Unifying and scalarizing vector optimization problems: a theoretical approach and optimality conditions

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Abstract

This paper introduces and analyses a general vector optimization problem which encompasses those related to efficiency, weak efficiency, and two kinds of strict efficiency, among others, in a unified framework. A corresponding approximate vector problem is also studied, and new optimality conditions for both problems are established via a nonlinear scalarizing function and subdifferentials. Generalized convexity of vector functions are characterized through generalized convexity of scalar functions. The approach sheds new light and offers an alternative to obtain several existing results in the literature.

Key words. vector optimization, approximate optimality, scalarization, efficiency, weak efficiency, strict efficiency, optimality conditions; subdifferential

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1 Introduction

In most real-life problems, optimization problems concern the minimization of several criterion functions simultaneously. Very often, no single point minimizing all criteria at once may be found, and therefore the concept of optimality must be modified.

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Usually, the notion of efficient or weakly efficient solution is considered. A point is called efficient or Pareto-optimal, if there does not exist a different point with smaller or equal objective function values, such that there is a decrease in at least one objective function value; a point is called weakly efficient or weakly Pareto-optimal, if there exists no other point with strictly smaller objective function value. Certainly, both notions may be described in terms of the nonnegative orthant of some finite dimensional space. However, in several circumstances the previous notions may be described by means of a *preference relation* determined by a convex cone with nonempty interior. Thus, an optimization problem is formulated according to the decision maker's preferences.

On the other hand, it is well-known that the scalarization techniques in optimization theory are very useful from the practical point of view. Moreover, we need to convert vector problems into appropriate scalar ones in the sense that the latter problem must inherit properties providing a good representation of the solutions to the vector problem.

Moreover, the approximate solutions of optimization problems are very interesting since most of usual resolution methods, (for instance, the iterative and heuristic methods), give as solution feasible points near to the theoretical solution. See [44] and references therein for more details. Very recent, motivated by a new approximate efficient concept, some scalarizations for vector optimization problems have been established in [21] and [23].
For more history, detailed background information and motivations about the before concepts we refer the reader to [28] and [32].

In this paper, we introduce a general vector optimization problem defined in vector spaces which encompasses the classical ones: efficiency, weak efficiency and a kind of strict efficiency. We also study an approximate vector optimization. Then we scalarize both problems by using a well-known nonlinear function. Moreover, we show that such a function allows us to characterize several notions of generalized quasiconvexity. As an application, we give necessary (and in some situations also sufficient) optimality conditions for (approximate) efficient solutions via subdifferentials of scalar functions.

The outline of the paper is as follows. In Section 2 we provide the notions and notations to present and discuss the general optimization problem. In Section 3, we introduce the scalarizing function and study its useful properties. Section 4 is devoted to describe the scalarization procedure for (approximate) efficiency by establishing complete scalarizations for both problems. In Section 5 we provide conditions regarding lower semicontinuity. Section 6 deals with some characterizations of convexity or generalized quasiconvexity of a vector function in terms of scalar functions. Finally, in

Section 7, we present optimality conditions under convexity via approximate subdifferentials in the sense of convex analysis; and when nonconvexity is assumed via the Mordukhovich subdifferential.

2 Preliminaries and formulation of problem

Let Y be a real normed vector space and let X be a Banach space with topological dual spaces Y^* and X^* respectively.

Given a nonempty set $S \subsetneq Y$, a nonempty set $K \subseteq X$ and a function $f : K \to Y$, we are interested in the problem

$$(\mathcal{P}) \qquad \text{find } \bar{x} \in K \quad f(x) - f(\bar{x}) \in S \quad \forall\, x \in K,\ x \neq \bar{x}.$$

The set of such vectors $\bar{x} \in K$ is denoted by $E_S = E_S(K)$, and each one of its elements is called a (global) S-minimal of f on K.
As mentioned above several notions of optimality require a proper convex cone $P \subseteq Y$ (by proper we mean that $\{0\} \neq P \neq Y$). In such a situation, $(\mathcal{P})$ subsumes several vector optimization problems as we shall see now.

In what follows, given any $\emptyset \neq A \subseteq Y$, we denote by $\mathcal{C}(A)$, $\operatorname{int} A$, $\operatorname{cl} A$ and ∂A the complement, the topological interior, the topological closure and the boundary of A respectively:

- if $S = P$, the solutions are termed "*ideal*" o "*strong*" minima of f (on K) and the solution set is denoted by E_P;
- ($\operatorname{int} P \neq \emptyset$) if $S = \mathcal{C}(-\operatorname{int} P)$, the solutions are called "*weakly efficient*" minima of f and the solution set is denoted by E_W;
- if $S = \mathcal{C}(-P) \cup l(P)$ where $l(P) = P \cap (-P)$, the solutions are said to be "*efficient*" minima of f and the solution set is denoted by E.
- if $S = \mathcal{C}(-P) \cup \{0\}$, such solutions are named "*weakly strict efficient*" minima of f and the solution set is denoted by E_{W1}.
- if $S = \mathcal{C}(-P)$, such solutions are named "*strict efficient*" minima of f and the solution set is denoted by E_1.
- if $0 \neq D \subsetneq Y$ is a convex cone with nonempty interior such that $P \setminus l(P) \subseteq \operatorname{int} D$ and $S = l(D) \cup \mathcal{C}(-D)$, the solutions are called "*proper efficient*" minima of Henig type of f, and the solution set is denoted by E_2.

Since $l(P) \cup \mathcal{C}(-P) \subseteq \mathcal{C}(-\operatorname{int} P)$ and $l(D) \cup \mathcal{C}(-D) \subseteq l(P) \cup \mathcal{C}(-P)$ we have $E_2 \subseteq E \subseteq E_W$. In this case, if $E_P \neq \emptyset$ then $E_P = E$. On the other hand: $E_1 \subseteq E_{W1} \subseteq E$; $E = E_{W1}$ whenever P is pointed; $E_W = E_P$ provided P is a closed halfspace (see Lemma 2.5 in [13]); $E_1 = E_{W1}$ whenever f is injective.

The notion of strict efficient minimum is further developed in [15].

Motivated by the previous specializations of S, we impose the following basic assumption on S.

Assumption (**A**): $P \subseteq Y$ is a proper (not necessarily closed or pointed) convex cone with nonempty interior, and $S \subsetneq Y$ is any set such that $0 \in \partial S$, $S + \operatorname{int} P \subseteq S$.

When $\operatorname{int} P \neq \emptyset$, P is closed and $0 \in \partial S$ this assumption is related to the free-disposal assumption (**P**): $S \subsetneq Y$ is closed and $S + P = S$, and to the strong free-disposal assumption ($\mathbf{P_S}$): $S \subsetneq Y$ is closed and $S + (P \setminus \{0\}) = \operatorname{int} S$, or equivalently, $S + (P \setminus \{0\}) \subseteq \operatorname{int} S$ (see [4, 42]). Originally $\operatorname{int} P \neq \emptyset$ is nor required in Assumptions (**P**) and ($\mathbf{P_S}$)).

Obviously if $0 \in \partial S$ then $(\mathbf{P_S}) \Longrightarrow (\mathbf{P}) \Longrightarrow (\mathbf{A})$, but certainly the set $S = \mathcal{C}(-P) \cup l(P)$ satisfies $S + P = S$ although S is not closed, whereas $S = \mathcal{C}(-P) \cup \{0\}$ is not closed, and if P is not pointed, the equality $S + P = S$ does not hold.
However, problem $(\mathcal{P})$ categorizes more general optimization problems. Exactly problem $(\mathcal{P})$ includes other problems given by a not necessarily pre-order relation. For instance, if S is a cone not necessarily convex we can similarly define the above efficient concepts w.r.t S. These efficient notions have been studied by using strongly star-shaped conic sets in [41] not for a optimization problem but for a closed set.

It is well-known that preferences which are not pre-order relations are very important in mathematical economics see [36] and references therein.

Given $\varepsilon \geq 0$ and $y \in Y$, an approximate problem associated to $(\mathcal{P})$ is:

$$(\mathcal{P}(\varepsilon y)) \qquad \text{find } \bar{x} \in K \quad f(x) - f(\bar{x}) \in -\varepsilon y + S \quad \forall\, x \in K,\ x \neq \bar{x},$$

where S is any set satisfying Assumption **(A)**, when a proper convex cone P is prescribed. We denote by $E_S(\varepsilon y)$ the solution set to $(\mathcal{P}(\varepsilon y))$. Thus, Assumption **(A)** implies that for all $q \in \operatorname{int} P$:

$$0 \leq \varepsilon_1 < \varepsilon_2 \Longrightarrow E_S(\varepsilon_1 q) \subseteq E_S(\varepsilon_2 q);$$

$$E_S = E_S(0) \subseteq E_S(\varepsilon q) \ \ \forall\, \varepsilon > 0.$$

Consequently,

$$E_S \subseteq \bigcap_{\varepsilon > 0} E_S(\varepsilon q) \subseteq E_{\operatorname{cl} S} \quad \forall\, q \in \operatorname{int} P.$$

When f is a real function we denote by $E(f, K, \varepsilon)$ the set of ε-solutions, that is, $\bar{x} \in E(f, K, \varepsilon)$ if and only if $f(x) - f(\bar{x}) \geq -\varepsilon$ for all $x \in K$.

A recent notion of approximate vector problem may be found in [23] (see also [21]). We emphasize that there is no relationship between any of those notions and the one presented here.

3 A nonlinear scalarizing function

It is well documented the scalarizing procedure is very important in vector optimization. This approach requires a suitable scalar function which allows us to substitute the vector problem by a scalar one, and hopefully most of the properties (like convexity, lower semicontinuity) of the vector objective function are inherited in its scalar representation. It is well known that linear scalarization functions (giving rise to the weighting method) are employed to describe weakly efficient minima when the vector function to be optimized is convex, or under a generalized convexity assumption as discussed in [30, 14]; see [10] for quadratic scalarization.

A nonlinear scalarizing function that nowadays is having a great impact in the development of a theoretical and algorithmic treatment of vector optimization problems, is that function which (seems to be) appeared for the first time in [40, Example 2, p. 139] and rediscovered, among others, in [38, 19]. Since then several authors continue to use that or some variant, see for instance Bonnisseau and Cornet [3, p. 139], Luc [32], Gerth and Weidner [20], Luenberger [35] (where it is called the shortage function in connection to economics), Hamel and Löhne [24], Hernández and Rodríguez-Marín [25], Tammer and Zalinescu [42]. Regarding nonlinear scalarization for approximate efficiency, we refer to [21] and [23] and references therein. A good account is given in [7] and [8].

This nonlinear scalar function will be used in Section 4 to characterize some notions of relaxed convexity for vector functions. The reader can find in [33, 1] various characterizations of quasiconvexity by using linear scalarizations and in [32] by using such a type of nonlinear scalar function.

¿From now on, we assume the convex (not necessarily closed or pointed) cone P has nonempty interior. Let $e \in \operatorname{int} P$ be fixed.

Definition 3.1. *Let $a \in Y$. Let $\xi_a \doteq \xi_{e,a} \colon Y \longrightarrow \mathbb{R} \cup \{-\infty\}$ be defined by*

$$\xi_a(y) \doteq \inf\{t \in \mathbb{R} \colon y \in te + a - P\} \quad (y \in Y).$$

This function is a nonlinear Minkowski-type functional and has many separation properties (see [32], [20], [31]) and plays an important role in many areas, for instance, mathematical finance, see [4] and [18].

We also consider the following function used in [32] to separate nonconvex sets, see also [34] and [25].

