APPROXIMATION OF THE BUCKLING PROBLEM FOR REISSNER-MINDLIN PLATES

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Abstract. This paper deals with the approximation of the buckling coefficients and modes of a clamped plate modeled by the Reissner-Mindlin equations. These coefficients are related with the eigenvalues of a non-compact operator. We give a spectral characterization of this operator and show that the relevant buckling coefficients correspond to isolated nondefective eigenvalues. Then we consider the numerical solution of the buckling problem. For the finite element approximation of Reissner-Mindlin equations, it is well known that some kind of reduced integration or mixed interpolation has to be used to avoid locking. In particular we consider Duran-Liberman elements, which have been already proved to be locking-free for load and vibration problems. We adapt the classical approximation theory for non-compact operators to obtain optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. These estimates are valid with constants independent of the plate thickness. We report some numerical experiments confirming the theoretical results. Finally, we refine the analysis in the case of a uniformly compressed plate.

Key words. Buckling, Reissner-Mindlin plates, finite elements, non-compact spectral problems.

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1. Introduction. This paper deals with the analysis of the elastic stability of plates, in particular the so-called *buckling problem*. This problem has attracted much interest, since it is frequently encountered in engineering applications, such as bridge, ship, and aircraft design. It can be formulated as a spectral problem whose solution is related with the limit elastic stability of the plate (i.e., eigenvalues-critical loads and eigenfunctions-buckling modes).

This problem has been studied for years by many researchers, being the Kirchhoff-Love and Reissner-Mindlin plate theories the most used. For the Kirchhoff-Love theory, there exists a thorough mathematical analysis; let us mention, for instance, [5, 13, 16, 17, 18]. This is not the case for the Reissner-Mindlin theory, for which only numerical experiments (cf. [15, 22]) or analytical solutions in particular cases (cf. [23]) have been reported so far. Recently, Dauge and Suri gave in [7] a mathematical spectral analysis of a problem of this kind based on three-dimensional elasticity which we will adapt to our case.

The Reissner-Mindlin theory is the most used model to approximate the deformation of an elastic thin or moderately thick plate. It is very well understood that standard finite element methods applied to this model lead to wrong results when the thickness is small with respect to the other dimensions of the plate, due to the *locking* phenomenon. Several families of methods have been rigorously shown to be free from locking and optimally convergent. We mention the recent monograph by Falk [12] for a thorough description of the state of the art and further references.

The aim of this paper is to analyze one of these methods applied to compute the critical load and buckling modes of a clamped plate. We choose the low-order,

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nonconforming finite elements introduced by Duran and Liberman in [11] (see also [10] for the analysis of this method applied to the plate vibration problem). However, the developed framework could be useful to analyze other methods, as well.

One disadvantage of the Reissner-Mindlin formulation for plate buckling is the fact that the corresponding resolvent operator is non-compact. This is the reason why the essential spectrum no longer reduces to zero (as is the case for compact operators). This means that the spectrum may now contain nonzero eigenvalues of infinite multiplicity, accumulation points, continuous spectrum, etc. Thus, our first task is to prove that the eigenvalue corresponding to the critical load can be isolated from the essential spectrum, at least for sufficiently thin plates.

On the other hand, the abstract spectral theory for non-compact operators introduced by Descloux, Nassif, and Rappaz in [8, 9] cannot be directly applied to analyze the numerical method, because we look for error estimates valid uniformly in the plate thickness. However, using the optimal order of convergence for the Duran-Liberman elements (cf. [10, 11]), the standard theory used to prove regularity results for Reissner-Mindlin equations (cf. [1]), and assuming that the family of meshes is *quasi-uniform*, we can adapt the theory from [8, 9] to obtain optimal order error estimates for the approximation of the buckling modes and a double order for the critical load. Moreover, these estimates are shown to be valid with constants independent of the plate thickness.

An outline of the paper is as follows. In the next section we recall the buckling problem and introduce the non-compact linear operator whose spectrum is related with the solution of this problem. In Section 3 we provide a thorough spectral characterization of this operator. In Section 4 we introduce a finite element discretization of the problem based on Duran-Liberman elements. In Section 5 we prove that the proposed numerical scheme is free of spurious modes and that optimal order error estimates hold true. In Section 6 we report some numerical tests which confirm the theoretical results. We include in this section a benchmark with a known analytical solution for a simply supported plate, which shows the efficiency of the method under other kind of boundary conditions, as well. Finally, in an appendix, we show that the results of Sections 3, 4, and 5 can be refined when considering a uniformly compressed plate.

Throughout the paper we will use standard notations for Sobolev norms and seminorms. Moreover, we will denote with C a generic constant independent of the mesh parameter h and the plate thickness t, which may take different values in different occurrences.

2. The buckling problem. Consider an elastic plate of thickness t with reference configuration $\Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)$, where Ω is a convex polygonal domain of \mathbb{R}^2 occupied by the midsection of the plate. The deformation of the plate is described by means of the Reissner-Mindlin model in terms of the rotations $\beta = (\beta_1, \beta_2)$ of the fibers initially normal to the plate midsurface and the transverse displacement w. Assuming that the plate is clamped on its whole boundary $\partial\Omega$ and subjected to a plane stress tensor field $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{2\times 2}$, the buckling problem reads as follows (see for instance [15, 22]):

Find $\lambda \in \mathbb{R}$ and $0 \neq (\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ such that (2.1) $a(\beta, \eta) + \frac{\kappa}{t^2} (\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = \lambda (\boldsymbol{\sigma} \nabla w, \nabla v)_{0,\Omega} \qquad \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega).$

Above, $\kappa := Ek/(2(1 + \nu))$ is the shear modulus, with E being the Young modulus, ν the Poisson ratio, and k a correction factor usually taken as 5/6 for clamped plates.

The applied stress tensor field is assumed to satisfy

(2.2)
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}} \quad \text{in } \Omega,$$

(2.3)
$$\boldsymbol{\sigma} \in \mathbf{W}^{1,\infty}(\Omega)^{2 \times 2}$$

Finally, a is the $H_0^1(\Omega)^2$ elliptic bilinear form defined by

$$a(\beta,\eta) := \frac{E}{12(1-\nu^2)} \int_{\Omega} \left[(1-\nu) \,\varepsilon(\beta) : \varepsilon(\eta) + \nu \operatorname{div}(\beta) \operatorname{div}(\eta) \right],$$

with $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{1 \leq i,j \leq 2}$ being the standard strain tensor with components $\varepsilon_{ij}(\beta) := \frac{1}{2} (\partial_i \beta_j + \partial_j \beta_i), 1 \leq i, j \leq 2$. Moreover, $(\cdot, \cdot)_{0,\Omega}$ denotes the usual L² inner product. *Remark* 2.1. Problem (2.1) is obtained by *scaling* the *physical* buckling problem:

$$t^{3}a(\beta,\eta) + t\kappa \left(\nabla w - \beta, \nabla v - \eta\right)_{0,\Omega} = \lambda_{b}t \left(\boldsymbol{\sigma}\nabla w, \nabla v\right)_{0,\Omega} \qquad \forall (\eta,v) \in \mathrm{H}_{0}^{1}(\Omega)^{2} \times \mathrm{H}_{0}^{1}(\Omega).$$

Accordingly, the buckling coefficients are related with the eigenvalues of problem (2.1)

by $\lambda_b = t^2 \lambda$, while the eigenfunctions (β, w) are exactly the same. Introducing the shear strain $\gamma := \frac{\kappa}{t^2} (\nabla w - \beta)$, problem (2.1) can be written as follows:

Find
$$\lambda \in \mathbb{R}$$
 and $0 \neq (\beta, w) \in \mathrm{H}_{0}^{1}(\Omega)^{2} \times \mathrm{H}_{0}^{1}(\Omega)$ such that
(2.4)
$$\begin{cases} a(\beta, \eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = \lambda (\boldsymbol{\sigma} \nabla w, \nabla v)_{0,\Omega} \qquad \forall (\eta, v) \in \mathrm{H}_{0}^{1}(\Omega)^{2} \times \mathrm{H}_{0}^{1}(\Omega), \\ \gamma = \frac{\kappa}{t^{2}} (\nabla w - \beta). \end{cases}$$

The *source* problem associated with the problem above reads: Given $f \in H^1_0(\Omega)$, find $(\beta, w) \in H^1_0(\Omega)^2 \times H^1_0(\Omega)$ such that

(2.5)
$$\begin{cases} a(\beta,\eta) + (\gamma,\nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f, \nabla v)_{0,\Omega} \qquad \forall (\eta,v) \in \mathrm{H}^{1}_{0}(\Omega)^{2} \times \mathrm{H}^{1}_{0}(\Omega), \\ \gamma = \frac{\kappa}{t^{2}} (\nabla w - \beta). \end{cases}$$

Using the Helmholtz decomposition

(2.6)
$$\gamma = \nabla \psi + \operatorname{curl} p$$

with $\psi \in \mathrm{H}^{1}_{0}(\Omega)$ and $p \in \mathrm{H}^{1}(\Omega)/\mathbb{R}$, we have that problem (2.5) is equivalent to the following one (see [1]):

Given $f \in H_0^1(\Omega)$, find $(\psi, \beta, p, w) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times H^1(\Omega) / \mathbb{R} \times H_0^1(\Omega)$ such that

$$(2.7) \qquad \begin{cases} (\nabla\psi, \nabla v)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f, \nabla v)_{0,\Omega} & \forall v \in \mathrm{H}_{0}^{1}(\Omega), \\ a(\beta, \eta) - (\operatorname{curl} p, \eta)_{0,\Omega} = (\nabla\psi, \eta)_{0,\Omega} & \forall \eta \in \mathrm{H}_{0}^{1}(\Omega)^{2}, \\ -(\beta, \operatorname{curl} q)_{0,\Omega} - \kappa^{-1}t^{2} \left(\operatorname{curl} p, \operatorname{curl} q\right)_{0,\Omega} = 0 & \forall q \in \mathrm{H}^{1}(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla\xi)_{0,\Omega} = (\beta, \nabla\xi)_{0,\Omega} + \kappa^{-1}t^{2} \left(\nabla\psi, \nabla\xi\right)_{0,\Omega} & \forall \xi \in \mathrm{H}_{0}^{1}(\Omega). \end{cases}$$

