Analysis of a velocity-pressure-pseudostress formulation for the stationary Stokes equations*

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Abstract

We consider a non-standard mixed approach for the Stokes problem in which the velocity, the pressure, and the pseudostress are the main unknowns. Alternatively, the pressure can be eliminated from the original equations, thus yielding an equivalent formulation with only two unknowns. In this paper we develop a priori and a posteriori error analyses of both approaches. We first apply the Babuška-Brezzi theory to prove the well-posedness of the continuous and discrete formulations. In particular, we show that Raviart-Thomas elements of order $k \geq 0$ for the pseudostresses, and piecewise polynomials of degree k for the velocities and the pressures, ensure unique solvability and stability of the associated Galerkin schemes. Then, we derive reliable and efficient residual-based a posteriori error estimators for both schemes, without and with pressure unknown. Finally, we provide several numerical results illustrating the good performance of the resulting mixed finite element methods, confirming the theoretical properties of the estimators, and showing the behaviour of the associated adaptive algorithms.

Key words: mixed finite element, incompressible flow, a posteriori error estimator

Mathematics Subject Classifications (1991): 65N15, 65N30, 65N50, 74B05

1 Introduction

The velocity-pressure-stress formulation for computational incompressible flows has gained considerable attention in recent years due to its natural applicability to non-Newtonian flows. Indeed, since in this case the constitutive equation is nonlinear, the stress can not be eliminated, and hence it becomes an unavoidable unknown in the corresponding solvability analysis. Actually, the main advantage of this formulation is that it allows for a unified analysis for linear and nonlinear flows. Another interesting feature of the velocity-pressure-stress approach is given by the fact that it yields direct finite element approximations of other quantities of physical interest. In particular, an accurate direct calculation of the stresses is very desirable for flow problems involving interaction with solid structures. Nevertheless, the increase in the number

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of unknowns and the symmetry requirement for the stress tensor constitute the main drawbacks of this formulation. Moreover, the difficulty in deriving and using finite element subspaces of symmetric tensors in the Stokes and Lamé systems is already well known (see, e.g. [9], [5]).

In order to circumvent these disadvantages, at least partially, there are two main approaches. A first idea, which goes back to [3], consists of imposing the symmetry of the stress in a weak sense through the introduction of a suitable Lagrange multiplier (rotation in elasticity and vorticity in fluid mechanics). However, a more appealing idea nowadays is given by the use of the pseudostress instead of the stress in the corresponding setting of the Stokes equations. Indeed, this procedure has become very popular lately, specially in the context of least-squares and augmented methods, thus yielding two new approaches for incompressible flows: the velocity-pressure-pseudostress and velocity-pseudostress formulations. In particular, the pseudostress is introduced in [10] and [22] as an additional unknown that replaces the original symmetric stress tensor. In this way, nonsymmetric tensor finite element subspaces can be employed and the stress and other quantities such as velocity gradient and vorticity can be obtained easily via a postprocessing computation. More precisely, following [12], two least-squares methods are developed and analyzed in [10] for the numerical solution of linear, stationary incompressible Newtonian fluid flow in two and three dimensions. One method, which applies to general boundary conditions, is based on the stress-velocity formulation, whereas the other one, restricted to pure velocity Dirichlet boundary conditions, is based on the equivalent velocity-pseudostress approach. Further least-squares methods for the steady Stokes problem, based on formulations with two or three fields among velocity, velocity gradient, pressure, vorticity, stress, and pseudostress, can be found in [7], [8], [11], [14], [17], and the references therein.

Similarly, augmented mixed finite element methods for both pseudostress-based formulations of the stationary Stokes equations are introduced and analyzed in [22]. The approach there, which extends analogue results for linear elasticity problems (see [25], [26], [29]), relies on the introduction of the Galerkin least-squares type terms arising from the constitutive and equilibrium equations, and the Dirichlet boundary condition for the velocity, all of them multiplied by suitable stabilization parameters. It is shown that these parameters can be chosen so that the resulting augmented variational formulations are defined by strongly coercive bilinear forms, whence the associated Galerkin schemes become well posed for any choice of finite element subspaces. In particular, continuous piecewise linear velocities, piecewise constant pressures, and Raviart-Thomas elements of lowest order for the stresses can be employed, which yields a number of unknowns behaving asymptotically as 5 times the number of triangles of the triangulation. Alternatively, the above factor reduces to 4 when the augmented variational formulation involving only the velocity and the pseudostress as unknowns, is considered. In addition, reliable and efficient residual-based a posteriori error estimators for both augmented mixed finite element schemes are also derived in [22]. The corresponding augmented mixed finite element schemes for the velocity-pressure-stress formulation of the Stokes problem, in which the vorticity is introduced as the Lagrange multiplier taking care of the weak symmetry of the stress, are studied in [21].

On the other hand, and as expected, the velocity-pressure-pseudostress formulation has also been applied to nonlinear Stokes problems. In fact, a new mixed finite element method for a class of models arising in quasi-Newtonian fluids is presented in [27]. The approach is based on the introduction of both the pseudostress and the velocity gradient as further unknowns, which yields a twofold saddle point operator equation in Hilbert spaces as the resulting variational formulation. The abstract theory for this kind of operator equation (see, e.g. [23], [28]), which constitutes a generalization of the Babuška-Brezzi theory, is then applied to prove that the continuous and discrete formulations are well posed. The results in [27] are extended in [20] to a setting in reflexive Banach spaces, thus allowing other nonlinear models such as the Carreau law for viscoplastic flows. In addition, the dual-mixed approach from [27] and [20] is reformulated in [32] by restricting the space for the velocity gradient to that of trace-free tensors. As a consequence the pressure is eliminated and a three-field formulation with the pseudostress, the velocity, and its gradient as unknowns, is obtained.

In spite of the numerous contributions mentioned above, it is surprising to realize, up to our knowledge, that mixed finite element methods for the pure velocity-pseudostress formulation, i.e. without any least-squares or augmented techniques, had not been studied until the recent work [13]. In this article it is shown that Raviart-Thomas elements of index $k \geq 0$ for the pseudostress and piecewise discontinuous polynomials of degree k for the velocity lead to a stable Galerkin scheme with quasi-optimal accuracy. A suitable combination of the penalty method with a preconditioned multigrid method is then applied in [13] to solve the indefinite system resulting from the associated Galerkin scheme. The pressure and other physical quantities (if needed) can be computed in a postprocessing procedure without affecting the accuracy of approximation.

The purpose of the present paper is to additionaly contribute in the direction of the results provided in [13]. More precisely, we recast the pure velocity-pseudostress formulation from [13], incorporate the pressure unknown into the discrete analysis, which does not necessarily yield an equivalent formulation at that level, and derive reliable and efficient residual-based a posteriori error estimators for both Galerkin schemes. The idea of reintroducing the pressure is to allow further flexibility in approximating this unknown. To this respect, we show that a Galerkin scheme for the velocity-pressure-pseudostress formulation only makes sense for pressure finite element subspaces not containing the traces of the pseudostresses subspace. In particular, this is the case when Raviart-Thomas elements of index $k \geq 0$ for the pseudostress, and piecewise discontinuous polynomials of degree k for the velocity and the pressure, are utilized. Otherwise, both discrete schemes coincide and hence one obviously stays with the simplest one. The rest of this work is organized as follows. In Section 2 we describe the boundary value problem of interest, and establish and analyze its dual-mixed variational formulations (without and with pressure). Then, in Section 3 we introduce and analyze the associated Galerkin schemes. In particular, we show that a suitable chosen stabilization parameter is needed for the well-posedness of the discrete system involving the pressure. Some components of the analysis in Sections 2 and 3 for the velocity-pseudostress formulation, though developed differently, coincide with the corresponding discussion in [13]. Next, in Section 4 we develop the residual-based a posteriori error analysis for both schemes. Finally, several numerical results illustrating the performance of the mixed finite element methods, confirming the reliability and efficiency of the a posteriori estimators, and showing the good behaviour of the associated adaptive algorithms, are provided in Section 5.

We end this section with some notations to be used below. Given any Hilbert space U, U^2 and $U^{2\times 2}$ denote, respectively, the space of vectors and square matrices of order 2 with entries in U. In addition, $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we write as usual

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji}), \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}\,.$$

Also, in what follows we utilize the standard terminology for Sobolev spaces and norms, employ $\mathbf{0}$ to denote a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The problem and its dual-mixed formulations

2.1 The boundary value problem

Let Ω be a bounded and simply connected polygonal domain in $\mathbb{R}^2$ with boundary Γ. Our goal is to determine the velocity $\mathbf{u}$, the pseudostress tensor $\boldsymbol{\sigma}$, and the pressure p of a steady flow occupying the region Ω, under the action of external forces. More precisely, given a volume force $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we seek a tensor field $\boldsymbol{\sigma}$, a vector field $\mathbf{u}$, and a scalar field p such that

$$\begin{gathered} \boldsymbol{\sigma} = 2\,\mu\,\nabla\mathbf{u} - p\,\mathbf{I} \quad \text{in} \quad \Omega\,, \qquad \mathbf{div}(\boldsymbol{\sigma}) = -\,\mathbf{f} \quad \text{in} \quad \Omega\,, \\ \mathrm{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \end{gathered} \tag{2.1}$$

where μ is the kinematic viscosity and $\mathbf{div}$ stands for the usual divergence operator div acting along each row of the tensor. As required by the incompressibility condition, we assume from now on that the datum $\mathbf{g}$ satisfies the compatibility condition

$$\int_{\Gamma} \mathbf{g}\cdot\boldsymbol{\nu} = 0\,, \tag{2.2}$$

where $\boldsymbol{\nu}$ stands for the unit outward normal at Γ.

It follows from the first equation in (2.1), using that $\mathrm{tr}(\nabla\mathbf{u}) = \mathrm{div}(\mathbf{u})$ in Ω, that the incompressibility condition $\mathrm{div}(\mathbf{u}) = 0$ in Ω can be stated in terms of the pseudostress tensor and the pressure as follows

$$p + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}) = 0 \quad \text{in} \quad \Omega\,. \tag{2.3}$$

Conversely, starting from (2.3), and using the first equation in (2.1), we recover the incompressibility condition $\mathrm{div}(\mathbf{u}) = 0$ in Ω. In other words, the pair of equations given by

$$\boldsymbol{\sigma} = 2\,\mu\,\nabla\mathbf{u} - p\,\mathbf{I} \;\text{ in }\; \Omega \quad \text{and} \quad \mathrm{div}(\mathbf{u}) = 0 \;\text{ in }\; \Omega\,, \tag{2.4}$$

is equivalent to

$$\boldsymbol{\sigma} = 2\,\mu\,\nabla\mathbf{u} - p\,\mathbf{I} \;\text{ in }\; \Omega \quad \text{and} \quad p + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}) = 0 \;\text{ in }\; \Omega\,, \tag{2.5}$$

and therefore, instead of (2.1), we now consider:

$$\begin{gathered} \boldsymbol{\sigma} = 2\,\mu\,\nabla\mathbf{u} - p\,\mathbf{I} \quad \text{in} \quad \Omega\,, \qquad \mathbf{div}(\boldsymbol{\sigma}) = -\,\mathbf{f} \quad \text{in} \quad \Omega\,, \\ p + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}) = 0 \;\text{ in }\; \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,. \end{gathered} \tag{2.6}$$

2.2 The dual-mixed formulations

We adopt the usual procedure and test the three field equations of (2.6) with $\boldsymbol{\tau} \in H(\mathbf{div};\Omega)$, $\mathbf{v} \in [L^2(\Omega)]^2$, and $q \in L^2(\Omega)$, respectively. In this way, integrating by parts the expression $\displaystyle\int_\Omega \nabla\mathbf{u} : \boldsymbol{\tau}$ and using the Dirichlet boundary condition, we arrive at the formulation: Find $(\boldsymbol{\sigma}, p, \mathbf{u})$ in $H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$ such that

$$\begin{aligned} \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} + \frac{1}{2\mu}\int_\Omega p\,\mathrm{tr}(\boldsymbol{\tau}) + \frac{1}{2\mu}\int_\Omega q\,\mathrm{tr}(\boldsymbol{\sigma}) + \frac{1}{\mu}\int_\Omega p\,q + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) &= \langle \boldsymbol{\tau\nu}, \mathbf{g}\rangle, \\ \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) &= -\int_\Omega \mathbf{f}\cdot\mathbf{v}, \end{aligned}$$

for all $(\boldsymbol{\tau}, q, \mathbf{v}) \in H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$, where

$$H(\mathbf{div};\Omega) := \left\{ \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2 \right\},$$

and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$, with respect to the $[L^2(\Gamma)]^2$-inner product. Next, noting that

$$\boldsymbol{\sigma} : \boldsymbol{\tau} \,+\, p\,\mathrm{tr}(\boldsymbol{\tau}) \,+\, q\,\mathrm{tr}(\boldsymbol{\sigma}) \,+\, 2\,pq \,=\, \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \,+\, 2\,\left(p + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})\right)\left(q + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\right),$$

the last system can be written in the more compact form: Find $(\boldsymbol{\sigma}, p, \mathbf{u})$ in $H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$ such that

$$\begin{aligned} \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{\mu}\int_\Omega \left(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\right)\left(q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\right) + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) &= \langle \boldsymbol{\tau\nu}, \mathbf{g}\rangle, \\ \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) &= -\int_\Omega \mathbf{f}\cdot\mathbf{v}, \end{aligned} \tag{2.7}$$

for all $(\boldsymbol{\tau}, q, \mathbf{v}) \in H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$.

LEMMA 2.1 *The set of solutions of the homogeneous version of system* (2.7) *is given by*

$$\Big\{ (c\,\mathbf{I}, -c, \mathbf{0}) : \quad c \in \mathbb{R} \Big\}.$$

Proof. Let $(\boldsymbol{\sigma}, p, \mathbf{u})$ in $H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$ such that

$$\begin{aligned} \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{\mu}\int_\Omega \left(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\right)\left(q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\right) + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) &= 0\,, \\ \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) &= 0\,, \end{aligned} \tag{2.8}$$

for all $(\boldsymbol{\tau}, q, \mathbf{v}) \in H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$. It is clear from the second equation of (2.8) that $\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{0}$, and taking $\boldsymbol{\tau} = \mathbf{0}$ in the first one we find that $p = -\dfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$. Next, taking $\boldsymbol{\tau} = \boldsymbol{\sigma}$ in the first equation of (2.8), we deduce that $\boldsymbol{\sigma}^{\mathrm{d}} = \mathbf{0}$, which yields $\boldsymbol{\sigma} = c\,\mathbf{I}$ and hence $p = -\,c$, with $c \in \mathbb{R}$. Finally, thanks to the surjectivity of the operator $\mathbf{div} : H(\mathbf{div};\Omega) \to [L^2(\Omega)]^2$, we conclude from the first equation of (2.8) that $\mathbf{u} = \mathbf{0}$ in Ω. In fact, it suffices to take $\boldsymbol{\tau} = \nabla \mathbf{z}$, where $\mathbf{z} \in [H_0^1(\Omega)]^2$ is the unique solution of the problem: $\Delta \mathbf{z} = \mathbf{u}$ in Ω, $\mathbf{z} = \mathbf{0}$ on Γ. □

In order to avoid the non-uniqueness given by Lemma 2.1 we consider the decomposition

$$H(\mathbf{div};\Omega) = H_0 \oplus \mathbb{R}\,\mathbf{I}, \tag{2.9}$$

where

$$H_0 := \left\{ \boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0 \right\},$$

and require from now on that $\boldsymbol{\sigma} \in H_0$. The following lemma guarantees that the corresponding test space can also be restricted to H_0, which throughout the rest of the paper is endowed with $\|\cdot\|_{\mathrm{div},\Omega}$, the norm of $H(\mathbf{div};\Omega)$.

Lemma 2.2 *Any solution of* (2.7) *with* $\boldsymbol{\sigma} \in H_0$ *is also solution of: Find* $(\boldsymbol{\sigma}, p, \mathbf{u},) \in H_0 \times L^2(\Omega) \times [L^2(\Omega)]^2$ *such that*

$$\begin{aligned} \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{\mu}\int_\Omega \Big(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\Big)\Big(q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\Big) + \int_\Omega \mathbf{u}\cdot \mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{g}\rangle, \\ \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) = -\int_\Omega \mathbf{f}\cdot\mathbf{v}, \end{aligned} \tag{2.10}$$

for all $(\boldsymbol{\tau}, q, \mathbf{v}) \in H_0 \times L^2(\Omega) \times [L^2(\Omega)]^2$. *Conversely, any solution of* (2.10) *is also a solution of* (2.7).