Let $\xi_A\colon Y \longrightarrow \mathbb{R}\cup\{-\infty\}$ be defined by

$$\xi_A(y) = \inf\{t \in \mathbb{R}\colon y \in te + A - P\} \quad \text{for } y \in Y.$$

The function ξ_A is continuous and satisfies the following equality

$$\xi_A(y) = \inf_{a\in A}\{\xi_a(y)\}. \tag{1}$$

This infimum is attained when $A - P$ is closed, that is,

$$\xi_A(y) = \min_{a\in A}\{\xi_a(y)\}.$$

From definition of ξ_A we immediately obtain $\xi_A = \xi_{A-P}$.

The next lemma collects some basic results on convex cones.

Lemma 3.2. *Let $\emptyset \neq A \subseteq Y$ and P be as above. The following assertions hold.*

(a) $\operatorname{int}\operatorname{cl}(A+P) = \operatorname{int}(A+P) = A + \operatorname{int} P$, $\operatorname{cl}(A+P) = \operatorname{cl}(A+\operatorname{int} P)$,

$$\partial(A+P) = \partial(A+\operatorname{int} P).$$

(b) $\operatorname{cl}(\mathcal{C}(A)) = \mathcal{C}(\operatorname{int} A)$.

(c) $A + \operatorname{int} P = \operatorname{int} A \iff A + \operatorname{int} P \subseteq A \iff \mathcal{C}(-A) + \operatorname{int} P \subseteq \mathcal{C}(-A)$.

(d) *If* $A + \operatorname{int} P \subseteq A$ *then* $\operatorname{cl}(\operatorname{int} A) = \operatorname{cl} A = \operatorname{cl} A + P = \operatorname{cl} A + \operatorname{cl} P = \operatorname{cl}(A+\operatorname{int} P) = \operatorname{cl}(A+P)$ *and* $\operatorname{int}(\operatorname{cl} A) = \operatorname{int} A = \operatorname{int}(A+\operatorname{int} P)$. *Consequently,*

$$\partial A = \partial(A+\operatorname{int} P) = \partial(\operatorname{cl} A) = \partial(\operatorname{int} A).$$

Proof. (a): It can be found for instance, in [5, Lemma 2.5].
(b): It is clear that $\operatorname{cl}(\mathcal{C}(A)) \subseteq \mathcal{C}(\operatorname{int} A)$. Let $x \in \mathcal{C}(\operatorname{int} A)$. If $x \notin A$ then we conclude $x \in \operatorname{cl}(\mathcal{C}(A))$. If $x \in A$ and $x \notin \operatorname{int} A$ then $x \in \partial A$. By taking into account that $\partial A = \partial(\mathcal{C}(A))$ we finish the proof.
(c): The second equivalence is trivial; the remaining implication is a consequence of $A \subseteq A + P$ implies $\operatorname{int} A \subseteq \operatorname{int}(A+P) = A + \operatorname{int} P \subseteq \operatorname{int} A$.
(d): The first four equalities follow from (a) and

$$\operatorname{cl}(\operatorname{int} A) \subseteq \operatorname{cl} A \subseteq \operatorname{cl}(A) + P \subseteq \operatorname{cl} A + \operatorname{cl} P \subseteq \operatorname{cl}(A+P) = \operatorname{cl}(A+\operatorname{int} P) \subseteq \operatorname{cl}(\operatorname{int} A).$$

From the previous equality and (a), we get

$$\operatorname{int}(\operatorname{cl} A) = \operatorname{int}(\operatorname{cl}(A+P)) = \operatorname{int}(A+P) = A + \operatorname{int} P \subseteq \operatorname{int} A \subseteq \operatorname{int}(\operatorname{cl} A).$$

The last equalities are easy consequences from the previous equalities and (a). □

Lemma 3.3. [*25, Lemma* 2.16]. *Let* $\emptyset \neq A \subseteq Y$. *Then,*

$$A - P \neq Y \iff \xi_A(y) > -\infty \quad \forall\, y \in Y.$$

The next result was proved in [25, Lemma 2.17]. See [15, Lemma 4.4] for the general case ($r \in \mathbb{R} \cup \{-\infty\}$).

Lemma 3.4. *Let* $A \subseteq Y$, $r \in \mathbb{R}$ *and* $y \in Y$. *Then*

(a) $\xi_A(y) < r \Leftrightarrow y \in re + A - \operatorname{int} P$;

(b) $\xi_A(y) \leq r \Leftrightarrow y \in re + \operatorname{cl}(A - P)$;

(c) $\xi_A(y) = r \Leftrightarrow y \in re + \partial(A - P)$.

By using the previous lemmas, one deduces the following simple but important result.

Proposition 3.5. *Let* $\emptyset \neq A \subseteq Y$. *Then*

$$\xi_A(y) = \inf\{t \in \mathbb{R} \colon y \in te + A - \operatorname{int} P\} = \inf\{t \in \mathbb{R} \colon y \in te + A - \operatorname{cl} P\}.$$

Lemma 3.6. *Let* $\emptyset \neq A \subseteq Y$. *The following conditions hold.*

(a) $\xi_{A-P} = \xi_A = \xi_{\operatorname{cl} A}$. *Consequently,*

$$\xi_A = \xi_{A - \operatorname{int} P} = \xi_{A - \operatorname{cl} P} = \xi_{\operatorname{cl}(A-P)} = \xi_{A - P \setminus l(P)} = \xi_{A - P \setminus \{0\}}.$$

(b) *Let* $\emptyset \neq B \subseteq Y$. *Then*

$$\operatorname{cl}(B + P) = \operatorname{cl} P \implies \xi_A(y) = \xi_{A-B}(y) \quad \forall\, y \in Y.$$

Proof. (a): The first equality results from the definition. Obviously $\xi_{\operatorname{cl} A}(y) \leq \xi_A(y)$ for all $y \in Y$. Suppose that $\xi_{\operatorname{cl} A}(y) < \xi_A(y)$ and choose any $t \in \mathbb{R}$ satisfying $\xi_{\operatorname{cl} A}(y) < t < \xi_A(y)$. Then, there is $t_0 \in \mathbb{R}$ such that $t_0 < t$ and

$$y \in t_0 e + \operatorname{cl}(A) - P \subseteq t_0 e + \operatorname{cl}(A - P).$$

Thus, by Lemma 3.4, $\xi_A(y) \leq t_0 < t$, yielding a contradiction.
The last part follows from the equalities:

$$\operatorname{cl}(A-P) = \operatorname{cl}(A - \operatorname{cl} P)) = \operatorname{cl}(A - \operatorname{cl}(\operatorname{int} P)) = \operatorname{cl}(A - \operatorname{cl}(P \setminus l(P))) = \operatorname{cl}(A - \operatorname{cl}(P \setminus \{0\})),$$

because of $\operatorname{cl} P = \operatorname{cl}(P \setminus l(P)) = \operatorname{cl}(P \setminus \{0\}) = \operatorname{cl}(\operatorname{int} P)$.
(b): From (a) we obtain

$$\xi_A = \xi_{A-\operatorname{cl} P} = \xi_{A-\operatorname{cl}(B+P)} = \xi_{\operatorname{cl}(A-B-P)} = \xi_{A-B-P} = \xi_{A-B}.$$

□

The following lemma, being important by itself, will play a central rol in the scalarization procedure to be presented later on.

Lemma 3.7. *Suppose that A, $B \subseteq Y$, $0 \in \partial(B)$ such that $B + B \subseteq B$.*

(a) If y, $y' \in Y$ and $y - y' \in -B$, then $\xi_{A-B}(y) \leq \xi_{A-B}(y')$;

(b) Assume that $B + \operatorname{int} P \subseteq B$. If y, $y' \in Y$ and $y - y' \in -\operatorname{int} B$, then

$$\xi_{A-B}(y) < \xi_{A-B}(y').$$

Proof. By definition $\xi_{A-B}(y) = \inf\{t \in \mathbb{R} \colon y \in te + A - B - P\}$ for every $y \in Y$.
(a) Suppose that y, $y' \in Y$ and $y-y' \in -B$. Take any $t \in \mathbb{R}$ such that $y' \in te+A-B-P$, then

$$y \in -B + te + A - B - P \subseteq te + A - B - P$$

since $B + B \subseteq B$. Thus $\xi_{A-B}(y) \leq t$ and therefore $\xi_{A-B}(y) \leq \xi_{A-B}(y')$.
(b) Suppose that y, $y' \in Y$ and $y - y' \in -\operatorname{int} B$. Since $\operatorname{int} B = B + \operatorname{int} P$ (by Lemma 3.2) and $e \in \operatorname{int} P$, there exists $\varepsilon < 0$ such that $y - y' \in \varepsilon e - B - \operatorname{int} P$. Thus, if $t \in \mathbb{R}$ is such that $y' \in te + A - B - P$, then

$$y \in \varepsilon e - B - \operatorname{int} P + te + A - B - P \subseteq (\varepsilon + t)e + A - B - \operatorname{int} P$$

since $B + B \subseteq B$. It follows that $\xi_{A-B}(y) < \varepsilon + t$ and hence $\xi_{A-B}(y) < \xi_{A-B}(y')$. □

Since

$$\xi_A(y) = \xi_{A-\operatorname{int} P}(y) = \inf_{a \in A - \operatorname{int} P} \xi_a(y),$$

one can deduce immediately the following result (see also Lemma 3.6(a)).

Lemma 3.8. *The function $\xi_a \colon Y \longrightarrow \mathbb{R}$ is convex. Moreover, if $\operatorname{cl}(A - P)$ is convex (or equivalently $A - \operatorname{int} P$ is convex) if and only if $\xi_A \colon Y \longrightarrow \mathbb{R}$ is convex.*

4 The scalarization procedure for (approximate) efficiency

In this section we proceed to scalarize the problems

$$(\mathcal{P}) \qquad \text{find } \bar{x} \in K \quad f(x) - f(\bar{x}) \in S \quad \forall\, x \in K,\ x \neq \bar{x},$$

and

$$(\mathcal{P}(\varepsilon q)) \qquad \text{find } \bar{x} \in K \quad f(x) - f(\bar{x}) \in -\varepsilon q + S \quad \forall\, x \in K,\ x \neq \bar{x},$$

where $\varepsilon \geq 0$, $q \in \operatorname{int} P$, $\emptyset \neq K \subseteq X$ and $f : K \to Y$ by introducing families of scalar optimization problems which will describe the solution set to $(\mathcal{P})$ and $(\mathcal{P}(\varepsilon q))$, denoted by E_S and $E_S(\varepsilon q)$ respectively. This will be carried out through the scalarizing function discussed in the previous section. Obviously $E_S = E_S(0)$.

According to [32, Definition 3.1, pag. 95], given a family G of functions $g : Y \to \mathbb{R}$, we say that G is a complete scalarization for $(\mathcal{P})$ if for every $x \in E_S$ there exists $g \in G$ such that $x \in E(g \circ f, K)$, solution to $(\mathcal{SP})$ corresponding to g, and $E(g \circ f, K) \subseteq E_S$, where $E(g \circ f, K)$ denotes the solution set to $(\mathcal{SP})$:

$$(\mathcal{SP}) \qquad \min\{(g \circ f)(x) \colon x \in K\}.$$

In other words, G is a complete scalarization for $(\mathcal{P})$ if and only if there exists $G' \subseteq G$ such that

$$E_S = \bigcup_{g \in G'} E(g \circ f, K).$$

Similar representations will be established for $(\mathcal{P}(\varepsilon q))$.

Through this section we will impose the following basic assumption on P and S.

Assumption (A) : $P \subseteq Y$ is a (not necessarily closed or pointed) convex cone with nonempty interior, and $Y \neq S \subseteq Y$ satisfies $0 \in \partial S$ and $S + \operatorname{int} P \subseteq S$.

