V}}$ be a nonempty family of subsets of \mathcal{V} . $M \in \mathcal{F}$ is maximal in \mathcal{F} with respect to the inclusion relation (maximal for short) if $M = M'$ for all $M' \in \mathcal{F}$ such that $M \subset M'$. Let $\emptyset \neq M \subset \mathcal{V}$. A point $\bar{v} \in \mathcal{V}$ is a lower (order) bound of M if $\bar{v} \leq v$ for all $v \in M$. M is said to be lower bounded if there exists some lower bound of M . The set of lower bounds of M is denoted by $\text{LB}(M)$.

Consider the following family of sets, where $\varphi \in \text{H}(\mathcal{V}, \mathcal{G})$:

$$\begin{aligned} \mathcal{F}_\varphi := \{M \subset \mathcal{V} : M \text{ is totally ordered,} \\ \varphi(v_1) \triangleleft \varphi(v_2) \forall v_1, v_2 \in M, v_1 < v_2\}. \end{aligned}$$

Here M totally ordered means that for each $v_1, v_2 \in M$, we have $v_1 \trianglelefteq v_2$ or $v_2 \trianglelefteq v_1$. The followings statements are clear:

$$\begin{aligned} \mathcal{V} \text{ totally ordered} &\Rightarrow \mathcal{F}_\varphi = \{M \subset \mathcal{V} : \varphi(v_1) \triangleleft \varphi(v_2) \forall v_1, v_2 \in M, v_1 < v_2\}, \\ \varphi \in \text{H}^s(\mathcal{V}, \mathcal{G}) &\Rightarrow \mathcal{F}_\varphi = \{M \subset \mathcal{V} : M \text{ is totally ordered}\}, \\ \mathcal{V} \text{ totally ordered, } \varphi \in \text{H}^s(\mathcal{V}, \mathcal{G}) &\Rightarrow \mathcal{F}_\varphi = 2^{\mathcal{V}}. \end{aligned}$$

The following result is a direct consequence of Zorn's lemma and it was obtained implicitly in the proof of [36, Theorem 2.1].

Lemma 4.1. \mathcal{F}_φ has at least one maximal element.

Proof. As $\mathcal{V} \neq \emptyset$ there exists $v \in \mathcal{V}$ and so $\{v\} \in \mathcal{F}_\varphi$. Thus the family \mathcal{F}_φ is nonempty. Let us check that \mathcal{F}_φ is inductively ordered through the inclusion relation. For this aim, let $\mathcal{M} \subset \mathcal{F}_\varphi$ be totally ordered and consider

$$L = \bigcup \{M : M \in \mathcal{M}\} \subset \mathcal{V}.$$

It follows that $L \in \mathcal{F}_\varphi$. Indeed, let $v_1, v_2 \in L$ such that $v_1 \neq v_2$. There exist $M_1, M_2 \in \mathcal{M}$ such that $v_i \in M_i, i = 1, 2$ and we can suppose that $M_1 \subset M_2$ since \mathcal{M} is totally ordered. Therefore $v_1, v_2 \in M_2$ and $v_1 < v_2, \varphi(v_1) \triangleleft \varphi(v_2)$ or reciprocally, from which we have that $L \in \mathcal{F}_\varphi$. It is obvious that L is an upper bound of \mathcal{M} and so the inclusion relation is inductively ordered in \mathcal{F}_φ . The result follows by applying Zorn's lemma. \square

Theorem 4.1 (GENERALIZED BRÉZIS–BROWDER PRINCIPLE). Suppose that some maximal element M^* of \mathcal{F}_φ is lower bounded and let $\bar{v} \in \text{LB}(M^*)$. Then $\varphi(v) = \varphi(\bar{v})$ for all $v \in S(\mathcal{V}, \bar{v})$.

Proof. First, let us observe that $M^* \cup \{v\}$ is totally ordered for all $v \in S(\mathcal{V}, \bar{v})$, since M^* is totally ordered and each element $v \in S(\mathcal{V}, \bar{v})$ is a lower bound of M^* .

Consider $v \in S(\mathcal{V}, \bar{v})$. As \bar{v} is a lower bound of M^* and φ is nondecreasing we see that $\varphi(v) \trianglelefteq \varphi(\bar{v}) \trianglelefteq \varphi(z)$ for all $z \in M^*$. If $\varphi(v) \triangleleft \varphi(\bar{v})$ then $\varphi(v) \triangleleft \varphi(z)$ for all $z \in M^*$ and $v \notin M^*$, i.e., $v < z$ for all $z \in M^*$. Therefore, $M^* \cup \{v\} \in \mathcal{F}_\varphi$, which is a contradiction since M^* is maximal in \mathcal{F}_φ . Thus $\varphi(v) = \varphi(\bar{v})$ and the proof is completed. \square

In view of Theorem 4.1, we can say that the central role in Brézis–Browder principle is the lower boundness of maximal sets in which the monotone mapping φ is increasing. To illustrate this fact, consider $\emptyset \neq \mathcal{V} \subset \mathbb{R}$ and $\varphi(x) = \exp(x)$ for all $x \in \mathbb{R}$. In this case, \mathcal{V} is the unique maximal element of \mathcal{F}_φ and it is clear that there exists $\bar{v} \in \mathcal{V}$ such that $\varphi(v) = \varphi(\bar{v})$ for all $v \in S(\mathcal{V}, \bar{v})$ if and only if there exists $\bar{v} \in \mathcal{V}$ such that $S(\mathcal{V}, \bar{v}) = \{\bar{v}\}$, and this last condition is equivalent to say that $-\infty < \inf\{v : v \in \mathcal{V}\} \in \mathcal{V}$, i.e., \mathcal{V} is lower bounded.

Theorem 4.1 collapses several versions of Brézis–Browder principle proved in the literature, as it is showed in Corollary 4.1. Let us recall that \mathcal{V} is countably inductive (CIO in short form, see [13]) if for all nonincreasing sequence $(v_n) \subset \mathcal{V}$ there exist $v \in \mathcal{V}$ and a subsequence (v_{n_k}) satisfying $v \leq v_{n_k} \forall n_k$. This notion is slightly more general than the usual CIO concept, in which v is a lower bound for the whole sequence (v_n) (see [13]).

Let us recall that \mathcal{G} is said to be totally ordered lower-separable (see [36]) if for any nonempty totally ordered set $N \subset \mathcal{G}$ there exists a nonincreasing sequence $(y_n) \subset N$ such that, for any $y \in N$, there exists n_0 satisfying $y_{n_0} \trianglelefteq y$.

Corollary 4.1. [36, Theorem 2.1] Consider $\varphi \in \mathbf{H}(\mathcal{V}, \mathcal{G})$ and suppose that \mathcal{V} is CIO and \mathcal{G} is totally ordered lower-separable. Then, for each $z \in \mathcal{V}$ there exists $\bar{v} \in \mathcal{V}$, $\bar{v} \leq z$, such that $\varphi(v) = \varphi(\bar{v})$ for all $v \in S(\mathcal{V}, \bar{v})$.

Proof. Let $z \in \mathcal{V}$. As $S(\mathcal{V}, z)$ is CIO, without loss of generality we can assume that $\mathcal{V} = S(\mathcal{V}, z)$. Let us check that M is lower bounded for all $M \in \mathcal{F}_\varphi$. Indeed, since φ is nondecreasing and M is totally ordered it follows that $\varphi(M)$ is totally ordered. By using that \mathcal{G} is totally ordered lower-separable we deduce that there exists a sequence $(v_n) \subset M$ such that $\varphi(v_n)$ is nonincreasing and for any $v \in M$, there exists n satisfying $\varphi(v_n) \trianglelefteq \varphi(v)$.

As $M \in \mathcal{F}_\varphi$ we see that (v_n) is nonincreasing and $v_n \leq v$ if $v \in M$ and $\varphi(v_n) \trianglelefteq \varphi(v)$. Moreover, since \mathcal{V} is CIO there exists $\bar{w} \in \mathcal{V}$ such that $\bar{w} \leq v_n$ for all n and \bar{w} is a lower bound of M .