Proof. It is immediate that any solution of (2.7) with $\boldsymbol{\sigma} \in H_0$ is also a solution of (2.10). Conversely, let $(\boldsymbol{\sigma}, p, \mathbf{u})$ be a solution of (2.10). Because of (2.9) it suffices to prove that $(\boldsymbol{\sigma}, p, \mathbf{u})$ also satisfies (2.7) if tested with $(\mathbf{I}, 0, \mathbf{0})$. In fact, according to the compatibility condition (2.2), this requires that $\int_\Omega \big(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\big)$ vanishes which can be seen to be true by selecting $(\boldsymbol{\tau}, q, \mathbf{v}) = (\mathbf{0}, 1, \mathbf{0}) \in H_0 \times L^2(\Omega) \times [L^2(\Omega)]^2$ in (2.10). □

Furthermore, we now let $H := H_0 \times L^2(\Omega)$, $Q := [L^2(\Omega)]^2$, consider a constant $\kappa > 0$, and introduce a generalized version of (2.10): Find $((\boldsymbol{\sigma}, p), \mathbf{u})$ in $H \times Q$ such that

$$\begin{aligned} a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b(\boldsymbol{\tau}, \mathbf{u}) = \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{g}\rangle \qquad & \forall\, (\boldsymbol{\tau}, q) \in H, \\ b(\boldsymbol{\sigma}, \mathbf{v}) = -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad & \forall\, \mathbf{v} \in Q, \end{aligned} \tag{2.11}$$

where $a : H \times H \longrightarrow \mathbb{R}$ and $b : H_0 \times Q \longrightarrow \mathbb{R}$ are the bounded bilinear forms defined by

$$a((\boldsymbol{\zeta}, r), (\boldsymbol{\tau}, q)) := \frac{1}{2\mu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{\kappa}{\mu}\int_\Omega \Big(r + \frac{1}{2}\mathrm{tr}(\boldsymbol{\zeta})\Big)\Big(q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\Big) \tag{2.12}$$

and

$$b(\boldsymbol{\zeta}, \mathbf{v}) := \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\zeta}) \tag{2.13}$$

for $(\boldsymbol{\zeta}, r)$, $(\boldsymbol{\tau}, q)$ in H and $\mathbf{v}$ in Q. Note that (2.10) corresponds to (2.11) with $\kappa = 1$.

In order to show that the formulations (2.11) are independent of $\kappa > 0$, we prove next that they are all equivalent to the simplified version arising after replacing (2.3) into (2.11) (equivalently, taking $\kappa = 0$ in (2.11)), that is: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in H_0 \times Q$ such that

$$\begin{aligned} a_0(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \mathbf{u}) &= \langle \boldsymbol{\tau\nu}, \mathbf{g} \rangle \qquad && \forall\, \boldsymbol{\tau} \in H_0 \,, \\ b(\boldsymbol{\sigma}, \mathbf{v}) &= -\int_\Omega \mathbf{f} \cdot \mathbf{v} && \forall\, \mathbf{v} \in Q \,, \end{aligned} \tag{2.14}$$

where $a_0 : H_0 \times H_0 \longrightarrow \mathbb{R}$ is the bounded bilinear form defined by

$$a_0(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \frac{1}{2\mu} \int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \qquad \forall\, (\boldsymbol{\zeta}, \boldsymbol{\tau}) \in H_0 \times H_0 \,.$$

LEMMA 2.3 *Problems* (2.11) *and* (2.14) *are equivalent. Indeed,* $((\boldsymbol{\sigma}, p), \mathbf{u}) \in H \times Q$ *is a solution of* (2.11) *if and only if* $(\boldsymbol{\sigma}, \mathbf{u}) \in H_0 \times Q$ *is a solution of* (2.14) *and* $p = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$.

Proof. It suffices to take $\boldsymbol{\tau} = \mathbf{0}$ in (2.11) and then use that the traces of the tensor-valued functions in $H(\mathbf{div}; \Omega)$ live in $L^2(\Omega)$ as the pressure test functions do. □

Another way of seeing the equivalence between (2.11) and (2.14) is the following. We observe that eliminating the pressure unknown from (2.6), that is replacing p by $-\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$ in its first equation, we are lead to the reduced problem:

$$\frac{1}{2\mu}\boldsymbol{\sigma}^{\mathrm{d}} = \nabla \mathbf{u} \quad \text{in} \quad \Omega \,, \qquad \mathbf{div}(\boldsymbol{\sigma}) \;=\; -\,\mathbf{f} \quad \text{in} \quad \Omega \,, \quad \mathbf{u} \,=\, \mathbf{g} \quad \text{on} \quad \Gamma \,, \tag{2.15}$$

whose variational formulation is precisely (2.14). Hence, (2.11) can also be considered as the equivalent augmented formulation arising from (2.14) after adding the equation

$$\frac{\kappa}{\mu} \int_\Omega \Big(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}) \Big) \Big(q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau}) \Big) \;=\; 0 \qquad \forall\, (\boldsymbol{\tau}, q) \in H \,.$$

Certainly, if we had to choose, we would stay with (2.14) since it is simpler than (2.11). However, the interest in (2.11) lies in the corresponding Galerkin scheme, which, as we show below in Section 3, provides more flexibility for choosing the pressure finite element subspace.

2.3 Analysis of the dual-mixed formulations

In this section we prove that (2.11) and (2.14) are well-posed. To this end, we first recall the following well known estimate.

LEMMA 2.4 *There exists* $c_1 > 0$*, depending only on* Ω*, such that*

$$c_1 \, \|\boldsymbol{\tau}\|_{0,\Omega}^2 \;\leq\; \big\|\boldsymbol{\tau}^{\mathrm{d}}\big\|_{0,\Omega}^2 \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \quad \forall\, \boldsymbol{\tau} \in H_0 \,. \tag{2.16}$$

Proof. See Lemma 3.1 in [4] or Proposition 3.1 of Chapter IV in [9]. □

Then we have the following main result.

THEOREM 2.1 *Problem* (2.14) *has a unique solution* $(\boldsymbol{\sigma}, \mathbf{u}) \in H_0 \times Q$*. Moreover, there exists a positive constant* C*, depending only on* Ω*, such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u})\|_{H_0 \times Q} \,\leq\, C \,\Big\{ \|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\} \,.$$

Proof. It suffices to prove that the bilinear forms a_0 and b satisfy the hypotheses of the Babuška-Brezzi theory. Indeed, given $\mathbf{v}$ in $Q := [L^2(\Omega)]^2$, we proceed as in the proof of Lemma 2.1 and let $\mathbf{z} \in [H_0^1(\Omega)]^2$ be the unique weak solution of the boundary value problem:

$$\Delta \mathbf{z} \;=\; \mathbf{v} \quad \text{in} \quad \Omega\,, \quad \mathbf{z} = \mathbf{0} \quad \text{on} \quad \Gamma\,. \tag{2.17}$$

Then, we let $\bar{\boldsymbol{\tau}} := \nabla \mathbf{z}$, note that $\bar{\boldsymbol{\tau}} \in H(\mathbf{div};\Omega)$, and decompose $\bar{\boldsymbol{\tau}} = \bar{\boldsymbol{\tau}}_0 + c_0\,\mathbf{I}$, with $\bar{\boldsymbol{\tau}}_0 \in H_0$ and $c_0 \in \mathbb{R}$. It follows that $\mathbf{div}(\bar{\boldsymbol{\tau}}_0) = \mathbf{div}(\bar{\boldsymbol{\tau}}) = \mathbf{v}$, which proves that the bounded linear operator $\mathbf{div} : H_0 \to [L^2(\Omega)]^2$ is surjective, as well. Equivalently, the bilinear form b satisfies the continuous inf-sup condition, which means that there exists $\beta > 0$ such that

$$\sup_{\substack{\boldsymbol{\tau}\in H_0\\ \boldsymbol{\tau}\neq 0}} \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}} \;\geq\; \beta\,\|\mathbf{v}\|_{0,\Omega} \qquad \forall\,\mathbf{v}\in[L^2(\Omega)]^2\,. \tag{2.18}$$

Now, let V be the kernel of the operator induced by b, that is

$$V \;:=\; \{\boldsymbol{\tau}\in H_0 : \quad b(\boldsymbol{\tau},\mathbf{v}) = 0 \quad \forall\,\mathbf{v}\in Q\} \;=\; \{\boldsymbol{\tau}\in H_0 : \quad \mathbf{div}(\boldsymbol{\tau}) = 0\}\,.$$

Then, applying Lemma 2.4, we find that for each $\boldsymbol{\tau} \in V$ there holds

$$a_0(\boldsymbol{\tau},\boldsymbol{\tau}) \;=\; \frac{1}{2\,\mu}\,\left\|\boldsymbol{\tau}^{\mathrm{d}}\right\|_{0,\Omega}^2 \;\geq\; \frac{c_1}{2\,\mu}\,\|\boldsymbol{\tau}\|_{0,\Omega}^2 \;=\; \frac{c_1}{2\,\mu}\,\|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}^2\,, \tag{2.19}$$

which shows that the bilinear form a_0 is strongly coercive in V. Finally, a direct application of Theorem 4.1 in Chapter I of [30] completes the proof. □

The unique solvability of (2.11) is now straightforward.

THEOREM 2.2 *Problem* (2.11) *has a unique solution* $((\boldsymbol{\sigma},p),\mathbf{u}) \in H \times Q$, *independent of* κ, *and there holds* $p = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$. *Moreover, there exists a constant* $C > 0$, *depending only on* Ω, *such that*

$$\|((\boldsymbol{\sigma},p),\mathbf{u})\|_{H\times Q} \;\leq\; C\,\left\{\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma}\right\}.$$

Proof. It is a direct consequence of Lemma 2.3, which gives the equivalence between (2.11) and (2.14), and Theorem 2.1, which yields the well-posedness of (2.14). □

3 The mixed finite element methods

3.1 The Galerkin schemes

We now let $H_{0,h}^{\boldsymbol{\sigma}}$, H_h^p and Q_h be arbitrary finite element subspaces of H_0, $L^2(\Omega)$ and Q, respectively, and define $H_h := H_{0,h}^{\boldsymbol{\sigma}} \times H_h^p$. Then, the Galerkin schemes associated with (2.11) and (2.14) read: Find $((\boldsymbol{\sigma}_h,p_h),\mathbf{u}_h) \in H_h \times Q_h$ such that

$$\begin{aligned} a((\boldsymbol{\sigma}_h,p_h),(\boldsymbol{\tau},q)) + b(\boldsymbol{\tau},\mathbf{u}_h) \;&=\; \langle\boldsymbol{\tau\nu},\mathbf{g}\rangle \qquad \forall\,(\boldsymbol{\tau},q)\,\in\,H_h\,,\\ b(\boldsymbol{\sigma}_h,\mathbf{v}) \;&=\; -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v}\,\in\,Q_h\,, \end{aligned} \tag{3.1}$$

and: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_{0,h}^{\boldsymbol{\sigma}} \times Q_h$ such that

$$\begin{aligned} a_0(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \mathbf{u}_h) &= \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{g} \rangle \qquad \forall\, \boldsymbol{\tau} \in H_{0,h}^{\boldsymbol{\sigma}}\,, \\ b(\boldsymbol{\sigma}_h, \mathbf{v}) &= -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\, \mathbf{v} \in Q_h\,. \end{aligned} \tag{3.2}$$

The discrete analogue of Lemma 2.3, which gives a sufficient condition for the equivalence between (3.1) and (3.2), is as follows.

Lemma 3.1 *Assume that the pressure finite element subspace H_h^p contains the traces of the members of the pseudostress tensor finite element subspace $H_{0,h}^{\boldsymbol{\sigma}}$, that is,*

$$\operatorname{tr}(H_{0,h}^{\boldsymbol{\sigma}}) \subseteq H_h^p\,. \tag{3.3}$$

Then, problems (3.1) *and* (3.2) *are equivalent, that is $((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h) \in H_h \times Q_h$ is a solution of* (3.1) *if and only if $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_{0,h}^{\boldsymbol{\sigma}} \times Q_h$ is a solution of* (3.2) *and $p_h = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)$.*

Proof. Let $((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h) \in H_h \times Q_h$ be a solution of (3.1). It is clear from the assumption (3.3) that $p_h + \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)$ belongs to H_h^p. Then, taking $((\boldsymbol{\tau}, q), \mathbf{v}) = ((\mathbf{0}, p_h + \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)), \mathbf{0}) \in H_h \times Q_h$ in (3.1), we find that

$$\frac{\kappa}{\mu}\int_\Omega \Big(p_h + \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)\Big)^2 = 0\,,$$

which yields $p_h = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)$. Conversely, given $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_{0,h}^{\boldsymbol{\sigma}} \times Q_h$ a solution of (3.2), we let $p_h := -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)$ and see that $((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h) \in H_h \times Q_h$ becomes a solution of (3.1). □

It is clear from Lemma 3.1 that the discrete formulation (3.1) makes sense only if the condition (3.3) does not hold. Otherwise, it suffices to look for the solution of (3.2).

3.2 The finite element subspaces

In order to introduce explicit finite element subspaces guaranteeing the unique solvability and stability of (3.1) and (3.2), we now let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the polygonal region $\bar{\Omega}$ by triangles T of diameter h_T such that $\bar{\Omega} = \cup\{T : T \in \mathcal{T}_h\}$ and define $h := \max\{h_T : T \in \mathcal{T}_h\}$. Given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathbb{P}_\ell(S)$ the space of polynomials of total degree at most ℓ defined on S. Then, for each integer $k \geq 0$ and for each $T \in \mathcal{T}_h$, we define the local Raviart-Thomas space of order k (see, e.g. [33], [9])

$$\mathbb{RT}_k(T) = [\mathbb{P}_k(T)]^2 \oplus \mathbb{P}_k(T)\,\mathbf{x}\,, \tag{3.4}$$

where $\mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a generic vector of $\mathbb{R}^2$, and let $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global space, that is

$$\mathbb{RT}_k(\mathcal{T}_h) := \big\{ \boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \quad (\tau_{i1}, \tau_{i2})^{\mathrm{t}} \in \mathbb{RT}_k(T) \quad \forall\, i \in \{1,2\}, \quad \forall\, T \in \mathcal{T}_h \big\}\,. \tag{3.5}$$

We also let $\mathbb{P}_k(\mathcal{T}_h)$ be the global space of piecewise polynomials of degree $\leq k$, that is

$$\mathbb{P}_k(\mathcal{T}_h) := \big\{ v \in L^2(\Omega) : \quad v|_T \in \mathbb{P}_k(T) \quad \forall\, T \in \mathcal{T}_h \big\}\,. \tag{3.6}$$

Next, we let $\mathcal{E}_h^k : [H^1(\Omega)]^{2\times 2} \longrightarrow \mathbb{RT}_k(\mathcal{T}_h)$ be the usual equilibrium interpolation operator (see, e.g. [33], [9]), which, given $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2}$, is characterized by the following identities:

$$\int_e \mathcal{E}_h^k(\boldsymbol{\tau})\boldsymbol{\nu}\cdot\mathbf{r} \,=\, \int_e \boldsymbol{\tau}\boldsymbol{\nu}\cdot\mathbf{r} \qquad \forall \text{ edge } e \in \mathcal{T}_h\,, \quad \forall\, \mathbf{r} \in [\mathbb{P}_k(e)]^2\,, \quad \text{when} \quad k \geq 0\,, \tag{3.7}$$

and

$$\int_T \mathcal{E}_h^k(\boldsymbol{\tau}) : \mathbf{r} \,=\, \int_T \boldsymbol{\tau} : \mathbf{r} \qquad \forall\, T \in \mathcal{T}_h\,, \quad \forall\, \mathbf{r} \in [\mathbb{P}_{k-1}(T)]^{2\times 2}\,, \quad \text{when} \quad k \geq 1\,. \tag{3.8}$$

It is easy to show, using (3.7) and (3.8), that

$$\mathbf{div}(\mathcal{E}_h^k(\boldsymbol{\tau})) \,=\, \mathcal{P}_h^k(\mathbf{div}(\boldsymbol{\tau}))\,, \tag{3.9}$$

where $\mathcal{P}_h^k$ is the orthogonal projector from $[L^2(\Omega)]^2$ into $[\mathbb{P}_k(\mathcal{T}_h)]^2$. It is well known (see, e.g. [18]) that for each $\mathbf{v} \in [H^m(\Omega)]^2$, with $0 \leq m \leq k+1$, there holds

$$\|\mathbf{v} - \mathcal{P}_h^k(\mathbf{v})\|_{0,T} \,\leq\, C\, h_T^m\, |\mathbf{v}|_{m,T} \qquad \forall T \in \mathcal{T}_h\,. \tag{3.10}$$