We recall that by Lemma 3.2(c) we have

$$S + \operatorname{int} P = \operatorname{int} S \iff S + \operatorname{int} P \subseteq S \iff \mathcal{C}(-S) + \operatorname{int} P \subseteq \mathcal{C}(-S).$$

Remark 4.1. *Assumption* **(A)** *holds for a wide class of (not necessarily closed) sets including those classical models:*

$$S = P,\ S = \mathcal{C}(-\operatorname{int} P),\ S = \mathcal{C}(-P) \cup l(P),\ S = \mathcal{C}(-P) \cup \{0\},\ S = \mathcal{C}(-P).$$

Notice that any set S satisfying $0 \in \partial S$ and $S + P = S$ fulfills Assumption **(A)***, but this equality is not verified by $S = \mathcal{C}(-P) \cup \{0\}$ when P is not pointed.*

The next two theorems, which are new in the literature, characterize when a point $\bar{x} \in K$ belongs to E_S (resp. $E_S(\varepsilon q)$) in terms of $E(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f, K)$ (resp. $E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon)$) under Assumption **(A)**.

Theorem 4.2. *Suppose that Assumption* **(A)** *holds. Let $\bar{x} \in K$, the following assertions are equivalent:*

(*a*) $\bar{x} \in E_S$;

(*b*) $\bar{x} \in E(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f, K)$ *and*

$$E(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f, K) \setminus \{\bar{x}\} = \{x \in K : \ x \neq \bar{x}, \ f(x) - f(\bar{x}) \in -(\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S))\}$$

$$= \{x \in K : \ x \neq \bar{x}, \ f(x) - f(\bar{x}) \in S \setminus \operatorname{int} S\}.$$

Proof. (*a*) $\Longrightarrow$ (*b*): It is clear that $(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$ since $0 \in \partial(\mathcal{C}(S))$. From $\bar{x} \in E_S$, we have $f(x) - f(\bar{x}) \notin -\mathcal{C}(-S)$ for all $x \in K$, $x \neq \bar{x}$. Then $f(x) - f(\bar{x}) \notin -(\mathcal{C}(-S) + \operatorname{int} P)$ by assumption. The latter implies $(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) \geq 0$ by Lemma 3.4, which turns out $\bar{x} \in E(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f, K)$. On the other hand, take any $x \in K$, $x \neq \bar{x}$, such that

$$(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) = (\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) = 0.$$

Then $f(x) - f(\bar{x}) \in \partial(-\mathcal{C}(-S) - \operatorname{int} P) = \partial(-\mathcal{C}(-S))$ by Lemma 3.2(*d*). We also have $f(x) - f(\bar{x}) \in S$. From both relations, we obtain $f(x) - f(\bar{x}) \in [\partial(-\mathcal{C}(-S))] \cap S$. By simplifying, we get

$$f(x) - f(\bar{x}) \in -(\mathrm{cl}(\mathcal{C}(-S)) \backslash \mathcal{C}(-S))$$

which proves one inclusion in (*b*).
For the other inclusion simply observe that if $x \in K \setminus \{\bar{x}\}$ is such that $f(x) - f(\bar{x}) \in -(\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S))$, then $f(x) - f(\bar{x}) \in -\partial(\mathcal{C}(-S))$. Hence,

$$(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) = 0.$$

Thus, $x \in E(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f, K)$.
The remaining equality follows from Lemma 3.2(*b*).
(*b*) $\Longrightarrow$ (*a*): Let $x \in K$, $x \neq \bar{x}$. We distinguish two cases. If x is such that

$$f(x) - f(\bar{x}) \in -(\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S)) \subseteq S,$$

we are done. If x is not in the set of the right hand side of (*b*), then $x \notin E(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f, K)$ by assumption. Thus $(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) > 0$ since $\bar{x} \in E(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f, K)$ and $(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$. Whence $f(x) - f(\bar{x}) \notin -\mathcal{C}(-S)$ (since $-\mathcal{C}(-S) \subseteq \mathrm{cl}(-\mathcal{C}(-S) - P)$), proving that $f(x) - f(\bar{x}) \in S$. Hence $\bar{x} \in E_S$. $\square$

Before continuing, some remarks are in order.

Remark 4.3. (i) *It may happen that the set of the right-hand side in* (b) *be empty* (*this occurs for instance when* P *is closed and* $S = \mathcal{C}(-P)$)*: in such a situation Theorem 4.2 reduces*

$$\bar{x} \in E_S \Longleftrightarrow (\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) > 0 \quad \forall\, x \in K,\ x \neq \bar{x}.$$

We will discuss related points later on.
(ii) *When* $0 \in S$ (*some models are described in Remark 4.1*), (b) *of the previous theorem admits the following formulation:*

$$E(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f, K) = \{x \in K : \ f(x) - f(\bar{x}) \in S \setminus \operatorname{int} S\}.$$

Now, we establish a similar characterization for the problem $(\mathcal{P}(\varepsilon q))$. Notice that it also provides another characterization for $\varepsilon = 0$. Theorem 4.4 not only unifies Theorems 4.5, 5.1(a) in [22] and extends them to more general situations, but also provides sharper results.

Theorem 4.4. *Suppose that Assumption* **(A)** *holds. Let us consider problem* $(\mathcal{P}(\varepsilon q))$ *with* $\varepsilon \geq 0$, *and* $\bar{x} \in K$. *The following assertions are equivalent:*

(a) $\bar{x} \in E_S(\varepsilon q)$;

(b) $\bar{x} \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon)$ *and*

$$E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon) \setminus \{\bar{x}\} \subseteq \{x \in K : \ x \neq \bar{x},\ f(x) - f(\bar{x}) \in (-\varepsilon q + S) \cap (\varepsilon q - \operatorname{cl}(\mathcal{C}(-S)))\}.$$

Proof. $(a) \Longrightarrow (b)$: Obviously $(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$ since $0 \in \partial S$. From $\bar{x} \in E_S(\varepsilon q)$, we have $f(x) - f(\bar{x}) \notin -\varepsilon q - \mathcal{C}(-S)$ for all $x \in K$, $x \neq \bar{x}$. By Lemma 3.4, $(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) \geq -\varepsilon$, which turns out $(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) - (\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) \geq -\varepsilon$. Thus $\bar{x} \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon)$.
Let us prove the inclusion in (b). If $x' \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon)$, $x' \neq \bar{x}$, then

$$(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) - (\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x') \geq -\varepsilon \quad \forall\, x \in K.$$

Since $(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$ we have $(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x') \leq \varepsilon$. Therefore, $f(x') \in f(\bar{x}) + \varepsilon q - \operatorname{cl}(\mathcal{C}(-S))$ by Lemma 3.2(d). On the other hand, by hypothesis, we have $f(x') - f(\bar{x}) \in -\varepsilon q + S$. Thus, $f(x') - f(\bar{x}) \in (-\varepsilon q + S) \cap (\varepsilon q - \operatorname{cl}(\mathcal{C}(-S)))$.
$(b) \Longrightarrow (a)$: Let $\bar{x} \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon)$. Then,

$$(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) - (\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) \geq -\varepsilon \quad \forall\, x \in K.$$

Since $\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$ we have $(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) \geq -\varepsilon$ for all $x \in K$. If there exists $x' \in K$, $x' \neq \bar{x}$, such that $f(x') - f(\bar{x}) \in -\varepsilon q - \mathcal{C}(-S)$, then $(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x') \leq -\varepsilon$. From the above inequality we obtain

$$(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x') = -\varepsilon.$$

Thus, $x' \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon) \setminus \{\bar{x}\}$, which implies by (b) that $f(x') - f(\bar{x}) \in -\varepsilon q + S$, contradicting a previous relation. Hence $\bar{x} \in E(\varepsilon q)$. $\square$

The next example shows the inclusion in Theorem 4.4(b) for $\varepsilon > 0$ may be strict.

Example 4.5. *Take $K = [-\frac{5}{2}, 2]$ and $f : K \to \mathbb{R}^2$, $f(x) = (x, x+2)$ if $-\frac{5}{2} \leq x < 0$ and $f(x) = (x, 0)$ if $0 \leq x \leq 2$. Let $S = \mathcal{C}(-\operatorname{int}\mathbb{R}^2_+)$, $q = (\frac{1}{2}, \frac{1}{2})$ and $\varepsilon = 2$. It is clear that $0 \in E_S(\varepsilon q)$, in addition,*

$$-1, -\frac{6}{5} \in \{x \in K : x \neq 0,\ f(x) - f(0) \in (-\varepsilon q + S) \cap (\varepsilon q - \operatorname{cl}(\mathcal{C}(-S)))\} =$$

$$\{x \in K : x \neq 0,\ f(x) \in ((-1,-1) + S) \cap ((1,1) - \operatorname{cl}(\mathcal{C}(-S)))\}.$$

However

$$-1, -\frac{6}{5} \notin E(\xi_{q,f(0)-\mathcal{C}(-S)} \circ f, K, \varepsilon) = E(\xi_{q,-\mathcal{C}(-S)} \circ f, K, 2)$$

since

$$(\xi_{q,-\mathcal{C}(-S)} \circ f)(-\frac{5}{2}) - (\xi_{q,-\mathcal{C}(-S)} \circ f)(-1) \ngeq -2$$

and

$$(\xi_{q,-\mathcal{C}(-S)} \circ f)(-\frac{5}{2}) - (\xi_{q,-\mathcal{C}(-S)} \circ f)(-\frac{6}{5}) \ngeq -2$$

taking into account that $(\xi_{q,-\mathcal{C}(-S)} \circ f)(-\frac{5}{2}) = -1$, $(\xi_{q,-\mathcal{C}(-S)} \circ f)(-1) = 2$ and $1 < (\xi_{q,-\mathcal{C}(-S)} \circ f)(-\frac{6}{5}) < 2$.

Next theorem extends also Theorem 5.1(b) in [22], where pointedness is imposed.

A simpler equivalence than those in Theorems 4.2 and 4.4 can be obtained under an additional assumption on S.

Theorem 4.6. *Consider problem $(\mathcal{P}(\varepsilon q))$, $\varepsilon \geq 0$, and suppose that Assumption* **(A)** *holds. Let $\bar{x} \in K$. Then,*

$$\bar{x} \in E_S(\varepsilon q) \Longrightarrow \bar{x} \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon) \Longrightarrow \bar{x} \in E_S(\delta q)\ \forall\, \delta > \varepsilon \Longrightarrow \bar{x} \in E_{\operatorname{cl} S}(\varepsilon q).$$

Consequently if, in addition, S is closed then
$\bar{x} \in E_S(\varepsilon q) \Longleftrightarrow \bar{x} \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon)$; *and* $E_S(\varepsilon q) = \bigcap_{\delta > \varepsilon} E_S(\delta q)$.

Proof. The first implication is in Theorem 4.4.
For the second we proceed as follows. If on the contrary $\bar{x} \notin E_S(\delta q)$, then $f(x) - f(\bar{x}) \notin -\delta q + S$ for some $x \in K$, $x \neq \bar{x}$. Then, $f(x) - f(\bar{x}) \in -\delta q - \mathcal{C}(-S) \subseteq -\delta q - \mathrm{cl}(\mathcal{C}(-S) + P)$. Thus, $(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) \leq -\delta$. By assumption,

$$(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x') - (\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) \geq -\varepsilon \quad \forall\, x' \in K.$$

Hence, if $\delta > \varepsilon$ then $-\varepsilon \leq (\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) \leq -\delta < -\varepsilon$, a contradiction.
The third implication is obtained by taking the limit as δ goes to ε. □

The second part of the previous theorem can be applied when P is any (not necessarily closed or pointed) convex cone and $S = \mathcal{C}(-\operatorname{int} P)$; or when P is a closed halfspace, to $S = P$, and when $P = Q \cup \{0\}$ with Q being open and convex satisfying $tQ \subseteq Q$ for all $t > 0$, to $S = \mathcal{C}(-P \setminus \{0\}) = \mathcal{C}(-P) \cup \{0\}$. This last particular case extends Theorems 4.5, 5.1 and 5.2 in [22].