We recall the following result for the solution of problem (2.7) (see [1]):

THEOREM 2.1. Let Ω be a convex polygon or a smoothly bounded domain in the plane. For any $t \in (0,1]$, $\boldsymbol{\sigma} \in L^{\infty}(\Omega)^{2 \times 2}$, and $f \in H^1_0(\Omega)$, there exists a unique solution of problem (2.7). Moreover, $\beta \in H^2(\Omega)^2$, $p \in H^2(\Omega)$ and there exists a constant C, independent of t and f, such that

$$\|\psi\|_{1,\Omega} + \|\beta\|_{2,\Omega} + \|p\|_{1,\Omega} + t \|p\|_{2,\Omega} + \|w\|_{1,\Omega} \le C \|f\|_{1,\Omega}$$

As a consequence of Theorem 2.1, by virtue of (2.6) and the equivalence between problems (2.5) and (2.7), we have that problem (2.5) is well-posed and there exists a constant C, independent of t and f, such that

(2.8)
$$\|\beta\|_{2,\Omega} + \|w\|_{1,\Omega} + \|\gamma\|_{0,\Omega} \le C \|f\|_{1,\Omega}.$$

Let T_t be the following bounded linear operator:

(2.9)
$$\begin{aligned} T_t: \, \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega) \\ f \mapsto w, \end{aligned}$$

where (β, w) is the solution of problem (2.5). It is easy to see that (μ, w) , with $\mu \neq 0$, is an eigenpair of T_t (i.e. $T_t w = \mu w$, $w \neq 0$) if and only if (λ, β, w) is a solution of problem (2.4), with $\lambda = 1/\mu$ and a suitable $\beta \in H_0^1(\Omega)^2$.

3. Spectral properties. The aim of this section is to prove a spectral characterization for the operator T_t defined above, to study the convergence of T_t and the behavior of its spectrum as t goes to zero, and to prove additional regularity for the eigenfunctions of T_t .

3.1. Spectral characterization. Given a generic linear bounded operator $T : X \to X$, defined on a Hilbert space X, we denote the spectrum of T by $\operatorname{Sp}(T) := \{z \in \mathbb{C} : (zI - T) \text{ is not invertible}\}$ and by $\rho(T) := \mathbb{C} \setminus \operatorname{Sp}(T)$ the resolvent set of T. Moreover, for any $z \in \rho(T)$, $R_z(T) := (zI - T)^{-1} : X \to X$ denotes the resolvent operator of T corresponding to z.

We recall the definitions of the following components of the spectrum.

• Discrete spectrum:

 $\operatorname{Sp}_{d}(T) := \{ z \in \mathbb{C} : \operatorname{Ker}(zI - T) \neq \{ 0 \} \text{ and } (zI - T) : X \to X \text{ is Fredholm} \}.$

• Essential spectrum:

$$\operatorname{Sp}_{e}(T) := \{ z \in \mathbb{C} : (zI - T) : X \to X \text{ is not Fredholm} \}.$$

The main result of this section is the following theorem which provides a suitable spectral characterization for the operator T_t defined in (2.9).

- THEOREM 3.1. The spectrum $Sp(T_t)$ decomposes into:
 - $\operatorname{Sp}_{d}(T_{t})$, which consists of real isolated eigenvalues of finite multiplicity and ascent one,
 - $Sp_e(T_t)$, essential spectrum.

Moreover, $\operatorname{Sp}_{e}(T_{t}) \subset \left\{ z \in \mathbb{C} : |z| \leq \kappa^{-1} t^{2} \|\boldsymbol{\sigma}\|_{\infty,\Omega} \right\}.$

Here and thereafter, we denote $\|\boldsymbol{\sigma}\|_{\infty,\Omega} := \max_{x \in \overline{\Omega}} |\boldsymbol{\sigma}(x)|$, with $|\cdot|$ being the matrix norm induced by the standard Euclidean norm in \mathbb{R}^2 . Notice that the maximum above is well defined because of (2.3) and the fact that $W^{1,\infty}(\Omega) \subset C(\overline{\Omega})$.

As a consequence of this theorem we know that, although T_t may have essential spectrum, all the points of $Sp(T_t)$ outside a ball centered at the origin of the complex plane are nondefective isolated eigenvalues. Moreover, the thinner the plate, the smaller the ball containing the essential spectrum.

The proof of Theorem 3.1 will be an immediate consequence of the results that follow. Consider the continuous bilinear forms defined in $H_0^1(\Omega)^2 \times H_0^1(\Omega)$

$$(3.1) A((\beta, w), (\eta, v)) := a(\beta, \eta) + \frac{\kappa}{t^2} \left(\nabla w - \beta, \nabla v - \eta\right)_{0,\Omega}$$

(3.2)
$$B((g, f), (\eta, v)) := (\boldsymbol{\sigma} \nabla f, \nabla v)_{0,\Omega}$$

We notice that $A(\cdot, \cdot)$ is symmetric and elliptic (cf. [4]). Moreover, from the symmetry of $\boldsymbol{\sigma}$ (cf. (2.2)), it follows that $B(\cdot, \cdot)$ is symmetric too. Consider the bounded linear operator

(3.3)
$$\widetilde{T}_t: \operatorname{H}^1_0(\Omega)^2 \times \operatorname{H}^1_0(\Omega) \to \operatorname{H}^1_0(\Omega)^2 \times \operatorname{H}^1_0(\Omega), (g, f) \mapsto (\beta, w),$$

where $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ is the solution of

$$A((\beta,w),(\eta,v)) = B((g,f),(\eta,v)) \qquad \forall (\eta,v) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega).$$

By virtue of the symmetry of $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$, we have

$$A(\widetilde{T}_{t}(g,f),(\eta,v)) = B((g,f),(\eta,v)) = B((\eta,v),(g,f)) = A((g,f),\widetilde{T}_{t}(\eta,v)),$$

for every $(g, f), (\eta, v) \in \mathrm{H}_{0}^{1}(\Omega)^{2} \times \mathrm{H}_{0}^{1}(\Omega)$. Therefore, \widetilde{T}_{t} is self-adjoint with respect to the inner product $A(\cdot, \cdot)$. As a consequence, we have the following theorem (see, for instance, [7, Theorem 3.3]).

THEOREM 3.2. There holds $\operatorname{Sp}(\widetilde{T}_t) \subset \mathbb{R}$. Moreover, the spectrum of \widetilde{T}_t decomposes as follows: $\operatorname{Sp}(\widetilde{T}_t) = \operatorname{Sp}_{d}(\widetilde{T}_t) \cup \operatorname{Sp}_{e}(\widetilde{T}_t)$. Finally, if $\mu \in \operatorname{Sp}_{d}(\widetilde{T}_t)$, then μ is an isolated eigenvalue of finite multiplicity.

The following result shows that the essential spectrum of \widetilde{T}_t is confined in a neighborhood of the origin of diameter proportional to t_{\perp}^2 .

PROPOSITION 3.3. Let $\mu \in \mathbb{R}$ be such that $\mu \in \operatorname{Sp}(\widetilde{T}_t)$ and $|\mu| > \kappa^{-1}t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$. Then $\mu \in \operatorname{Sp}_d(\widetilde{T}_t)$.

Proof. Let $\mu \in \mathbb{R}$ be such that $\mu \in \operatorname{Sp}(\widetilde{T}_t)$ and $|\mu| > \kappa^{-1}t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$. By virtue of Theorem 3.2, we only have to prove that $(\mu \widetilde{I} - \widetilde{T}_t)$ is a Fredholm operator. To this end, it is enough to show that there exists a compact operator \widetilde{G} such that $(\mu \widetilde{I} - \widetilde{T}_t + \widetilde{G})$ is invertible. Let us introduce the operator S as follows:

$$S: \operatorname{H}_{0}^{1}(\Omega) \to \operatorname{H}_{0}^{1}(\Omega)^{2},$$
$$f \mapsto \beta,$$

where (β, w) is the unique solution of problem (2.5). Notice that

(3.4)
$$\widetilde{T}_t(g, f) = (Sf, T_t f).$$

According to (2.8), we have that $\beta \in \mathrm{H}^2(\Omega)^2$ and hence S is compact. Let us now define the operator G as follows:

(3.5)
$$\begin{aligned} G: \ \mathrm{H}^{1}_{0}(\Omega) \to \mathrm{H}^{1}_{0}(\Omega), \\ f \mapsto u, \end{aligned}$$

where $u \in \mathrm{H}_{0}^{1}(\Omega)$ is the unique solution of

$$(\nabla u, \nabla \xi)_{0,\Omega} = (Sf, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}^1_0(\Omega).$$

The operator G is compact as a consequence of the compactness of S. Next, we define \widetilde{G} as follows:

$$\begin{split} \widetilde{G}: \, \mathrm{H}^{1}_{0}(\Omega)^{2} \times \mathrm{H}^{1}_{0}(\Omega) \to \mathrm{H}^{1}_{0}(\Omega)^{2} \times \mathrm{H}^{1}_{0}(\Omega), \\ (g,f) \mapsto (Sf,Gf). \end{split}$$

Since S and G are compact, \widetilde{G} is compact, too. In addition,

$$(\mu \tilde{I} - \tilde{T}_t + \tilde{G})(g, f) = ((\mu g - Sf + Sf), (\mu I - T_t + G)f) = (\mu g, (\mu I - T_t + G)f).$$