In addition, the operator $\mathcal{E}_h^k$ satisfies the following approximation properties (see, e.g. [9], [33]):

$$\|\boldsymbol{\tau} - \mathcal{E}_h^k(\boldsymbol{\tau})\|_{0,T} \,\leq\, C\, h_T^m\, |\boldsymbol{\tau}|_{m,T} \qquad \forall T \in \mathcal{T}_h\,, \tag{3.11}$$

for each $\boldsymbol{\tau} \in [H^m(\Omega)]^{2\times 2}$, with $1 \leq m \leq k+1$,

$$\|\mathbf{div}(\boldsymbol{\tau} - \mathcal{E}_h^k(\boldsymbol{\tau}))\|_{0,T} \,\leq\, C\, h_T^m\, |\mathbf{div}(\boldsymbol{\tau})|_{m,T} \qquad \forall T \in \mathcal{T}_h\,, \tag{3.12}$$

for each $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2}$ such that $\mathbf{div}(\boldsymbol{\tau}) \in [H^m(\Omega)]^2$, with $0 \leq m \leq k+1$, and

$$\|\boldsymbol{\tau}\,\boldsymbol{\nu} - \mathcal{E}_h^k(\boldsymbol{\tau})\,\boldsymbol{\nu}\|_{0,e} \,\leq\, C\, h_e^{1/2}\, \|\boldsymbol{\tau}\|_{1,T_e} \qquad \forall \text{ edge } e \in \mathcal{T}_h\,, \tag{3.13}$$

for each $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2}$, where $T_e \in \mathcal{T}_h$ contains e on its boundary. In particular, note that (3.12) follows easily from (3.9) and (3.10). Moreover, it turns out (see, e.g. Theorem 3.16 in [31]) that $\mathcal{E}_h^k$ can also be defined as a bounded linear operator from the larger space $[H^s(\Omega)]^{2\times 2} \cap H(\mathbf{div};\Omega)$ into $\mathbb{RT}_k(\mathcal{T}_h)$ for all $s \in (0,1]$, and that in this case there holds the following interpolation error estimate

$$\|\boldsymbol{\tau} - \mathcal{E}_h^k(\boldsymbol{\tau})\|_{0,T} \,\leq\, C\, h_T^s \left\{ \|\boldsymbol{\tau}\|_{s,T} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,T} \right\} \qquad \forall T \in \mathcal{T}_h\,. \tag{3.14}$$

We now introduce the following finite element subspaces of H_0, $L^2(\Omega)$, and Q, respectively,

$$\begin{aligned} H_{0,h}^{\boldsymbol{\sigma}} &:= \left\{ \boldsymbol{\tau} \in \mathbb{RT}_k(\mathcal{T}_h) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \,=\, 0 \right\}, \\ H_h^p &:= \mathbb{P}_k(\mathcal{T}_h)\,, \\ Q_h &:= [\mathbb{P}_k(\mathcal{T}_h)]^2\,. \end{aligned} \tag{3.15}$$

Then, as a consequence of (3.10), (3.11), (3.12), (3.13), (3.14), and the usual interpolation estimates, we find that $H_{0,h}^{\boldsymbol{\sigma}}$, H_h^p, and Q_h satisfy the following approximation properties:

($\mathrm{AP}^{\boldsymbol{\sigma}}_{0,h}$) *For each* $s \in (0, k+1]$ *and for each* $\boldsymbol{\tau} \in [H^s(\Omega)]^{2\times 2} \cap H_0$ *with* $\mathbf{div}(\boldsymbol{\tau}) \in [H^s(\Omega)]^2$ *there exists* $\boldsymbol{\tau}_h \in H^{\boldsymbol{\sigma}}_{0,h}$ *such that*

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathrm{div},\Omega} \,\le\, C\,h^s \,\Big\{ \|\boldsymbol{\tau}\|_{s,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{s,\Omega} \Big\}\,.$$

(AP^{p}_{h}) *For each* $s \in [0, k+1]$ *and for each* $q \in H^s(\Omega)$ *there exists* $q_h \in H^p_h$ *such that*

$$\|q - q_h\|_{0,\Omega} \,\le\, C\,h^s\, \|q\|_{s,\Omega}\,.$$

($\mathrm{AP}^{\mathbf{u}}_{h}$) *For each* $s \in [0, k+1]$ *and for each* $\mathbf{v} \in [H^s(\Omega)]^2$ *there exists* $\mathbf{v}_h \in Q_h$ *such that*

$$\|\mathbf{v} - \mathbf{v}_h\|_{0,\Omega} \,\le\, C\,h^s\, \|\mathbf{v}\|_{s,\Omega}\,.$$

3.3 Analysis of the Galerkin schemes

In what follows we establish the unique solvability, stability, and convergence of the Galerkin schemes (3.1) and (3.2) with the finite element subspaces given by (3.15). Note that in this case the condition (3.3) does not hold, and hence the Galerkin scheme (3.1) becomes meaningful. We begin the analysis with the discrete inf-sup condition for the bilinear form b.

Lemma 3.2 *Let* $H^{\boldsymbol{\sigma}}_{0,h}$ *and* Q_h *be given by* (3.15). *Then, there exists* $\beta > 0$, *independent of* h, *such that*

$$\sup_{\substack{\boldsymbol{\tau}\in H^{\boldsymbol{\sigma}}_{0,h}\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{b(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}} \,\ge\, \beta\,\|\mathbf{v}\|_{0,\Omega} \qquad \forall\,\mathbf{v}\,\in\, Q_h\,. \tag{3.16}$$

Proof. Since b satisfies the continuous inf-sup condition (cf. (2.18) in the proof of Theorem 2.1), we just need to construct a Fortin operator. To this end, we first let G be a bounded convex polygonal domain containing $\bar{\Omega}$. Then, given $\boldsymbol{\tau} \in H_0$, we let $\mathbf{z} \in [H^1_0(G)]^2$ be the unique weak solution of the boundary value problem:

$$\Delta z \,=\, \begin{cases} \operatorname{div}\boldsymbol{\tau} & \text{in} \quad \Omega \\ 0 & \text{in} \quad G\backslash\Omega \end{cases}\,, \qquad \mathbf{z} \,=\, 0 \quad \text{on} \quad \partial G\,. \tag{3.17}$$

Thanks to the elliptic regularity result of (3.17) we have that $\mathbf{z} \in [H^2(\Omega_0)]^2$ and

$$\|\mathbf{z}\|_{2,\Omega} \,\le\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}\,. \tag{3.18}$$

Also, it is clear that $\big(\nabla\mathbf{z}\big)|_\Omega \,\in\, [H^1(\Omega)]^{2\times 2}$, $\mathbf{div}(\nabla\mathbf{z}) \,=\, \Delta\mathbf{z} \,=\, \mathbf{div}(\boldsymbol{\tau})$ in Ω, and

$$\|\nabla\mathbf{z}\|_{1,\Omega} \,\le\, \|\mathbf{z}\|_{2,\Omega} \,\le\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}\,. \tag{3.19}$$

Next, we introduce the linear operator $\Pi^k_h : H_0 \longrightarrow H^{\boldsymbol{\sigma}}_{0,h}$, where $\Pi^k_h(\boldsymbol{\tau})$ is the H_0-component of $\mathcal{E}^k_h(\nabla\mathbf{z})$ determined by the decomposition (2.9), that is

$$\Pi^k_h(\boldsymbol{\tau}) \,:=\, \mathcal{E}^k_h(\nabla\mathbf{z}) \,-\, \left\{\frac{1}{2\,|\Omega|}\int_\Omega \operatorname{tr}(\mathcal{E}^k_h(\nabla\mathbf{z}))\right\}\mathbf{I}\,.$$

It follows, using (3.9), that

$$\mathbf{div}(\Pi^k_h(\boldsymbol{\tau})) \,=\, \mathbf{div}(\mathcal{E}^k_h(\nabla\mathbf{z})) \,=\, \mathcal{P}^k_h(\mathbf{div}(\nabla\mathbf{z})) \,=\, \mathcal{P}^k_h(\mathbf{div}(\boldsymbol{\tau})) \quad \text{in} \quad \Omega\,,$$

and hence for each $\boldsymbol{\tau} \in H_0$ and $\mathbf{v} \in Q_h$ there holds

$$b(\Pi_h^k(\boldsymbol{\tau}), \mathbf{v}) \,=\, \int_\Omega \mathbf{v} \cdot \mathbf{div}(\Pi_h^k(\boldsymbol{\tau})) \,=\, \int_\Omega \mathbf{v} \cdot \mathcal{P}_h^k(\mathbf{div}(\boldsymbol{\tau})) \,=\, \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \,=\, b(\boldsymbol{\tau}, \mathbf{v})\,. \qquad (3.20)$$

In addition, using the stability of the decomposition (2.9), and applying (3.11) (with $m = 1$) and (3.19), we find that for each $\boldsymbol{\tau} \in H_0$ there holds

$$\begin{aligned} \|\Pi_h^k(\boldsymbol{\tau})\|_{\mathrm{div},\Omega}^2 \,&\leq\, \|\mathcal{E}_h^k(\nabla \mathbf{z})\|_{\mathrm{div},\Omega}^2 \,=\, \|\mathcal{E}_h^k(\nabla \mathbf{z})\|_{0,\Omega}^2 \,+\, \|\mathcal{P}_h^k(\mathbf{div}(\boldsymbol{\tau}))\|_{0,\Omega}^2 \\ &\leq\, C \left\{ \|\nabla \mathbf{z} - \mathcal{E}_h^k(\nabla \mathbf{z})\|_{0,\Omega}^2 \,+\, \|\nabla \mathbf{z}\|_{0,\Omega}^2 \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \right\} \,\leq\, C\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2\,, \end{aligned}$$

which shows that Π_h^k is uniformly bounded. The above estimate and (3.20) prove that Π_h^k becomes a Fortin operator, which finishes the proof. $\square$

We are now in a position to establish the following theorems.

THEOREM 3.1 *The Galerkin scheme* (3.2) *has a unique solution* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_{0,h}^{\boldsymbol{\sigma}} \times Q_h$*, and there exist positive constants* C*,* $\tilde{C}$*, independent of* h*, such that*

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{H_0 \times Q} \,\leq\, C \left\{ \, \|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \right\},$$

and

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{H_0 \times Q} \,\leq\, \tilde{C} \inf_{(\boldsymbol{\tau}_h, \mathbf{v}_h) \,\in\, H_{0,h}^{\boldsymbol{\sigma}} \times Q_h} \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{H_0 \times Q}\,. \qquad (3.21)$$

Proof. Since $\mathrm{div}(H_{0,h}^{\boldsymbol{\sigma}}) \subseteq Q_h$, we find that the discrete kernel of b is given by

$$V_h := \left\{ \boldsymbol{\tau} \in H_{0,h}^{\boldsymbol{\sigma}} : \quad b(\boldsymbol{\tau}, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in Q_h \right\} = \left\{ \boldsymbol{\tau} \in H_{0,h}^{\boldsymbol{\sigma}} : \quad \mathbf{div}(\boldsymbol{\tau}) = 0 \quad \text{in} \quad \Omega \right\} \subseteq V\,,$$

which, thanks to (2.19), shows that a_0 is strongly coercive in V_h. This fact, Lemma 3.2, and a direct application of the classical Babuška-Brezzi theory (see, e.g. Theorem 1.1 in Chapter II of [30]) complete the proof. $\square$

THEOREM 3.2 *Let* $(\boldsymbol{\sigma}, \mathbf{u}) \in H_0 \times Q$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_{0,h}^{\boldsymbol{\sigma}} \times Q_h$ *be the unique solutions of the continuous and discrete formulations* (2.14) *and* (3.2)*, respectively. Assume that* $\boldsymbol{\sigma} \in [H^s(\Omega)]^{2\times 2}$*,* $\mathbf{div}(\boldsymbol{\sigma}) \in [H^s(\Omega)]^2$*, and* $\mathbf{u} \in [H^s(\Omega)]^2$*, for some* $s \in (0, k+1]$*. Then there exists* $C > 0$*, independent of* h*, such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u}) \,-\, (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{H_0 \times Q} \,\leq\, C\, h^s \, \left\{ \|\boldsymbol{\sigma}\|_{s,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{s,\Omega} \,+\, \|\mathbf{u}\|_{s,\Omega} \right\}.$$

Proof. It follows from the Cea estimate (3.21) and the approximation properties $(\mathrm{AP}_{0,h}^{\boldsymbol{\sigma}})$ and $(\mathrm{AP}_h^{\mathbf{u}})$. $\square$

We end this section with the analogues of Theorems 3.1 and 3.2 for the scheme (3.1). In this case, the well-posedness of (3.1) depends on a suitable choice of the parameter κ that defines the bilinear form a (cf. (2.12)).

THEOREM 3.3 *Let $c_1 > 0$ be the constant provided by Lemma 2.4 and assume that the parameter κ lies in $(0, c_1)$. Then the Galerkin scheme (3.1) has a unique solution $((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h) \in H_h \times Q_h$, and there exist positive constants C, $\tilde{C}$, independent of h, such that*

$$\|((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h)\|_{H\times Q} \le C \left\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \right\},$$

and

$$\|((\boldsymbol{\sigma}, p), \mathbf{u}) - ((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h)\|_{H\times Q} \le \tilde{C} \inf_{((\boldsymbol{\tau}_h, q_h), \mathbf{v}_h) \in H_h \times Q_h} \|((\boldsymbol{\sigma}, p), \mathbf{u}) - ((\boldsymbol{\tau}_h, q_h), \mathbf{v}_h)\|_{H\times Q} . \tag{3.22}$$

Proof. Using again that $\text{div}(H^{\boldsymbol{\sigma}}_{0,h}) \subseteq Q_h$, we find that the discrete kernel of b is given in this case by

$$W_h := \left\{ (\boldsymbol{\tau}, q) \in H_h := H^{\boldsymbol{\sigma}}_{0,h} \times H^p_h : \quad b(\boldsymbol{\tau}, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in Q_h \right\} = V_h \times H^p_h .$$

Note that b has to be considered here as a bilinear form acting from $H_h \times Q_h$ into $\mathbb{R}$ (instead of $H^{\boldsymbol{\sigma}}_{0,h} \times Q_h$ into $\mathbb{R}$ as in the formulation (3.2)). Then, according to the definition of a (cf. (2.12)) and Lemma 2.4, we deduce that for each $(\boldsymbol{\tau}, q) \in W_h$ there holds

$$\begin{aligned} a((\boldsymbol{\tau}, q), (\boldsymbol{\tau}, q)) &= \frac{1}{2\,\mu} \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega} + \frac{\kappa}{\mu} \|q + \frac{1}{2}\text{tr}(\boldsymbol{\tau})\|^2_{0,\Omega} \\ &\ge \frac{1}{2\,\mu} \|\boldsymbol{\tau}^{\mathtt{d}}\|^2_{0,\Omega} + \frac{\kappa}{2\,\mu} \|q\|^2_{0,\Omega} - \frac{\kappa}{2\,\mu} \|\boldsymbol{\tau}\|^2_{0,\Omega} \\ &\ge \frac{1}{2\,\mu} (c_1 - \kappa) \|\boldsymbol{\tau}\|^2_{0,\Omega} + \frac{\kappa}{2\,\mu} \|q\|^2_{0,\Omega} , \end{aligned}$$

which shows that a is strongly coercive in W_h whenever $\kappa \in (0, c_1)$. Hence, as in Theorem 3.1, the proof is completed having in mind Lemma 3.2 (with the above mentioned modification on the definition of the bilinear form b) and applying Theorem 1.1 in Chapter II of [30]. □

THEOREM 3.4 *Let $((\boldsymbol{\sigma}, p), \mathbf{u}) \in H \times Q$ and $((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h) \in H_h \times Q_h$ be the unique solutions of the continuous and discrete formulations (2.11) and (3.1), respectively. Assume that $\boldsymbol{\sigma} \in [H^s(\Omega)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^s(\Omega)]^2$, and $\mathbf{u} \in [H^s(\Omega)]^2$, for some $s \in (0, k+1]$. Then there exists $C > 0$, independent of h, such that*

$$\|((\boldsymbol{\sigma}, p), \mathbf{u}) - ((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h)\|_{H\times Q} \le C\, h^s \left\{ \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{s,\Omega} + \|\mathbf{u}\|_{s,\Omega} \right\} .$$

Proof. It follows from the Cea estimate (3.22) and the approximation properties $(\text{AP}^{\boldsymbol{\sigma}}_{0,h})$, (AP^p_h), and $(\text{AP}^{\mathbf{u}}_h)$, noting that $p = -\frac{1}{2}\,\text{tr}(\boldsymbol{\sigma}) \in H^s(\Omega)$. □

4 A posteriori error analysis

In this section we derive reliable and efficient residual based a posteriori error estimators for (3.1) and (3.2). Actually, the analysis is performed first for (3.2) and then extended to (3.1).