The examples below show that under the assumptions given in Theorem 4.6 the implication $\bar{x} \in E_S(\delta q) \;\; \forall\, \delta > \varepsilon \Longrightarrow \bar{x} \in E_S(\varepsilon q)$ may be false when S is not closed.

Example 4.7. *Here consider $S = \mathcal{C}(-P) \cup \{(0,0)\}$ where $P = \{(x,y) \in \mathbb{R}^2 \colon x > 0,\, y \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 \colon x \geq 0,\, y \leq 0\}$. Let f be a function from $K = \mathbb{R}$ to $Y = \mathbb{R}^2$ defined by*

$$f(x) = \begin{cases} (0,-x) & if \quad x < 0 \\ (0,1) & if \quad x = 0 \\ (x,x) & if \quad 0 < x < 1 \\ (x, 2x-1) & if \quad x \geq 1, \end{cases}$$

and take $q = (1,1)$, $\varepsilon = 1$. Then, it is easy to check that $1 = \bar{x} \notin E_S(\varepsilon q)$ since $f(0) - f(\bar{x}) \notin -\varepsilon q + S$. However

$$(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) - (\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) \geq -\varepsilon \quad \forall x \in \mathbb{R},$$

that is, $\bar{x} \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon)$, and therefore $1 = \bar{x} \in E_S(\delta q)$ $\forall\, \delta > \varepsilon = 1$. Note that $E_S(0) = E_S = \emptyset$.

Example 4.8. *Consider $S = P$ with $P = \{(x,y) \in \mathbb{R}^2 \colon x > 0,\, y \geq 0\} \cup \{(0,0)\}$. Let f, q and ε be as in the previous example. Then, we see that $1 = \bar{x} \notin E_S(\varepsilon q)$ since $f(0) - f(\bar{x}) \notin -q + P$. However, we can also check that $\bar{x} \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon)$ and so $1 = \bar{x} \in E_S(\delta q)$ $\forall\, \delta > \varepsilon = 1$. Note that $E_S(0) = E_S = \emptyset$.*

We can easily obtain the following characterizations from Theorem 4.2 and Remark 4.3.

Corollary 4.9. *Let* $\bar{x} \in K$. *Then,*

(*a*) $\bar{x} \in E_P \iff E(\xi_{f(\bar{x})-\mathcal{C}(-P)} \circ f, K) = \{x \in K : f(x) - f(\bar{x}) \in -(\mathrm{cl}(\mathcal{C}(-P)) \setminus \mathcal{C}(-P))\} = \{x \in K : f(x) - f(\bar{x}) \in P \setminus \mathrm{int}\, P\}$;

(*b*) $\bar{x} \in E_W \iff E(\xi_{f(\bar{x})} \circ f, K) = \{x \in K : f(x) - f(\bar{x}) \in -\partial P\}$;

(*c*) $\bar{x} \in E \iff E(\xi_{f(\bar{x})} \circ f, K) = \{x \in K : f(x) - f(\bar{x}) \in -(\mathrm{cl}\, P \setminus P) \cup l(P)\}$;

(*d*) $\bar{x} \in E_{W1} \iff E(\xi_{f(\bar{x})} \circ f, K) = \{x \in K : f(x) - f(\bar{x}) \in -(\mathrm{cl}\, P \setminus P) \cup \{0\}\}$.

(*e*) $\bar{x} \in E_1 \iff E(\xi_{f(\bar{x})} \circ f, K) \setminus \{\bar{x}\} = \{x \in K : f(x) - f(\bar{x}) \in -(\mathrm{cl}\, P \setminus P)\}$.

When P is closed and pointed, Part (*c*) was earlier proved in [25, Corollary 4.9].

In order to obtain complete scalarizations for E_S, we need the next theorem.

Theorem 4.10. *Suppose that Assumption* **(A)** *holds.*

(*a*) *If* $\emptyset \neq A \subseteq E_S$, *then* $A \subseteq E(\xi_{f(A)-\mathcal{C}(-S)} \circ f, K) \subseteq E(\xi_{f(E_S)-\mathcal{C}(-S)} \circ f, K)$ and

$$\min\{(\xi_{f(A)-\mathcal{C}(-S)} \circ f)(x) : \; x \in K\} = 0.$$

(*b*) *If* $0 \in S$ *and* $S + [\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S)] \subseteq S$ *then,*

$$\bar{x} \in E_S \iff \bar{x} \in E(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f, K) \subseteq E_S.$$

Proof. (*a*): Since for each $\bar{x} \in A$, $f(x) \in f(\bar{x}) + S$ for all $x \in K \setminus \{\bar{x}\}$ is equivalent to $f(x) \notin f(\bar{x}) - \mathcal{C}(-S)$ for all $x \in K \setminus \{\bar{x}\}$. Then, taking into account that $\mathcal{C}(-S) + \mathrm{int}\, P \subseteq \mathcal{C}(-S)$, we have $f(x) \notin f(\bar{x}) - \mathcal{C}(-S) - \mathrm{int}\, P$ for all $x \in K$, $x \neq \bar{x}$. Therefore, by Lemma 3.4, $(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) \geq 0$ for all $x \in K$. Thus, $(\xi_{f(A)-\mathcal{C}(-S)} \circ f)(x) \geq 0$ for all $x \in K$. Since $(\xi_{f(A)-\mathcal{C}(-S)} \circ f)(\bar{x}) \leq (\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$, we get

$$\bar{x} \in E(\xi_{f(A)-\mathcal{C}(-S)} \circ f, K) \quad \text{and} \quad \min\{(\xi_{f(A)-\mathcal{C}(-S)} \circ f)(x) : \; x \in K\} = 0.$$

The same reasoning also proves

$$\min\{(\xi_{f(E_S)-\mathcal{C}(-S)} \circ f)(x) : \; x \in K\} = 0.$$

(*b*): Let $\bar{x} \in E_S$ and $x' \in E(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f, K)$ with $x' \neq \bar{x}$. By Theorem 4.2, $f(\bar{x}) - f(x') \in \mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S)$. Hence, for every $x \in K$ with $x \neq x'$,

$$f(x) - f(x') = f(x) - f(\bar{x}) + f(\bar{x}) - f(x') \in S + [\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S)] \subseteq S,$$

and so $x' \in E_S$ since $0 \in S$.
The sufficient condition is immediate. □

Remark 4.11. *Taking into account Remark 4.1, we point out that Theorem 4.10(a) applies when P is any (not necessarily closed or pointed) convex cone, to $S = P$; $S = \mathcal{C}(-\operatorname{int} P)$; $\mathcal{C}(-P) \cup l(P)$; $\mathcal{C}(-P) \cup \{0\}$; $\mathcal{C}(-P)$; whereas (b) applies when P is any (not necessarily pointed) closed convex cone to $S = \mathcal{C}(-\operatorname{int} P)$; $\mathcal{C}(-P) \cup l(P)$; $\mathcal{C}(-P) \cup \{0\}$. Notice that $0 \in S \cap \partial S$ implies that $\operatorname{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S) \neq \emptyset$.*

Theorem 4.12. *Suppose that Assumption* **(A)** *holds and consider $(\mathcal{P}(\varepsilon q))$, $\varepsilon \geq 0$. Assume that $\mathcal{C}(-S) + \mathcal{C}(-S) \subseteq \mathcal{C}(-S)$ and S is closed. If $\emptyset \neq A \subseteq K$ then*

$$E(\xi_{q,f(A)-\mathcal{C}(-S)} \circ f, K, \varepsilon) \subseteq E_S(\varepsilon q).$$

Proof. Let $\bar{x} \in E(\xi_{q,f(A)-\mathcal{C}(-S)} \circ f, K, \varepsilon)$ and $\bar{x} \notin E_S(\varepsilon q)$. Then, there exists $x \in K$, $x \neq \bar{x}$, such that $f(x) - f(\bar{x}) \notin -\varepsilon q + S$ or equivalently $f(x) + \varepsilon q - f(\bar{x}) \in -\mathcal{C}(-S)$. By the closedness of S, Lemma 3.7(b) implies that

$$(\xi_{q,f(A)-\mathcal{C}(-S)})(f(x)) + \varepsilon = (\xi_{q,f(A)-\mathcal{C}(-S)})(f(x) + \varepsilon q) < (\xi_{q,f(A)-\mathcal{C}(-S)})(f(\bar{x})).$$

It follows that $\bar{x} \notin E(\xi_{q,f(A)-\mathcal{C}(-S)} \circ f, K, \varepsilon)$, which cannot happen. □

Remark 4.13. *When P is any (not necessarily closed or pointed) convex cone, the previous theorem can be applied to $S = \mathcal{C}(-\operatorname{int} P)$, and to $S = P$ provided P is a closed halfspace. In addition, it also applies when $S = \mathcal{C}(-P \setminus \{0\}) = \mathcal{C}(-P) \cup \{0\}$ where $P = Q \cup \{0\}$ is pointed with Q being open and convex set satisfying $tQ \subseteq Q$ for all $t > 0$.*

Next result, whose proof follows from Theorem 4.6 and Corollary 4.9, provides some characterizations for a point to be in E_S when $S = \mathcal{C}(-\operatorname{int} P)$, $S = P$, $S = \mathcal{C}(-P) \cup \{0\}$ or $S = \mathcal{C}(-P)$. In particular, we recover Corollary 5.5 in [22]. Moreover, we find a scalar minimization problem providing elements in E_W.

Corollary 4.14. *Assume that P is a convex cone with nonempty interior. The following assertions hold.*

(a) *Let $\varepsilon \geq 0$. Then,*

($a1$) $\bar{x} \in E_W(\varepsilon q) \iff \bar{x} \in E(\xi_{q,f(\bar{x})} \circ f, K, \varepsilon)$; *and*

$$E_W(\varepsilon q) = \bigcap_{\delta > \varepsilon} E(\delta q) = \bigcap_{\delta > \varepsilon} E_W(\delta q).$$

($a2$) $E(\xi_{q,f(K)} \circ f, K, \varepsilon) \subseteq E_W(\varepsilon q)$.

(b) *if, in addition, P is closed, then*

$(b1)$ $\bar{x} \in E \iff [x \in K,\ (\xi_{f(x)} \circ f)(\bar{x}) > 0 \implies (\xi_{f(\bar{x})} \circ f)(x) > 0]$;

$(b2)$ $\bar{x} \in E_{w1} \iff (\xi_{f(\bar{x})} \circ f)(x) > 0\ \ \forall\, x \in K$ *such that* $f(x) \neq f(\bar{x})$;

$(b3)$ $\bar{x} \in E_1 \iff (\xi_{f(\bar{x})} \circ f)(x) > 0\ \ \forall\, x \in K,\ x \neq \bar{x}$;

$(b4)$ $\bar{x} \in E_P \iff \bar{x} \in E(\xi_{f(\bar{x})-\mathcal{C}(-P)} \circ f, K)$.