By Lemma 4.1 and Theorem 4.1 we deduce that there exists $\bar{v} \in \mathcal{V}$ such that $\varphi(v) = \varphi(\bar{v})$ for all $v \in S(\mathcal{V}, \bar{v})$, which finishes the proof. \square

5 Existence of strong solutions and Bishop–Phelps Lemma

One can use Corollary 4.1 to prove strong solution existence results, Bishop–Phelps lemmas and new versions of Brézis–Browder principle. In what follows we establish an existence result of strong minimal solutions. We say that a Hausdorff uniform space is nonincreasing sequentially complete if every nonincreasing Cauchy sequence is convergent.

The novelty of the next result is that no boundedness of the scalar function ϕ is required. Moreover, let us observe that assumption (a) below is strictly weaker than the usual increasing assumption on the scalarization function ϕ , see Remark 5.1.

Theorem 5.1 (EXISTENCE OF STRONG MINIMAL POINTS). Let (X, \leq) be a quasi ordered space and consider two mappings $J \in \mathbf{H}(X, \mathcal{V})$ and $\phi \in \mathbf{H}(\mathcal{V}, \mathbb{R} \cup \{+\infty\})$. Then, for each $u \in X$ such that $S(X, u)$ is CIO there exists $\bar{x} \in S(X, u)$ satisfying $(\phi \circ J)(x) = (\phi \circ J)(\bar{x})$ for all $x \in S(X, \bar{x})$. If additionally the following condition (a) is true, then $S(X, \bar{x}) = \{\bar{x}\}$.

(a)

$$x \in S(X, u), \quad x \in \operatorname{argmin}_{S(X, x)} \phi \circ J \implies \begin{cases} \exists x' \in S(X, x) : \\ \operatorname{argmin}_{S(X, x')} \phi \circ J = \{x'\}. \end{cases}$$

If additionally (X, \mathcal{U}) is a nonincreasing sequentially complete Hausdorff uniform space then the same conclusions hold by considering the following assumptions (b) and (c) instead of assuming that $S(X, u)$ is CIO:

(b) For all nonincreasing sequence $(x_n) \subset S(X, u)$ there exists a Cauchy subsequence (x_{n_k}) .

(c) If $(x_n) \subset S(X, u)$ is nonincreasing and $x_n \rightarrow x$ then $x \leq x_n$ for all n .

Proof. Let $u \in X$ be such that $S(X, u)$ is CIO. We will apply Corollary 4.1 (where we denote \mathcal{V}' instead of \mathcal{V}) to $\mathcal{V}' = S(X, u)$, $\mathcal{G} = \mathbb{R} \cup \{+\infty\}$ and the function $\varphi = \phi \circ J : (\mathcal{V}', \leq) \rightarrow (\mathbb{R} \cup \{+\infty\}, \leq)$ which is nondecreasing. Notice that $\mathbb{R} \cup \{+\infty\}$ is totally ordered lower-separable and the standard partial ordering on $\mathbb{R} \cup \{+\infty\}$ has the antisymmetric property. By Corollary 4.1 there exists $\bar{x} \in X$, $\bar{x} \leq u$, such that $(\phi \circ J)(x) = (\phi \circ J)(\bar{x})$ for all $x \in S(\mathcal{V}', \bar{x})$, and the first part is proved since $S(\mathcal{V}', \bar{x}) = S(X, \bar{x})$.

Let us assume that hypothesis (a) is true. By applying it to \bar{x} we obtain that there exists $x' \in S(X, \bar{x})$ such that $\operatorname{argmin}_{S(X, x')} \phi \circ J = \{x'\}$. As φ is constant in $S(X, \bar{x})$ and $S(X, x') \subset S(X, \bar{x})$ it follows that φ is constant in $S(X, x')$ too. Therefore $\operatorname{argmin}_{S(X, x')} \phi \circ J = S(X, x')$ and the proof of the second part is completed.

The third part is trivial since (b) and (c) imply that $S(X, u)$ is CIO for all $u \in X$, and this implication is true because (X, \mathcal{U}) is nonincreasing sequentially complete. \square

Remark 5.1. (a) An assumption implying (a) is the following:

$$S(X, x) \cap S_0(\phi \circ J, \phi(J(x))) \neq \emptyset, \quad \forall x \in S(X, u), S(X, x) \setminus \{x\} \neq \emptyset, \quad (5)$$

which is equivalent to:

$$x \in S(X, u), \quad x \in \operatorname{argmin}_{S(X, x)} \phi \circ J \implies S(X, x) = \{x\}.$$

(b) If $\phi(J(z)) < \phi(J(x))$ for all $z \in S(J, J(x))$, $z \neq x$ (in particular, if J is injective and $\phi \in H^s(\mathcal{V}, \leq, \mathbb{R} \cup \{+\infty\})$), then $S(J, J(x)) \setminus \{x\} \subset S_0(\phi \circ J, \phi(J(x)))$ and so $S(J, J(x)) \cap S_0(\phi \circ J, \phi(J(x))) \neq \emptyset$ when we have $S(J, J(x)) \setminus \{x\} \neq \emptyset$, i.e., condition (5) and so (a) of Theorem 5.1 are satisfied.

Theorem 5.1 reduces to [10, Theorem 4] by assuming that \mathcal{V} is a topological vector space ordered through the relation \leq_D , where D is based, and $\phi \in D^{+s}$. Moreover, the original version of Brézis–Browder principle (see [7, Corollary 1]) can be obtained via Theorem 5.1 by considering $X = \mathcal{V}$, J equal to the identity mapping and $\phi \in H(\mathcal{V}, \mathbb{R})$. In this case, observe that the boundedness assumption on function ϕ is superfluous and ϕ could not be increasing.

Theorem 5.1 with assumption (b) encompasses the well known Brøndsted result [8, Theorem 1], as it is showed in the sequel, but before the following definition is necessary. We say that $F : X \rightarrow \mathcal{V}$ is nonincreasing sequentially lower closed in $S(F, F(u)), u \in X$, if for each sequence $(x_n) \subset S(F, F(u))$ converging to x such that $(F(x_n))$ is nonincreasing, one has

$$\exists n_0 \quad \forall n \geq n_0 : F(x) \leq F(x_n),$$

or equivalently (because $(F(x_n))$ is nonincreasing),

$$\forall n \in \mathbb{N} : F(x) \leq F(x_n).$$

Corollary 5.1. [8, Theorem 1], [22, Theorem 1] Let (X, \mathcal{U}) be a nonincreasing sequentially complete Hausdorff uniform space equipped with a quasi order \leq such that $i : X \rightarrow X$ is nonincreasingly sequentially lower closed in $S(i, u)$ for $u \in X$. Assume that $\phi \in H(X, \mathbb{R} \cup \{+\infty\})$ is bounded from below on X , proper and for each $U \in \mathcal{U}$ there exists $\delta > 0$ satisfying

$$x_1 \leq x_2, \phi(x_2) - \phi(x_1) < \delta \implies (x_1, x_2) \in U.$$

Then, for each $u \in \text{dom}(\phi)$ there exists $\bar{x} \in \text{dom}(\phi)$ such that $\bar{x} \leq u$, $S(X, \bar{x}) = \{\bar{x}\}$.

Proof. We will apply Theorem 5.1 to $u \in \text{dom}(\phi)$. Let us check assumption (b). Let $(x_n) \subset S(X, u)$ be satisfying $x_{n+1} \leq x_n$ for all n . Since $u \in \text{dom}(\phi)$ and ϕ is nondecreasing and bounded below, we may suppose that $\phi(x_n) \downarrow r \in \mathbb{R}$. Then, $r \leq \phi(x_n)$ for all n , and given any $U \in \mathcal{U}$ there exists n_0 such that $\phi(x_{n_0}) < r + (\delta/2)$ (δ as above). Thus, if $n \geq n_0$ then $\phi(x_{n_0}) - \phi(x_n) < r + (\delta/2) - r = \delta/2$. This implies that $\phi(x_m) - \phi(x_n) < \delta$ for all $m > n > n_0$. Hence $(x_n, x_m) \in U$, showing that (b) holds.