We begin by introducing several notations. We let $\mathcal{E}_h$ be the set of all edges of the triangulation $\mathcal{T}_h$, and given $T \in \mathcal{T}_h$, we let $\mathcal{E}(T)$ be the set of its edges. Then we write $\mathcal{E}_h = \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$, where $\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h : e \subseteq \Omega\}$ and $\mathcal{E}_h(\Gamma) := \{e \in \mathcal{E}_h : e \subseteq \Gamma\}$. In what follows, h_e stands for the length of the edge e. Also, for each edge $e \in \mathcal{E}_h$ we fix a unit normal vector $\boldsymbol{\nu}_e := (\nu_1, \nu_2)^{\mathtt{t}}$, and let $\mathbf{s}_e := (-\nu_2, \nu_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Then, given $e \in \mathcal{E}_h(\Omega)$ and $\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$ such that $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2}$ on each $T \in \mathcal{T}_h$, we let $[\boldsymbol{\tau}\, \mathbf{s}_e]$ be the corresponding jump across e, that is $[\boldsymbol{\tau}\, \mathbf{s}_e] := (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e\, \mathbf{s}_e$, where T and T' are the triangles of $\mathcal{T}_h$ having e as a common edge. Abusing notation, when $e \in \mathcal{E}_h(\Gamma)$, we also write $[\boldsymbol{\tau}\, \mathbf{s}_e] := \boldsymbol{\tau}|_e\, \mathbf{s}_e$. Similar definitions hold for the tangential jumps of scalar fields $v \in L^2(\Omega)$ such that $v|_T \in C(T)$ on each $T \in \mathcal{T}_h$. From now on, when no confusion arises, we simple write $\mathbf{s}$ and $\boldsymbol{\nu}$ instead of $\mathbf{s}_e$ and $\boldsymbol{\nu}_e$, respectively. Finally, given scalar, vector and tensor valued fields v, $\boldsymbol{\varphi} := (\varphi_1, \varphi_2)$ and $\boldsymbol{\tau} := (\tau_{ij})$, respectively, we let

$$\mathbf{curl}(v) := \begin{pmatrix} \frac{\partial v}{\partial x_2} \\ -\frac{\partial v}{\partial x_1} \end{pmatrix}, \quad \underline{\mathbf{curl}}(\boldsymbol{\varphi}) := \begin{pmatrix} \mathbf{curl}(\varphi_1)^{\mathtt{t}} \\ \mathbf{curl}(\varphi_2)^{\mathtt{t}} \end{pmatrix}, \quad \text{and } \operatorname{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \\ \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

Then, letting $(\boldsymbol{\sigma}, \mathbf{u}) \in H_0 \times Q$ and $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H^{\boldsymbol{\sigma}}_{0,h} \times Q_h$ be the unique solutions of the continuous and discrete formulations (2.14) and (3.2), respectively, we define for each $T \in \mathcal{T}_h$ a local error indicator θ_T as follows:

$$\begin{aligned} \theta_T^2 \,:=\, & \|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,T}^2 \,+\, h_T^2 \left\| \operatorname{curl}\Big\{ \frac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}} \Big\} \right\|_{0,T}^2 \,+\, h_T^2 \left\| \nabla \mathbf{u}_h - \frac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\|_{0,T}^2 \\ & + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} h_e \left\| \Big[\frac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}}\, \mathbf{s} \Big] \right\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \left\{ \left\| \frac{d\mathbf{g}}{d\mathbf{s}} - \frac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}}\, \mathbf{s} \right\|_{0,e}^2 + \|\mathbf{g} - \mathbf{u}_h\|_{0,e}^2 \right\}. \end{aligned} \tag{4.1}$$

Note that the above requires that $\left.\frac{d\mathbf{g}}{d\mathbf{s}}\right|_e \in [L^2(e)]^2$ for each $e \in \mathcal{E}_h(\Gamma)$. This is fixed below by assuming that $\mathbf{g} \in [H^1(\Gamma)]^2$. Similarly, letting $((\boldsymbol{\sigma}, p), \mathbf{u}) \in H \times Q$ and $((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h) \in H_h \times Q_h$ be the unique solutions of the continuous and discrete formulations (2.11) and (3.1), respectively, we define for each $T \in \mathcal{T}_h$ a local error indicator η_T as follows:

$$\begin{aligned} \eta_T^2 \,:=\, \theta_T^2 \,+\, & \left\| p_h + \frac{1}{2} \operatorname{tr}(\boldsymbol{\sigma}_h) \right\|_{0,T}^2 \,+\, h_T^2 \left\| \mathbf{curl}\Big(p_h + \frac{1}{2} \operatorname{tr}(\boldsymbol{\sigma}_h) \Big) \right\|_{0,T}^2 \\ & + \sum_{e \in \mathcal{E}(T)} h_e \left\| \Big[\Big(p_h + \frac{1}{2} \operatorname{tr}(\boldsymbol{\sigma}_h) \Big)\, \mathbf{s} \Big] \right\|_{0,e}^2. \end{aligned} \tag{4.2}$$

The residual character of each term on the right hand side of (4.1) and (4.2) is quite clear. As usual the expressions

$$\boldsymbol{\theta} \,:=\, \Big\{ \sum_{T \in \mathcal{T}_h} \theta_T^2 \Big\}^{1/2} \quad \text{and} \quad \boldsymbol{\eta} \,:=\, \Big\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \Big\}^{1/2}$$

are employed as the global residual error estimators.

The following theorems constitute the main results of this section.

THEOREM 4.1 *Let* $(\boldsymbol{\sigma},\mathbf{u}) \in H_0 \times Q$ *and* $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in H^{\boldsymbol{\sigma}}_{0,h} \times Q_h$ *be the unique solutions of* (2.14) *and* (3.2)*, respectively, and assume that* $\mathbf{g} \in [H^1(\Gamma)]^2$*. Then there exist positive constants* $C_{\tt eff}$ *and* $C_{\tt rel}$*, independent of* h*, such that*

$$C_{\tt eff}\,\boldsymbol{\theta} \,+\, h.o.t. \;\le\; \|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{H_0\times Q} \;\le\; C_{\tt rel}\,\boldsymbol{\theta}\,, \tag{4.3}$$

where h.o.t. stands for one or several terms of higher order.

THEOREM 4.2 *Let* $((\boldsymbol{\sigma},p),\mathbf{u}) \in H \times Q$ *and* $((\boldsymbol{\sigma}_h,p_h),\mathbf{u}_h) \in H_h \times Q_h$ *be the unique solutions of* (2.11) *and* (3.1)*, respectively, and assume that* $\mathbf{g} \in [H^1(\Gamma)]^2$*. Then there exist positive constants* $\tilde{C}_{\tt eff}$ *and* $\tilde{C}_{\tt rel}$*, independent of* h*, such that*

$$\tilde{C}_{\tt eff}\,\boldsymbol{\eta} \,+\, h.o.t. \;\le\; \|((\boldsymbol{\sigma},p),\mathbf{u}) - ((\boldsymbol{\sigma}_h,p_h),\mathbf{u}_h)\|_{H\times Q} \;\le\; \tilde{C}_{\tt rel}\,\boldsymbol{\eta}\,, \tag{4.4}$$

where h.o.t. stands for one or several terms of higher order.

The efficiency of the global error estimators (lower bounds in (4.3) and (4.4)) are proved below in Subsection 4.2, whereas the corresponding reliability (upper bounds in (4.3) and (4.4)) is derived now.

4.1 Reliability

We begin with the following preliminary estimate.

LEMMA 4.1 *Let* $(\boldsymbol{\sigma},\mathbf{u}) \in H_0 \times Q$ *and* $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in H^{\boldsymbol{\sigma}}_{0,h} \times Q_h$ *be the unique solutions of* (2.14) *and* (3.2)*, respectively. Then there exists* $C > 0$*, independent of* h*, such that*

$$C\,\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{H_0\times Q} \le \sup_{\boldsymbol{\tau}\in H_0\backslash\{\theta\}} \frac{R(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}} \,+\, \|f \,+\, \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,\Omega}\,, \tag{4.5}$$

where

$$R(\boldsymbol{\tau}) \,:=\, \langle(\boldsymbol{\tau}-\boldsymbol{\tau}_h)\boldsymbol{\nu},g\rangle \,-\, \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}_h^d : (\boldsymbol{\tau}-\boldsymbol{\tau}_h) \,-\, \int_\Omega \mathbf{u}_h\cdot\mathbf{div}(\boldsymbol{\tau}-\boldsymbol{\tau}_h) \quad \forall\,\boldsymbol{\tau}_h \,\in\, H^{\boldsymbol{\sigma}}_{0,h}\,. \tag{4.6}$$

Proof. We first recall that the continuous dependence result for the problem (2.14) (cf. Theorem 2.1) is equivalent to the global continuous inf-sup condition, which establishes the existence of $C > 0$ such that

$$C\,\|(\boldsymbol{\zeta},\mathbf{w})\|_{H_0\times Q} \le \sup_{(\boldsymbol{\tau},\mathbf{v})\in H_0\times Q\backslash\{\theta\}} \frac{a_0(\boldsymbol{\zeta},\boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau},\mathbf{w}) \,+\, b(\boldsymbol{\zeta},\mathbf{v})}{\|(\boldsymbol{\tau},\mathbf{v})\|_{H_0\times Q}} \quad \forall\,(\boldsymbol{\zeta},\mathbf{w}) \,\in\, H_0\times Q\,. \tag{4.7}$$

Then, applying (4.7) to the Galerkin error $(\boldsymbol{\zeta},\mathbf{w}) \,:=\, (\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)$, and using the second equation of (2.14), the fact that $\|(\boldsymbol{\tau},\mathbf{v})\|_{H_0\times Q} \,\ge\, \max\{\|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}, \|\mathbf{v}\|_{0,\Omega}\}$, and the Cauchy-Schwarz inequality, we find that

$$C\,\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{H_0\times Q} \,\le\, \sup_{(\boldsymbol{\tau},\mathbf{v})\in H_0\times Q\backslash\{\theta\}} \frac{a_0(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau},\mathbf{u}-\mathbf{u}_h) \,+\, b(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\mathbf{v})}{\|(\boldsymbol{\tau},\mathbf{v})\|_{H_0\times Q}}$$

$$= \sup_{(\boldsymbol{\tau},\mathbf{v})\in H_0\times Q\backslash\{\theta\}} \frac{R(\boldsymbol{\tau}) \,-\, \displaystyle\int_\Omega (\mathbf{f}+\mathbf{div}(\boldsymbol{\sigma}_h))\,\mathbf{v}}{\|(\boldsymbol{\tau},\mathbf{v})\|_{H_0\times Q}} \,\le\, \sup_{\boldsymbol{\tau}\in H_0\backslash\{\theta\}} \frac{R(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}} \,+\, \|f + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,\Omega}\,,$$

where $R(\boldsymbol{\tau}) = a_0(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \mathbf{u} - \mathbf{u}_h)$. But, from the first equations of (2.14) and (3.2) we have that $a_0(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{u} - \mathbf{u}_h) = 0 \quad \forall \boldsymbol{\tau}_h \in H_{0,h}^{\boldsymbol{\sigma}}$, and hence

$$R(\boldsymbol{\tau}) = a_0(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau} - \boldsymbol{\tau}_h) + b(\boldsymbol{\tau} - \boldsymbol{\tau}_h, \mathbf{u} - \mathbf{u}_h) \quad \forall \boldsymbol{\tau}_h \in H_{0,h}^{\boldsymbol{\sigma}},$$

which, using again the first equation of (2.14) and the definitions of a_0 and b, becomes (4.6), thus completing the proof. □

We now aim to bound the supremum on the right hand side of (4.5). To this end, and in order to choose below a suitable $\boldsymbol{\tau}_h \in H_{0,h}^{\boldsymbol{\sigma}}$ for the definition of $R(\boldsymbol{\tau})$ (cf. (4.6)), we now let $I_h : H^1(\Omega) \longrightarrow X_h$ be the Clément interpolation operator (cf. [19]), where

$$X_h := \left\{ v_h \in C(\bar{\Omega}) : \quad v_h|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h \right\} . \tag{4.8}$$

The following lemma establishes the local approximation properties of I_h.

Lemma 4.2 *There exist constants $C_1, C_2 > 0$, independent of h, such that for all $v \in H^1(\Omega)$ there hold*

$$\|v - I_h(v)\|_{0,T} \le C_1\, h_T\, \|v\|_{1,\Delta(T)} \quad \forall T \in \mathcal{T}_h, \tag{4.9}$$

and

$$\|v - I_h(v)\|_{0,e} \le C_2\, h_e^{1/2}\, \|v\|_{1,\Delta(e)} \quad \forall e \in \mathcal{E}_h, \tag{4.10}$$

where $\Delta(T)$ and $\Delta(e)$ are the union of all elements intersecting with T and e, respectively.

Proof. See [19]. □

Next, given $\boldsymbol{\tau} \in H_0$, we proceed as in the proof of Lemma 3.2, and let $\mathbf{z} \in [H_0^1(G)]^2$ be the unique weak solution of the boundary value problem (3.17), where G is a bounded convex polygonal domain containing $\bar{\Omega}$. Since $\mathbf{div}(\boldsymbol{\tau} - \nabla\mathbf{z}) = 0$ in Ω, and Ω is connected, there exists $\boldsymbol{\varphi} := (\varphi_1, \varphi_2)^{\mathrm{t}} \in [H^1(\Omega)]^2$, with $\displaystyle\int_\Omega \varphi_1 = \int_\Omega \varphi_2 = 0$, such that

$$\boldsymbol{\tau} = \underline{\mathbf{curl}}(\boldsymbol{\varphi}) + \nabla\mathbf{z}. \tag{4.11}$$

This identity is known as a Helmholtz decomposition of $\boldsymbol{\tau}$. Note that the equivalence between $\|\boldsymbol{\varphi}\|_{1,\Omega}$ and $|\boldsymbol{\varphi}|_{1,\Omega}$ (which is consequence of the generalized Poincaré inequality), together with (4.11) and (3.19), imply that

$$\|\boldsymbol{\varphi}\|_{1,\Omega} \le c\,|\boldsymbol{\varphi}|_{1,\Omega} = c\,\|\underline{\mathbf{curl}}(\boldsymbol{\varphi})\|_{0,\Omega} \le c\left\{\|\boldsymbol{\tau}\|_{0,\Omega} + \|\nabla\mathbf{z}\|_{0,\Omega}\right\} \le C\,\|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}. \tag{4.12}$$

Now, we let $\boldsymbol{\varphi}_h := (I_h(\varphi_1), I_h(\varphi_2))^{\mathrm{t}}$ and define

$$\boldsymbol{\tau}_h := \underline{\mathbf{curl}}(\boldsymbol{\varphi}_h) + \mathcal{E}_h^k(\nabla\mathbf{z}) + c\,\mathbf{I}, \tag{4.13}$$

where $\mathcal{E}_h^k$ is the Raviart-Thomas interpolation operator introduced before (cf. (3.7), (3.8)), and the constant c is chosen so that $\boldsymbol{\tau}_h$, which is already in $\mathbb{RT}_k(\mathcal{T}_h)$, belongs to $H_{h,0}^{\sigma}$. Equivalently, $\boldsymbol{\tau}_h$ is the H_0-component of $\underline{\mathbf{curl}}(\boldsymbol{\varphi}_h) + \mathcal{E}_h^k(\nabla\mathbf{z}) \in \mathbb{RT}_k(\mathcal{T}_h)$. We refer to (4.13) as a discrete Helmholtz descomposition of $\boldsymbol{\tau}_h$.

Then, replacing $\boldsymbol{\tau}$ (cf. (4.11)) and $\boldsymbol{\tau}_h$ (cf. (4.13)) into (4.6), observing that the expression $c\,\mathbf{I}$ cancells out from the three terms defining R, and noting, according to (3.6) and (3.9), that

$$\int_\Omega \mathbf{u}_h \cdot \mathbf{div}(\nabla \mathbf{z} - \mathcal{E}_h^k(\nabla \mathbf{z})) \,=\, \int_\Omega \mathbf{u}_h \cdot (\mathbf{div}(\boldsymbol{\tau}) \,-\, \mathcal{P}_h^k(\mathbf{div}(\boldsymbol{\tau}))) \,=\, 0\,,$$

we find that R can be decomposed as $R(\boldsymbol{\tau}) \,=\, R_1(\boldsymbol{\varphi}) \,+\, R_2(\mathbf{z})$, where

$$R_1(\boldsymbol{\varphi}) \,:=\, \langle \underline{\mathbf{curl}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\,\boldsymbol{\nu}, \mathbf{g}\rangle \,-\, \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}_h^d : \underline{\mathbf{curl}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\,, \tag{4.14}$$

and

$$R_2(\mathbf{z}) \,:=\, \langle (\nabla \mathbf{z} - \mathcal{E}_h^k(\nabla \mathbf{z}))\,\boldsymbol{\nu}, \mathbf{g}\rangle \,-\, \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}_h^d : (\nabla \mathbf{z} - \mathcal{E}_h^k(\nabla \mathbf{z}))\,. \tag{4.15}$$

The following two lemmas provide upper bounds for $|R_1(\varphi)|$ and $|R_2(\mathbf{z})|$.