Proof. $(a1)$ follows from Theorem 4.6 and $(a2)$ results by particularizing $S = \mathcal{C}(-\operatorname{int} P)$ in Theorem 4.12.
$(b1)$ is a consequence of the following equivalence:

$$\bar{x} \in E \iff [x \in K,\ f(x) - f(\bar{x}) \in -P \implies f(\bar{x}) - f(x) \in -P],$$

and the closedness of P, along with Lemma 3.4; $(b2)$ results from (d) of Corollary 4.9; $(b3)$ is Remark 4.3(i).
In order to prove $(b4)$, we write

$$\begin{aligned} \bar{x} \in E_P &\iff f(x) - f(\bar{x}) \in P\ \ \forall\, x \in K \iff f(x) - f(\bar{x}) \notin -\mathcal{C}(-P)\ \ \forall\, x \in K \\ &\iff f(x) \notin f(\bar{x}) - \mathcal{C}(-P) - P = f(\bar{x}) - \mathcal{C}(-P) - \operatorname{int} P,\ \ \forall\, x \in K \end{aligned}$$

since P is closed. We now use Lemma 3.4 to conclude with the desired result. □

The next example shows that the closedness of P is necessary in $(b1)$, $(b2)$, $(b3)$ and $(b4)$.

Example 4.15. *Let f be a function from $\mathbb{R}$ to $\mathbb{R}^2$ defined by*

$$f(x) = \begin{cases} (-1, -x-1) & if \quad x \leq -1 \\ (x, 0) & if \quad x \in (-1, 0) \\ (x, x) & if \quad x \geq 0 \end{cases}$$

Let $P = \{(x, y) \in \mathbb{R}^2\colon x,\ y > 0\} \cup \{(0, 0)\}$ and $e = (1, 1)$. It is clear that $E_P = \emptyset$ and $E = E_1 = E_{W1} = (-\infty, 0]$. However $(b4)$ is false because $-1 \in E(\xi_{f(-1)-\mathcal{C}(-P)} \circ f, K)$ since $(\xi_{f(-1)-\mathcal{C}(-P)} \circ f)(x) = 0$ if $x \leq 0$ and $(\xi_{f(-1)-\mathcal{C}(-P)} \circ f)(x) > 0$ if $x > 0$. In addition, $(b1)$, $(b2)$ and $(b3)$ do not hold since $(\xi_{f(0)} \circ f)(-1) = 0$, $(\xi_{f(-1)} \circ f)(0) > 0$ and $f(0) \neq f(-1)$.

¿From Corollary 4.14 we deduce Corollary 4.8(a) in [15].

We are ready to state our main result of complete scalarization for $(\mathcal{P})$ which is a consequence of Theorems 4.10 and 4.12. This result encompasses our classical models described at the introduction.

Theorem 4.16. *Suppose that Assumption* **(A)** *holds. Assume that $E_S \neq \emptyset$.*

(a) *If* $0 \in S$ *and* $S + [\mathrm{cl}(\mathcal{C}(-S)) \setminus \mathcal{C}(-S)] \subseteq S$, *then*

$$E_S = \bigcup_{x \in E_S} E(\xi_{f(x)-\mathcal{C}(-S)} \circ f, K) \subseteq E(\xi_{f(E_S)-\mathcal{C}(-S)} \circ f, K).$$

(b) *If* S *is closed and* $\mathcal{C}(-S) + \mathcal{C}(-S) \subseteq \mathcal{C}(-S)$, *then*

$$E_S = E(\xi_{f(E_S)-\mathcal{C}(-S)} \circ f, K) = \bigcup_{x \in E_S} E(\xi_{f(x)-\mathcal{C}(-S)} \circ f, K) = \bigcup_{x \in K} E(\xi_{f(x)-\mathcal{C}(-S)} \circ f, K),$$

$$E_S(\varepsilon q) = \bigcup_{x \in K} E(\xi_{q,f(x)-\mathcal{C}(-S)} \circ f, K, \varepsilon) \quad \forall\, \varepsilon > 0.$$

Now, by particularizing the previous result to our classical models, we obtain complete scalarization for E_W, E, E_1 and E_{w1}. The first part in (a) of the next result was established in the proof of [32, Theorem 3.4, pag. 96]; whereas the second part was proved in [22, Theorem 5.11] for P pointed.

Corollary 4.17. *Let* $P \subseteq Y$ *be a (not necessarily pointed) convex cone with* $\mathrm{int}\, P \neq \emptyset$.

(a) *If* $E_W \neq \emptyset$, *then*

$$E_W = E(\xi_{f(E_W)} \circ f, K) = \bigcup_{x \in E_W} E(\xi_{f(x)} \circ f, K) = \bigcup_{x \in K} E(\xi_{f(x)} \circ f, K),$$

$$E_W(\varepsilon q) = \bigcup_{x \in K} E(\xi_{q,f(x)} \circ f, K, \varepsilon) \quad \forall\, \varepsilon > 0;$$

(b) *If* P *is closed and* $E \neq \emptyset$ *then*

$$E = \bigcup_{x \in E} E(\xi_{f(x)} \circ f, K) \subseteq E(\xi_{f(E)} \circ f, K);$$

(c) *If* P *is closed and* $E_{W1} \neq \emptyset$ *then*

$$E_{W1} = \bigcup_{x \in E_{W1}} E(\xi_{f(x)} \circ f, K) \subseteq E(\xi_{f(E_{W1})} \circ f, K);$$

(d) *If* P *is closed and* $E_1 \neq \emptyset$ *then*

$$E_1 = \bigcup_{x \in E_1} E(\xi_{f(x)} \circ f, K) \subseteq E(\xi_{f(E_1)} \circ f, K);$$

Proof. By taking into account Remark 4.11, the corollary is a consequence of the previous theorem. Notice the equality of (c) may be also obtained from Corollary 4.9(d) since $\bar{x} \in E_1$ if and only if $E(\xi_{f(\bar{x})} \circ f, K) = \{x \in K : f(x) = f(\bar{x})\}$. Part ($d$) trivially holds by Remark 4.3(i) since $\bar{x} \in E_1$ if and only if $E(\xi_{f(\bar{x})} \circ f, K) = \{\bar{x}\}$. □

We cannot expect an equality in Corollary 4.17 for E_P even when P is closed. Indeed, take $P = \mathbb{R}^2_+$, $e = (1,1)$ and $f\colon \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by $f(x) = (-1, -x-1)$ if $x \leq -1$, $f(x) = (x, 0)$ if $-1 < x < 0$ and $f(x) = (x, x)$ if $x \geq 0$. We have $E_P = \{-1\}$ and $E(\xi_{f(-1)-\mathcal{C}(-P)} \circ f, K) = \,]\infty, 0]$.

According to Corollaries 4.16 and 4.17 under certain assumptions on S or P the existence of complete scalarizations for problem $(\mathcal{P})$ or $(\mathcal{P}(\varepsilon q))$ is guaranteed without any convexity assumption.

Now the following question arises: what are the conditions on f implying some kind of convexity or continuity of $\xi_{f(\bar{x})} \circ f$? Partial answers will be given in Sections 5 and 6.

5 Lower semicontinuity of $\xi_{a-\mathcal{C}(-S)} \circ f$

This section is devoted to establish conditions on f under which the lower semicontinuity of $\xi_{q,a-\mathcal{C}(-S)} \circ f$ is obtained for any $a \in Y$ and $q \in \operatorname{int} P$. To that end, we recall the following definition.

Definition 5.1. ([*32*]) *$f : K \to Y$ is P-lower semicontinuous (P-lsc) at $x_0 \in K$ if for any open set $V \subseteq Y$ such that $f(x_0) \in V$ there exists an open neighborhood $U \subseteq X$ of x_0 such that $f(U \cap K) \subseteq V + P$. We shall say that f is P-lsc (on K) if it is at every $x_0 \in K$.*

We point out that $f = (f_1, \ldots, f_m)$ is $\mathbb{R}^m_+$-lsc if and only if each f_i is lsc.

Concerning this definition the following proposition, whose first part is Lemma 2.4 in [11] and second one is taken from [12, Lemma 2.7] and [2] (note that proof of both results does not requiere the closedness of P), takes place.

Proposition 5.2. *Let $K \subseteq X$ be closed and let P be a convex cone with $\operatorname{int} P \neq \emptyset$. The following assertions hold:*

- (a) *if $A \subseteq Y$ is closed such that $A + P \subseteq A$ and f is P-lsc then $\{x \in K : \ f(x) - y \in -A\}$ is closed for all $y \in Y$;*
- (b) *f is P-lsc if and only if $\{x \in K : \ f(x) - y \notin \operatorname{int} P\}$ is closed for all $y \in Y$.*

If $a \in Y$ and $e \in \operatorname{int} P$, by taking into account Lemma 3.4, we write

$$\begin{aligned}\{x \in K : \ \xi_{a-\mathcal{C}(-S)} \circ f(x) \leq t\} &= \{x \in K : \ f(x) - a \in te - \operatorname{cl}(\mathcal{C}(-S) - P)\} \\ &= \{x \in K : \ f(x) - a \in te - \operatorname{cl}(\mathcal{C}(-S))\}\end{aligned}$$

provided $S + \operatorname{int} P \subseteq S$ because of Lemma 3.2(d). Consequently the previous results allow us to obtain the next lemma.

Lemma 5.3. *Let $a \in Y$, $Y \neq S \subseteq Y$ be such that $S + \operatorname{int} P \subseteq S$. If K is closed and $f\colon K \to Y$ is P-lsc then $\xi_{a-\mathcal{C}(-S)} \circ f : K \to \mathbb{R}$ is lsc.*

6 Convexity and (semistrict) quasiconvexity of $\xi_{a-\mathcal{C}(-S)} \circ f$

Motivated by Section 3, we now provide conditions on f implying the convexity, pointwise quasiconvexity and quasiconvexity of $\xi_{q,a-\mathcal{C}(-S)} \circ f$ for any $a \in Y$ and $q \in \operatorname{int} P$. To that purpose, we start by recalling some notions of quasiconvexity for vector functions, see [28] for more details.

Definition 6.1. *Let $\emptyset \neq S \subseteq Y$, $\emptyset \neq K \subseteq X$ convex, P be convex cone, and let $f : K \to Y$. We say that f is*

(a) *[13, 16] semistrictly (S)-quasiconvex at $\bar{x} \in K$ if*

$$x \in K,\, x \neq \bar{x},\, f(x) - f(\bar{x}) \in -S \quad \Longrightarrow \quad f(\xi) - f(\bar{x}) \in -S \quad \forall\, \xi \in\ ]x, \bar{x}[,$$

we say that f is semistrictly (S)-quasiconvex $(on\ K)$ if it is at every $x \in K$;

(b) *(S)-quasiconvex on K if,*

$$x_1,\, x_2 \in K,\ f(x_1),\, f(x_2) \in y - S \quad \Longrightarrow \quad f(\xi) \in y - S \quad \forall\, \xi \in [x_1, x_2],$$

(c) *$(\operatorname{int} S \neq \emptyset)$ semistrongly (S)-quasiconvex at $\bar{x} \in K$ if*

$$x \in K,\ x \neq \bar{x},\, f(x) - f(\bar{x}) \in -S \quad \Longrightarrow \quad f(\xi) - f(\bar{x}) \in -\operatorname{int} S \quad \forall\, \xi \in\ ]x, \bar{x}[,$$

we say that f is semistrongly (S)-quasiconvex $(on\ K)$ if it is at every $x \in K$.