We now check that (a) holds as well with $J(x) = x$ for all $x \in X$. For this aim we use Remark 5.1. Indeed, if there exist $x \in S(X, u)$ and $x' \in S(X, x) \setminus \{x\}$, such that $\phi(x') = \phi(x)$ then the assumption on ϕ implies that $(x', x) \in U$ for all $U \in \mathcal{U}$ and since (X, \mathcal{U}) is Hausdorff we conclude that $x = x'$, a contradiction.

Then the result follows by Theorem 5.1. □

Next we use Theorem 5.1 to prove an existence result on strong solutions of problem (2) via scalarization.

For each $u \in X$ we denote $\text{Im}(F|_{S(X,u)}) := \{F(x) : x \in S(X,u)\}$.

Theorem 5.2 (EXISTENCE OF STRONG MINIMAL SOLUTIONS). Consider problem (2), $\phi \in \text{H}(\mathcal{V}, \mathbb{R} \cup \{+\infty\})$, $u \in X$ and assume that $\text{Im}(F|_{S(F,F(u))})$ is CIO. Then there exists $\bar{x} \in X$, $F(\bar{x}) \leq F(u)$, such that $(\phi \circ F)(x) = (\phi \circ F)(\bar{x})$ for all $x \in S(F, F(\bar{x}))$. If additionally the following condition (a') is satisfied, then $\bar{x} \in \text{SMin}(F, \leq)$.

(a')

$$x \in S(F, F(u)), x \in \text{argmin}_{S(F,F(x))} \phi \circ F \implies \begin{cases} \exists x' \in S(F, F(x)) : \\ \text{argmin}_{S(F,F(x'))} \phi \circ F = \{x'\}. \end{cases}$$

If additionally (X, \mathcal{U}) is a nonincreasing (with the quasi order (6) below) sequentially complete Hausdorff uniform space and F is nonincreasing sequentially lower closed in $S(F, F(u))$, then the same conclusions hold by changing assumption (CIO) on the set $\text{Im}(F|_{S(F,F(u))})$, by:

(b') For all sequence $(x_n) \subset S(F, F(u))$ such that $(F(x_n))$ is nonincreasing there exists a Cauchy subsequence (x_{n_k}) .

Proof. To apply Theorem 5.1, let us order the set X through the following quasi order:

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \iff F(x_1) \leq F(x_2). \quad (6)$$

With this relation the objective mapping F is nondecreasing. Moreover, $S(X, u)$ is CIO if and only if $\text{Im}(F|_{S(F,F(u))})$ so is, $S(X, x) = S(F, F(x))$ for all $x \in X$, (a) and (a') are equivalent, (b) and (b') are equivalent and statement (c) is equivalent to say that F is nonincreasing sequentially lower closed in $S(F, F(u))$. Then Theorem 5.2 is a direct consequence of Theorem 5.1. \square

Remark 5.2. We actually have that Theorem 5.2 and Theorem 5.1 are equivalent, in the sense that one can be obtained from the other. Indeed, that Theorem 5.1 implies Theorem 5.2 is proved above. The other implication is obtained by applying Theorem 5.2 to the mappings

$$F = (i, J) : X \rightarrow X \times \mathcal{V}, \quad \tilde{\phi}(x, v) = \phi(v),$$

with the quasi order \preceq on $X \times \mathcal{V}$ defined by

$$(x_1, v_1) \preceq (x_2, v_2) \iff x_1 \leq x_2 \quad \text{and} \quad v_1 \leq v_2.$$

By combining Theorem 5.2 with the quasi order $\preceq_{g,C}$ we obtain the following strong solution existence theorem. We write $C = 0$ (resp. $0 \prec C$) if $C(v_1, v_2) = 0 \forall v_1, v_2 \in \mathcal{V}$ (resp. $0 \prec C(v_1, v_2) \forall v_1, v_2 \in \mathcal{V}, v_1 \neq v_2$).

Theorem 5.3 (SECOND STRONG SOLUTION EXISTENCE RESULT). Consider problem (2), where the objective space is ordered via the quasi order $\preceq_{g,C}$ and $0 \preceq C(v_1, v_2)$ for all $v_1, v_2 \in \mathcal{V}$. Let $u \in X$ be such that $\text{Im}(F|_{S(F, F(u))})$ is CIO with respect to $\preceq_{g,C}$. If there exists $\xi \in \text{H}(\mathcal{E}, \preceq, \mathbb{R} \cup \{+\infty\})$ satisfying one of the following conditions, then there exists $\bar{x} \in X$, $F(\bar{x}) \preceq_{g,C} F(u)$, such that $\bar{x} \in \text{SMin}(F, \preceq_{g,C})$.

- (a) $C = 0$, $g \circ F$ is injective in $S(F, F(u))$ and $\xi \in \text{H}^s(\mathcal{E}, \preceq, \mathbb{R} \cup \{+\infty\})$.
- (b) $0 \prec C$, F is injective in $S(F, F(u))$ and $\xi(y_1 + y_2) > \xi(y_1), \forall y_1, y_2 \in \mathcal{E}, 0 \prec y_2$.

Proof. The result follows by applying Theorem 5.2 and Remark 5.1 to $\phi = \xi \circ g$. \square

By combining Theorem 5.3 with the domination structures of Remark 3.1 it is possible to derive a lot of strong point/solution existence results in different frameworks, as it is showed in Proposition 5.1. In the rest of this section we assume that (Y, τ, \leq_D) is a complete topological quasi ordered linear space. Consider a nonempty set $M \subset Y$. The following existence theorem for strong points of M is a direct consequence of Theorem 5.3 and the assumption:

- Assumptions (A1)** (a) There exists $\varphi \in \text{H}^s(Y, \leq_D, \mathbb{R} \cup \{+\infty\})$,
(b) For each nonincreasing sequence $(y_n) \subset M$ there exists a Cauchy subsequence (y_{n_k}) .

Proposition 5.1. Assume that M and D are closed and assumption (A1) is satisfied. Then, for each $y \in M$, $S(Y, y) \cap \text{SMin}(M, \leq_D) \neq \emptyset$.

Proof. By Remark 3.1(a) we see that (M, \leq_D) is the same as $(\mathcal{V}, \preceq_{g,C})$, where $\mathcal{V} = M$, $(\mathcal{E}, +, \preceq) = (Y, +, \leq_D)$, $g(v) = v$ and $C(v_1, v_2) = 0$ for all $v, v_1, v_2 \in \mathcal{V}$. Property CIO is satisfied with respect to $\preceq_{g,C}$ and $F = \text{id}$ by assumption (b), since (Y, τ) is complete and M, D are closed. Then the result follows by applying Theorem 5.3. \square

In order to apply Proposition 5.1 let us observe that $\text{H}^s(Y, \leq_D, \mathbb{R} \cup \{+\infty\})$ must be nonempty. Next we show some conditions in order to check the hypotheses of Proposition 5.1.

Proposition 5.2. The set $\text{H}^s(Y, \leq_D, \mathbb{R} \cup \{+\infty\})$ is nonempty if one of the following conditions is true:

(a) D is based.

(b) There exists a proper closed solid convex cone D' such that $D \subset \text{int}(D')$.

Proof. Part (a) is obvious since $\emptyset \neq D^{+s} \subset H^s(Y, \leq_D, \mathbb{R} \cup \{+\infty\})$. To prove part (b), let us take $q \in \text{int}(D')$ and consider the mapping $\varphi_q : Y \rightarrow \mathbb{R}$ defined by

$$\varphi_q(y) := \inf\{t \in \mathbb{R} : y \in tq - D'\} \quad \forall y \in Y.$$

It is easy to check that $\varphi_q \in H^s(Y, \leq_D, \mathbb{R} \cup \{+\infty\})$ (see, for instance, [16, Corollary 2.3.5]). \square

Theorem 5.4. Consider a nonempty set $M \subset Y$. Assumption (A1)(b) is satisfied if one of the following conditions is true:

(a) M is compact.

(b) Y is a locally convex space, D is w -normal (i.e., normal with respect to the weak topology) and ξ is bounded from below on M for all $\xi \in D^+$.