From the fourth equation in (2.7), we notice that $v := (\mu I - T_t + G) f$ satisfies

$$\begin{aligned} \left(\nabla v, \nabla \xi\right)_{0,\Omega} &= \mu \left(\nabla f, \nabla \xi\right)_{0,\Omega} - \left(\nabla w, \nabla \xi\right)_{0,\Omega} + (\beta, \nabla \xi)_{0,\Omega} \\ &= \left(\left(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma}\right) \nabla f, \nabla \xi\right)_{0,\Omega} \quad \forall \xi \in \mathrm{H}_0^1(\Omega) \end{aligned}$$

Consequently, the operator $(\mu I - T_t + G)$ will be invertible if and only if, given $v \in H_0^1(\Omega)$, there exists a unique $f \in H_0^1(\Omega)$ solution of

(3.6)
$$\left(\left(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma}\right) \nabla f, \nabla \xi\right)_{0,\Omega} = \left(\nabla v, \nabla \xi\right)_{0,\Omega} \quad \forall \xi \in \mathrm{H}_0^1(\Omega)$$

Now, because of the symmetry of $\boldsymbol{\sigma}(x)$, there exists an orthogonal tensor $\boldsymbol{P}(x)$ such that $\boldsymbol{\sigma}(x) = \boldsymbol{P}(x)\boldsymbol{D}(x)\boldsymbol{P}^{t}(x)$, where

$$\boldsymbol{D}(x) := \begin{bmatrix} \overline{\omega}(x) & 0 \\ 0 & \underline{\omega}(x) \end{bmatrix},$$

with $\underline{\omega}(x) \leq \overline{\omega}(x)$ being the eigenvalues of $\boldsymbol{\sigma}(x)$. Hence, we write

$$\left(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma}\right) = \boldsymbol{P}(x) \begin{bmatrix} \mu - \kappa^{-1} t^2 \overline{\omega}(x) & 0\\ 0 & \mu - \kappa^{-1} t^2 \underline{\omega}(x) \end{bmatrix} \boldsymbol{P}(x)^{\mathrm{t}}.$$

Let us denote $\omega_{\max} := \max_{x \in \overline{\Omega}} \overline{\omega}(x)$ and $\omega_{\min} := \min_{x \in \overline{\Omega}} \underline{\omega}(x)$. Since $\|\boldsymbol{\sigma}\|_{\infty,\Omega} = \max_{x \in \overline{\Omega}} |\boldsymbol{\sigma}(x)| = \max\{|\omega_{\max}|, |\omega_{\min}|\}, \text{ if } |\mu| > \kappa^{-1}t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}, \text{ then there holds either } \mu > \kappa^{-1}t^2 \omega_{\max} \text{ or } \mu < \kappa^{-1}t^2 \omega_{\min}.$ Hence, $(\mu \boldsymbol{I} - \kappa^{-1}t^2 \boldsymbol{\sigma})$ is uniformly positive definite in the first case or uniformly negative definite in the second one. Therefore, in both cases, there exists a unique solution $f \in \mathrm{H}^1_0(\Omega)$ of (3.6). Consequently, $(\mu \boldsymbol{I} - T_t + G)$ is invertible. Consequently, $(\mu \boldsymbol{I} - \tilde{T}_t)$ is Fredholm and $\mu \in \mathrm{Sp}_{\mathrm{d}}(\tilde{T}_t)$. \Box

The following result shows that T_t and \tilde{T}_t have the same spectrum.

LEMMA 3.4. If T_t and \tilde{T}_t are the operators defined in (2.9) and (3.3), respectively, then $\operatorname{Sp}(\tilde{T}_t) = \operatorname{Sp}(T_t)$.

Proof. We will prove that $\rho(\widetilde{T}_t) = \rho(T_t)$. Let z be such that $(z\widetilde{I} - \widetilde{T}_t)$ is invertible. We will prove that $(zI - T_t)$ is invertible, too. By hypothesis, for every $(\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ there exists a unique $(g, f) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ such that

(3.7)
$$(z\widetilde{I} - \widetilde{T}_t)(g, f) = (\beta, w)$$

Recalling (3.4), we infer that there is a unique (g, f) such that $zg - Sf = \beta$ and $(zI - T_t) f = w$. Hence, we deduce that the operator $(zI - T_t) : H_0^1(\Omega) \to H_0^1(\Omega)$

is onto. Now, let us assume that there exists another \hat{f} such that $(zI - T_t) \hat{f} = w$. Taking $\hat{g} = \frac{1}{z}(S\hat{f} + \beta)$, we see that $(z\tilde{I} - \tilde{T}_t)(\hat{g}, \hat{f}) = (\beta, w)$. Since, by assumption, $(z\tilde{I} - \tilde{T}_t)$ is invertible, from (3.7) it follows that $f = \hat{f}$. Therefore, $(zI - T_t)$ is also one-to-one and thus it is invertible.

Conversely, let z be such that $(zI - T_t)$ is invertible. We will prove that $(z\tilde{I} - \tilde{T}_t)$ is invertible, too. Recalling (3.4) again, we have to show that for every $(\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$, there exists a unique $(g, f) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ such that

$$\begin{cases} zg - Sf = \beta, \\ zf - T_t f = w. \end{cases}$$

Let $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ be given. There exists a unique $f \in \mathrm{H}_0^1(\Omega)$ such that $(zI - T_t) f = w$. Therefore, taking $g := \frac{1}{z} (Sf + \beta)$, we obtain $(z\widetilde{I} - \widetilde{T}_t)(g, f) = (\beta, w)$. The uniqueness of g follows immediately from the uniqueness of f and the first equation of the system above. The proof is complete. \Box

The following result shows that the eigenvalues of T_t are nondefective.

LEMMA 3.5. Suppose that $\mu \neq 0$ is an isolated eigenvalue of T_t . Then its ascent is one.

Proof. By contradiction. Let (μ, w) be an eigenpair of T_t , $\mu \neq 0$, and let us assume that T_t has a corresponding generalized eigenfunction, namely, $\exists \hat{w} \neq 0$ such that $T_t \hat{w} = \mu \hat{w} + w$. Since (μ, w) is an eigenpair of T_t , there exists $\beta \in \mathrm{H}^1_0(\Omega)^2$ such that (cf. (2.9) and (2.4)) (3.8)

$$a(\beta,\eta) + \frac{\kappa}{t^2} \left(\nabla w - \beta, \nabla v - \eta \right)_{0,\Omega} = \frac{1}{\mu} \left(\boldsymbol{\sigma} \nabla w, \nabla v \right)_{0,\Omega} \qquad \forall (\eta,v) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega).$$

On the other hand, since $T_t \hat{w} = \mu \hat{w} + w$, the definition of T_t implies the existence of $\hat{\beta} \in \mathrm{H}^1_0(\Omega)^2$ such that

$$\begin{aligned} a(\hat{\beta},\eta) + \frac{\kappa}{t^2} (\nabla (w + \mu \hat{w}) - \hat{\beta}, \nabla v - \eta)_{0,\Omega} &= (\boldsymbol{\sigma} \nabla \hat{w}, \nabla v)_{0,\Omega} \\ \forall (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega). \end{aligned}$$

Defining $\bar{\beta} := (\hat{\beta} - \beta)/\mu$, the equation above can be written as follows:

$$\mu a(\bar{\beta},\eta) + a(\beta,\eta) + \frac{\kappa\mu}{t^2} (\nabla \hat{w} - \bar{\beta}, \nabla v - \eta)_{0,\Omega} + \frac{\kappa}{t^2} (\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma} \nabla \hat{w}, \nabla v)_{0,\Omega}.$$

We now take $(\eta, v) = \mu(\bar{\beta}, \hat{w})$ in (3.8) and $(\eta, v) = (\beta, w)$ in the equation above and we subtract the resulting equations. Using also the symmetry of $a(\cdot, \cdot)$ and σ , we obtain

$$a(\beta,\beta) + \frac{\kappa}{t^2} \left\| \nabla w - \beta \right\|_{0,\Omega}^2 = 0.$$

Thus, from the ellipticity of $a(\cdot, \cdot)$, we infer $\beta = 0$ and hence w = 0, which is a contradiction since w is an eigenfunction of T_t . The proof is complete. \Box

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. The proof follows easily by combining Lemma 3.4 with Theorem 3.2, Proposition 3.3, and Lemma 3.5. \Box

3.2. Limit problem. In this section we study the convergence properties of the operator T_t as t goes to zero. First, let us recall that it is well-known (see [4]) that, when t goes to zero, the solution (β, w, γ) of problem (2.5) converges to the solution $(\beta_0, w_0, \gamma_0) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega) \times \mathrm{H}_0(\mathrm{rot}; \Omega)'$ of

(3.9)
$$\begin{cases} a(\beta_0,\eta) + \langle \gamma_0, \nabla v - \eta \rangle = (\boldsymbol{\sigma} \nabla f, \nabla v)_{0,\Omega} \qquad \forall (\eta,v) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega), \\ \nabla w_0 - \beta_0 = 0, \end{cases}$$

where, $\langle \cdot, \cdot \rangle$ stands for the duality pairing in H₀(rot; Ω). Problem (3.9) is a mixed formulation for the following well-posed problem:

Find $w_0 \in \mathrm{H}^2_0(\Omega)$ such that

$$\frac{E}{12(1-\nu^2)} \left(\Delta w_0, \Delta v\right)_{0,\Omega} = \left(\boldsymbol{\sigma} \nabla f, \nabla v\right)_{0,\Omega} \qquad \forall v \in \mathrm{H}^2_0(\Omega)$$

Let T_0 be the bounded linear operator defined by

$$T_0: \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega),$$
$$f \mapsto w_0,$$

where (β_0, w_0, γ_0) is the solution of problem (3.9). Since $w_0 \in H^2_0(\Omega)$, the operator T_0 is compact. Hence, apart from $\mu_0 = 0$, the spectrum of T_0 consists of a sequence of finite multiplicity isolated eigenvalues converging to zero. The following lemma, which yields the convergence in norm of T_t to T_0 has been essentially proved in [10, Lemma 3.1].