LEMMA 4.3 *Assume that $\mathbf{g} \in [H^1(\Gamma)]^2$. Then there exists $C > 0$, independent of h, such that*

$$|R_1(\varphi)| \,\le\, C \left\{ \sum_{T\in\mathcal{T}_h} \theta_{1,T}^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega} \tag{4.16}$$

where

$$\theta_{1,T}^2 \,:=\, h_T^2 \left\| \mathrm{curl}\Big\{\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathrm{d}}\Big\} \right\|_{0,T}^2 \,+\, \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega)} h_e \left\| \Big[\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathrm{d}}\,\mathbf{s}\Big] \right\|_{0,e}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e \left\| \frac{d\mathbf{g}}{d\mathbf{s}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathrm{d}}\,\mathbf{s} \right\|_{0,e}^2 .$$

Proof. Using that $\underline{\mathbf{curl}}(\varphi - \varphi_h)\boldsymbol{\nu} \,=\, \frac{d}{d\mathbf{s}}(\varphi - \varphi_h)$ and then integrating by parts on Γ, we find that

$$\langle \underline{\mathbf{curl}}(\varphi - \varphi_h)\,\boldsymbol{\nu}, \mathbf{g}\rangle \,=\, -\,\langle\, \varphi - \varphi_h, \frac{d\mathbf{g}}{d\mathbf{s}}\,\rangle \,=\, -\sum_{e\in\mathcal{E}_h(\Gamma)} \int_e (\varphi - \varphi_h)\,\frac{d\mathbf{g}}{d\mathbf{s}}\,.$$

Next, integrating by parts on each $T \in \mathcal{T}_h$, we obtain that

$$\int_\Omega \boldsymbol{\sigma}_h^d : \underline{\mathbf{curl}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \,=\, \sum_{T\in\mathcal{T}_h} \left\{ \int_T \mathrm{curl}\{\boldsymbol{\sigma}_h^d\}\cdot(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \,-\, \int_{\partial T} \boldsymbol{\sigma}_h^d\,\mathbf{s}\cdot(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \right\}$$
$$= \sum_{T\in\mathcal{T}_h} \int_T \mathrm{curl}\{\boldsymbol{\sigma}_h^d\}\cdot(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \,-\, \sum_{e\in\mathcal{E}_h(\Omega)} \int_e [\boldsymbol{\sigma}_h^d\,\mathbf{s}]\cdot(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \,-\, \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \boldsymbol{\sigma}_h^d\,\mathbf{s}\cdot(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\,.$$

Hence, replacing the above expressions into (4.14), we deduce that

$$R_1(\boldsymbol{\varphi}) \,=\, \sum_{T\in\mathcal{T}_h} \int_T \mathrm{curl}\Big\{\frac{1}{2\mu}\,\boldsymbol{\sigma}_h^d\Big\}\cdot(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \,+\, \sum_{e\in\mathcal{E}_h(\Omega)} \int_e \Big[\frac{1}{2\mu}\,\boldsymbol{\sigma}_h^d\,\mathbf{s}\Big]\cdot(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)$$
$$-\, \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \Big\{\frac{d\mathbf{g}}{d\mathbf{s}} \,-\, \frac{1}{2\mu}\,\boldsymbol{\sigma}_h^d\,\mathbf{s}\Big\}\cdot(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)\,.$$

Finally, applying the Cauchy-Schwarz inequality, the approximation properties provided by Lemma 4.2 together with the fact that the number of triangles in $\Delta(T)$ and $\Delta(e)$ are bounded, and then the estimate (4.12), we derive the upper bound (4.16). $\square$

LEMMA 4.4 *There exists $C > 0$, independent of h, such that*

$$|R_2(\mathbf{z})| \,\le\, C \left\{ \sum_{T\in\mathcal{T}_h} \theta_{2,T}^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}\,, \tag{4.17}$$

where

$$\theta_{2,T}^2 \,=\, h_T^2 \left\| \nabla \mathbf{u}_h - \frac{1}{2\mu}\,\boldsymbol{\sigma}_h^d \right\|_{0,T}^2 \,+\, \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\,\|\mathbf{g} - \mathbf{u}_h\|_{0,e}^2\,.$$

Proof. Since $\mathbf{u}_h|_e \in [P_k(e)]^2$ for each edge $e \in \mathcal{E}_h$ (in particular for each edge $e \in \mathcal{E}_h(\Gamma)$), and $\nabla\mathbf{u}_h|_T \in [\mathbb{P}_{k-1}(T)]^{2\times 2}$ for each $T \in \mathcal{T}_h$, the identities (3.7) and (3.8) characterizing $\mathcal{E}_h^k$, yield, respectively,

$$\int_e (\nabla\mathbf{z} - \mathcal{E}_h^k(\nabla\mathbf{z}))\,\boldsymbol{\nu}\cdot\mathbf{u}_h \,=\, 0 \qquad \forall\, e \,\in\, \mathcal{E}_h(\Gamma)\,,$$

and

$$\int_T (\nabla\mathbf{z} - \mathcal{E}_h^k(\nabla\mathbf{z})) : \nabla\mathbf{u}_h \,=\, 0 \qquad \forall\, T \,\in\, \mathcal{T}_h\,.$$

Hence, introducing the above expressions into the definition of R_2 (cf. (4.15)), we obtain that

$$R_2(\mathbf{z}) \,:=\, \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e (\nabla\mathbf{z} - \mathcal{E}_h^k(\nabla\mathbf{z}))\,\boldsymbol{\nu}\cdot(\mathbf{g} - \mathbf{u}_h) \,+\, \sum_{T\in\mathcal{T}_h} \int_T \left\{ \nabla\mathbf{u}_h - \frac{1}{2\mu}\,\boldsymbol{\sigma}_h^d \right\} : (\nabla\mathbf{z} - \mathcal{E}_h^k(\nabla\mathbf{z}))\,,$$

from which, applying the Cauchy-Schwarz inequality, the approximation properties (3.13) and (3.11) (with $m = 1$), and then the estimate (3.19), we deduce the upper bound (4.17). $\square$

As a straightforward corollary of Lemmas 4.3 and 4.4 we deduce that

$$|R(\boldsymbol{\tau})| \,\le\, \left\{ \sum_{T\in\mathcal{T}_h} \left(\theta_{1,T}^2 + \theta_{2,T}^2\right) \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega} \qquad \forall\,\boldsymbol{\tau} \,\in\, H_0\,, \tag{4.18}$$

which gives an upper bound for the supremum on the right hand side of (4.5). In this way, and noting that

$$\|f \,+\, \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,\Omega}^2 \,=\, \sum_{T\in\mathcal{T}_h} \|f \,+\, \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,T}^2\,,$$

we conclude from Lemma 4.1 the reliability of the a posteriori error estimator $\boldsymbol{\theta}$ (upper bound in (4.3)).

In what follows we show the reliability of $\boldsymbol{\eta}$. We begin with the analogue of Lemma 4.1.

LEMMA 4.5 *Let $((\boldsymbol{\sigma}, p), \mathbf{u}) \in H \times Q$ and $((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h) \in H_h \times Q_h$ be the unique solutions of* (2.11) *and* (3.1)*, respectively. Then there exists $C > 0$, independent of h, such that*

$$\begin{aligned} C\,\|((\boldsymbol{\sigma}, p), \mathbf{u}) - ((\boldsymbol{\sigma}_h, p_h), \mathbf{u}_h)\|_{H\times Q} \,&\le\, \sup_{\boldsymbol{\tau}\in H_0\setminus\{\theta\}} \frac{S(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}} \\ &+\, \|f \,+\, \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,\Omega} \,+\, \left\| p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h) \right\|_{0,\Omega}\,, \end{aligned} \tag{4.19}$$

where

$$S(\boldsymbol{\tau}) \,:=\, R(\boldsymbol{\tau}) \,-\, \frac{\kappa}{2\mu}\int_\Omega \left\{ \left(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\mathbf{I}\right\} : (\boldsymbol{\tau} - \boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \,\in\, H^{\boldsymbol{\sigma}}_{0,h}\,. \tag{4.20}$$

Proof. It is similar to the proof of Lemma 4.1 and hence we omit further details here. □

Next, in order to estimate the supremum on the right hand side of (4.19), and thanks to (4.18), it only remains to bound the extra-term given by

$$T(\boldsymbol{\tau}) \,:=\, \int_\Omega \left\{ \left(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\mathbf{I}\right\} : (\boldsymbol{\tau} - \boldsymbol{\tau}_h)\,.$$

To this end, we apply again the Helmholtz decompositions (4.11) and (4.13), and obtain that

$$T(\boldsymbol{\tau}) \,=\, \int_\Omega \left\{ \Big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\Big)\mathbf{I}\right\} : \underline{\mathbf{curl}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \,+\, \int_\Omega \left\{ \Big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\Big)\mathbf{I}\right\} : (\nabla\mathbf{z} \,-\, \mathcal{E}_h^k(\nabla\mathbf{z}))\,.$$

Note here that the expression $c\,\mathbf{I}$ (cf. (4.13)) cancells out from the definition of T since, as shown by (3.1) with $\boldsymbol{\tau} = \mathbf{0}$ and $q = 1$, there holds $\displaystyle\int_\Omega \Big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\Big) \,=\, 0.$

Lemma 4.6 *There exists $C > 0$, independent of h, such that*

$$|T(\boldsymbol{\tau})| \,\le\, C \left\{ \sum_{T\in\mathcal{T}_h} \eta^2_{1,T} \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega}\,, \tag{4.21}$$

where

$$\eta^2_{1,T} \,=\, h_T^2 \left\| \mathbf{curl}\Big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\Big)\right\|^2_{0,T} \,+\, \sum_{e\in\mathcal{E}(T)} h_e \left\| \Big[\Big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\Big)\,\mathbf{s}\Big]\right\|^2_{0,e}$$
$$+\;\; h_T^2 \left\| p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right\|^2_{0,T}\,.$$

Proof. It follows with the same techniques employed in the proofs of Lemmas 4.3 and 4.4. □

As a consequence of (4.18), (4.20), and (4.21), we deduce that

$$|S(\boldsymbol{\tau})| \,\le\, \left\{ \sum_{T\in\mathcal{T}_h} \left(\theta^2_{1,T} + \theta^2_{2,T} + \eta^2_{1,T}\right) \right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathrm{div},\Omega} \qquad \forall\,\boldsymbol{\tau} \,\in\, H_0\,, \tag{4.22}$$

which gives an upper bound for the supremum on the right hand side of (4.19). In this way, and similarly as for $\boldsymbol{\theta}$, we conclude from Lemma 4.5 the reliability of the a posteriori error estimator $\boldsymbol{\eta}$ (upper bound in (4.4)). Note that the term $h_T^2 \left\| p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right\|^2_{0,T}$ is not included in the final estimation since it is dominated by $\left\| p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right\|^2_{0,T}$ (cf. (4.19)).

4.2 Efficiency of the a posteriori error estimators

In this section we prove the efficiency of our a posteriori error estimators $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ (lower bounds in (4.3) and (4.4), respectively). In other words, we derive suitable upper bounds for the six terms defining the local error indicator θ_T^2 (cf. (4.1)), and for the remaining three terms completing the definition of the local error indicator η_T^2 (cf. (4.2)).

We first notice, using that $\mathbf{f} = -\,\mathbf{div}(\boldsymbol{\sigma})$ and $p + \dfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}) = 0$ in Ω, that there hold

$$\|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,T}^2 = \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,T}^2 \tag{4.23}$$

and

$$\left\|p_h + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}_h)\right\|_{0,T}^2 \leq 2\left\{\|p - p_h\|_{0,T}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2\right\}. \tag{4.24}$$

Next, in order to bound the terms involving the mesh parameters h_T and h_e, we proceed similarly as in [15] and [16] (see also [24]), and apply the localization technique based on bubble functions, together with inverse inequalities. To this end, we now introduce further notations and preliminary results. Given $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T)$, we let ψ_T and ψ_e be the usual triangle-bubble and edge-bubble functions, respectively (see (1.5) and (1.6) in [34]), which satisfy:

i) $\psi_T \in \mathbb{P}_3(T)$, $\mathrm{supp}(\psi_T) \subseteq T$, $\psi_T = 0$ on ∂T, and $0 \leq \psi_T \leq 1$ in T.

ii) $\psi_e|_T \in \mathbb{P}_2(T)$, $\mathrm{supp}(\psi_e) \subseteq \omega_e := \cup\{T' \in \mathcal{T}_h : \quad e \in \mathcal{E}(T')\}$, $\psi_e = 0$ on $\partial T \backslash e$, and $0 \leq \psi_e \leq 1$ in ω_e.

We also recall from [35] that, given $k \in \mathbb{N} \cup \{0\}$, there exists a linear operator $L : C(e) \to C(T)$ that satisfies $L(p) \in \mathbb{P}_k(T)$ and $L(p)|_e = p \;\; \forall p \in \mathbb{P}_k(e)$. A corresponding vectorial version of L, that is the componentwise application of L, is denoted by $\mathbf{L}$. Additional properties of ψ_T, ψ_e and L are collected in the following lemma.

LEMMA 4.7 *Given $k \in \mathbb{N} \cup \{0\}$, there exist positive constants c_1, c_2, c_3, and c_4 , depending only on k and the shape regularity of the triangulations (minimum angle condition), such that for each $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T)$, there hold*

$$\|\psi_T\, q\|_{0,T}^2 \leq \|q\|_{0,T}^2 \leq c_1\, \|\psi_T^{1/2}\, q\|_{0,T}^2 \qquad \forall\, q \in \mathbb{P}_k(T)\,, \tag{4.25}$$

$$\|\psi_e\, L(p)\|_{0,T}^2 \leq \|p\|_{0,e}^2 \leq c_2\, \|\psi_e^{1/2}\, p\|_{0,e}^2 \qquad \forall\, p \in \mathbb{P}_k(e)\,, \tag{4.26}$$

and

$$c_3\, h_e\, \|p\|_{0,e}^2 \leq \|\psi_e^{1/2}\, L(p)\|_{0,T}^2 \leq c_4\, h_e\, \|p\|_{0,e}^2 \qquad \forall\, p \in \mathbb{P}_k(e)\,. \tag{4.27}$$

Proof. See Lemma 1.3 in [35]. □

The following inverse estimate will also be used.

LEMMA 4.8 *Let l, $m \in \mathbb{N} \cup \{0\}$ such that $l \leq m$. Then, there exists $c > 0$, depending only on k, l, m and the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h$ there holds*

$$|q|_{m,T} \leq c\, h_T^{l-m}\, |q|_{l,T} \quad \forall\, q \in \mathbb{P}_k(T). \tag{4.28}$$

Proof. See Theorem 3.2.6 in [18]. □

The following two lemmas are required for the terms involving the curl operator and the tangential jumps across the edges of $\mathcal{T}_h$. Their proofs, which make use of Lemmas 4.7 and 4.8, can be found in [6].

LEMMA 4.9 *Let $\boldsymbol{\rho}_h \in [L^2(\Omega)]^{2\times 2}$ be a piecewise polynomial of degree $k \geq 0$ on each $T \in \mathcal{T}_h$. In addition, let $\boldsymbol{\rho} \in [L^2(\Omega)]^{2\times 2}$ be such that* $\mathrm{curl}(\boldsymbol{\rho}) = \mathbf{0}$ *on each $T \in \mathcal{T}_h$. Then, there exists $c > 0$, independent of h, such that*

$$\|\mathrm{curl}(\boldsymbol{\rho}_h)\|_{0,T} \leq c\, h_T^{-1}\, \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,T} \qquad \forall\, T \in \mathcal{T}_h\,.$$

Proof. See Lemma 4.3 in [6]. □

LEMMA 4.10 *Let $\boldsymbol{\rho}_h \in [L^2(\Omega)]^{2\times 2}$ be a piecewise polynomial of degree $k \geq 0$ on each $T \in \mathcal{T}_h$, and let $\boldsymbol{\rho} \in [L^2(\Omega)]^{2\times 2}$ be such that* $\mathrm{curl}(\boldsymbol{\rho}) = \mathbf{0}$ *in Ω. Then, there exists $c > 0$, independent of h, such that*

$$\|[\boldsymbol{\rho}_h\, \mathbf{s}]\|_{0,e} \leq c\, h_e^{-1/2}\, \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\omega_e} \qquad \forall\, e \in \mathcal{E}_h\,.$$

Proof. It is a slight modification of the proof of Lemma 4.4 in [6]. We omit the details here. □

We now apply Lemmas 4.9 and 4.10 to bound other four terms defining θ_T^2 and η_T^2.

LEMMA 4.11 *There exist C_1, $C_2 > 0$, independent of h, such that*

$$h_T^2 \left\| \mathrm{curl}\Big\{ \frac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}} \Big\} \right\|_{0,T}^2 \leq C_1 \left\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \right\|_{0,T}^2 \qquad \forall\, T \in \mathcal{T}_h\,, \tag{4.29}$$

and

$$h_e \left\| \Big[\frac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}}\, \mathbf{s} \Big] \right\|_{0,e}^2 \leq C_2 \left\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \right\|_{0,\omega_e}^2 \qquad \forall\, e \in \mathcal{E}_h(\Omega)\,. \tag{4.30}$$

Proof. Applying Lemmas 4.9 and 4.10 to $\boldsymbol{\rho}_h := \dfrac{1}{2\mu}\boldsymbol{\sigma}_h^d$ and $\boldsymbol{\rho} := \dfrac{1}{2\mu}\boldsymbol{\sigma}^d = \nabla \mathbf{u}$, and then using the continuity of $\boldsymbol{\tau} \longrightarrow \boldsymbol{\tau}^d$, we obtain (4.29) and (4.30), respectively. □

LEMMA 4.12 *There exist C_3, $C_4 > 0$, independent of h, such that*

$$h_T^2 \left\| \mathbf{curl}\big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\big) \right\|_{0,T}^2 \leq C_3 \Big\{ \|p - p_h\|_{0,T}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \Big\} \qquad \forall\, T \in \mathcal{T}_h\,, \tag{4.31}$$

and

$$h_e \left\| \Big[\big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\big)\, \mathbf{s} \Big] \right\|_{0,e}^2 \leq C_4 \Big\{ \|p - p_h\|_{0,\omega_e}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\omega_e}^2 \Big\} \qquad \forall\, e \in \mathcal{E}_h\,. \tag{4.32}$$

Proof. It suffices to apply Lemmas 4.9 and 4.10 to the tensors $\boldsymbol{\rho}_h := \big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\big)\,\mathbf{I}$ and $\boldsymbol{\rho} := \big(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\big)\,\mathbf{I} = \mathbf{0}$, and then use the triangle inequality. □

The remaining three terms are bounded next applying Lemmas 4.7 and 4.8.