(c) (Y, τ) is a locally convex space, where τ is generated by the seminorms $p \in \mathcal{P}$, D is supernormal and ξ_p is lower bounded on M for all $p \in \mathcal{P}$.

(d) $(Y, \|\cdot\|)$ is normed, D has the angle property and ξ_D is lower bounded on M .

Proof. Part (a). It is obvious that (A1)(b) is hold when M is compact.

Part (b). Assume that Y is a locally convex space and D is w -normal. Then $Y^* = D^+ - D^+$ (see, for instance, [16, Proposition 2.2.9]). In order to prove that (A1)(b) is satisfied, let us consider a nonincreasing sequence $(y_n) \subset M$. By hypothesis $(\xi(y_n))$ is bounded $\forall \xi \in D^+$. Fix $\varepsilon > 0$, $\xi_1, \xi_2 \in D^+$, $g = \xi_1 - \xi_2$ and the weak neighborhood

$$U(g, \varepsilon) = \{y \in Y : |g(y)| \leq \varepsilon\}.$$

The sequences $(\xi_1(y_n))$ and $(\xi_2(y_n))$ converge since they are nonincreasing and bounded below. Therefore they are Cauchy sequences and there exist $n_1, n_2 \in \mathbb{N}$ such that

$$|\xi_i(y_m - y_n)| \leq \varepsilon/2, \quad n, m \geq n_i, \quad \forall i = 1, 2.$$

It follows that

$$\begin{aligned} |g(y_m - y_n)| &= |\xi_1(y_m - y_n) - \xi_2(y_m - y_n)| \leq |\xi_1(y_m - y_n)| + |\xi_2(y_m - y_n)| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall m, n \geq n_0 = \max\{n_1, n_2\} \end{aligned}$$

and so $y_m - y_n \in U(g, \varepsilon) \forall m, n \geq n_0$. Thus (y_n) is a weak Cauchy sequence.

Part (c). Let $(y_n) \subset M$ be a nonincreasing sequence. By the hypotheses we see that $(\xi_p(y_n))$ is nonincreasing and bounded $\forall p \in \mathcal{P}$. Then, for each $p \in \mathcal{P}$ the sequence $(\xi_p(y_n))$ converges and so is a Cauchy sequence. From here it is clear that (y_n) is a Cauchy sequence, since

$$p(y_m - y_n) \leq \xi_p(y_n - y_m), \quad \forall p \in \mathcal{P}, \forall m > n.$$

Part (d) is a consequence of part (c) and the proof is completed. \square

Remark 5.3. It is well-known that if a convex cone D is supernormal then is w -normal too (see [16]) and from this point of view, one could think that part (c) of Theorem 5.4 is a consequence of part (b). However, let us note that the boundedness assumption in (c) is weaker than (b). Similar reasonings can be made with the other parts and in this sense it is clear that the boundedness hypotheses on the set M and the assumptions on the order cone D are complementary, since the first ones are weaker when the second ones are stronger and reciprocally.

By combining the hypotheses in Proposition 5.2 and Theorem 5.4 one can deduce different strong minimal point existence theorems via Proposition 5.1. In this sense, this approach allows us to generalize Bishop–Phelps extremal principle to different contexts and under weaker assumptions. Indeed, Bishop–Phelps extremal principle (see [4, Theorem 2.5]) is a consequence of Proposition 5.1, Proposition 5.2(a) and Theorem 5.4(d).

In the following corollary, Bishop–Phelps extremal principle is extended to locally convex spaces. Its proof is a direct consequence of Proposition 5.2(a), Theorem 5.4(b) and Proposition 5.1 applied to the weak topology.

Corollary 5.2. Let (Y, τ) be a Hausdorff locally convex space and let $\emptyset \neq M \subset Y$ be complete with respect to the weak topology. Suppose that $D \subset Y$ is a based w -normal closed convex cone. If ξ is bounded below on M , $\forall \xi \in D^+$, then $\text{SMin}(M, \leq_D) \neq \emptyset$.

In [32, Theorem 3.1], a Bishop–Phelps principle to topological linear spaces was obtained via well-based order cones. From the previous corollary we deduce that this hypothesis can be weakened to convex cones which are based w -normal and closed when the topological linear space is Hausdorff locally convex.

6 A quasi ordered Ekeland variational principle and relatives

It is well-known that Ekeland variational principle is a strong element existence result based on a particular ordering in the epigraph of the objective mapping (see [4, 12]). This idea is exploited here to prove general Ekeland variational principles on quasi ordered spaces via Theorem 5.2.

In this section, we assume that $F : X \rightarrow \mathcal{E}$, where X is any nonempty set and (\mathcal{E}, \preceq) is a preordered space. Consider the problem

$$\preceq_e \text{-Min } (i, F)(x) \text{ subject to } x \in X, \quad (7)$$

where $i : X \rightarrow X$ denotes the identity mapping and \preceq_e is a quasi order defined on $X \times \mathcal{E}$ (actually, it needs to be defined only on $\text{Graph}(F)$) in terms of the preorder on \mathcal{E} .

Different versions of the Ekeland variational principle can be obtained via existence results of strong solution to problem (7). The following theorem shows this fact. Let us consider first some assumptions:

Assumptions (A2) There exists $x \in X$ and $\varphi \in \text{H}(X \times \mathcal{E}, \preceq_e, \mathbb{R} \cup \{+\infty\})$ such that

$$\begin{aligned} z \in S((i, F), (x, F(x))), \quad z \in \underset{S((i, F), (z, F(z)))}{\text{argmin}} \varphi \circ (i, F) \\ \implies \left\{ \begin{array}{l} \exists x' \in S((i, F), (z, F(z))) : \\ \underset{S((i, F), (x', F(x')))}{\text{argmin}} \varphi \circ (i, F) = \{x'\}. \end{array} \right. \quad (8) \end{aligned}$$

Moreover, $\text{Graph}(F|_{S((i, F), (x, F(x)))})$ is CIO with respect to \preceq_e , i.e., for each sequence $(x_n) \subset S((i, F), (x, F(x)))$ such that $(x_n, F(x_n))$ is nonincreasing with respect to the quasi order \preceq_e , there exists $u \in S((i, F), (x, F(x)))$ and a subsequence (x_{n_k}) satisfying $(u, F(u)) \preceq_e (x_{n_k}, F(x_{n_k}))$ for all n_k .

Theorem 6.1. Consider that assumptions (A2) are satisfied. Then there exists $\bar{x} \in X$ such that

$$(i) \quad (\bar{x}, F(\bar{x})) \preceq_e (x, F(x));$$

$$(ii) \quad x' \in X, (x', F(x')) \preceq_e (\bar{x}, F(\bar{x})) \implies x' = \bar{x}.$$

Proof. The result follows by applying Theorem 5.2 to problem (7). \square

In what follows, we give two main quasi orders \preceq_e encompassing well-known ordering appearing in the literature. We start with that when the

perturbation is given by a family of mappings K_λ , $\lambda \in \Lambda$ that satisfy the “triangle inequality”. Throughout we assume that $(\mathcal{E}, +, \preceq)$ is a preordered magma with zero element satisfying (1).

Definition 6.1. We say that a family of mappings $K_\lambda : X \times X \rightarrow \mathcal{E}$ parametrized by $\lambda \in \Lambda$ satisfies the “triangle inequality” property (property TI for short) if for each $x_i \in X$, $i = 1, 2, 3$, and $\lambda \in \Lambda$ there exist $\mu, \gamma \in \Lambda$ such that $K_\lambda(x_1, x_3) \preceq K_\mu(x_1, x_2) + K_\gamma(x_2, x_3)$.