LEMMA 3.6. There exists a constant C, independent of t, such that

$$\|(T_t - T_0) f\|_{1,\Omega} \le Ct \|f\|_{1,\Omega} \qquad \forall f \in \mathrm{H}^1_0(\Omega).$$

As a consequence of this lemma, standard properties about the separation of isolated parts of the spectrum (see [14], for instance) yield the following result.

LEMMA 3.7. Let μ_0 be an eigenvalue of T_0 of multiplicity m. Let D be any disc in the complex plane centered at μ_0 and containing no other element of the spectrum of T_0 . Then there exists $t_0 > 0$ such that, $\forall t < t_0$, D contains exactly m isolated eigenvalues of T_t (repeated according to their respective multiplicities). Consequently, each eigenvalue μ_0 of T_0 is a limit of isolated eigenvalues μ_t of T_t , as t goes to zero.

Our next goal is to show that the largest eigenvalues of T_t converge to the largest eigenvalues of T_0 as t goes to zero. With this aim, we prove first the following lemma. Here and thereafter, we will use $\|\cdot\|$ to denote the operator norm induced by the $\mathrm{H}^1(\Omega)$ norm.

LEMMA 3.8. Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \operatorname{Sp}(T_0) = \emptyset$. Then there exist strictly positive constants t_0 and C such that, $\forall t < t_0$, $F \cap \operatorname{Sp}(T_t) = \emptyset$ and

$$||R_{z}(T_{t})|| := \sup_{\substack{w \in \mathrm{H}_{0}^{1}(\Omega) \\ w \neq 0}} \frac{||R_{z}(T_{t})w||_{1,\Omega}}{||w||_{1,\Omega}} \le C \qquad \forall z \in F.$$

Proof. The mapping $z \mapsto || (zI - T_0)^{-1} ||$ is continuous for all $z \in \rho(T_0)$ and goes to zero as $|z| \to \infty$. Consequently, it attains its maximum on any closed subset $F \subset \rho(T_0)$. Let $C_1 := 1/\max_{z \in F} || (zI - T_0)^{-1} ||$; there holds

$$\|(zI - T_0)w\|_{1,\Omega} \ge \frac{1}{C_1} \|w\|_{1,\Omega} \qquad \forall w \in \mathrm{H}^1_0(\Omega) \quad \forall z \in F.$$

Now, according to Lemma 3.6, there exists $t_1 > 0$ such that, for all $t < t_1$,

$$\|(T_t - T_0) w\|_{1,\Omega} \le \frac{1}{2C_1} \|w\|_{1,\Omega} \qquad \forall w \in \mathrm{H}^1_0(\Omega).$$

Therefore, for all $w \in H_0^1(\Omega)$, for all $z \in F$, and for all $t < t_1$,

(3.10)
$$\|(zI - T_t)w\|_{1,\Omega} \ge \|(zI - T_0)w\|_{1,\Omega} - \|(T_t - T_0)w\|_{1,\Omega} \ge \frac{1}{2C_1} \|w\|_{1,\Omega}$$

and, consequently, $z \notin \text{Sp}_{d}(T_{t})$.

On the other hand, since $\operatorname{Sp}(T_0) \ni 0$, $F \cap \operatorname{Sp}(T_0) = \emptyset$, and F is closed, we have that $d := \min_{z \in F} |z|$ is strictly positive. Let $t_2 > 0$ be such $\kappa^{-1}t_2^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega} < d$. Hence, for all $z \in F$ and for all $t < t_2$, $|z| > \kappa^{-1}t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$ and, by virtue of Theorem 3.1, either $z \in \operatorname{Sp}_d(T_t)$ or $z \notin \operatorname{Sp}(T_t)$.

Altogether, if $t_0 := \min\{t_1, t_2\}$, then $(zI - T_t)$ is invertible for all $t < t_0$ and all $z \in F$. Moreover, because of (3.10),

$$|R_z(T_t)|| = ||(zI - T_t)^{-1}|| \le 2C_1$$

and we conclude the proof. \Box

Since T_0 is a compact operator, its nonzero eigenvalues are isolated and we can order them as follows:

$$|\mu_0^{(1)}| \ge |\mu_0^{(2)}| \ge \dots \ge |\mu_0^{(k)}| \ge \dots$$

where each eigenvalue is repeated as many times as its corresponding multiplicity. According to Lemma 3.7, for t sufficiently small there exist eigenvalues of T_t close to each $\mu_0^{(k)}$. On the other hand, according to Theorem 3.1, the essential spectrum of T_t is confined in a ball centered at the origin of the complex plane with radius proportional to t^2 . Therefore, at least for t sufficiently small, the points of the spectrum of T_t largest in modulus have to be isolated eigenvalues. Hence we order them as we did with those of T_0 :

$$|\mu_t^{(1)}| \ge |\mu_t^{(2)}| \ge \dots \ge |\mu_t^{(k)}| \ge \dots$$

The following theorem shows that the k-th eigenvalue of T_t converge to the k-th eigenvalue of T_0 as t goes to zero.

THEOREM 3.9. Let $\mu_t^{(k)}$, $k \in \mathbb{N}$, $t \ge 0$, be as defined above. For all $k \in \mathbb{N}$, $\mu_t^{(k)} \to \mu_0^{(k)}$ as $t \to 0$.

Proof. We will prove the result for the first eigenvalue $\mu_t^{(1)}$. The proof for the others is a straightforward modification of this one.

Let D_0 be an open disk in the complex plane centered at $\mu_0^{(1)}$ with radius $r_0 < (|\mu_0^{(1)}| - |\mu_0^{(k)}|)/2$, where $\mu_0^{(k)}$ is the first eigenvalue of T_0 such that $|\mu_0^{(1)}| > |\mu_0^{(k)}|$. Let D_1 be another open disk in the complex plane centered at the origin with

Let D_1 be another open disk in the complex plane centered at the origin with radius $r_1 := |\mu_0^{(1)}| - r_0$. Therefore $\operatorname{Sp}(T_0) \subset D_0 \cup D_1$. Let $F := \mathbb{C} \setminus (D_0 \cup D_1)$. F is a closed set and $F \cap \operatorname{Sp}(T_0) = \emptyset$. Hence, according to Lemma 3.8, there exists $t_0 > 0$ such that, for all $t < t_0$, $F \cap \operatorname{Sp}(T_t) = \emptyset$, too, and hence $\operatorname{Sp}(T_t) \subset D_0 \cup D_1$, as well.

On the other hand, because of Lemma 3.7, there exists $t_1 > 0$ such that, for all $t < t_1$, D_0 contains eigenvalues of T_t . Therefore, for all $t < \min\{t_0, t_1\}$, the eigenvalue of T_t largest in modulus, $\mu_t^{(1)}$, has to lie in D_0 . Since D_0 can be taken arbitrarily small, we conclude that $\mu_t^{(1)}$ converges to $\mu_0^{(1)}$ as t goes to zero. Thus, we conclude the proof. \Box

3.3. Additional regularity of the eigenfunctions. The aim of this section is to prove a regularity result for the eigenfunctions of problem (2.4). More precisely, we have the following proposition.

PROPOSITION 3.10. Let $\mu_t^{(k)}$, $k \in \mathbb{N}$, $t \ge 0$, be as in Theorem 3.9. Let $(\lambda, \beta, w, \gamma)$ be a solution of problem (2.4) with $\lambda = 1/\mu_t^{(k)}$. Then there exists $t_0 > 0$ such that, for all $t < t_0$, $\beta \in \mathrm{H}^2(\Omega)^2$, $w \in \mathrm{H}^2(\Omega)$, div $\gamma \in \mathrm{L}^2(\Omega)$, and there holds

$$\|\beta\|_{2,\Omega} \le C |\lambda| \|w\|_{1,\Omega}$$

$$\|w\|_{2,\Omega} \le C |\lambda| \|w\|_{1,\Omega}$$

 $(3.13) \qquad \qquad \|\operatorname{div} \gamma\|_{0,\Omega} \le C |\lambda| \, \|w\|_{2,\Omega} \,,$

with C a positive constant independent of t.

Proof. Using the Helmholtz decomposition (2.6), problem (2.4) is equivalent to the following one:

 $\operatorname{Find} \lambda \stackrel{\scriptstyle \smile}{\in} \mathbb{R} \ \operatorname{and} \ 0 \neq (\psi,\beta,p,w) \in \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1(\Omega) / \mathbb{R} \times \mathrm{H}^1_0(\Omega) \ \operatorname{such} \ \operatorname{that}$

$$\begin{split} (\nabla\psi,\nabla v)_{0,\Omega} &= \lambda \left(\boldsymbol{\sigma} \nabla w, \nabla v \right)_{0,\Omega} \quad \forall v \in \mathrm{H}_{0}^{1}(\Omega), \\ a(\beta,\eta) - (\operatorname{curl} p,\eta)_{0,\Omega} &= (\nabla\psi,\eta)_{0,\Omega} \quad \forall \eta \in \mathrm{H}_{0}^{1}(\Omega)^{2}, \\ - (\beta,\operatorname{curl} q)_{0,\Omega} - \kappa^{-1}t^{2} \left(\operatorname{curl} p,\operatorname{curl} q\right)_{0,\Omega} = 0 \quad \forall q \in \mathrm{H}^{1}(\Omega)/\mathbb{R}, \\ (\nabla w,\nabla\xi)_{0,\Omega} &= (\beta,\nabla\xi)_{0,\Omega} + \kappa^{-1}t^{2} \left(\nabla\psi,\nabla\xi\right)_{0,\Omega} \quad \forall \xi \in \mathrm{H}_{0}^{1}(\Omega). \end{split}$$

From Theorem 2.1 applied to the problem above, we immediately obtain that $\beta \in H^2(\Omega)^2$ and the estimate (3.11).