LEMMA 4.13 *There exist $C_5 > 0$, independent of h, such that*

$$h_T^2 \left\| \nabla \mathbf{u}_h - \frac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\|_{0,T}^2 \le C_5 \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 + h_T^2 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \right\} \qquad \forall\, T \in \mathcal{T}_h \,. \tag{4.33}$$

Proof. It is a slight modification of the proof of Lemma 6.3 in [15] (see also Lemma 5.5 in [24]). For sake of completeness we now provide the details. Given $T \in \mathcal{T}_h$ we denote $\chi_T := \nabla \mathbf{u}_h - \dfrac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}}$ in T. Then, applying the right hand side of (4.25), using that $\nabla \mathbf{u} = \frac{1}{2\mu} \boldsymbol{\sigma}^{\mathtt{d}}$ in Ω, and integrating by parts, we find that

$$\begin{aligned}
\|\chi_T\|_{0,T}^2 \le c_1 \, \|\psi_T^{1/2} \, \chi_T\|_{0,T}^2 \;&=\; c_1 \int_T \psi_T \, \chi_T : \Big(\nabla \mathbf{u}_h - \frac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}} \Big) \\
&=\; c_1 \int_T \psi_T \, \chi_T : \Big\{ \nabla (\mathbf{u}_h - \mathbf{u}) - \frac{1}{2\mu} \Big(\boldsymbol{\sigma}_h^{\mathtt{d}} - \boldsymbol{\sigma}^{\mathtt{d}} \Big) \Big\} \\
&=\; c_1 \left\{ \int_T \operatorname{div} \big(\psi_T \, \chi_T \big) \cdot (\mathbf{u} - \mathbf{u}_h) \,+\, \frac{1}{2\mu} \int_T \psi_T \, \chi_T : \big(\boldsymbol{\sigma}^{\mathtt{d}} - \boldsymbol{\sigma}_h^{\mathtt{d}} \big) \right\} .
\end{aligned}$$

Then, applying the Cauchy-Schwarz inequality, the inverse estimate (4.28), the left hand side of (4.25), and the continuity of $\boldsymbol{\tau} \longrightarrow \boldsymbol{\tau}^d$, we get

$$\|\chi_T\|_{0,T}^2 \;\le\; C \left\{ h_T^{-1} \, \|\mathbf{u} - \mathbf{u}_h\|_{0,T} \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} \right\} \|\chi_T\|_{0,T} \,,$$

which yields

$$\|\chi_T\|_{0,T} \;\le\; C \left\{ h_T^{-1} \, \|\mathbf{u} - \mathbf{u}_h\|_{0,T} \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} \right\} .$$

This inequality implies (4.33) and completes the proof. □

In order to bound the boundary terms given by $h_e \, \|\mathbf{g} - \mathbf{u}_h\|_{0,e}^2$, $e \in \mathcal{E}_h(\Gamma)$, we need to recall a discrete trace inequality. In fact, as established by Theorem 3.10 in [1] (see also eq. (2.4) in [2]), there exists $c > 0$, depending only on the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T)$, there holds

$$\|v\|_{0,e}^2 \;\le\; c \left\{ h_e^{-1} \, \|v\|_{0,T}^2 \,+\, h_e \, |v|_{1,T}^2 \right\} \qquad \forall\, v \in H^1(T) \,. \tag{4.34}$$

LEMMA 4.14 *There exist $C_6 > 0$, independent of h, such that*

$$h_e \, \|\mathbf{g} - \mathbf{u}_h\|_{0,e}^2 \;\le\; C_6 \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \,+\, h_T^2 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \right\} \qquad \forall\, e \in \mathcal{E}_h(\Gamma) \,, \tag{4.35}$$

where T is the triangle of $\mathcal{T}_h$ having e as en edge.

Proof. Applying the inequality (4.34) together with the fact that $\mathbf{u} = \mathbf{g}$ on Γ and $\nabla \mathbf{u} = \dfrac{1}{2\mu} \boldsymbol{\sigma}^{\mathtt{d}}$ in Ω, we easily obtain that

$$\|\mathbf{g} - \mathbf{u}_h\|_{0,e}^2 \;=\; \|\mathbf{u} - \mathbf{u}_h\|_{0,e}^2 \;\le\; c \left\{ h_e^{-1} \, \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \,+\, h_e \, |\mathbf{u} - \mathbf{u}_h|_{1,T}^2 \right\}$$

$$
\begin{aligned}
&= c\left\{ h_e^{-1}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,T}^2 \,+\, h_e\left\|\frac{1}{2\mu}\,\boldsymbol{\sigma}^{\mathtt{d}} - \nabla\mathbf{u}_h\right\|_{0,T}^2 \right\}\\
&\le C\left\{ h_e^{-1}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,T}^2 \,+\, h_e\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, h_e\left\|\nabla\mathbf{u}_h - \frac{1}{2\mu}\,\boldsymbol{\sigma}_h^{\mathtt{d}}\right\|_{0,T}^2 \right\},
\end{aligned}
$$

which, using that $h_e \le h_T$, gives

$$
h_e\,\|\mathbf{g}-\mathbf{u}_h\|_{0,e}^2 \le C\left\{ \|\mathbf{u}-\mathbf{u}_h\|_{0,T}^2 \,+\, h_T^2\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, h_T^2\left\|\nabla\mathbf{u}_h - \frac{1}{2\mu}\,\boldsymbol{\sigma}_h^{\mathtt{d}}\right\|_{0,T}^2 \right\}.
$$

This estimate and the upper bound for $h_T^2\left\|\nabla\mathbf{u}_h - \frac{1}{2\mu}\,\boldsymbol{\sigma}_h^{\mathtt{d}}\right\|_{0,T}^2$ (cf. Lemma 4.13) yield (4.35), which ends the proof. □

LEMMA 4.15 *Assume that* $\mathbf{g}$ *is piecewise polynomial. Then there exist* $C_7 > 0$*, independent of* h*, such that*

$$
h_e\left\|\frac{d\mathbf{g}}{d\mathbf{s}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\,\mathbf{s}\right\|_{0,e}^2 \le C_7\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T_e}^2 \qquad \forall\, e \in \mathcal{E}_h(\Gamma)\,, \tag{4.36}
$$

where T_e *is the triangle of* $\mathcal{T}_h$ *having* e *as en edge.*

Proof. Given $e \in \mathcal{E}_h(\Gamma)$ we denote $\chi_e := \dfrac{d\mathbf{g}}{d\mathbf{s}} - \dfrac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\,\mathbf{s}$ on e. Then, applying (4.26), the extension operator $\mathbf{L} : [C(e)]^2 \to [C(T)]^2$, and the fact that $\dfrac{d\mathbf{g}}{d\mathbf{s}} = \nabla\mathbf{u}\,\mathbf{s}$, we obtain that

$$
\begin{aligned}
\|\chi_e\|_{0,e}^2 &\le c_2\,\|\psi_e^{1/2}\,\chi_e\|_{0,e}^2 = c_2\int_e \psi_e\,\chi_e\cdot\left(\frac{d\mathbf{g}}{d\mathbf{s}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\,\mathbf{s}\right)\\
&= c_2\int_{\partial T_e} \psi_e\,\mathbf{L}(\chi_e)\cdot\left\{\left(\nabla\mathbf{u} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\right)\mathbf{s}\right\}.
\end{aligned} \tag{4.37}
$$

Now, integrating by parts and using that $\nabla\mathbf{u} = \dfrac{1}{2\mu}\boldsymbol{\sigma}^{\mathtt{d}}$ in Ω, we find that

$$
\begin{aligned}
&\int_{\partial T_e} \psi_e\,\mathbf{L}(\chi_e)\cdot\left\{\left(\nabla\mathbf{u} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\right)\mathbf{s}\right\}\\
&\quad = \int_{T_e} \underline{\mathbf{curl}}(\psi_e\,\mathbf{L}(\chi_e)) : \left(\frac{1}{2\mu}\boldsymbol{\sigma}^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\right) + \int_{T_e} \psi_e\,\mathbf{L}(\chi_e)\cdot \mathrm{curl}\left\{\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\right\}.
\end{aligned} \tag{4.38}
$$

Then, applying the Cauchy-Schwarz inequality, the inverse estimate (4.28), the continuity of $\boldsymbol{\tau} \longrightarrow \boldsymbol{\tau}^d$, and noting, thanks to the fact that $0 \le \psi_e \le 1$ and the right-hand side of (4.27), that

$$
\|\psi_e\,\mathbf{L}(\chi_e)\|_{0,T_e} \le \|\psi_e^{1/2}\,\mathbf{L}(\chi_e)\|_{0,T_e} \le c\,h_e^{1/2}\,\|\chi_e\|_{0,e}\,,
$$

we deduce from (4.37) and (4.38) that

$$
\|\chi_e\|_{0,e}^2 \le C\left\{h_{T_e}^{-1}\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T_e} + \left\|\mathrm{curl}\left\{\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\right\}\right\|_{0,T_e}\right\} h_e^{1/2}\,\|\chi_e\|_{0,e}\,,
$$

which, using that $h_e \leq h_{T_e}$, yields

$$h_e \, \|\chi_e\|_{0,e}^2 \leq C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T_e}^2 + h_{T_e}^2 \left\| \mathrm{curl}\Big\{ \frac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}} \Big\} \right\|_{0,T_e}^2 \right\} .$$

This inequality and the upper bound for $h_{T_e}^2 \left\| \mathrm{curl}\Big\{ \frac{1}{2\mu} \boldsymbol{\sigma}_h^{\mathtt{d}} \Big\} \right\|_{0,T_e}^2$ (cf. Lemma 4.11) imply (4.36), which completes the proof. □

If $\mathbf{g}$ were not piecewise polynomial but sufficiently smooth, then higher order terms given by the errors arising from suitable polynomial approximations would appear in (4.36). This explains the eventual expression *h.o.t.* in (4.3) and (4.4).

Finally, the efficiency of $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ follows straightforwardly from estimates (4.23) and (4.24), together with Lemmas 4.11 throughout 4.15, after summing up over $T \in \mathcal{T}_h$ and using that the number of triangles on each domain ω_e is bounded by two.

5 Numerical results

In this section we present three numerical examples illustrating the performance of the mixed finite element schemes (3.1) and (3.2), confirming the reliability and efficiency of the a posteriori error estimators $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$ derived in Section 4, and showing the behaviour of the associated adaptive algorithms. In all the computations we consider the specific finite element subspaces $H_{0,h}^{\boldsymbol{\sigma}}$, H_h^p, and Q_h given by (3.15) with $k = 0$. As in [25] and [22], the zero integral mean condition for functions of the space $H_{0,h}^{\boldsymbol{\sigma}}$ is implemented in (3.1) and (3.2) by introducing a real Lagrange multiplier.

In what follows, N stands for the total number of degrees of freedom (unknowns) of (3.1) and (3.2), which can be proved to behave asymptotically as 6 and 5 times, respectively, the number of elements of each triangulation. Also, the individual and total errors are given by

$$e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{div},\Omega} , \quad e(p) := \|p - p_h\|_{0,\Omega} , \quad e(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} ,$$

$$e(\boldsymbol{\sigma},\mathbf{u}) := \big\{ (e(\boldsymbol{\sigma}))^2 + (e(\mathbf{u}))^2 \big\}^{1/2} , \quad \text{and} \quad e(\boldsymbol{\sigma},p,\mathbf{u}) := \big\{ (e(\boldsymbol{\sigma}))^2 + (e(p))^2 + (e(\mathbf{u}))^2 \big\}^{1/2} ,$$

whereas the effectivity indexes with respect to $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are defined, respectively, by

$$eff(\boldsymbol{\theta}) := e(\boldsymbol{\sigma},\mathbf{u})/\boldsymbol{\theta} \quad \text{and} \quad eff(\boldsymbol{\eta}) := e(\boldsymbol{\sigma},p,\mathbf{u})/\boldsymbol{\eta} .$$

Then, we define the experimental rates of convergence

$$r(\boldsymbol{\sigma}) := \frac{\log(e(\boldsymbol{\sigma})/e'(\boldsymbol{\sigma}))}{\log(h/h')} , \quad r(p) := \frac{\log(e(p)/e'(p))}{\log(h/h')} , \quad r(\mathbf{u}) := \frac{\log(e(\mathbf{u})/e'(\mathbf{u}))}{\log(h/h')} ,$$

$$r(\boldsymbol{\sigma},\mathbf{u}) := \frac{\log(e(\boldsymbol{\sigma},\mathbf{u})/e'(\boldsymbol{\sigma},\mathbf{u}))}{\log(h/h')} , \quad \text{and} \quad r(\boldsymbol{\sigma},p,\mathbf{u}) := \frac{\log(e(\boldsymbol{\sigma},p,\mathbf{u})/e'(\boldsymbol{\sigma},p,\mathbf{u}))}{\log(h/h')} ,$$

where e and e' denote the corresponding errors at two consecutive triangulations with mesh sizes h and h', respectively. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by

$-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. In all of them we take the kinematic viscosity $\mu = 1$. The first example is employed to illustrate the performance of the mixed finite element schemes and to confirm the reliability and efficiency of the a posteriori error estimators. Then, Example 2 is utilized to show the behaviour of the adaptive algorithm associated with $\boldsymbol{\eta}$, which apply the following procedure from [34]:

1) Start with a coarse mesh $\mathcal{T}_h$.

2) Solve the discrete problem (3.1) for the actual mesh $\mathcal{T}_h$.

3) Compute η_T for each triangle $T \in \mathcal{T}_h$.

4) Evaluate stopping criterion and decide to finish or go to next step.

5) Use *blue-green* procedure to refine each $T' \in \mathcal{T}_h$ whose indicator $\eta_{T'}$ satisfies

$$\eta_{T'} \geq \frac{1}{2}\max\{\eta_T : \quad T \in \mathcal{T}_h\}$$

6) Define resulting mesh as actual mesh $\mathcal{T}_h$ and go to step 2.

Finally, in Example 3 we consider the standard test case given by a driven cavity.

In Example 1 we consider $\Omega =]0,1[^2$ and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) := \frac{1}{8\pi\mu}\left\{-\log \mathbf{r}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\mathbf{r}^2}\begin{pmatrix} (x_1-2)^2 \\ (x_1-2)(x_2-2) \end{pmatrix}\right\} \quad \text{and} \quad p(\mathbf{x}) = \frac{(x_1-2)}{2\pi\mathbf{r}^2} - p_0\,,$$

with $\mathbf{r} = \sqrt{(x_1-2)^2 + (x_2-2)^2}$, for all $\mathbf{x} := (x_1, x_2) \in \Omega$, where $p_0 \in \mathbb{R}$ is such that $\int_\Omega p = 0$ holds. At this point we recall from (2.5) and the fact that $\boldsymbol{\sigma} \in H_0$, that an admissible solution p must satisfy $\int_\Omega p = 0$. Note in this example that $(\mathbf{u}, p)$ corresponds to the fundamental solution located at the point $(2,2)$, which is outside $\bar{\Omega}$. Hence, $\mathbf{f} = \mathbf{0}$, $\mathbf{u}$ is divergence free, and $(\mathbf{u}, p)$ is regular in the whole domain Ω.

In Example 2 we consider Ω as the L-shaped domain $]-1,1[^2 -]0,1[^2$ and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) := \left((x_1-0.1)^2 + (x_2-0.1)^2\right)^{-1/2}\begin{pmatrix} x_2 - 0.1 \\ 0.1 - x_1 \end{pmatrix} \quad \text{and} \quad p(\mathbf{x}) = \frac{1}{x_2 - 1.1} - p_0\,,$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega$. We note that $\mathbf{u}$ is divergence free in Ω. In addition, it is clear that $\mathbf{u}$ and p are singular at $(0.1, 0.1)$ and along the line $x_2 = 1.1$, respectively. Hence, we should expect regions of high gradients around the origin and along the line $x_2 = 1.0$.

In Example 3 we take $\Omega =]0,1[^2$, $\mathbf{f} = \mathbf{0}$ in Ω, and

$$\mathbf{g}(x_1, x_2) := \begin{cases} (\sin(\pi x_1), 0) & \text{if} \quad 0 < x_1 < 1,\ x_2 = 1 \\ 0 & \text{otherwise}. \end{cases}$$

Our numerical results were obtained in a *Pentium Xeon computer with dual processors*, using a MATLAB code. Since the choice of κ in (3.1) depends on the unknown constant c_1 from Lemma 2.4 we simply take $\kappa = \mu$, which worked out well in the computations shown below.

In Tables 5.1 - 5.2 and Figure 5.1, we summarize the convergence history of the mixed finite element schemes (3.1) and (3.2) as applied to Example 1, for sequences of quasi-uniform triangulations of the domain. We observe there that the rate of convergence $O(h)$ predicted by Theorems 3.4 and 3.2 (when $s = 1$) is attained by all the unknowns. In addition, though we do not present the corresponding results here, we remark that the same Table 5.1 (up to the first 6 and 7 digits) is obtained with $\kappa \in \{\mu/2, 2\,\mu, \mu/10, 10\,\mu, \mu/100, 100\,\mu\}$, which suggests the robustness of (3.1) with respect to the choice of this parameter. According to this fact, the hypothesis on κ established by Theorem 3.3, that is $0 < \kappa < c_1$, seems to be more technical than truly necessary for practical computations. Next, we remark the good behaviour of the a posteriori error estimators in sequences of quasi-uniform meshes. Indeed, we notice from Tables 5.1 and 5.2 that the effectivity indexes $eff(\boldsymbol{\eta})$ and $eff(\boldsymbol{\theta})$ remain always in a neighborhood of 0.45 and 0.42, respectively, which illustrates the reliability and efficiency of $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$. Finally, in order to emphasize the good performance of our schemes, in Figures 5.2 - 5.4 we display some components of the approximate and exact solutions for Example 1.