If the mappings $(K_\lambda)_{\lambda \in \Lambda}$ satisfy property TI then it is not difficult to prove that the relation

$$(x_1, y_1) \preceq_e (x_2, y_2) \iff \begin{cases} (x_1, y_1) = (x_2, y_2) \text{ or} \\ y_1 + K_\lambda(x_1, x_2) \preceq y_2, \quad \forall \lambda \in \Lambda \end{cases} \quad (9)$$

defines a quasi order on $X \times \mathcal{E}$. Moreover, if $K_\lambda(x, x) \preceq 0$ for all $x \in X$ and $\lambda \in \Lambda$ then (9) is equivalent to say

$$(x_1, y_1) \preceq_e (x_2, y_2) \iff y_1 + K_\lambda(x_1, x_2) \preceq y_2, \quad \forall \lambda \in \Lambda. \quad (10)$$

Theorem 6.2. [FIRST QUASI ORDERED EKELAND VARIATIONAL PRINCIPLE] Let $(K_\lambda)_{\lambda \in \Lambda}$ be satisfying property TI and assume that assumptions (A2) are true with the quasi order given by (9). Then there exists $\bar{x} \in X$ such that, for x as in (A2),

- (i) $x \neq \bar{x} \Rightarrow F(\bar{x}) + K_\lambda(\bar{x}, x) \preceq F(x) \quad \forall \lambda \in \Lambda$;
- (ii) For each $x' \in X \setminus \{\bar{x}\}$ there exists $\lambda \in \Lambda$ such that $F(x') + K_\lambda(x', \bar{x}) \not\preceq F(\bar{x})$.

Proof. By applying Theorem 6.1 with the quasi order \preceq_e given by (9) we deduce that there exists $\bar{x} \in X$ satisfying

$$\begin{cases} (i) & (\bar{x}, F(\bar{x})) \preceq_e (x, F(x)), \\ (ii) & x' \in X, (x', F(x')) \preceq_e (\bar{x}, F(\bar{x})) \Rightarrow x' = \bar{x}. \end{cases} \quad (11)$$

By (11)(i) we obtain $(\bar{x}, F(\bar{x})) = (x, F(x))$ or $F(\bar{x}) + K_\lambda(\bar{x}, x) \preceq F(x)$ for all $\lambda \in \Lambda$ and then Part (i) is proved. Analogously, by (11)(ii) we have

$$x' \in X, \quad \begin{cases} (x', F(x')) = (\bar{x}, F(\bar{x})) \text{ or} \\ F(x') + K_\lambda(x', \bar{x}) \preceq F(\bar{x}) \quad \forall \lambda \in \Lambda \end{cases} \Rightarrow x' = \bar{x}. \quad (12)$$

Therefore, if $x' \neq \bar{x}$ then there exists $\lambda \in \Lambda$ such that $F(x') + K_\lambda(x', \bar{x}) \not\preceq F(\bar{x})$ and the proof is complete. \square

Remark 6.1. If we have

$$K_\lambda(x, x) \preceq 0 \quad \forall x \in X, \quad \forall \lambda \in \Lambda \quad (13)$$

then by (10) we have the following relation instead of (12):

$$x' \in X, F(x') + K_\lambda(x', \bar{x}) \preceq F(\bar{x}) \quad \forall \lambda \in \Lambda \Rightarrow x' = \bar{x}$$

and therefore Part (ii) of Theorem 6.2 can be rewritten as in the following standard form:

$$x' \in X, F(x') + K_\lambda(x', \bar{x}) \preceq F(\bar{x}) \quad \forall \lambda \in \Lambda \Rightarrow x' = \bar{x}. \quad (14)$$

Obviously, this statement is also true if relation (13) is not satisfied. However, the expression in Part (ii) of Theorem 6.2 is more suitable because statement (14) could be useless when (13) is false. Indeed, in this last case, it could happen that for each $x' \in X$ there exists $\lambda \in \Lambda$ such that $F(x') + K_\lambda(x', \bar{x}) \not\preceq F(\bar{x})$, i.e., just the expression used in Part (ii) of Theorem 6.2. For example, let us apply Theorem 6.2 with the following data: $X = \mathbb{R}$, $\mathcal{E} = \mathbb{R}_+$, $F(z) = |z|$, $\varphi(z, y) = y$ for all $z \in \mathbb{R}$, $y \in \mathbb{R}_+$ and

$$K(x_1, x_2) = \begin{cases} -x_2 & \text{if } x_2 \leq -1 \\ 1 & \text{if } x_2 > -1 \end{cases} \quad \forall x_1, x_2 \in \mathbb{R}. \quad (15)$$

Let $x = 1$. It is clear that assumptions (A2) hold. Part (i) of Theorem 6.2 is satisfied by $\bar{x} = 0$, but relation (14) cannot be checked. However, $\bar{x} = 0$ satisfies Part (ii) of Theorem 6.2.

Theorem 6.2 extends [22, Theorems 4 and 9], [23, Corollaries 6.13, 6.15 and Theorems 6.1, 6.8, 6.9] and [24, Corollaries 3.1, 4.3 and Theorem 4.2] to a space $(\mathcal{E}, +, \preceq)$ which could not be linear and \preceq could not be a set relation. By taking $F(x) = 0$, $x \in X$, and

$$K_\lambda(x, y) = K(x, y) = F(x, y) + \varepsilon w(x, y)e$$

we get [3, Theorem 3.1]. Furthermore, we have also obtained [36, Theorems 3.1, 3.2]. For this last result, take $\preceq = \leq_K$, $F(x) = 0$, $x \in X$, and

$$K_\lambda(x, y) = \Phi(x, y) + k_\lambda^0 d_\lambda(x, y), \quad \lambda \in \Lambda := (0, 1],$$

in Theorem 6.2. A consequence of obtaining Ekeland variational principles via Brézis-Browder principles is that the usual nonincreasing monotone completeness hypothesis of these variational principle is showed to be a CIO type

assumption (see, for instance, [23, Corollaries 6.13, 6.15 and Theorems 6.1, 6.8, 6.9]).

In [1], the following quasi order on $X \times (\mathbb{R} \cup \{+\infty\})$, which is not of the form (9), is considered:

$$(x_1, y_1) \preceq_e (x_2, y_2) \iff (x_1, y_1) = (x_2, y_2) \text{ or } q(x_1, x_2) \leq \varphi(y_2)(y_2 - y_1), \quad (16)$$

where $q : X \times X \rightarrow [0, +\infty)$ satisfies

$$q(x_1, x_3) \leq q(x_1, x_2) + q(x_2, x_3) \quad \forall x_1, x_2, x_3 \in X,$$

and $\varphi : \mathbb{R} \cup \{+\infty\} \rightarrow (0, +\infty)$ is a nondecreasing function.

In order to generalize such kind of quasi order we assume that a distributive external operation is defined on \mathcal{E} ($ty \in \mathcal{E}$ and $t(y_1 + y_2) = ty_1 + ty_2$ for all $t > 0$ and $y, y_1, y_2 \in \mathcal{E}$), which is compatible with the quasi order \preceq ($y_1 \preceq y_2$ implies that $ty_1 \preceq ty_2$ for all $t > 0$), and satisfies $0 \preceq t0$, $t_1y \preceq t_2y$, for all $t, t_1, t_2 > 0$, $t_1 < t_2$ and $y \in \mathcal{E}$. Then, given $\varphi \in \mathbf{H}(\mathcal{E}, \preceq, (0, +\infty))$ and a family $(K_\lambda)_{\lambda \in \Lambda}$ satisfying property TI such that $0 \preceq K_\lambda(x_1, x_2)$ for all $x_1, x_2 \in X$ and $\lambda \in \Lambda$, the quasi order on $X \times \mathcal{E}$ induced by (16) is

$$(x_1, y_1) \preceq_\varphi (x_2, y_2) \iff \begin{cases} (x_1, y_1) = (x_2, y_2) \text{ or} \\ y_1 + \frac{1}{\varphi(y_2)} K_\lambda(x_1, x_2) \preceq y_2 \quad \forall \lambda \in \Lambda \end{cases} \quad (17)$$

and the following Ekeland variational principle can be obtained by applying Theorem 6.1.