On the other hand, the first and the last equations from the system above lead to

$$\left(\left(\boldsymbol{I}-\lambda\kappa^{-1}t^{2}\boldsymbol{\sigma}\right)\nabla w,\nabla\xi\right)_{0,\Omega}=\left(\beta,\nabla\xi\right)_{0,\Omega}\qquad\forall\xi\in\mathrm{H}_{0}^{1}(\Omega).$$

Since $\mu_t^{(k)} \to \mu_0^{(k)} \neq 0$ as $t \to 0$, there exists $t_1 > 0$ such that $|\mu_t^{(k)}| > |\mu_0^{(k)}|/2 \quad \forall t < t_1$. Hence $\lambda = 1/\mu_t^{(k)} < 2/|\mu_0^{(k)}|$. We take $t_0 < t_1$ such that $\kappa^{-1}t_0^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega} < |\mu_0^{(k)}|/2$. Therefore, for all $t < t_0$, $(\boldsymbol{I} - \lambda \kappa^{-1}t^2\boldsymbol{\sigma})$ is uniformly positive definite. Thus, since w is the solution of the problem

$$\begin{cases} \operatorname{div} \left[\left(\boldsymbol{I} - \lambda \kappa^{-1} t^2 \boldsymbol{\sigma} \right) \nabla w \right] = \operatorname{div} \beta & \text{in } \Omega, \\ w = 0 & \text{on } \partial \Omega, \end{cases}$$

using a standard regularity result (see [21]), we have that $w \in \mathrm{H}^{2}(\Omega)$ and

$$\left\|w\right\|_{2,\Omega} \le C \left\|\operatorname{div}\beta\right\|_{0,\Omega} \le C \left\|\beta\right\|_{1,\Omega} \le C \left|\lambda\right| \left\|w\right\|_{1,\Omega},$$

the last inequality because of (3.11).

Furthermore, taking $\eta = 0$ in (2.4), using the estimate above and (2.3), it follows that

$$\operatorname{div} \gamma = \lambda \operatorname{div}(\boldsymbol{\sigma} \nabla w) \in \mathrm{L}^2(\Omega).$$

and

$$\|\operatorname{div} \gamma\|_{0,\Omega} \le C |\lambda| \|w\|_{2,\Omega}.$$

The proof is complete. \Box

4. Spectral approximation. For the numerical approximation, we focus on the finite element method proposed and studied in [11]. In what follows we introduce briefly this method (see this reference for further details). Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangular meshes of $\overline{\Omega}$. We will define finite element spaces H_h , W_h , and Γ_h for the rotations, the transverse displacement, and the shear strain, respectively.

For $K \in \mathcal{T}_h$, let $\alpha_1, \alpha_2, \alpha_3$ be its barycentric coordinates. We denote by τ_i a unit vector tangent to the edge $\alpha_i = 0$ and define

$$p_1^K = \alpha_2 \alpha_3 \tau_1, \qquad p_2^K = \alpha_1 \alpha_3 \tau_2, \qquad p_3^K = \alpha_1 \alpha_2 \tau_3.$$

The finite element space for the rotations is defined by

$$H_h := \left\{ \eta_h \in \mathrm{H}^1_0(\Omega)^2 : \, \eta_h|_K \in \mathbb{P}^2_1 \oplus \langle p_1^K, p_2^K, p_3^K \rangle \quad \forall K \in \mathcal{T}_h \right\}.$$

To approximate the transverse displacements, we use the usual piecewise-linear continuous finite element space:

$$W_h := \left\{ v_h \in \mathrm{H}^1_0(\Omega) : v_h |_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h \right\}.$$

Finally, for the shear strain, we use the lowest-order rotated Raviart-Thomas space:

$$\Gamma_h := \left\{ \phi \in \mathcal{H}_0(\operatorname{rot}; \Omega) : \phi |_K \in \mathbb{P}_0^2 \oplus (x_2, -x_1) \mathbb{P}_0 \quad \forall K \in \mathcal{T}_h \right\}.$$

We consider the reduction operator

$$R: \operatorname{H}^{1}(\Omega)^{2} \cap \operatorname{H}_{0}(\operatorname{rot}; \Omega) \to \Gamma_{h},$$

which is uniquely determined by

$$\int_{\ell} R\phi \cdot \tau_{\ell} = \int_{\ell} \phi \cdot \tau_{\ell},$$

for every edge ℓ of the triangulation. Above, τ_{ℓ} denotes a unit vector tangent to ℓ . It is well-known that

(4.1)
$$\|R\phi\|_{0,\Omega} \le C \, \|\phi\|_{1,\Omega} \qquad \forall \phi \in \mathrm{H}^1(\Omega)^2,$$

(4.2)
$$\|\phi - R\phi\|_{0,\Omega} \le Ch \|\phi\|_{1,\Omega} \qquad \forall \phi \in \mathrm{H}^1(\Omega)^2.$$

Moreover, the operator R can be extended continuously to $\mathrm{H}^{s}(\Omega)^{2} \cap \mathrm{H}_{0}(\mathrm{rot};\Omega)$ for any s > 0 and it is also well known that, for all $v \in \mathrm{H}^{1+s}(\Omega) \cap \mathrm{H}_{0}^{1}(\Omega)$,

$$(4.3) R(\nabla v) = \nabla v_{\rm I},$$

where $v_{\mathbf{I}} \in W_h$ is the standard piecewise-linear Lagrange interpolant of v (which is well defined because $\mathrm{H}^{1+s}(\Omega) \subset \mathcal{C}(\bar{\Omega})$ for all s > 0).

The discretization of problem (2.4) is as follows:

Find $\lambda_h \in \mathbb{R}$ and $0 \neq (\beta_h, w_h) \in H_h \times W_h$ such that (4.4) $\begin{cases}
a(\beta_h, \eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = \lambda_h \left(\boldsymbol{\sigma} \nabla w_h, \nabla v_h\right)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\
\gamma_h = \frac{\kappa}{t^2} \left(\nabla w_h - R\beta_h\right).
\end{cases}$

The corresponding discrete source problem reads:

Given $f \in H_0^1(\Omega)$, find $(\beta_h, w_h) \in H_h \times W_h$ such that (4.5) $\begin{cases}
a(\beta_h, \eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\boldsymbol{\sigma} \nabla f, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\
\gamma_h = \frac{\kappa}{t^2} (\nabla w_h - R\beta_h).
\end{cases}$

The method is nonconforming, since consistency terms arise because of the reduction operator R. Existence and uniqueness of the solution to problem (4.5) follow easily (see [11]).

Remark 4.1. Given a generic $f \in H_0^1(\Omega)$, the usual convergence rate in terms of positive powers of the mesh-size h does not hold for problems (2.5) and (4.5). Indeed, the right-hand sides of both problems are not regular enough, since div $(\boldsymbol{\sigma} \nabla f) \notin L^2(\Omega)$. However, whenever f is more regular, for instance assuming $f \in H^2(\Omega)$ and taking into account the regularity of $\boldsymbol{\sigma}$ (cf. (2.3)), the convergence results of [11] can be applied, to obtain

$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} + \|w - w_h\|_{1,\Omega} \le Ch \|f\|_{2,\Omega}.$$

As for the continuous case, we introduce the operator

$$T_{th}: \operatorname{H}^{1}_{0}(\Omega) \to W_{h} \hookrightarrow \operatorname{H}^{1}_{0}(\Omega),$$
$$f \mapsto w_{h},$$

where (β_h, w_h) is the solution of problem (4.5). The following lemma shows that this operator is bounded uniformly in t and h.

LEMMA 4.1. There exists C > 0 such that $||T_{th}|| \leq C$ for all t > 0 and all h > 0.

Proof. Let $f \in H_0^1(\Omega)$ and (β_h, w_h) be the solution of problem (4.5). Taking $(\eta_h, v_h) = (\beta_h, w_h)$ as test function in (4.5), we obtain

$$a(\beta_h, \beta_h) + \kappa^{-1} t^2 \left\| \gamma_h \right\|_{0,\Omega}^2 \le \left\| \boldsymbol{\sigma} \right\|_{\infty,\Omega} \left\| \nabla f \right\|_{0,\Omega} \left\| \nabla w_h \right\|_{0,\Omega},$$

where $\gamma_h = \frac{\kappa}{t^2} \left(\nabla w_h - R\beta_h \right)$ as in (4.5). Hence, from the ellipticity of $a(\cdot, \cdot)$,

$$\|\beta_{h}\|_{1,\Omega}^{2} + \kappa^{-1}t^{2} \|\gamma_{h}\|_{0,\Omega}^{2} \leq C \|\boldsymbol{\sigma}\|_{\infty,\Omega} \|\nabla f\|_{0,\Omega} \|\nabla w_{h}\|_{0,\Omega}.$$

Therefore, using (4.1),

$$\left\|\nabla w_{h}\right\|_{0,\Omega}^{2} = \left\|\kappa^{-1}t^{2}\gamma_{h} + R\beta_{h}\right\|_{0,\Omega}^{2} \leq C \left\|\boldsymbol{\sigma}\right\|_{\infty,\Omega} \left\|\nabla f\right\|_{0,\Omega} \left\|\nabla w_{h}\right\|_{0,\Omega},$$

which allows us to conclude the proof. \Box

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Clearly, the nonzero eigenvalues of T_{th} are given by $\mu_h := 1/\lambda_h$, with λ_h being the eigenvalues of (4.4), and the corresponding eigenfunctions coincide. In what follows, we will prove a characterization of the discrete spectral problem (4.4).

LEMMA 4.2. Let $Y_h := \left\{ w_h \in W_h : (\boldsymbol{\sigma} \nabla w_h, \nabla v_h)_{0,\Omega} = 0 \quad \forall v_h \in W_h \right\}$. Then problem (4.4) has exactly dim W_h – dim Y_h eigenvalues, repeated according to their respective multiplicities. All of them are real and nonzero.