(d) *P-convex if,*

$$x_1,\, x_2 \in K,\ f(tx_1 + (1-t)x_2) \in tf(x_1) + (1-t)f(x_2) - P \quad \forall\, t \in\ ]0, 1[.$$

Remark 6.2. *When $S = P$ is a convex cone, Definition 6.1(a) $(on\ K)$ is considered in [28, Chapter 7]. Likewise, Definition 6.1(b) with $S = P$ is the classical notion of P-quasiconvexity discussed by Luc [32] and Ferro [9].*

Remark 6.3. *For scalar functions, that is, when $Y = \mathbb{R}$ and $S = \mathbb{R}_+ \doteq [0, +\infty[$, the semistrict $(\mathbb{R}_+)$-quasiconvexity $(on\ K)$ and $(\mathbb{R}_+)$-quasiconvexity reduce to quasiconvexity in the usual sense; however, semistrict $(\mathbb{R}_+)$-quasiconvexity at $\bar{x}$ is known as quasiconvexity at $\bar{x}$ which means:*

$$f(x) \leq f(\bar{x}) \Longrightarrow f(\xi) \leq f(\bar{x}) \quad \forall\, \xi \in\ [x, \bar{x}].$$

When $S = \mathbb{R}_{++} \doteq\ ]0, +\infty[$, semistrict $(\mathbb{R}_{++})$-quasiconvexity coincides with the standar definition of semistrict quasiconvexity well-known in mathematical programming.

Note that (P)-quasiconvexity implies semistrict $(\mathcal{C}(-\operatorname{int} P))$-quasiconvexity and the converse is, in general, false. The reader can find examples and relationships between several concepts of quasiconvex vector functions in [32, 13, 16, 17].

We point out that several generalizations of the quasiconvexity notion have been considered to give optimality conditions. For instance, in [29] is presented a weaker notion than Definition 6.1(a) (where $S = P$) to give optimality conditions in terms of multiplier rules.

On the other hand, some existence results for problem $(\mathcal{P})$ are established in [17] under semistrict (S)-quasiconvexity: among other results, it was used to characterize the nonemptiness and boundedness of the solution set to vector optimization problems on the real-line. The class of vector functions that are semistrictly (S) and $(\mathcal{C}(-S))$-quasiconvex introduced in [13] and called explicitly (S)-quasiconvex was employed in [13, 16] to obtain characterizations of the nonemptiness of the (possibly unbounded) solution set.

We must emphasize that the semistrict quasiconvexity is associated with problem $(\mathcal{P})$ in a natural way as the following results show.

Proposition 6.4. *If $x \in E_S$, then f is semistrictly $(\mathcal{C}(-S))$-quasiconvex and semistrictly $(-S)$-quasiconvex at x;*

Moreover we have the following result which extends the well-known real case, which was stated without proof in [16, Theorem 4.1]. It establishes that semistrict $(\mathcal{C}(-S))$-quasiconvexity characterizes the local-global property which is very interesting in numerical computing to reduce the algorithmic cost. See also the characterizations of the local-global property presented in [29, Theorems 3.8, 3.9] and [28, Theorem 7.15] when $S = \mathcal{C}(-\operatorname{int} P)$ or $S = \mathcal{C}(-P) \cup l(P)$.

Proposition 6.5. *Let $\bar{x} \in K$ be a local solution of $(\mathcal{P})$, $\bar{x} \in E_S$ if and only if f is semistrictly $(\mathcal{C}(-S))$-quasiconvex at $\bar{x}$.*

Proof. Suppose that $\bar{x}$ is a local solution. Then there exists an open neighborhood U of $\bar{x}$ such that $f(x) - f(\bar{x}) \in S \quad \forall x \in U \cap K,\ x \neq \bar{x}$, or equivalently,

$$f(\bar{x}) - f(x) \notin \mathcal{C}(-S) \quad \forall\, x \in U \cap K,\ x \neq \bar{x}. \tag{2}$$

Let $x_0 \in K$, $x_0 \neq \bar{x}$, be such that $f(x_0) - f(\bar{x}) \notin S$, that is, $f(\bar{x}) - f(x_0) \in \mathcal{C}(-S)$. Since f is semistrictly $(\mathcal{C}(-S))$-quasiconvex at $\bar{x}$ we have $f(\bar{x}) - f(\xi) \in \mathcal{C}(-S)$ for all $\xi \in\ ]x_0, \bar{x}[$ which contradicts (2). Thus, $f(x) - f(\bar{x}) \in S$ for all $x \in K$, $x \neq \bar{x}$.
The necessary condition follows from Proposition 6.4. □

The next result assures the uniqueness of the solution set to the scalar problem.

Proposition 6.6. *Let $\bar{x} \in K$ and let P be closed. Suppose that $\bar{x} \in E(\xi_{f(\bar{x})} \circ f, K)$ and f is semistrongly (P)-quasiconvex at $\bar{x}$. Then $E(\xi_{f(\bar{x})} \circ f, K) = \{\bar{x}\}$, i.e., $\bar{x} \in E_1$.*

Proof. It follows from Definition 6.1 and Corollary 4.14($b3$). □

We recall that $\xi_a \colon Y \longrightarrow \mathbb{R}$ is convex for any $a \in Y$ (see Lemma 3.8).

In the next subsections we give characterizations of (S)-quasiconvexity and semistrict (S)-quasiconvexity in terms of quasiconvex scalar functions. Firstly we observe that the composition of a convex increasing function g and a (P)-quasiconvex function f, $g \circ f$, in general, is not (P)-quasiconvex as is shown for instance in [32, Remark 6.9, pag. 32]. However, the function ξ_A has good behavior with respect to the composition.

6.1 Convexity of $\xi_{a-\mathcal{C}(-S)} \circ f$

The convexity of $\xi_{a-\mathcal{C}(-S)} \circ f$ is obtained under the P-convexity of f.

Proposition 6.7. *Let $a \in Y$, K be convex and $f : K \to Y$ be P-convex. If* $\mathrm{cl}(\mathcal{C}(-S) + P)$ (*or equivalently* $\mathcal{C}(-S) + \mathrm{int}\, P$ *is convex*), *then*

$$\xi_{a-\mathcal{C}(-S)} \circ f : K \to \mathbb{R} \ \text{ is convex.}$$

Proof. Since for all $x_1, x_2 \in K$, $f(tx_1 + (1-t)x_2) - tf(x_1) - (1-t)f(x_2) \in -P$ for all $t \in \ ]0, 1[$, we apply Lemma 3.7(a) with $B = P$ to obtain

$$\xi_{a-\mathcal{C}(-S)}(f(tx_1 + (1-t)x_2)) \leq \xi_{a-\mathcal{C}(-S)}(tf(x_1) + (1-t)f(x_2))$$

and then Lemma 3.8 yields the result. □

Note that in the above result we can replace $\mathcal{C}(-S)$ by $\mathrm{cl}(\mathcal{C}(-S))$ according to Lemma 3.6(a). In particular, taking into account Remark 4.1, the previous proposition can be applied when

$$S = \mathcal{C}(-\,\mathrm{int}\, P),\ S = \mathcal{C}(-P) \cup l(P),\ S = \mathcal{C}(-P) \cup \{0\},\ S = \mathcal{C}(-P), \tag{3}$$

since in all of these instances $\mathrm{cl}(\mathcal{C}(-S)) = \mathrm{cl}\, P$, and therefore, $\xi_{a-\mathcal{C}(-S)} \circ f = \xi_a \circ f$ by Lemma 3.6.

6.2 Semistrict quasiconvexity of $\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f$

Given $\bar{x} \in K$, the semistrict quasiconvexity of $\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f$ at $\bar{x}$ is obtained under the semistrict $(\mathrm{int}\, \mathcal{C}(-S))$-quasiconvexity at $\bar{x}$ of f as the following characterization shows.

Proposition 6.8. *Suppose that Assumption* **(A)** *holds. The following assertions are equivalent:*

(*a*) *f is semistrictly* $(\operatorname{int}\mathcal{C}(-S))$*-quasiconvex at $\bar{x} \in K$;*

(*b*) *$\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f$ is semistrictly quasiconvex at $\bar{x} \in K$.*

Proof. By Lemma 3.4(c) we have $(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) = 0$ because $f(\bar{x}) \in f(\bar{x}) - \partial(S)$. Hence,

$$(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) < (\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) \Longleftrightarrow (\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(x) < 0$$
$$\Longleftrightarrow f(x) \in f(\bar{x}) - \mathcal{C}(-S) - \operatorname{int} P = f(\bar{x}) - \operatorname{int}\mathcal{C}(-S).$$

On the other hand, by Lemma 3.4 we have

$$(\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f)(\xi) < 0 \Longleftrightarrow f(\xi) \in f(\bar{x}) - \mathcal{C}(-S) - \operatorname{int} P = f(\bar{x}) - \operatorname{int}\mathcal{C}(-S).$$

□

Observe that $\operatorname{int}\mathcal{C}(-S) = \operatorname{int} P$ if S is any of the sets appearing in (3) and, in all of these cases, we have again $\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f = \xi_{f(\bar{x})} \circ f$.
¿From now on we denote by P^* the (positive) polar cone of P and by $\operatorname{extrd} P^*$ its extremal directions: here $q^* \in \operatorname{extrd} P^*$ if and only if $q^* \in P^* \backslash \{0\}$ and for all $q_1^*, q_2^* \in P^*$ such that $q^* = q_1^* + q_2^*$ we actually have $q_1^*, q_2^* \in \mathbb{R}_{++} q^*$. See [1] for more details.

In view of these classical models, the following proposition arises,

Proposition 6.9. [*17*] *Let $\emptyset \neq P \subseteq Y$ be a convex cone with nonempty interior and $f : K \to Y$ be given with K being convex. Then, f is semistrictly* $(\operatorname{int} P)$*-quasiconvex under any of the following circumstances:*

(*a*) *if f is P-convex;*

(*b*) *if P is also closed and for all $p^* \in P^*$,*

$$x \in K \mapsto \langle p^*, f(x)\rangle \quad is \quad semistrictly\ quasiconvex;$$

(*c*) *if P is also closed, P^* is polyhedral, and for all $p^* \in \operatorname{extrd} P^*$,*

$$x \in K \mapsto \langle p^*, f(x)\rangle \quad is \quad semistrictly\ quasiconvex;$$

(*d*) *if P is also closed and for all $p^* \in P^*$ such that $||p^*|| = 1$,*

$$x \in K \mapsto \langle p^*, f(x)\rangle \quad is \quad semistrictly\ quasiconvex.$$

Proof. (a), (b), (c) and (d) are, respectively, Propositions 2.6, 2.7, 2.8(a) and 2.8(b3) in [17].

□

6.3 Quasiconvexity and strong quasiconvexity of $\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f$

Given $\bar{x} \in K$, the quasiconvexity of $\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f$ at $\bar{x}$ is obtained under the semistrict $(\mathrm{cl}(\mathcal{C}(-S)))$-quasiconvexity at $\bar{x}$ of f.

Proposition 6.10. *Suppose that Assumption* **(A)** *holds. The following assertions are equivalent*

(*a*) f *is semistrictly* $(\mathrm{cl}(\mathcal{C}(-S))$*-quasiconvex at* $\bar{x} \in K$;

(*b*) $\xi_{f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))} \circ f = \xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f$ *is quasiconvex at* $\bar{x} \in K$.

Note that Proposition 3.1 in [33] is a particular case of Proposition 6.10. The quasiconvexity of $\xi_{q,a-\mathcal{C}(-S)} \circ f$ on K requires the stronger notion of quasiconvexity since $(\mathrm{cl}(\mathcal{C}(-S)))$-quasiconvexity implies semistrict $(\mathrm{cl}(\mathcal{C}(-S)))$-quasiconvexity.

Proposition 6.11. *Suppose that Assumption* **(A)** *holds. The following assertions are equivalent:*

(*a*) f *is* $(\mathrm{cl}(\mathcal{C}(-S)))$*-quasiconvex on* K;

(*b*) $\xi_{a-\mathrm{cl}(\mathcal{C}(-S))} \circ f = \xi_{a-\mathcal{C}(-S)} \circ f$ *is quasiconvex on* K *for every* $a \in Y$.