Theorem 6.3 (SECOND QUASI ORDERED EKELAND VARIATIONAL PRINCIPLE). Let $(K_\lambda)_{\lambda \in \Lambda}$ be satisfying property TI and assume that assumptions (A2) are true with the quasi order given by (17). Then there exists $\bar{x} \in X$ such that, for x as in (A2),

$$(i) \quad x \neq \bar{x} \Rightarrow F(\bar{x}) + \frac{1}{\varphi(F(x))} K_\lambda(\bar{x}, x) \preceq F(x) \quad \forall \lambda \in \Lambda;$$

(ii) For each $x' \in X \setminus \{\bar{x}\}$ there exists $\lambda \in \Lambda$ such that

$$F(x') + \frac{1}{\varphi(F(\bar{x}))} K_\lambda(x', \bar{x}) \not\preceq F(\bar{x}).$$

Next, Theorem 6.3 is applied to prove a slight generalization of a recent scalar Ekeland-type variational principle.

Definition 6.2. Let (X, d) be a quasi-metric space. A function $q : X \times X \rightarrow [0, \infty]$ is called a (G)eneralized Q-function on X (GQ-function for short) if the following conditions are satisfied:

- (a) For each $x_1, x_2, x_3 \in X$, $q(x_1, x_3) \leq q(x_1, x_2) + q(x_2, x_3)$.
- (b) For each $x_1, x_2, x_3 \in X$, if $q(x_1, x_2) = 0$ and $q(x_1, x_3) = 0$ then $x_2 = x_3$.
- (c) Let (x_n) be a sequence in X and $(\alpha_n) \subset \mathbb{R}$, $\alpha_n \downarrow 0$. If $q(x_n, x_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$, $m > n$, then (x_n) is a Cauchy sequence.

Remark 6.2. The class of GQ-functions is wider than the class of Q-functions (see [1, Definition 2.1]). For example, take the quasi-metric in \mathbb{R} , $d(x, y) = y - x$ if $y \geq x$ and $d(x, y) = 1$ otherwise, and then consider $q(x, y) = d(x, y)$ for all $x, y \in \mathbb{R}$: this GQ-function does not satisfy (Q3) of [1, Definition 2.1].

Theorem 6.4. Let X be a quasi-metric space and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper bounded below function. Consider a GQ-function $q : X \times X \rightarrow [0, \infty]$, a nondecreasing function $\varphi : (-\infty, \infty] \rightarrow (0, \infty)$ and assume that $X \times (\mathbb{R} \cup \{+\infty\})$ is ordered through the relation \preceq_φ given by $K = q$. Suppose that there exists $x \in \text{dom}(f)$ such that for every $y \in S((i, f), (x, f(x)))$ and every Cauchy sequence $(x_n) \subset S((i, f), (y, f(y)))$ with $((x_n, f(x_n)))$ nonincreasing it follows that (x_n) converges to a point in $S((i, f), (y, f(y)))$. Then there exists $\bar{x} \in X$ such that

- (i) $x \neq \bar{x} \Rightarrow q(\bar{x}, x) \leq \varphi(f(x))(f(x) - f(\bar{x}))$;
- (ii) $q(x', \bar{x}) > \varphi(f(\bar{x}))(f(\bar{x}) - f(x')) \quad \forall x' \neq \bar{x}$.

Proof. Let us check that Theorem 6.3 (where we consider the notation φ' instead of φ) can be applied with the following data: $\mathcal{E} = \mathbb{R} \cup \{+\infty\}$, $F = f$, $K = q$ and $\varphi'(x, t) = t$ for all $(x, t) \in X \times \mathbb{R} \cup \{+\infty\}$. As q is a GQ-function then q satisfies property TI and \preceq_φ is a quasi order in $X \times (\mathbb{R} \cup \{+\infty\})$. Let us see that assumptions (A2) are satisfied.

First we show that (8) is true. For this aim observe that $\varphi' \in H(X \times \mathcal{E}, \preceq_\varphi, \mathbb{R} \cup \{+\infty\})$ and consider $z, y \in X$, $z \neq y$, such that

$$\{z, y\} \subset \underset{S((i, f), (y, f(y)))}{\text{argmin}} \varphi' \circ (i, f).$$

Therefore $f(z) = f(y)$ and $q(z, y) = 0$. Suppose that

$$\underset{S((i, f), (z, f(z)))}{\text{argmin}} \varphi' \circ (i, f) \neq \{z\}.$$

Then there exists $u \in S((i, f), (z, f(z)))$, $u \neq z$, such that $\varphi'(u, f(u)) = \varphi'(z, f(z))$, i.e., $f(u) = f(z)$ and then $q(u, z) = 0$. If $u = y$ then

$$q(u, u) \leq q(u, z) + q(z, u) = q(u, z) + q(z, y) = 0$$

and $z = u$, that is a contradiction. Thus $u \neq y$ and as $u \in S((i, f), (y, f(y)))$ with $f(u) = f(y)$ we have $q(u, y) = 0$ and so $y = z$, which is a contradiction. Then

$$\operatorname{argmin}_{S((i,f),(z,f(z)))} \varphi' \circ (i, f) = \{z\}$$

and the first part of assumptions (A2) is checked.

In order to prove that $\operatorname{Graph}(f|_{S((i,f),(x,f(x)))})$ is CIO, consider a sequence $(x_n) \subset S((i, f), (x, f(x)))$ such that $((x_n, f(x_n)))$ is nonincreasing with respect to the quasi order \preceq_φ . We can suppose that $x_n \neq x_m$ for all $n \neq m$ since in the other case it is obvious that (x_n) satisfies property CIO. Then $(f(x_n)) \subset \mathbb{R}$ is nonincreasing and since f is bounded below there exists $r \in \mathbb{R}$ such that $f(x_n) \downarrow r$. Thus, $\forall n, k \in \mathbb{N}$, $n, k \geq 1$, we have that:

$$\begin{aligned} q(x_n, x_{n+k}) &\leq \sum_{i=0}^{k-1} q(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{k-1} (\varphi(f(x_{n+i}))(f(x_{n+i}) - f(x_{n+i+1}))) \\ &\leq \varphi(f(x_n)) \sum_{i=0}^{k-1} (f(x_{n+i}) - f(x_{n+i+1})) \\ &= \varphi(f(x_n))(f(x_n) - f(x_{n+k})) \leq \varphi(f(x_n))(f(x_n) - r). \end{aligned}$$

By applying property (c) of Definition 6.2 we deduce that (x_n) is a Cauchy sequence and by the hypotheses there exists $\bar{x} \in S((i, f), (x, f(x)))$ such that $x_n \rightarrow \bar{x}$ with $(\bar{x}, f(\bar{x})) \preceq_\varphi (x_n, f(x_n))$ for all n . Thus $\operatorname{Graph}(f|_{S((i,f),(x,f(x)))})$ is CIO. From here, the result follows by applying Theorem 6.3. \square

Remark 6.3. Theorem 6.4 extends properly Theorem 3.1 of [1] since every Q-function is GQ-function, and there are GQ-functions which are not Q-functions (see Remark 6.2). As a by-product, Theorem 6.4 corrects a mistake of Theorem 3.1 of [1], since statement $q(\bar{x}, x) \leq \varphi(f(x))(f(x) - f(\bar{x}))$ can be false when $x = \bar{x}$.

Next we deduce a version of Theorem 6.2 in the framework of a set-valued optimization problem with set relations. Let (Y, \leq_D) be a partially ordered locally convex space and consider problem (2) with objective space $(2^Y, \leq_D^l)$ and $\operatorname{dom}(F) := \{x \in X : F(x) \neq \emptyset\} \neq \emptyset$. Let $K_\lambda^D : X \times X \rightrightarrows 2^D \setminus \{\emptyset\}$, $\lambda \in \Lambda$, be a family of set-valued mappings and consider the following assumptions:

Assumptions (A3) There exists $\xi \in D^+ \setminus \{0\}$, $\lambda_0 \in \Lambda$ and $x \in \operatorname{dom}(F)$

such that

$$\inf\{\xi(y) : y \in \text{Im}(F)\} > -\infty, \quad (18)$$

$$\inf\{\xi(q) : q \in K_{\lambda_0}^D(x_1, x_2)\} > 0, \quad \forall x_1, x_2 \in X, x_1 \neq x_2 \quad (19)$$

and for each sequence $(x_n) \subset S((i, F), (x, F(x)))$ (we assume that $X \times 2^Y$ is ordered via \leq_D^l by (9)) such that

$$F(x_{n+1}) + K_\lambda^D(x_{n+1}, x_n) \leq_D^l F(x_n), \quad \forall n, \forall \lambda \in \Lambda$$

there exists $u \in X$ and a subsequence (x_{n_k}) satisfying

$$F(u) + K_\lambda^D(u, x_{n_k}) \leq_D^l F(x_{n_k}), \quad \forall n_k, \forall \lambda \in \Lambda.$$