Proof. We eliminate γ_h in problem (4.4) to write it as follows:

(4.6)
$$a(\beta_h, \eta_h) + \frac{\kappa}{t^2} \left(\nabla w_h - R\beta_h, \nabla v_h - R\eta_h \right)_{0,\Omega} = \lambda_h \left(\boldsymbol{\sigma} \nabla w_h, \nabla v_h \right)_{0,\Omega} \\ \forall (\eta_h, v_h) \in H_h \times W_h.$$

Taking particular bases of H_h and W_h , this problem can be written in matrix form as follows:

(4.7)
$$\boldsymbol{\mathcal{A}} \begin{bmatrix} \boldsymbol{\beta}_h \\ \boldsymbol{w}_h \end{bmatrix} = \lambda_h \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_h \\ \boldsymbol{w}_h \end{bmatrix},$$

where β_h and w_h denote the vectors whose entries are the components in those basis of β_h and w_h , respectively. The matrix \mathcal{A} is symmetric and positive definite because the bilinear form on the left-hand side of (4.6) is elliptic on $\mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ (cf. [11]). Consequently, $\lambda_h \neq 0$ and, since \boldsymbol{E} is also symmetric, $\lambda_h \in \mathbb{R}$. Now, (4.7) holds true if and only if

$$egin{bmatrix} oldsymbol{0} & oldsymbol{0} \ oldsymbol{0} & oldsymbol{E} \end{bmatrix} egin{bmatrix} oldsymbol{eta}_h \ oldsymbol{w}_h \end{bmatrix} = \mu_h oldsymbol{\mathcal{A}} egin{bmatrix} oldsymbol{eta}_h \ oldsymbol{w}_h \end{bmatrix}$$

with $\lambda_h = 1/\mu_h$ and $\mu_h \neq 0$. The latter is a well-posed generalized eigenvalue problem with dim W_h – dim Ker(\boldsymbol{E}) nonzero eigenvalues. Thus, we conclude the lemma by noting that $\boldsymbol{E}\boldsymbol{w}_h = \boldsymbol{0}$ if and only if $w_h \in Y_h$. \Box

Remark 4.2. If $(\lambda_h, \beta_h, w_h)$ is a solution of problem (4.4), then

$$\boldsymbol{w}_h^T \boldsymbol{E} \boldsymbol{w}_h = (\boldsymbol{\sigma} \nabla w_h, \nabla w_h)_{0,\Omega} \neq 0.$$

In fact, this follows by left multiplying both sides of (4.7) by $(\boldsymbol{\beta}_h, \boldsymbol{w}_h)$ and using the positive definiteness of $\boldsymbol{\mathcal{A}}$.

We will adapt the theory in [8, 9] to prove convergence of our spectral approximation and to obtain error estimates. To this end, we will use the following properties:

P1.
$$||T_0 - T_{th}||_h := \sup_{\substack{f_h \in W_h \\ f_h \neq 0}} \frac{||(T_0 - T_{th}) f_h||_{1,\Omega}}{||f_h||_{1,\Omega}} \to 0, \quad \text{as } (h, t) \to (0, 0);$$

P2. $\forall u \in \mathrm{H}_0^1(\Omega), \quad \inf_{v_h \in W_h} ||u - v_h||_{1,\Omega} \to 0, \quad \text{as } h \to 0.$

From now on, we will use the operator norm $\|\cdot\|_h$ as defined in property P1.

Since property P2 follows from standard approximation results, in the sequel we focus on property P1. We first notice that

(4.8)
$$\|T_0 - T_{th}\|_h \le \|T_0 - T_t\|_h + \|T_t - T_{th}\|_h,$$

where T_t is the operator defined in (2.9). Since $W_h \subset H_0^1(\Omega)$, from Lemma 3.6 we deduce that for all h > 0

(4.9)
$$||T_0 - T_t||_h \le Ct.$$

Regarding the other term in the right-hand side of (4.8), we aim at proving the following result.

PROPOSITION 4.3. Suppose that the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform. Then we have

$$\left\|T_t - T_{th}\right\|_h \le C \left(h + t\right).$$

To give the proof of Proposition 4.3, we consider problems (2.5) and (4.5) with source term in W_h , namely:

Given $f_h \in W_h$, find $(\beta, w) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega)$ such that (4.10) $\begin{cases}
a(\beta, \eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma} \nabla f_h, \nabla v)_{0,\Omega} & \forall (\eta, v) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega), \\
\gamma = \frac{\kappa}{t^2} (\nabla w - \beta).
\end{cases}$

Given $f_h \in W_h$, find $(\beta_h, w_h) \in H_h \times W_h$ such that (4.11) $\begin{cases} a(\beta_h, \eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\boldsymbol{\sigma} \nabla f_h, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h = \frac{\kappa}{t^2} (\nabla w_h - R\beta_h). \end{cases}$

We need some results concerning the solutions of problems (4.10) and (4.11). First, we apply the Helmholtz decomposition (2.6) to the term γ from (4.10):

(4.12)
$$\gamma = \nabla \psi + \operatorname{curl} p, \quad \psi \in \mathrm{H}^{1}_{0}(\Omega), \ p \in \mathrm{H}^{1}(\Omega)/\mathbb{R}.$$

Then we apply Theorem 2.1 and (2.8), to obtain the following a priori estimate for the solution to problem (4.10):

$$(4.13) \|\psi\|_{1,\Omega} + \|\beta\|_{2,\Omega} + \|w\|_{1,\Omega} + \|p\|_{1,\Omega} + t \|p\|_{2,\Omega} + \|\gamma\|_{0,\Omega} \le C \|f_h\|_{1,\Omega}.$$

The following result shows that w and ψ are actually smoother, because $f_h \in W_h$, and an inverse estimate which will be used to prove Proposition 4.3.

LEMMA 4.4. Let w be defined by problem (4.10) and ψ as in (4.12). Then $w, \psi \in \mathrm{H}^{1+s}(\Omega)$ for all $s \in (0, \frac{1}{2})$. Moreover, if the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, then

$$\|\psi\|_{1+s,\Omega} \le Ch^{-s} \|f_h\|_{1,\Omega}.$$

Proof. Recalling the equivalence between problems (4.10) and (2.7), the latter with source term f_h instead of f, from the first equation of (2.7) we have that ψ is the weak solution of

(4.14)
$$\begin{cases} \Delta \psi = \operatorname{div}(\boldsymbol{\sigma} \nabla f_h), \\ \psi = 0 \quad \text{on } \partial \Omega. \end{cases}$$

Since f_h is a continuous piecewise linear function, we have that $f_h \in \mathrm{H}^{1+s}(\Omega) \ \forall s \in (0, \frac{1}{2})$. Therefore, the assumption (2.3) implies $\boldsymbol{\sigma} \nabla f_h \in \mathrm{H}^s(\Omega)^2$. Hence, $\operatorname{div}(\boldsymbol{\sigma} \nabla f_h) \in \mathrm{H}^{s-1}(\Omega)$. Then, from standard regularity results for problem (4.14), $\psi \in \mathrm{H}^{1+s}(\Omega) \ \forall s \in (0, \frac{1}{2})$ and

$$\left\|\psi\right\|_{1+s,\Omega} \leq C \left\|\operatorname{div}(\boldsymbol{\sigma}\nabla f_h)\right\|_{s-1,\Omega} \leq C \left\|f_h\right\|_{1+s,\Omega}.$$

If the family of meshes is quasi-uniform, then the inverse inequality $||f_h||_{1+s,\Omega} \leq Ch^{-s} ||f_h||_{1,\Omega}$ holds true and from this and the estimate above we obtain

$$\left\|\psi\right\|_{1+s,\Omega} \le Ch^{-s} \left\|f_h\right\|_{1,\Omega}$$

On the other hand, from the last equation of (2.7) we have that

$$\left(\nabla\left(w-\kappa^{-1}t^{2}\psi\right),\nabla\xi\right)_{0,\Omega}=\left(\beta,\nabla\xi\right)_{0,\Omega}\qquad\forall\xi\in\mathrm{H}_{0}^{1}(\Omega)$$

Therefore, $(w - \kappa^{-1}t^2\psi)$ is the weak solution to the problem

$$\begin{cases} \Delta \left(w - \kappa^{-1} t^2 \psi \right) = \operatorname{div} \beta \in \mathrm{L}^2(\Omega), \\ \left(w - \kappa^{-1} t^2 \psi \right) = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Hence, $(w - \kappa^{-1}t^2\psi) \in \mathrm{H}^2(\Omega)$ (recall Ω is convex) and $w = (w - \kappa^{-1}t^2\psi) + \kappa^{-1}t^2\psi \in \mathrm{H}^{1+s}(\Omega)$ for all $s \in (0, \frac{1}{2})$. Thus the proof is complete. \Box

The following lemma is the key point to prove Proposition 4.3.

LEMMA 4.5. If (β, w) and (β_h, w_h) are the solutions of (4.10) and (4.11), respectively, then

$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} \le C (h+t) \|f_h\|_{1,\Omega}.$$

Proof. It has been proved in [11] (see Example 4.1 from this reference) that there exists $\widetilde{\beta} \in H_h$ such that

$$\begin{split} R\widetilde{\boldsymbol{\beta}} &= R\boldsymbol{\beta}, \\ \|\boldsymbol{\beta} - \widetilde{\boldsymbol{\beta}}\|_{1,\Omega} \leq Ch \, \|\boldsymbol{\beta}\|_{2,\Omega} \end{split}$$

Let

$$\widetilde{\gamma} := \frac{\kappa}{t^2} (\nabla w_{\mathrm{I}} - R\widetilde{\beta}),$$

where the Lagrange interpolant $w_{I} \in W_{h}$ is well defined because of Lemma 4.4. Because of (4.3) and the equation above,

$$\widetilde{\gamma} = R\gamma.$$

It has also been proved in [11] that

$$\|\widetilde{\beta} - \beta_h\|_{1,\Omega} + t \,\|\widetilde{\gamma} - \gamma_h\|_{0,\Omega} \le C \left(\|\widetilde{\beta} - \beta\|_{1,\Omega} + t \,\|\widetilde{\gamma} - \gamma\|_{0,\Omega} + h \,\|\gamma\|_{0,\Omega} \right).$$