As usual, for the classical models we have $\mathrm{cl}(\mathcal{C}(-S)) = \mathrm{cl}\, P$. In such a case, if P is closed, Proposition 6.11 was established in [32, Proposition 6.3, pag. 30]. See also Proposition 3.2 in [33] when $Y = \mathbb{R}^n$ (recall that $\xi_{a-P} = \xi_a$). Thus, we are concerned with the semistrict $(\mathrm{cl}\, P)$-quasiconvexity of f. Again, if P is closed, the author in [6] considers instead, the term (P, P)-quasiconvexity, and various equivalent conditions are derived in the bicriteria case, that is, when P is polyhedral in $\mathbb{R}^2$. One is expressed in terms of the Jacobian matrix of the function involved [6, Theorem 3]. Moreover, it is also proved in [6, Theorem 1] that semistrict (P)-quasiconvexity is equivalent to (P)-quasiconvexity whenever the function is continuous and $P \subseteq \mathbb{R}^2$. Recall that in general, (P)-quasiconvexity implies semistrict (P)-quasiconvexity.

A very important characterization of (P)-quasiconvexity, when P is closed and $\mathrm{int}\, P \neq \emptyset$, is given in [1]: $f : K \to Y$ is (P)-quasiconvex if and only if for every $p^* \in \mathrm{extrd}\, P^*$,

$$x \in K \mapsto \langle p^*, f(x) \rangle \text{ is quasiconvex.}$$

A smaller class of vector functions that are (P)-quasiconvex is that called $* - P$-*quasiconvex* introduced in [30]. Such mappings are such that

$$x \in K \mapsto \langle p^*, f(x) \rangle \text{ is quasiconvex for every } p^* \in P^*.$$

This class coincides (proved in [17, 14]) with the one discussed in [43] called *naturally P-quasiconvex* defined as those functions f satisfying

$$f([x,y]) \subseteq [f(x), f(y)] - P \quad \forall\, x,\ y \in K.$$

Concerning the strong quasiconvexity of $\xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f$ the next proposition arises.

Proposition 6.12. *Suppose that Assumption* **(A)** *holds. The following assertions are equivalent:*

(a) *f is semistrongly* $(\mathrm{cl}(\mathcal{C}(-S)))$*-quasiconvex at $\bar{x} \in K$;*

(b) $\xi_{f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))} \circ f = \xi_{f(\bar{x})-\mathcal{C}(-S)} \circ f$ *is strongly quasiconvex at $\bar{x} \in K$.*

Proof. It follows from Lemma 3.4(a) and by noticing that $\mathrm{int}\,\mathcal{C}(-S) = \mathrm{int}(\mathrm{cl}(\mathcal{C}(-S))$ because of Lemma 3.2. □

When P is closed, the semistrongly (P)-quasiconvexity on K was discussed in [6] under the name of $(P, \mathrm{int}\, P)$-quasiconvexity. When P is polyhedral in $\mathbb{R}^2$, it was proved the equivalence to the semistrict quasiconvexity of $x \mapsto \langle p^*, f(x)\rangle$ for all $p^* \in \mathrm{extrd}\ P^*$ (see [6, Theorem 2]).

According to Corollary 4.14(a) each weakly efficient solution of a vector problem is always a solution of a scalar problem. Thus, by Propositions 6.11 and 6.10 we deduce that [32, Theorem 2.15, pag. 93] established for (P)-quasiconvexity on K is also valid under the weaker assumption of semistrict (P)-quasiconvexity at $\bar{x}$.

So, as the scalar case similar results concerning continuity and differentiability can be established.

7 Optimality conditions for (approximate) efficiency via subdifferentials

In this section we apply some before results to derive new optimality conditions for problem $(\mathcal{P})$ and $(\mathcal{P}(\varepsilon q))$ by using subdifferentials and approximate subdifferentials respectively. First under convexity on f we obtain necessary and/or sufficient optimality conditions for solutions to $(\mathcal{P}(\varepsilon q))$, $\varepsilon \geq 0$, in terms of subdifferentials in the sense of convex analysis. Afterwards, in the framework of Asplund spaces, we establish a necessary optimality condition for problem $(\mathcal{P})$ without any convexity assumption on f, via the Mordukhovich subdifferentials.
We emphasize that the results and proofs are adapted from [42].

7.1 The convex case

Given a set $\emptyset \neq A \subseteq Y$ we denote by bar$A$ the barrier cone of A which is the (effective) domain of the support function of A denoted by σ_A.

For a given $\varepsilon \geq 0$ and a function $h : Y \to \mathbb{R} \cup \{+\infty\}$, we define the ε-subdifferential of h at $\bar{y} \in \mathrm{dom}(f) \doteq \{x \in Y : f(x) < +\infty\}$ as follows

$$\partial_\varepsilon h(\bar{y}) \doteq \{y^* \in Y^* : \ h(y) \geq h(\bar{y}) + \langle y^*, y - \bar{y}\rangle - \varepsilon \ \ \forall\, y \in Y\}.$$

We set $\partial_\varepsilon h(y) = \emptyset$ if $y \notin \mathrm{dom}\; h$ and $\partial h(y) = \partial_0 h(y)$. It is clear that

$$\bar{y} \in E(h, Y, \varepsilon) \Longleftrightarrow 0 \in \partial_\varepsilon h(\bar{y}).$$

Following the notation in [42] we denote by $\varphi_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ the function,

$$\varphi_A(y) \doteq \inf\{t \in \mathbb{R} : \ y \in tq + A\}$$

where $q \in \mathrm{int}\, P$ and $A \subseteq Y$ satisfies Assumption (**P**). Furthermore, by Theorem 3.1(ii) in [42], the above function φ_A is finite, that is, $\mathrm{dom}\, \varphi_A = Y$ and Lipschitz on Y.

We should point out that in [42] the authors make a deep study about the function φ_A with $q \in P \setminus (-P)$ and P not necessarily solid. In particular, several Lipschitz continuity properties of such a scalarizing function and some applications are established.

Under the assumption $S + \mathrm{int}\, P \subseteq S$, Lemma 3.6 and definition of ξ_A (where $Y \neq A$), allow us to obtain the following relationships:

$$\begin{aligned}\xi_{q,a-\mathcal{C}(-S)}(y) =& \xi_{q,a-\mathrm{cl}(\mathcal{C}(-S))}(y) = \xi_{q,-\,\mathrm{cl}(\mathcal{C}(-S))}(y-a) \\ =& \varphi_{-\,\mathrm{cl}(\mathcal{C}(-S))}(y-a) = \varphi_{a-\mathrm{cl}(\mathcal{C}(-S))}(y). \end{aligned} \tag{4}$$

If, in addition $\mathrm{cl}(\mathcal{C}(-S))$ is convex, following a reasoning similar to that used in the proof of Corollary 4.2 in [42], we obtain, given $\varepsilon \geq 0$ and $a \in Y$,

$$\begin{aligned}\partial_\varepsilon \xi_{q,a-\mathcal{C}(-S)}(\bar{y}) = \{y^* \in& \mathrm{bar}(a - \mathrm{cl}(\mathcal{C}(-S))) : \ \langle q, y^*\rangle = 1,\ \langle \bar{y}, y^*\rangle - \xi_{q,a-\mathcal{C}(-S)}(\bar{y}) + \varepsilon \\ &\geq \langle y, y^*\rangle \ \ \forall\, y \in a - \mathrm{cl}(\mathcal{C}(-S))\}, \end{aligned} \tag{5}$$

In what follows we denote the ε normal cone (in the sense of convex analysis) of K at $\bar{x} \in K$ by

$$N_\varepsilon(K; \bar{x}) \doteq \{x^* \in X^* : \ \langle x^*, x - \bar{x}\rangle \leq \varepsilon \quad \forall\, x \in K\},$$

and we set $N(K; \bar{x}) \doteq N_0(K; \bar{x})$. Finally by ι_K we denote the indicator function of K, i.e., $\iota_K(x) = 0$ if $x \in K$ and $\iota_K(x) = +\infty$ elsewhere. We immediately obtain

$$\partial_\varepsilon \iota_K(\bar{x}) = N_\varepsilon(K; \bar{x}).$$

If $\bar{y} = f(\bar{x})$, (5) reduces to

$$\partial_\varepsilon \xi_{q,f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))}(f(\bar{x})) = \{y^* \in \mathrm{bar}(f(\bar{x}) - \mathrm{cl}(\mathcal{C}(-S))) : \langle q, y^* \rangle = 1,$$
$$\langle f(\bar{x}), y^* \rangle + \varepsilon \geq \langle y, y^* \rangle, \quad \forall\, y \in f(\bar{x}) - \mathrm{cl}(\mathcal{C}(-S))\}.$$

This implies that

$$\partial_\varepsilon \xi_{q,f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))}(f(\bar{x})) = \{y^* \in \mathrm{bar}(-\,\mathrm{cl}(\mathcal{C}(-S))) : \langle q, y^* \rangle = 1,$$
$$\varepsilon \geq \langle y, y^* \rangle, \quad \forall\, y \in -\,\mathrm{cl}(\mathcal{C}(-S))\}.$$

$$\partial_\varepsilon \xi_{q,f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))}(f(\bar{x})) = \{y^* \in N_\varepsilon(-\,\mathrm{cl}(\mathcal{C}(-S)); 0) : \langle q, y^* \rangle = 1\}, \tag{6}$$

since $N_\varepsilon(-\,\mathrm{cl}(\mathcal{C}(-S)); 0) \subseteq \mathrm{bar}(-\,\mathrm{cl}(\mathcal{C}(-S)))$.
In case $\varepsilon = 0$, we get

$$\partial \xi_{q,f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))}(f(\bar{x})) = \{y^* \in \mathrm{bar}(-\,\mathrm{cl}(\mathcal{C}(-S))) \cap (\mathcal{C}(-S))^* : \langle q, y^* \rangle = 1\}$$
$$= \{y^* \in (\mathcal{C}(-S))^* : \langle q, y^* \rangle = 1\}, \tag{7}$$

since $(\mathcal{C}(-S))^* \subseteq \mathrm{bar}(-\,\mathrm{cl}(\mathcal{C}(-S)))$.

The preceding results take a more precise formulation when S is as in our standard models. Indeed, when $\mathrm{cl}(\mathcal{C}(-S)) = \mathrm{cl}\, P$ (for instance $S = \mathcal{C}(-\,\mathrm{int}\, P), \mathcal{C}(-P) \cup l(P), \mathcal{C}(-P) \cup \{0\}), \mathcal{C}(-P))$ we have from (6) and (7)

$$\partial_\varepsilon \xi_{q,f(\bar{x})}(f(\bar{x})) = \{y^* \in N_\varepsilon(-P; 0) : \langle q, y^* \rangle = 1\}, \tag{8}$$

$$\partial \xi_{q,f(\bar{x})}(f(\bar{x})) = \{y^* \in P^* : \langle q, y^* \rangle = 1\}, \tag{9}$$

Notice that $B \doteq \{y^* \in P^*, \ \langle q, y^* \rangle = 1\}$ is a weak $*$ compact, convex base for P^*, that is, $P^* = \cup_{t \geq 0} tB$. From (7) we also get

$$(\mathcal{C}(-S))^* = \bigcup_{t \geq 0} t \partial \xi_{q,f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))}(f(\bar{x})).$$

Before establishing our optimality conditions, we need to compute the subdifferential of the composition $\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f$. To that end, we need the following assumptions and notions.