Theorem 6.5. Let $(K_\lambda^D)_{\lambda \in \Lambda}$ be satisfying property TI with respect to the relation \leq_D^l and assume that assumptions (A3) are true. Then there exists $\bar{x} \in \text{dom}(F)$ such that, for x as in (A3),

$$(i) \quad x \neq \bar{x} \Rightarrow F(\bar{x}) + K_\lambda^D(\bar{x}, x) \leq_D^l F(x) \quad \forall \lambda \in \Lambda;$$

$$(ii) \quad \text{For each } x' \in X \setminus \{\bar{x}\} \text{ there exists } \lambda \in \Lambda \text{ such that } F(x') + K_\lambda^D(x', \bar{x}) \not\leq_D^l F(\bar{x}).$$

Proof. We apply Theorem 6.2 for the particular case $(\mathcal{E}, \preceq) = (2^Y, \leq_D^l)$ and through the following scalarizing mapping $\varphi_\xi : X \times 2^Y \rightarrow \mathbb{R} \cup \{+\infty\}$,

$$\varphi_\xi(z, M) = \begin{cases} \inf\{\xi(y) : y \in M\} & \text{if } M \neq \emptyset; \\ +\infty & \text{if } M = \emptyset, \end{cases}$$

where $\xi \in D^+ \setminus \{0\}$ satisfies (18) and (19). We will check that φ_ξ is nondecreasing with respect to \preceq_e defined by

$$(x_1, M_1) \preceq_e (x_2, M_2) \iff \begin{cases} (x_1, M_1) = (x_2, M_2) \text{ or} \\ M_1 + K_\lambda^D(x_1, x_2) \leq_D^l M_2, \quad \forall \lambda \in \Lambda. \end{cases} \quad (20)$$

Indeed, let $(x_1, M_1), (x_2, M_2) \in X \times 2^Y$ such that $(x_1, M_1) \preceq_e (x_2, M_2)$ and $(x_1, M_1) \neq (x_2, M_2)$. If $M_1 = \emptyset$ or $M_2 = \emptyset$ then it is obvious that $\varphi_\xi(x_2, M_2) \geq \varphi_\xi(x_1, M_1)$. If $M_1 \neq \emptyset$ and $M_2 \neq \emptyset$, by (20) we have that

$$\begin{aligned} \varphi_\xi(x_2, M_2) &= \inf\{\xi(y) : y \in M_2\} \\ &\geq \inf\{\xi(y) : y \in M_1 + K_\lambda^D(x_1, x_2) + D\} \\ &\geq \inf\{\xi(y) : y \in M_1\} = \varphi_\xi(x_1, M_1), \end{aligned}$$

proving the desired result.

Now let us check assumptions (A2). The second part is clear by the second part of (A3). Consider $u \in S((i, F), (x, F(x)))$ and $u \neq x$. Then, for $\lambda_0 \in \Lambda$ satisfying (19) we obtain $F(u) + K_{\lambda_0}^D(u, x) \leq_D^l F(x)$ and it follows that

$$\begin{aligned} \varphi_\xi(x, F(x)) &= \inf\{\xi(y) : y \in F(x)\} \\ &\geq \inf\{\xi(y) : y \in F(u) + K_{\lambda_0}^D(u, x) + D\} \\ &> \inf\{\xi(y) : y \in F(u)\} = \varphi_\xi(u, F(u)). \end{aligned}$$

Thus, by Theorem 6.2 there exists $\bar{x} \in \text{dom}(F)$ satisfying properties (i) and (ii), which completes the proof. \square

Remark 6.4. The previous theorem extends the Ekeland variational principles obtained in [20] for vector functions with set-valued perturbations, as well as those found in the literature involving set relations. Furthermore, our results allow perturbations which are more general than those called D -metrics.

In the literature, it is usual to obtain Ekeland variational principles based on approximate solutions of the optimization problem. Next we prove one result in this line. For this aim, first it is necessary to define the notion of approximate solution of problem (2) when the objective space is $(\mathcal{E}, +, \preceq)$.

Definition 6.3. Let $\tilde{q} \in \mathcal{E}$, $0 \preceq \tilde{q}$. A point $x_0 \in X$ is a strong \tilde{q} -efficient solution of problem (2), denoted $x_0 \in \text{ASMin}(F, \preceq, \tilde{q})$, if

$$F(x) + \tilde{q} \not\preceq F(x_0), \quad \forall x \in X \setminus \{x_0\}.$$

Notice that in order to have $x_0 \in \text{ASMin}(F, \preceq, \tilde{q})$, we need only to check the relation for $x \in X \setminus \{x_0\}$ satisfying $F(x) \preceq F(x_0)$.

Let us show some properties of these approximate solutions.

Proposition 6.1. The following properties are true.

- (a) $\text{ASMin}(F, \preceq, 0) = \text{SMin}(F, \preceq) \subset \bigcap_{0 \prec \tilde{q}} \text{ASMin}(F, \preceq, \tilde{q})$. If the quasi order \preceq is a partial order and (1) is satisfied for \prec instead of \preceq then

$$\text{Min}(F, \prec) \subset \bigcap_{0 \prec \tilde{q}} \text{ASMin}(F, \preceq, \tilde{q}).$$

- (b) If $\tilde{q}_1 \preceq \tilde{q}_2$ then $\text{ASMin}(F, \preceq, \tilde{q}_1) \subset \text{ASMin}(F, \preceq, \tilde{q}_2)$.

(c) If for each $y_1, y_2 \in \mathcal{E}$, $y_1 \prec y_2$ there exists $0 \prec \tilde{q}$ such that $y_1 + \tilde{q} \preceq y_2$ then $\bigcap_{0 \prec \tilde{q}} \text{ASMin}(F, \preceq, \tilde{q}) \subset \text{Min}(F, \prec)$.

Proof. (a) It is obvious that $\text{ASMin}(F, \preceq, 0) = \text{SMin}(F, \preceq)$. Consider $x_0 \in \text{SMin}(F, \preceq)$ and suppose that $x_0 \notin \text{ASMin}(F, \preceq, \tilde{q})$ with $0 \prec \tilde{q}$. Then there exists $x \in X \setminus \{x_0\}$ such that $F(x) + \tilde{q} \preceq F(x_0)$ and by (1) we deduce via the transitive property that $F(x) \preceq F(x_0)$, which is a contradiction and the proof for the first statement of part (a) is complete. The second statement follows in a similar way since by the hypotheses, $F(x) \prec F(x) + \tilde{q} \preceq F(x_0)$ implies $F(x) \prec F(x_0)$.

(b) Let $\tilde{q}_1 \preceq \tilde{q}_2$, $x_0 \in \text{ASMin}(F, \preceq, \tilde{q}_1)$ and suppose that $x_0 \notin \text{ASMin}(F, \preceq, \tilde{q}_2)$. Then there exists $x \in X \setminus \{x_0\}$ such that $F(x) + \tilde{q}_2 \preceq F(x_0)$. As in the previous part we deduce that $F(x) + \tilde{q}_1 \preceq F(x_0)$, since $\tilde{q}_1 \preceq \tilde{q}_2$, which is a contradiction and the proof of part (b) is finished.