Hence, by adding and subtracting $\tilde{\beta}$ and $\tilde{\gamma} = R\gamma$, we obtain

$$\left\|\beta - \beta_{h}\right\|_{1,\Omega} + t \left\|\gamma - \gamma_{h}\right\|_{0,\Omega} \le C \left(\left\|\beta - \widetilde{\beta}\right\|_{1,\Omega} + t \left\|\gamma - R\gamma\right\|_{0,\Omega} + h \left\|\gamma\right\|_{0,\Omega}\right).$$

The first and last term in the right-hand side above are already bounded. To estimate the second one, we use (4.12), Lemma 4.4, and (4.3), to obtain

(4.15)
$$\|\gamma - R\gamma\|_{0,\Omega} \le \|\nabla\psi - \nabla\psi_{\mathbf{I}}\|_{0,\Omega} + \|\operatorname{curl} p - R(\operatorname{curl} p)\|_{0,\Omega}.$$

Next, from standard error estimates for the Lagrange interpolant, we have

$$\left\|\nabla\psi - \nabla\psi_{\mathbf{I}}\right\|_{0,\Omega} \le Ch^{s} \left\|\psi\right\|_{1+s,\Omega},$$

whereas from (4.2) and the fact that $p \in H^2(\Omega)$ (cf. (4.13))

$$\|\operatorname{curl} p - R(\operatorname{curl} p)\|_{0,\Omega} \le Ch \, \|p\|_{2,\Omega} \, .$$

Thus, by using Lemma 4.4, we conclude

$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} \le C \left(h \|\beta\|_{2,\Omega} + t \|f_h\|_{1,\Omega} + th \|p\|_{2,\Omega} + h \|\gamma\|_{0,\Omega} \right)$$

$$\le C (h+t) \|f_h\|_{1,\Omega},$$

where we have used (4.13) for the last inequality. The proof is complete. \Box

We are now in a position to prove Proposition 4.3.

Proof of Proposition 4.3. Under the same assumptions of Lemma 4.5, we need to prove that

$$||w - w_h||_{1,\Omega} \leq C (h+t) ||f_h||_{1,\Omega}.$$

From problems (4.10) and (4.11), we have

$$\nabla w - \nabla w_h = \kappa^{-1} t^2 \left(\gamma - \gamma_h \right) + \left(\beta - R \beta_h \right).$$

Adding and subtracting $R\beta$, we obtain

(4.16)
$$\|\nabla w - \nabla w_h\|_{0,\Omega} \le \kappa^{-1} t^2 \|\gamma - \gamma_h\|_{0,\Omega} + \|\beta - R\beta\|_{0,\Omega} + \|R(\beta - \beta_h)\|_{0,\Omega}$$

Hence, using Poincaré inequality, (4.1), Lemma 4.5, (4.2), and (4.13), we have

$$||w - w_h||_{1,\Omega} \leq C (h+t) ||f_h||_{1,\Omega}$$

The proof is complete. \Box

We end this section by proving property P1.

LEMMA 4.6. Suppose that the family $\{T_h\}_{h>0}$ is quasi-uniform. Then we have

$$||T_0 - T_{th}||_h \le C (h+t)$$

Proof. The assertion follows immediately from estimate (4.8), by using (4.9) and Proposition 4.3. \Box

5. Error estimates. In this section we will adapt the arguments from [9] to prove error estimates for the approximate eigenvalues and eigenfunctions. Throughout this section, we will assume that the family of meshes $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, so that property P1 holds true, although such assumption is not actually necessary in some particular cases (see the appendix below).

Our first goal is to prove that, provided the plate is sufficiently thin, the numerical method does not introduce spurious modes with eigenvalues interspersed among the relevant ones of T_t (namely, around $\mu_t^{(k)}$ for small k). Let us remark that such a spectral pollution could be in principle expected from the fact that T_t has a nontrivial essential spectrum. However, that this is not the case is an immediate consequence of the following theorem, which is essentially identical to Lemma 1 from [8].

THEOREM 5.1. Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \operatorname{Sp}(T_0) = \emptyset$. There exist strictly positive constants h_0 , t_0 , and C such that, $\forall h < h_0$ and $\forall t < t_0$, there holds $F \cap \operatorname{Sp}(T_{th}) = \emptyset$ and

$$\|R_z(T_{th})\|_h \le C \qquad \forall z \in F.$$

Proof. The same arguments used to prove Lemma 3.8 (but using Lemma 4.6 instead of Lemma 3.6) allow us to show an estimate analogous to (3.10), namely, for all $w_h \in W_h$ and all $z \in F$,

$$\|(zI - T_{th}) w_h\|_{1,\Omega} \ge \|(zI - T_0) w_h\|_{1,\Omega} - \|(T_0 - T_{th}) w_h\|_{1,\Omega} \ge \frac{1}{2C_1} \|w_h\|_{1,\Omega},$$

provided h and t are small enough. Since W_h is finite dimensional, we deduce from above that $(zI - T_{th})|_{W_h}$ is invertible and, hence, $z \notin \operatorname{Sp}(T_{th}|_{W_h})$. Now, $\operatorname{Sp}(T_{th}) =$ $\operatorname{Sp}(T_{th}|_{W_h}) \cup \{0\}$ (see, for instance, [3, Lemma 4.1]) and, for $z \in F$, $z \neq 0$. Thus, $z \notin \operatorname{Sp}(T_{th})$ either. Then $(zI - T_{th})$ is invertible too and

$$||R_z(T_{th})||_h = ||(zI - T_{th})^{-1}||_h \le 2C_1 \quad \forall z \in F.$$

The proof is complete. \Box

We have already proved in Theorem 3.1 that the essential spectrum of T_t is confined to the real interval $(-\kappa^{-1}t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}, \kappa^{-1}t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega})$. The spectrum of T_t outside this interval consists of finite multiplicity isolated eigenvalues of ascent one, which converge to eigenvalues of T_0 , as t goes to zero (cf. Theorem 3.9). The eigenvalue of T_t with physical significance is the largest in modulus, $\mu_t^{(1)}$, which corresponds to the critical load that leads to buckling effects. This eigenvalue is typically simple and converges to a simple eigenvalue of T_0 , as t tends to zero. Because of this, for simplicity, from now on we restrict our analysis to simple eigenvalues.

Let $\mu_0 \neq 0$ be an eigenvalue of T_0 with multiplicity m = 1. Let D be a closed disk centered at μ_0 with boundary Γ such that $0 \notin D$ and $D \cap \operatorname{Sp}(T_0) = {\mu_0}$. Let $t_0 > 0$ be small enough, so that for all $t < t_0$:

- D contains only one eigenvalue of T_t , which we already know is simple (cf. Lemma 3.7) and
- *D* does not intersect the real interval $(-\kappa^{-1}t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}, \kappa^{-1}t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega})$, which contains the essential spectrum of T_t .

According to Theorem 5.1 there exist $t_0 > 0$ and $h_0 > 0$ such that $\forall t < t_0$ and $\forall h < h_0, \Gamma \subset \rho(T_{th})$. Moreover, proceeding as in [8, Section 2], from properties P1 and P2 it follows that, for h small enough, T_{th} has exactly one eigenvalue $\mu_{th} \in D$. The theory in [9] could be adapted too, to prove error estimates for the eigenvalues and eigenfunctions of T_{th} to those of T_0 as h and t go to zero. However, our goal is not this, but to prove that μ_{th} converges to μ_t as h goes to zero, with $t < t_0$ fixed, and to provide the corresponding error estimates for eigenvalues and eigenfunctions. With this aim, we will modify accordingly the theory from [9].

Let $\Pi_h : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ be the projector with range W_h defined for all $u \in \mathrm{H}^1_0(\Omega)$ by

$$\left(\nabla \left(\Pi_h u - u\right), \nabla v_h\right)_{0,\Omega} = 0 \qquad \forall v_h \in W_h.$$

The projector Π_h is bounded uniformly on h, namely, $\|\Pi_h u\|_{1,\Omega} \leq \|u\|_{1,\Omega}$, and the following error estimate is well known:

(5.1)
$$\|\Pi_h u - u\|_{1,\Omega} \le Ch \|u\|_{2,\Omega} \qquad \forall u \in \mathrm{H}^2(\Omega).$$

Let us define

$$B_{th} := T_{th} \Pi_h : \operatorname{H}^1_0(\Omega) \to W_h \hookrightarrow \operatorname{H}^1_0(\Omega).$$

It is clear that T_{th} and B_{th} have the same nonzero eigenvalues and corresponding eigenfunctions. Furthermore, we have the following result (cf. [9, Lemma 1]).

LEMMA 5.2. There exist h_0 , t_0 , and C such that

$$||R_z(B_{th})|| \le C \qquad \forall h < h_0, \quad \forall t < t_0, \quad \forall z \in \Gamma.$$

Proof. Since B_{th} is compact it suffices to verify that $||(zI - B_{th}) u||_{1,\Omega} \ge C ||u||_{1,\Omega}$ for all $u \in H_0^1(\Omega)$ and $z \in \Gamma$. Taking into account that $0 \notin \Gamma$ and using Theorem 5.1, we have

$$\|u\|_{1,\Omega} \le \|\Pi_h u\|_{1,\Omega} + \|u - \Pi_h u\|_{1,\Omega} \le C \|(zI - T_{th}) \Pi_h u\|_{1,\Omega} + |z|^{-1} \|z (u - \Pi_h u)\|_{1,\Omega}.$$

By using properties of the projector Π_h , we obtain

$$\begin{aligned} \|u\|_{1,\Omega} &\leq C \,\|(zI - B_{th}) \,\Pi_h u\|_{1,\Omega} + |z|^{-1} \,\|z \,(u - \Pi_h u) - B_{th} (u - \Pi_h u)\|_{1,\Omega} \\ &= C \,\|\Pi_h (zI - B_{th}) u\|_{1,\Omega} + |z|^{-1} \,\|(I - \Pi_h) \,(zI - B_{th}) \,u\|_{1,\Omega} \\ &\leq C \,\|(zI - B_{th}) \,u\|_{1,\Omega} \,. \end{aligned}$$

Thus we end the proof. \Box

Next, we introduce:

• $E_t : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$, the spectral projector of T_t corresponding to the isolated eigenvalue μ_t , namely,

$$E_t := \frac{1}{2\pi i} \int_{\Gamma} R_z(T_t) \, dz;$$

• F_{th} : $\mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$, the spectral projector of B_{th} corresponding to the eigenvalue μ_{th} , namely,

$$F_{th} := \frac{1}{2\pi i} \int_{\Gamma} R_z(B_{th}) \, dz.$$

As a consequence of Lemma 5.2, the spectral projectors F_{th} are bounded uniformly in h and t, for h and t small enough. Notice that, under our assumptions, $E_t(\mathrm{H}_0^1(\Omega))$ and $F_{th}(\mathrm{H}_0^1(\Omega))$ are both one dimensional. We have the following error estimate (cf. [9, Lemma 3]).