Next, in Tables 5.3 - 5.4 and Figure 5.5, we provide the convergence history of the uniform and adaptive schemes (3.1) with $\kappa = \mu$, as applied to Example 2. We observe here, as expected, that the errors of the adaptive procedure decrease faster than those obtained by the quasi-uniform one. This fact is better illustrated in Figure 5.5 where we display the total errors $e(\boldsymbol{\sigma}, p, \mathbf{u})$ vs. the degrees of freedom N for both refinements. Actually, looking at the experimental rates of convergence provided in the tables, we confirm that the adaptive method is able to recover the quasi-optimal $\mathcal{O}(h)$ for the total error. In addition, the effectivity indexes remain again bounded from above and below, which confirms the reliability and efficiency of $\boldsymbol{\eta}$ for the associated adaptive algorithm. Some intermediate meshes obtained with the adaptive refinement are displayed in Figure 5.6. Note that the adapted meshes concentrate the refinements around the origin and the line $x_2 = 1$, which says that the method is able to recognize the regions with high gradients of the solutions. In order to illustrate the good quality of the solutions provided by the adaptive scheme, in Figures 5.7 - 5.8 we display two components of the approximate and exact solutions for Example 2.

Finally, in Table 5.5 we provide the *convergence* history of the adaptive scheme (3.1) with $\kappa = \mu$ as applied to the driven cavity (Example 3). The errors and experimental rates of convergence shown there are computed by considering the discrete solution obtained with the finest mesh ($N = 331176$) as the *exact solution*. Then, in Figure 5.9 we display the individual and total errors vs. the degrees of freedom N, and observe that they all behave approximately of the same order. Two intermediate meshes obtained with the adaptive refinement and some components of the approximate solutions are displayed in Figures 5.10 and 5.11, respectively.

We end this section by remarking that very similar numerical results to those presented here for Examples 2 and 3 are obtained with the mixed finite element scheme (3.2) and the a posteriori error estimator $\boldsymbol{\theta}$. Therefore, there is enough support to consider the mixed finite element schemes (3.1) and (3.2) together with the associated adaptive algorithms, as valid and competitive alternatives to solve the stationary Stokes equations.

Table 5.1: EXAMPLE 1, uniform scheme (3.1) with $\kappa = \mu$

N	h	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\sigma}, p, \mathbf{u})$	$r(\boldsymbol{\sigma}, p, \mathbf{u})$	$eff(\boldsymbol{\theta})$
3137	0.06250	1.751E-03	7.542E-04	3.989E-04	1.948E-03	–	0.472
3961	0.05556	1.551E-03	6.605E-04	3.546E-04	1.723E-03	1.045	0.467
4881	0.05000	1.392E-03	5.873E-04	3.191E-04	1.544E-03	1.040	0.464
5897	0.04545	1.262E-03	5.287E-04	2.901E-04	1.399E-03	1.036	0.461
7009	0.04167	1.154E-03	4.807E-04	2.659E-04	1.278E-03	1.032	0.458
8217	0.03846	1.064E-03	4.408E-04	2.454E-04	1.177E-03	1.029	0.457
9521	0.03571	9.865E-04	4.069E-04	2.279E-04	1.091E-03	1.026	0.455
10921	0.03333	9.196E-04	3.780E-04	2.127E-04	1.017E-03	1.024	0.454
12417	0.03125	8.612E-04	3.529E-04	1.994E-04	9.518E-04	1.022	0.453
14009	0.02941	8.098E-04	3.309E-04	1.876E-04	8.947E-04	1.020	0.452
15697	0.02778	7.643E-04	3.115E-04	1.772E-04	8.441E-04	1.019	0.451
19361	0.02500	6.869E-04	2.789E-04	1.595E-04	7.584E-04	1.017	0.450
27841	0.02083	5.714E-04	2.307E-04	1.329E-04	6.304E-04	1.014	0.448
37857	0.01786	4.892E-04	1.968E-04	1.139E-04	5.395E-04	1.011	0.446
49409	0.01562	4.277E-04	1.716E-04	9.967E-05	4.715E-04	1.008	0.446
77121	0.01250	3.418E-04	1.367E-04	7.974E-05	3.767E-04	1.006	0.444
110977	0.01042	2.847E-04	1.137E-04	6.645E-05	3.136E-04	1.005	0.444
150977	0.00892	2.439E-04	9.726E-05	5.695E-05	2.687E-04	1.004	0.443
197121	0.00781	2.133E-04	8.501E-05	4.983E-05	2.350E-04	1.003	0.443
249409	0.00694	1.896E-04	7.551E-05	4.430E-05	2.088E-04	1.002	0.443
307841	0.00625	1.706E-04	6.792E-05	3.987E-05	1.879E-04	1.002	0.443

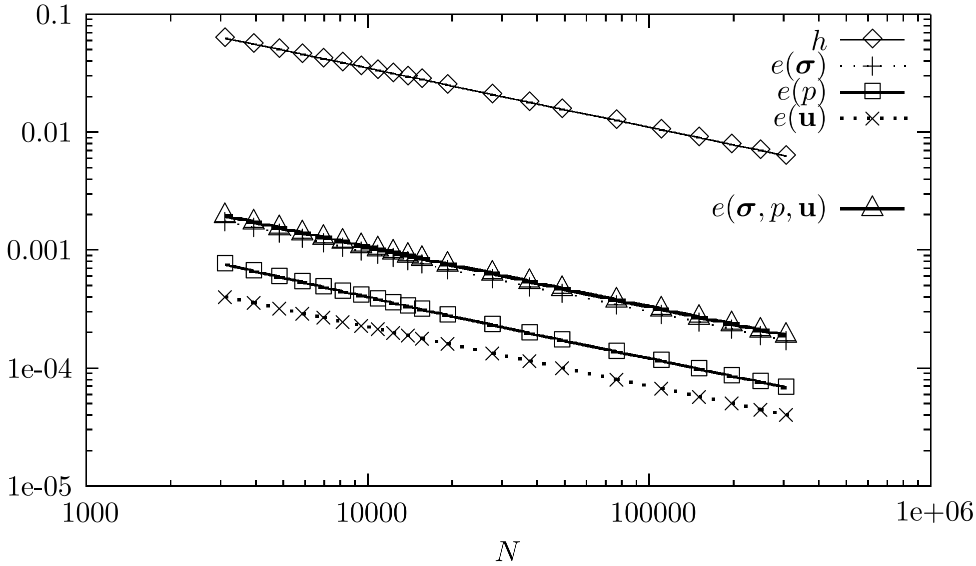

Figure 5.1: EXAMPLE 1, uniform scheme (3.1) with $\kappa = \mu$

Table 5.2: EXAMPLE 1, uniform scheme (3.2)

N	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\sigma},\mathbf{u})$	$r(\boldsymbol{\sigma},\mathbf{u})$	$eff(\boldsymbol{\theta})$
2625	0.06250	1.751E-03	–	3.989E-04	–	1.796E-03	–	0.435
3313	0.05556	1.551E-03	1.032	3.546E-04	1.003	1.591E-03	1.030	0.431
4081	0.05000	1.392E-03	1.027	3.191E-04	1.002	1.428E-03	1.027	0.429
4929	0.04545	1.262E-03	1.026	2.901E-04	1.002	1.295E-03	1.025	0.427
5857	0.04167	1.154E-03	1.023	2.659E-04	1.001	1.185E-03	1.022	0.425
6865	0.03846	1.064E-03	1.021	2.454E-04	1.001	1.092E-03	1.020	0.423
7953	0.03571	9.865E-04	1.019	2.279E-04	1.001	1.012E-03	1.018	0.422
9121	0.03333	9.196E-04	1.018	2.127E-04	1.001	9.439E-04	1.017	0.421
10369	0.03125	8.612E-04	1.016	1.994E-04	1.001	8.840E-04	1.015	0.420
11697	0.02941	8.098E-04	1.015	1.876E-04	1.001	8.313E-04	1.014	0.420
13105	0.02778	7.643E-04	1.014	1.772E-04	1.001	7.845E-04	1.013	0.419
16161	0.02500	6.869E-04	1.012	1.595E-04	1.000	7.052E-04	1.012	0.418
23233	0.02083	5.714E-04	1.010	1.329E-04	1.000	5.867E-04	1.009	0.417
31585	0.01786	4.892E-04	1.008	1.139E-04	1.000	5.023E-04	1.007	0.416
41217	0.01562	4.277E-04	1.006	9.967E-05	1.000	4.392E-04	1.006	0.415
64321	0.01250	3.418E-04	1.005	7.974E-05	1.000	3.510E-04	1.004	0.414
92545	0.01042	2.847E-04	1.003	6.645E-05	1.000	2.923E-04	1.003	0.414
125889	0.00892	2.439E-04	1.003	5.695E-05	1.000	2.504E-04	1.002	0.413
164353	0.00781	2.133E-04	1.002	4.983E-05	1.000	2.191E-04	1.002	0.413
207937	0.00694	1.896E-04	1.002	4.430E-05	1.000	1.947E-04	1.002	0.413
256641	0.00625	1.706E-04	1.001	3.987E-05	1.000	1.752E-04	1.001	0.413

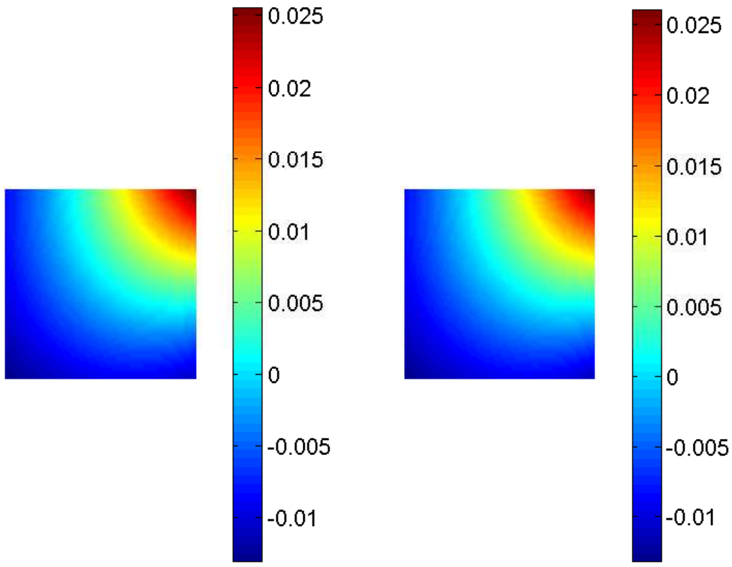

Figure 5.2: EXAMPLE 1, approximate (left) and exact σ_{22} for uniform scheme (3.1) with $\kappa = \mu$ and $N = 27841$

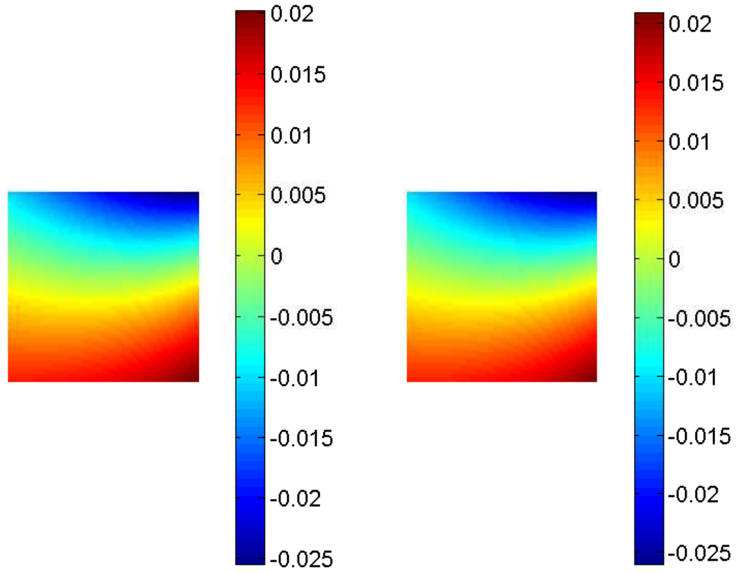

Figure 5.3: EXAMPLE 1, approximate (left) and exact p for uniform scheme (3.1) with $\kappa = \mu$ and $N = 27841$

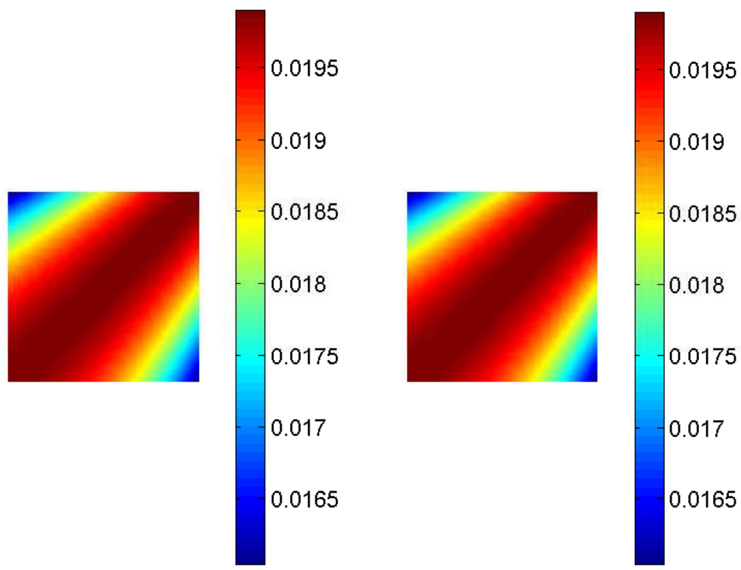

Figure 5.4: EXAMPLE 1, approximate (left) and exact u_2 for uniform scheme (3.1) with $\kappa = \mu$ and $N = 27841$

Table 5.3: EXAMPLE 2, uniform scheme (3.1) with $\kappa = \mu$

N	h	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\sigma}, p, \mathbf{u})$	$r(\boldsymbol{\sigma}, p, \mathbf{u})$	$eff(\boldsymbol{\eta})$
45	1.0000	1.358E+01	1.558E-00	5.903E-01	1.368E+01	–	0.816
245	0.5000	1.569E+01	1.368E-00	2.890E-01	1.575E+01	–	0.949
541	0.3333	1.401E+01	1.129E-00	1.942E-01	1.406E+01	0.280	0.967
993	0.2500	1.198E+01	9.174E-01	1.451E-01	1.202E+01	0.546	0.967
1457	0.2000	1.122E+01	8.247E-01	1.153E-01	1.125E+01	0.296	0.972
2161	0.1666	9.846E+00	6.603E-01	9.622E-02	9.869E+00	0.719	0.969
3057	0.1429	9.088E+00	6.122E-01	8.076E-02	9.109E+00	0.519	0.971
4037	0.1250	8.090E+00	5.325E-01	6.875E-02	8.107E+00	0.873	0.971
4897	0.1111	7.670E+00	5.258E-01	6.301E-02	7.688E+00	0.450	0.972
5889	0.1000	6.960E+00	4.809E-01	5.523E-02	6.977E+00	0.922	0.974
7433	0.0909	6.798E+00	4.381E-01	5.100E-02	6.812E+00	0.251	0.974
8881	0.0833	6.266E+00	3.923E-01	4.666E-02	6.279E+00	0.937	0.972
10557	0.0769	5.748E+00	3.583E-01	4.321E-02	5.759E+00	1.078	0.971
11981	0.0714	5.674E+00	3.587E-01	- 4.008E-02	5.685E+00	0.176	0.975
13729	0.0667	5.215E+00	3.297E-01	3.720E-02	5.226E+00	1.220	0.974
15465	0.0625	4.849E+00	2.977E-01	3.474E-02	4.858E+00	1.132	0.972
17729	0.0588	4.561E+00	2.833E-01	3.227E-02	4.569E+00	1.010	0.974
19933	0.0556	4.271E+00	2.655E-01	3.022E-02	4.280E+00	1.147	0.973
24929	0.0500	3.990E+00	2.484E-01	2.756E-02	3.998E+00	0.646	0.974
29721	0.0455	3.615E+00	2.197E-01	2.464E-02	3.622E+00	1.037	0.974
39213	0.0400	3.207E+00	1.903E-01	2.182E-02	3.212E+00	0.939	0.972
51425	0.0345	2.712E+00	1.693E-01	1.884E-02	2.717E+00	1.127	0.973
75665	0.0286	2.317E+00	1.394E-01	1.566E-02	2.321E+00	0.839	0.973
109405	0.0238	1.912E+00	1.156E-01	1.289E-02	1.916E+00	1.053	0.974
153017	0.0200	1.635E+00	9.575E-02	1.093E-02	1.638E+00	0.898	0.972
193337	0.0179	1.503E+00	8.728E-02	9.685E-03	1.506E+00	0.744	0.973
246529	0.0159	1.256E+00	7.517E-02	8.612E-03	1.258E+00	1.524	0.971