(c) Suppose that $x_0 \notin \text{Min}(F, \prec)$. Then there exists $x \in X \setminus \{x_0\}$ such that $F(x) \prec F(x_0)$ and by hypothesis we can find a point $0 \prec \tilde{q}_0$ satisfying $F(x) + \tilde{q}_0 \preceq F(x_0)$. Therefore $x_0 \notin \text{ASMin}(F, \preceq, \tilde{q}_0)$ and the proof of part (c) is complete. \square

Remark 6.5. Definition 6.3 encompasses the (C, ε) -efficient solution notion given by Gutiérrez et al. [18, 19, Definition 3.2], which collapses the main approximate efficiency concepts in vector optimization. Indeed, let (Y, \leq_D) be a partially ordered topological linear space, $f : X \rightarrow Y$ a vector mapping and consider the following vector optimization problem:

$$\leq_D \text{-Min } f(x) \text{ subject to } x \in X. \quad (21)$$

This problem is equivalent to problem (2) with objective space $(2^Y, \leq_D^l)$ and objective mapping $F : X \rightarrow 2^Y$, $F(x) = \{f(x)\}$ for all $x \in X$, since $\text{Min}(f, \leq_D) = \text{Min}(F, \leq_D^l)$. Let $C \subset D \setminus \{0\}$ be such that $C + D = C$ and $\varepsilon > 0$. By applying Definition 6.3 to $\tilde{q}_\varepsilon = \varepsilon C$ we obtain that $x_0 \in \text{ASMin}(F, \leq_D^l, \tilde{q}_\varepsilon)$ if and only if there is not $x \in X \setminus \{x_0\}$ such that $\{f(x)\} + \tilde{q}_\varepsilon \leq_D^l \{f(x_0)\}$. This condition is equivalent to say that $(f(X) - f(x_0)) \cap (-\varepsilon C) = \emptyset$, i.e., is equivalent to say that x_0 is a (C, ε) -efficient solution of problem (21).

Analogously, it is easy to check that Proposition 6.1 encompasses [19, Theorem 3.4(i)-(iii)] when $\cup_{\varepsilon \geq 0} \varepsilon C = D$.

The more general notion of ε -efficiency introduced in [21, Definition 2.1] is also recovered by setting $\tilde{q}_\varepsilon = Q(\varepsilon) + (D \setminus \{0\})$ instead of $\tilde{q}_\varepsilon = \varepsilon C$.

Next we show two versions of Ekeland variational principle for problem (2) based on approximate solutions and the objective space $(\mathcal{E}, +, \preceq)$.

Theorem 6.6 (THIRD QUASI ORDERED EKELAND VARIATIONAL PRINCIPLE). Let $(K_\lambda)_{\lambda \in \Lambda}$ be satisfying property TI and $K_\lambda(x, x) = 0$ for all $x \in X$ and all $\lambda \in \Lambda$. Assume that assumptions (A2) are true with the quasi order given by (9) and let $0 \preceq \tilde{q}$, $\tilde{q} \not\preceq 0$ and $x \in \text{ASMin}(F, \preceq, \tilde{q})$. Then, there exists $\bar{x} \in X$ such that:

- (i) $F(\bar{x}) + K_\lambda(\bar{x}, x) \preceq F(x) \forall \lambda \in \Lambda$;
- (ii) $\tilde{q} \not\preceq K_\lambda(\bar{x}, x) \forall \lambda \in \Lambda$;
- (iii) $x' \in X, F(x') + K_\lambda(x', \bar{x}) \preceq F(\bar{x}) \forall \lambda \in \Lambda \Rightarrow x' = \bar{x}$.

Proof. From Theorem 6.2 we obtain the existence of $\bar{x} \in X$ satisfying (i) and (iii). Part (ii) is obvious if $x = \bar{x}$ since $K_\lambda(\bar{x}, \bar{x}) = 0 \forall \lambda \in \Lambda$ and $\tilde{q} \not\preceq 0$. If $x \neq \bar{x}$ and $\tilde{q} \preceq K_\lambda(\bar{x}, x)$ for some $\lambda \in \Lambda$, then

$$F(\bar{x}) + \tilde{q} \preceq F(\bar{x}) + K_\lambda(\bar{x}, x) \preceq F(x),$$

which is a contradiction since $x \in \text{ASMin}(F, \preceq, \tilde{q})$. Then (ii) holds and the proof is completed. \square

One can also obtain a similar theorem under the quasi order (17).

We now establish the counterpart of Theorem 6.5 for approximate solutions, which can be proved as Theorem 6.6.

Theorem 6.7. Let $(K_\lambda^D)_{\lambda \in \Lambda}$ be satisfying property TI with respect to the relation \leq_D^l along with $K_\lambda^D(x, x) = \{0\}$ for all $x \in X$ and all $\lambda \in \Lambda$, and assume that assumptions (A3) are true. Let $\emptyset \neq \tilde{q} \subset D$, $0 \notin \tilde{q} + D$, and $x \in \text{ASMin}(F, \leq_D^l, \tilde{q})$. Then there exists $\bar{x} \in \text{dom}(F)$ such that

- (i) $F(\bar{x}) + K_\lambda^D(\bar{x}, x) \leq_D^l F(x) \forall \lambda \in \Lambda$;
- (ii) $\tilde{q} \not\leq_D^l K_\lambda^D(\bar{x}, x) \forall \lambda \in \Lambda$;
- (iii) $x' \in X, F(x') + K_\lambda^D(x', \bar{x}) \leq_D^l F(\bar{x}) \forall \lambda \in \Lambda \Rightarrow x' = \bar{x}$.

Remark 6.6. From Theorems 6.5 and 6.7, we recover all the results established in [20]. Indeed, Theorem 3.8 of [20] is obtained by taking $K(x_1, x_2) = \gamma F(x_1, x_2)$ in Theorem 6.5; by setting $K(x_1, x_2) = \gamma F(x_1, x_2)$, $\tilde{q} = C$ in Theorem 6.7, we get [20, Theorem 3.14]; if $K(x_1, x_2) = \gamma d(x_1, x_2)C_\Delta$ and $\tilde{q} = B_\Delta$, Proposition 5.1 is obtained; Theorem 5.4 is a special case of Theorem 6.7 when $K(x_1, x_2) = (\varepsilon/\gamma)d(x_1, x_2)C_q$ and $\tilde{q} = C_q$, $q \in D \setminus \{0\}$, $C_q = q + D \setminus \{0\}$; by taking $K(x_1, x_2) = \gamma d(x_1, x_2)(H + D)$, $\tilde{q} = H$, we obtain Theorem 5.11; by setting $K(x_1, x_2) = \gamma d(x_1, x_2)(B + D)$, $\tilde{q} = B$, Theorem 5.12 is recovered.

By particularizing the mappings K_λ^D in Theorem 6.7, we obtain the next result.

Theorem 6.8. Let (X, \mathcal{U}) be a Hausdorff uniform space generated by the quasimetrics $(q_\lambda)_{\lambda \in \Lambda}$ and consider a convex co-radiant set $C \subset D \setminus \{0\}$ such that $0 \notin C + D$. Let $\varepsilon > 0$, $x \in \text{ASMin}(F, \leq_D^l, \varepsilon C)$ and assume that assumptions (A3) are true with $K_\lambda^D(x_1, x_2) = q_\lambda(x_1, x_2)C$ for all $\lambda \in \Lambda$. Then there exists $\bar{x} \in \text{dom}(F)$ such that

$$(i) \quad F(\bar{x}) + q_\lambda(\bar{x}, x)C \leq_D^l F(x) \quad \forall \lambda \in \Lambda;$$

$$(ii) \quad q_\lambda(\bar{x}, x) < \varepsilon \quad \forall \lambda \in \Lambda;$$

$$(iii) \quad x' \in X, F(x') + q_\lambda(x', \bar{x})C \leq_D^l F(\bar{x}) \quad \forall \lambda \in \Lambda \Rightarrow x' = \bar{x}.$$

Proof. By Theorem 6.7 we have (i) and (iii). We now prove (ii). If on the contrary there exists $\lambda \in \Lambda$ such that $q_\lambda(\bar{x}, x) \geq \varepsilon$, then $\bar{x} \neq x$ and

$$K_\lambda^D(\bar{x}, x) = q_\lambda(\bar{x}, x)C \subset \varepsilon C + D,$$

since $C \subset D$. Such an inclusion means $\varepsilon C \leq_D^l K_\lambda^D(\bar{x}, x)$, which contradicts (ii) of the previous theorem. This proves $q_\lambda(\bar{x}, x) < \varepsilon$ for all $\lambda \in \Lambda$ and the proof is complete. \square

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