Let $g \colon Y \to \mathbb{R} \cup \{+\infty\}$. We denote by g^* and g^{**} the conjugate of g and the biconjugate of g respectively. Consider

$$(y^* \circ f)^{**}(x) = (y^* \circ f)(x) \quad \forall\, x \in \mathrm{dom}\, f, \forall\, y^* \in \mathrm{dom}\, g^*; \tag{10}$$

$$\text{for some } y_0^* \in \mathrm{dom}\, g^*, \quad \text{one has} \quad y_0^* \circ f = (y_0^* \circ f)^{**}, \tag{11}$$

where $f\colon X \to Y$ is P-convex and $g\colon Y \to \mathbb{R}\cup\{+\infty\}$ is nondecreasing proper convex lsc.

A formula for the conjugate of $g = \xi_{q,f(\bar{x})-\mathcal{C}(-S)}$ when $\mathrm{cl}(\mathcal{C}(-S))$ is convex and Assumption (**A**) holds may be found in [42, Proposition 4.1] as follows:

$$g^*(y^*) = \begin{cases} \sigma_{f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))}(y^*) & if\ \ y^* \in \mathrm{bar}(-\,\mathrm{cl}(\mathcal{C}(-S))),\ \langle q, y^*\rangle = 1, \\ +\infty & otherwise \end{cases}$$

¿From this it is easy to check that

$$\mathrm{dom}\, g^* \subseteq P^* \tag{12}$$

since $-\,\mathrm{cl}(\mathcal{C}(-S)) - P = -\,\mathrm{cl}(\mathcal{C}(-S))$.

Proposition 7.1. *Suppose that* $\mathrm{cl}(\mathcal{C}(-S))$ *is convex. Let* $q \in \mathrm{int}\, P$*,* $\bar{x} \in \mathrm{dom}\, f$ *and let* $f : X \to Y$ *be* P*-convex such that* (10) *and* (11) *hold for* $g \doteq \xi_{q,f(\bar{x})-\mathcal{C}(-S)}$*. Then,*

(a) *for every* $\varepsilon > 0$*, one has*

$$\partial_\varepsilon(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) = \mathrm{cl}\Big(\bigcup_{\substack{\eta_1\ge 0,\ \eta_2 \ge 0 \\ \eta_1+\eta_2=\varepsilon}} \ \bigcup_{y^*\in N_{\eta_1}(-\mathcal{C}(-S);0),\ \langle q,y^*\rangle=1} \partial_{\eta_2}(y^* \circ f)(\bar{x})\Big).$$

(b)

$$\partial(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) = \bigcap_{\mu>0} \mathrm{cl}\Big(\bigcup_{y^*\in N_\mu(-\mathcal{C}(-S);0),\ \langle q,y^*\rangle=1} \partial_\mu(y^* \circ f)(\bar{x})\Big).$$

Proof. (a) and (b) follow from Theorem 8.1 and Corollary 8.1 in [26] respectively along with (6). □

Conditions (10) and (11) holds trivially when $f : X \to Y$ is P-lsc and P-convex. More precisely, the P-convexity of f implies the convexity of

$$x \mapsto y^* \circ f(x) = \langle y^*, f(x)\rangle \in \mathbb{R} \ \ \forall\, y^* \in P^*$$

as one can check it directly; whereas the P-lower semicontinuity of f gives the lower semicontinuity of

$$x \mapsto y^* \circ f(x) = \langle y^*, f(x)\rangle \in \mathbb{R} \ \ \forall\, y^* \in P^*.$$

Hence, under our assumptions $(y^* \circ f)^{**} = y^* \circ f$ for all $y^* \in P^*$. In particular, $(y^* \circ f)^{**} = y^* \circ f$ for $y^* \in \mathrm{dom}\, g^*$ taking into account (12).

We are ready to establish our first optimality conditions for ε-efficiency, $\varepsilon > 0$. We can use (a) of the previous proposition to go further in writing (13).

Theorem 7.2. *Suppose that Assumption* (**A**) *holds and* $\mathrm{cl}(\mathcal{C}(-S))$ *is convex. Let* $q \in \operatorname{int} P$, $\varepsilon > 0$, $K \subseteq X$ *be convex and closed;* $f : X \to Y$ *be* P*-convex and* P*-lsc. If* $\bar{x} \in E_S(\varepsilon q)$ *then*

$$0 \in \mathrm{cl}\Big(\bigcap_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial_{\varepsilon_1}(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) + N_{\varepsilon_2}(K;\bar{x})\Big). \tag{13}$$

And if $\bar{x} \in K$ *satisfies* (13) *then* $\bar{x} \in E_{\mathrm{cl}\,S}(\varepsilon q)$.

Proof. If $\bar{x} \in E_S(\varepsilon q)$ then $\bar{x} \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K, \varepsilon) = E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f + \iota_K, X, \varepsilon)$ by Theorem 4.6. Taking into account Lemma 5.3 and Proposition 6.7, we obtain

$$0 \in \partial_\varepsilon(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f + \iota_K)(\bar{x}) = \mathrm{cl}\Big(\bigcap_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial_{\varepsilon_1}(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) + \partial_{\varepsilon_2}\iota_K(\bar{x})\Big)$$

by [26, Theorem 3.2]. The result follows from the previous proposition.
We apply Theorem 4.6 to conclude the proof. □

When $\varepsilon = 0$, a similar reasoning to the above proof along with [26, Theorem 3.1] allows us to obtain the next stronger result.

Theorem 7.3. *Suppose that Assumption* (**A**) *holds and* $\mathrm{cl}(\mathcal{C}(-S))$ *is convex. Let* $K \subseteq X$ *be convex and closed;* $f : X \to Y$ *be* P*-convex and* P*-lsc. If* $\bar{x} \in E_S$ *then*

$$0 \in \bigcap_{\mu > 0} \mathrm{cl}\Big(\partial_\mu(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f)(\bar{x}) + N_\mu(K;\bar{x})\Big). \tag{14}$$

And if $\bar{x} \in K$ *satisfies* (14) *then* $\bar{x} \in E_{\mathrm{cl}\,S}$.

7.2 The nonconvex case

We now proceed to establish a necessary optimality conditions for $\bar{x} \in E_S$ without convexity assumptions on $f\colon X \to Y$, when X and Y are Asplund spaces (cf [39, Definition 1.22]): we recall that the Banach spaces with separable dual and the reflexive Banach spaces are Asplund spaces. In this context we work with the Mordukhovich subdifferential ∂_M and the normal cone N_M considered in [37], where are denoted by ∂ and N.

Given a function $f : X \to Y$, it is said to be strictly Lipschitz at $\bar{x} \in X$ if f is Lipschitz on a neighbourhood U of the origin in X, such that the sequence $(t_k^{-1}(f(x_k + t_k u) - f(x_k)))_{k \in \mathbb{N}}$ contains a convergent subsequence (in norm) whenever $u \in U$, $x_k \to \bar{x}$, $t_k \downarrow 0$.

It is clear that this notion reduces to local Lipschitz continuity if Y is finite dimensional, see [37, Section 3.1.3] for more details.

The following lemma will be useful in the sequel.

Lemma 7.4. *Assume that X and Y Asplund spaces.*

(*a*) ([*37, Theorem* 3.36]) *If $f_1, f_2 : X \to \mathbb{R}\cup\{+\infty\}$ are proper functions and there exists a neighbourhood U of $\bar{x} \in \mathrm{dom}\ f_1 \cap \mathrm{dom}\ f_2$, such that f_1 is Lipschitz on U and f_2 is lsc on U, then*

$$\partial_M(f_1 + f_2)(\bar{x}) \subseteq \partial_M f_1(\bar{x}) + \partial_M f_2(\bar{x}).$$

(*b*) ([*37, Corollary* 3.43]) *If $f : X \to Y$ is strictly Lipschitz at $\bar{x}$ and $\varphi : Y \to \mathbb{R}\cup\{+\infty\}$ is finite and Lipschitz on a neighbourhood of $f(\bar{x})$, then*

$$\partial_M(\varphi \circ f)(\bar{x}) \subseteq \bigcup\{\partial_M(y^* \circ f)(\bar{x}) :\ y^* \in \partial_M\varphi(f(\bar{x}))\}.$$

In what follows we establish a necessary optimality condition for problem $(\mathcal{P})$ under Assumption $(\mathbf{A})$.

Theorem 7.5. *Suppose that X and Y are Asplund spaces and Assumption $(\mathbf{A})$ holds. Let $\mathrm{cl}(\mathcal{C}(-S))$ be convex, $f : X \to Y$ be strictly Lipschitz and $q \in \mathrm{int}\, P$. If $\bar{x} \in E_S$ then there exists $y^* \in (\mathcal{C}(-S))^*$, $\langle q, y^*\rangle = 1$ such that*

$$0 \in \partial_M(y^* \circ f)(\bar{x}) + N_M(K; \bar{x}). \tag{15}$$

Moreover, if f is strictly differentiable at $\bar{x}$ then $(f'(\bar{x}))^* y^* \in -N_M(K; \bar{x})$.

Proof. If $\bar{x} \in E_S$ then $\bar{x} \in E(\xi_{q,f(\bar{x})-\mathcal{C}(-S)} \circ f, K)$ by Theorem 4.6.
One can proceed as in [42, Theorem 5.4] to check all the assumptions of the previous lemma are satisfied. Thus, by applying it, we get

$$0 \in \partial_M(y^* \circ f)(\bar{x}) + N_M(K; \bar{x})$$

for some $y^* \in \partial_M \xi_{q,f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))}(f(\bar{x}))$. Due to the convexity of $\mathrm{cl}(\mathcal{C}(-S))$ by Lemma 3.8, we get

$$\partial_M \xi_{q,f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))}(f(\bar{x})) = \partial \xi_{q,f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))}(f(\bar{x})).$$

¿From (7) we know

$$\partial \xi_{q,f(\bar{x})-\mathrm{cl}(\mathcal{C}(-S))}(f(\bar{x})) = \{y^* \in (\mathcal{C}(-S))^* :\ \langle q, y^*\rangle = 1\}.$$

We conclude the proof taking into account that $\partial_M f(\bar{x}) = \{f'(\bar{x})\}$ if f is strictly differentiable at $\bar{x}$. □

We have observed that $S + \mathrm{int}\, P \subseteq S \iff \mathcal{C}(-S) + \mathrm{int}\, P \subseteq \mathcal{C}(-S)$, and therefore $P \subseteq \mathrm{cl}(\mathcal{C}(-S))$ provided $0 \in \partial S = \partial\mathcal{C}(S)$. Thus $(\mathrm{cl}(\mathcal{C}(-S)))^* = (\mathcal{C}(-S))^* \subseteq P^*$. By virtue of these facts, if S is closed, $0 \in \partial S$, $\mathcal{C}(-S)$ is convex and satisfies the following stronger inclusion

$$\mathrm{cl}(\mathcal{C}(-S)) + (P \setminus \{0\}) \subseteq \mathrm{int}(\mathrm{cl}(\mathcal{C}(-S)))$$

than that in (**A**), one can shows that y^* given in Theorem 7.5 is actually in $P^{\#} \doteq \{p^* \in Y^* : \langle p^*, p\rangle > 0 \ \ \forall\, p \in P \setminus \{0\}\}$, (cf. Tammer and Zalinescu [42, Theorem 5.4]).

Finally, we once again, by considering our standard models, $S = \mathcal{C}(-\operatorname{int} P), \mathcal{C}(-P) \cup l(P), \mathcal{C}(-P) \cup \{0\}, \mathcal{C}(-P)$ more manageable formulations can be obtained.

Remark 7.6. *In* [*15, Corollary* 4.14] *a free boundary Stefan problem is discussed taking into account the definitions introduced in* [*27*]*. Exactly, the scalarizing function* $\xi_{q,f(\bar{x})}$ *is computed. We point out that according to previous results* (*see, for instance, Theorem 4.6 and Corollary 4.14*) *we may obtain optimality conditions for the* (*approximate*) *free boundary Stefan problem.*

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