Table 5.4: EXAMPLE 2, adaptive scheme (3.1) with $\kappa = \mu$

N	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\sigma}, p, \mathbf{u})$	$r(\boldsymbol{\sigma}, p, \mathbf{u})$	$eff(\boldsymbol{\eta})$
45	1.358E+01	5.903E-01	1.558E-00	1.368E+01	–	0.816
133	1.484E+01	3.934E-01	1.734E-00	1.495E+01	–	0.936
264	1.298E+01	2.946E-01	1.306E-00	1.305E+01	0.396	0.949
431	1.112E+01	2.256E-01	1.144E-00	1.118E+01	0.631	0.942
1007	7.810E+00	1.701E-01	8.085E-01	7.853E+00	0.833	0.947
1805	5.362E+00	1.668E-01	6.791E-01	5.408E+00	1.278	0.927
3906	3.428E+00	1.131E-01	4.232E-01	3.456E+00	1.160	0.911
8326	2.394E+00	7.307E-02	2.916E-01	2.412E+00	0.950	0.908
14367	1.837E+00	5.602E-02	2.035E-01	1.849E+00	0.975	0.905
32400	1.206E+00	3.684E-02	1.338E-01	1.214E+00	1.035	0.898
55793	9.374E−01	2.722E-02	9.699E-02	9.428E−01	0.930	0.902
125406	6.200E−01	1.883E-02	6.727E-02	6.239E−01	1.019	0.893
222070	4.753E−01	1.368E-02	4.816E-02	4.779E−01	0.933	0.897
500376	3.115E−01	9.513E-03	3.332E-02	3.134E−01	1.039	0.886

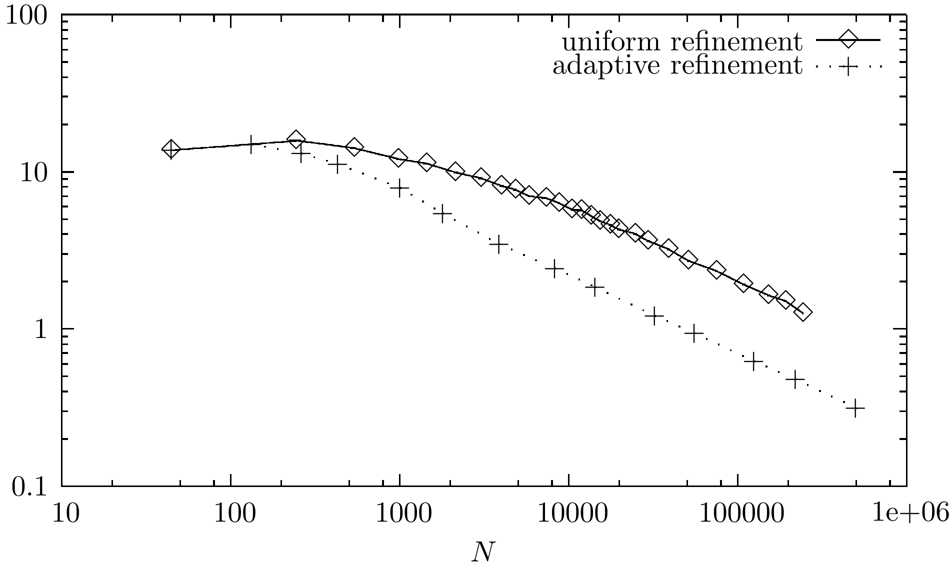

Figure 5.5: EXAMPLE 2, $e(\boldsymbol{\sigma}, p, \mathbf{u})$ vs. N for uniform/adaptive schemes (3.1) with $\kappa = \mu$

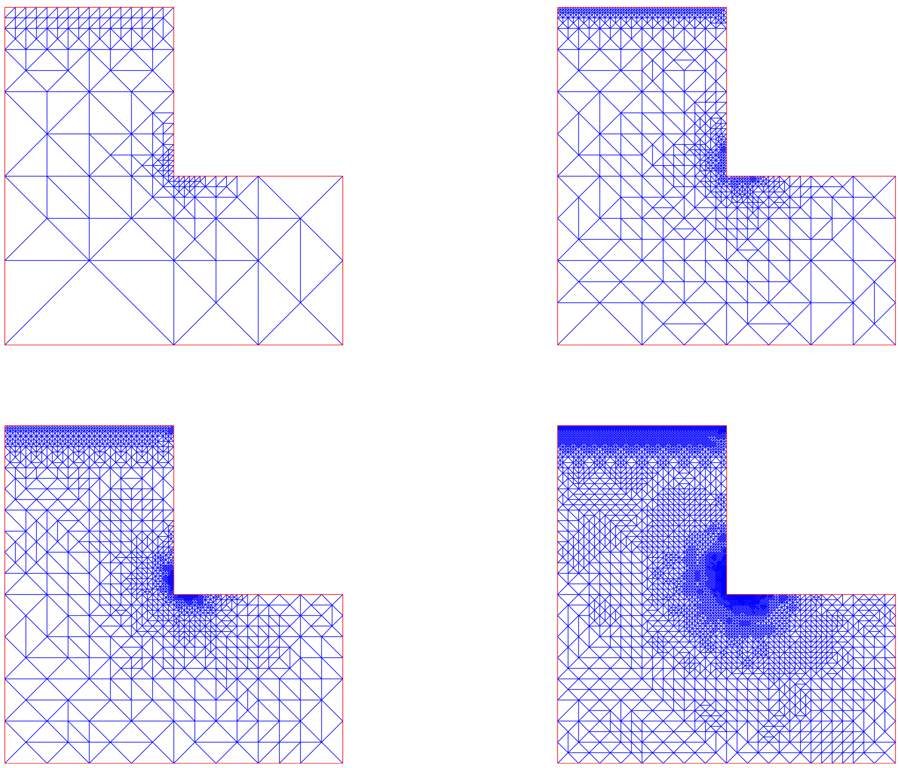

Figure 5.6: EXAMPLE 2, adapted intermediate meshes with 1805, 8326, 14367, and 55793 degrees of freedom for scheme (3.1) with $\kappa = \mu$

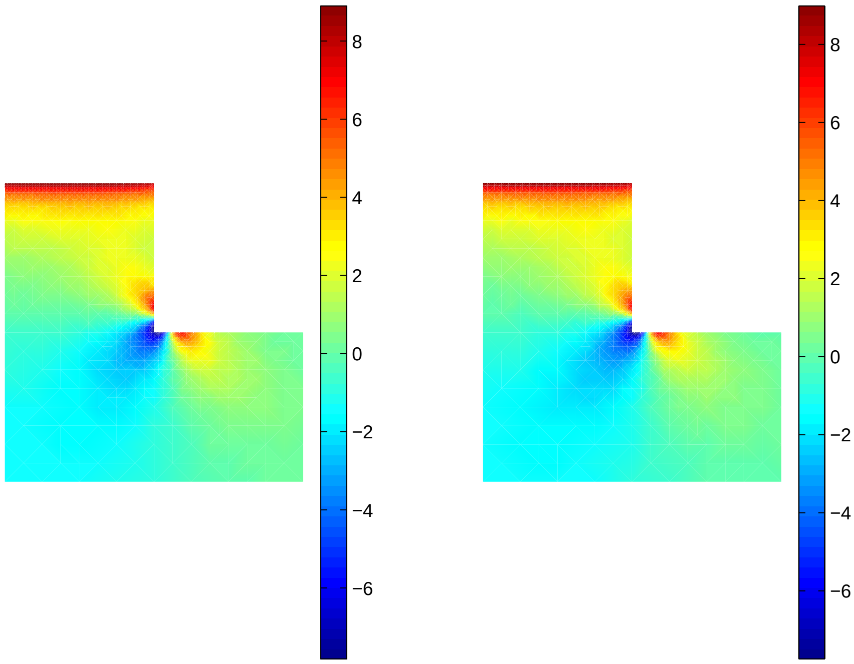

Figure 5.7: EXAMPLE 2, approximate (left) and exact σ_{11} for adaptive scheme (3.1) with $\kappa = \mu$ and $N = 14367$

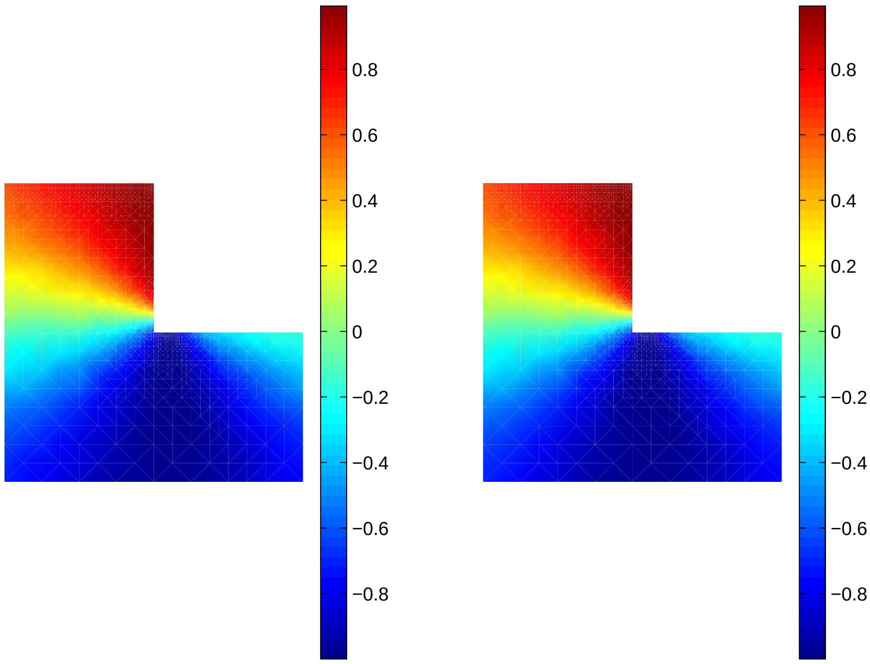

Figure 5.8: EXAMPLE 2, approximate (left) and exact u_1 for adaptive scheme (3.1) with $\kappa = \mu$ and $N = 14367$

Table 5.5: EXAMPLE 3, adaptive scheme (3.1) with $\kappa = \mu$

N	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\sigma}, p, \mathbf{u})$	$r(\boldsymbol{\sigma}, p, \mathbf{u})$
57	8.496E-00	5.492E-00	2.039E-01	1.012E+01	–
169	6.127E-00	3.898E-00	1.338E-01	7.263E+00	0.610
370	4.294E-00	2.681E-00	7.800E-02	5.063E+00	0.921
1035	2.605E-00	1.573E-00	4.586E-02	3.043E+00	0.990
1853	1.681E-00	9.560E-01	3.127E-02	1.934E+00	1.558
4750	9.902E-01	5.306E-01	1.986E-02	1.124E+00	1.154
8949	6.824E-01	3.418E-01	1.459E-02	7.633E−01	1.221
20382	4.127E-01	1.936E-01	9.596E-03	4.559E−01	1.252
39849	2.848E-01	1.264E-01	6.642E-03	3.116E−01	1.135
84440	1.755E-01	7.511E-02	4.257E-03	1.909E−01	1.305
163886	1.060E-01	4.393E-02	2.513E-03	1.147E−01	1.536

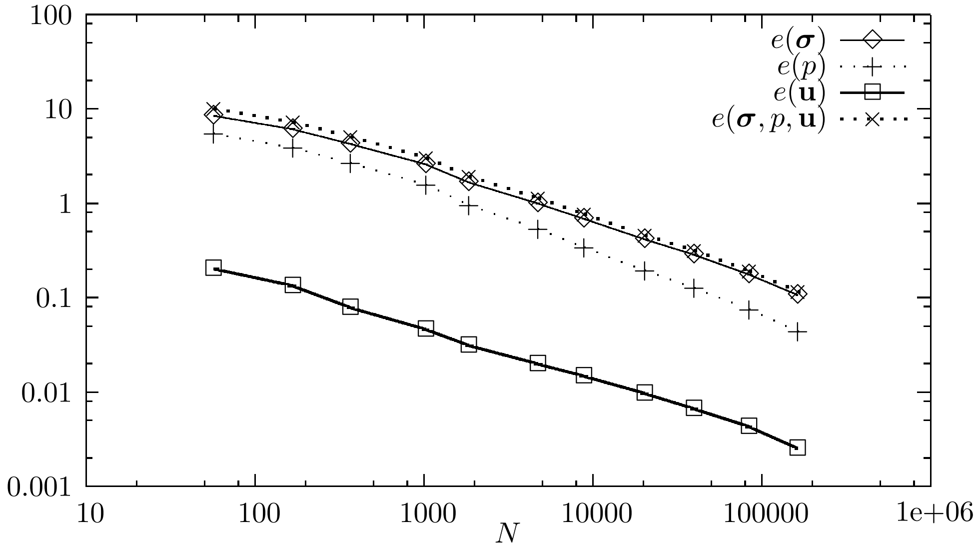

Figure 5.9: EXAMPLE 3, adaptive scheme (3.1) with $\kappa = \mu$

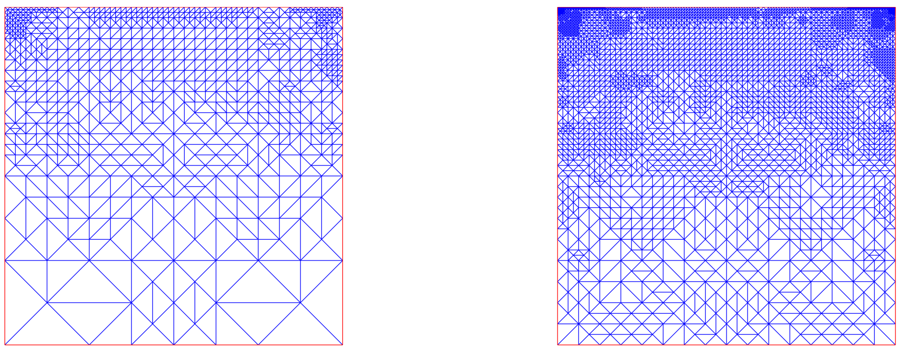

Figure 5.10: EXAMPLE 3, adapted intermediate meshes with 8949 and 39849 degrees of freedom for scheme (3.1) with $\kappa = \mu$

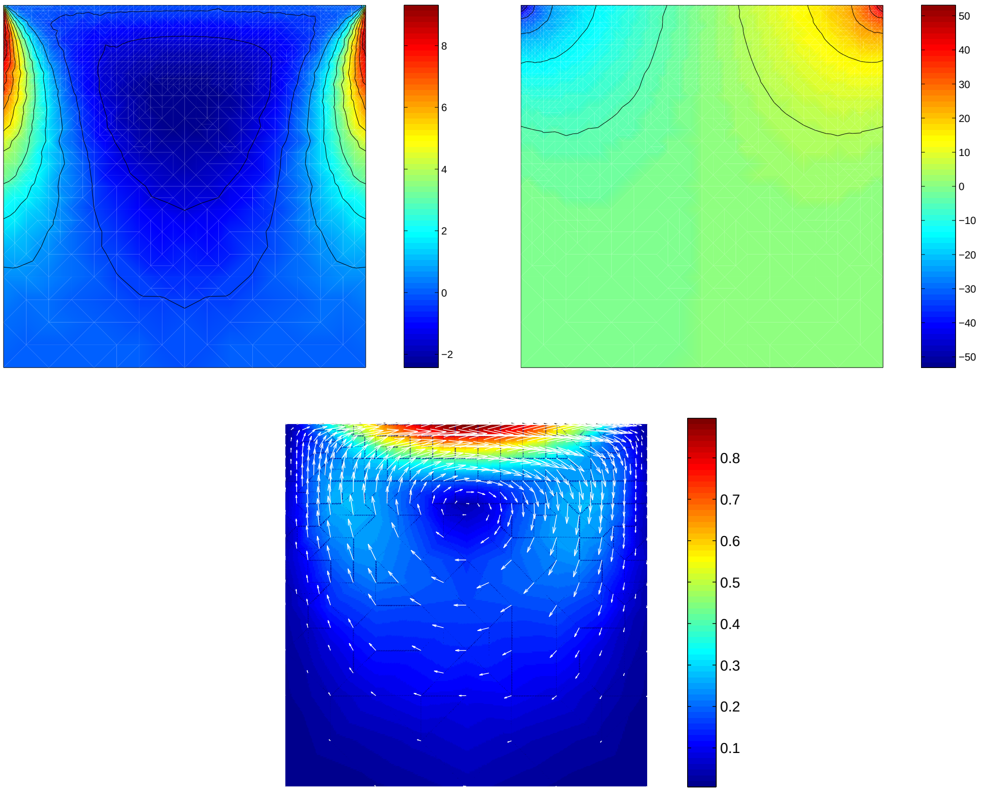

Figure 5.11: EXAMPLE 3, approximate σ_{21} (top left), p (top right), and $\mathbf{u}$ (bottom) for adaptive scheme (3.1) with $\kappa = \mu$ and $N = 20382$

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