LEMMA 5.3. There exist positive constants h_0 , t_1 , and C, such that for all $h < h_0$ and for all $t < t_1$,

$$\| (E_t - F_{th}) |_{E_t(\mathrm{H}^1_0(\Omega))} \| \le C \| (T_t - B_{th}) |_{E_t(\mathrm{H}^1_0(\Omega))} \| \le Ch.$$

Proof. The first inequality is proved using the same arguments of [9, Lemma 3] and Lemmas 3.8 and 5.2. For the other estimate, fix $w \in E_t(\mathrm{H}^1_0(\Omega))$. From Proposition 3.10, Remark 4.1, Lemma 4.1, and (5.1), we have

$$\begin{aligned} \| (T_t - B_{th}) w \|_{1,\Omega} &\leq \| (T_t - T_{th}) w \|_{1,\Omega} + \| (T_{th} - B_{th}) w \|_{1,\Omega} \\ &\leq \| (T_t - T_{th}) w \|_{1,\Omega} + \| T_{th} \| \, \| (I - \Pi_h) \, w \|_{1,\Omega} \\ &\leq Ch \, \| w \|_{2,\Omega} \,. \end{aligned}$$

Therefore, by using (3.12), we conclude the proof. \Box

Our next goal is to prove an optimal order error estimate for the eigenfunctions. With this aim, we first need a preliminary result.

LEMMA 5.4. Let

$$\Lambda_{th} := F_{th}|_{E_t(\mathrm{H}^1_0(\Omega))} : E_t(\mathrm{H}^1_0(\Omega)) \to F_{th}(\mathrm{H}^1_0(\Omega)).$$

For h and t small enough, the operator Λ_{th} is invertible and

$$\left\|\Lambda_{th}^{-1}\right\| \le C,$$

with C independent of h and t.

Proof. See the proof of Theorem 1 in [9]. \Box

We recall the definition of the gap $\hat{\delta}$ between two closed subspaces Y and Z of $\mathrm{H}^{1}_{0}(\Omega)$:

$$\hat{\delta}(Y, Z) := \max \left\{ \delta(Y, Z), \delta(Z, Y) \right\},$$

where

$$\delta(Y, Z) := \sup_{\substack{y \in Y \\ \|y\|_{1,\Omega} = 1}} \left(\inf_{z \in Z} \|y - z\|_{1,\Omega} \right).$$

THEOREM 5.5. There exist constants h_0 , t_1 , and C, such that, for all $h < h_0$ and for all $t < t_1$, there holds

$$\hat{\delta}\left(F_{th}(\mathrm{H}^{1}_{0}(\Omega)), E_{t}(\mathrm{H}^{1}_{0}(\Omega))\right) \leq Ch.$$

Proof. It follows by arguing exactly as in the proof of Theorem 1 from [9], and using Lemmas 5.3 and 5.4. \Box

Next, we prove a preliminary sub-optimal error estimate for $|\mu_t - \mu_{th}|$, which will be improved below (cf. Theorem 5.8).

LEMMA 5.6. There exists a positive constant C such that, for h and t small enough,

$$|\mu_t - \mu_{th}| \le Ch.$$

Proof. We define the following operators:

$$\begin{aligned} \widehat{T}_t &:= T_t|_{E_t(\mathrm{H}_0^1(\Omega))} : E_t(\mathrm{H}_0^1(\Omega)) \to E_t(\mathrm{H}_0^1(\Omega)), \\ \widehat{B}_{th} &:= \Lambda_{th}^{-1} B_{th} \Lambda_{th} : E_t(\mathrm{H}_0^1(\Omega)) \to E_t(\mathrm{H}_0^1(\Omega)). \end{aligned}$$

The operator \hat{T}_t has a unique eigenvalue μ_t of multiplicity m = 1, while the unique eigenvalue of \hat{B}_{th} is μ_{th} .

Let $v \in E_t(\mathrm{H}_0^1(\Omega))$. Since $(\Lambda_{th}^{-1}F_{th} - I) T_t|_{E_t(\mathrm{H}_0^1(\Omega))} = 0$ and B_{th} commutes with its spectral projector F_{th} , we have

$$(\hat{T}_t - \hat{B}_{th})v = (T_t - B_{th})v + (\Lambda_{th}^{-1}F_{th} - I)(T_t - B_{th})v.$$

Therefore, using Lemmas 5.3 and 5.4 and the fact that $||F_{th}||$ is bounded uniformly in h and t, for h and t small enough, we obtain

$$\|(\hat{T}_t - \hat{B}_{th})v\|_{1,\Omega} \le \|(T_t - B_{th})v\|_{1,\Omega} + \|(\Lambda_{th}^{-1}F_{th} - I)(T_t - B_{th})v\|_{1,\Omega} \le Ch \|v\|_{1,\Omega}.$$

Hence, the lemma follows from the fact that $\hat{T}_t = \mu_t I$ and $\hat{B}_{th} = \mu_{th} I$.

Since the eigenvalue $\mu_t \neq 0$ of T_t corresponds to an eigenvalue $\lambda = 1/\mu_t$ of problem (2.4), Lemma 5.6 leads to a sub-optimal error estimate for the approximation of λ as well. We now aim at improving that result. Let $\lambda_h := 1/\mu_{th}, w_h, \beta_h$ and γ_h be such that $(\lambda_h, w_h, \beta_h, \gamma_h)$ is a solution of problem (4.4), with $||w_h||_{1,\Omega} = 1$. According to Theorem 5.5, there exists a solution $(\lambda, w, \beta, \gamma)$ of problem (2.4), with $||w||_{1,\Omega} = 1$, such that

$$\|w - w_h\|_{1,\Omega} \le Ch.$$

The following lemma will be used to prove a double order of convergence for the corresponding eigenvalues.

LEMMA 5.7. Let $(\lambda, w, \beta, \gamma)$ be a solution of (2.4), with $||w||_{1,\Omega} = 1$, and let $(\lambda_h, w_h, \beta_h, \gamma_h)$ be a solution of (4.4), with $||w_h||_{1,\Omega} = 1$, such that

$$\|w - w_h\|_{1,\Omega} \le Ch.$$

Then for h and t small enough there holds

(5.3)
$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} \le Ch$$

Proof. Let $\hat{w}_h \in W_h$, $\hat{\beta}_h \in H_h$ and $\hat{\gamma}_h$ be the solution of the auxiliary problem:

$$\begin{cases} a(\hat{\beta}_h, \eta_h) + (\hat{\gamma}_h, \nabla v_h - R\eta_h)_{0,\Omega} = \lambda \left(\boldsymbol{\sigma} \nabla w, \nabla v_h\right)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \hat{\gamma}_h = \frac{\kappa}{t^2} (\nabla \hat{w}_h - R \hat{\beta}_h). \end{cases}$$

This problem is the finite element discretization of (2.4), with source term $f = \lambda w \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$. Then from Remark 4.1, (3.12), and the fact that $||w_h||_{1,\Omega} = 1$, we obtain the following error estimate:

(5.4)
$$\|\beta - \hat{\beta}_h\|_{1,\Omega} + t \|\gamma - \hat{\gamma}_h\|_{0,\Omega} + \|w - \hat{w}_h\|_{1,\Omega} \le Ch |\lambda| \|w\|_{2,\Omega} \le Ch |\lambda|$$

On the other hand, from (4.4), we have that $(\beta_h - \hat{\beta}_h, w_h - \hat{w}_h) \in H_h \times W_h$ satisfies

$$\begin{cases} a(\beta_h - \hat{\beta}_h, \eta_h) + (\gamma_h - \hat{\gamma}_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\boldsymbol{\sigma} \nabla (\lambda_h w_h - \lambda w), \nabla v_h)_{0,\Omega} \\ \forall (\eta_h, v_h) \in H_h \times W_h, \end{cases}$$
$$\forall (\eta_h, v_h) \in H_h \times W_h, \end{cases}$$

Taking $\eta_h = \beta_h - \hat{\beta}_h$ and $v_h = w_h - \hat{w}_h$ in the system above, from the ellipticity of $a(\cdot, \cdot)$, we obtain

$$\begin{aligned} \|\beta_{h} - \hat{\beta}_{h}\|_{1,\Omega}^{2} + \kappa^{-1}t^{2} \|\gamma_{h} - \hat{\gamma}_{h}\|_{0,\Omega}^{2} \\ &\leq C \|\lambda_{h}w_{h} - \lambda w\|_{1,\Omega} \|w_{h} - \hat{w}_{h}\|_{1,\Omega} \\ &\leq C \left(|\lambda_{h}| \|w - w_{h}\|_{1,\Omega} + |\lambda - \lambda_{h}| \|w\|_{1,\Omega} \right) \left(\|w - w_{h}\|_{1,\Omega} + \|w - \hat{w}_{h}\|_{1,\Omega} \right) \\ &\leq Ch^{2}, \end{aligned}$$

where we have used Lemma 5.6 and estimates (5.2) and (5.4) for the last inequality. Therefore, we have

$$\|\beta_h - \hat{\beta}_h\|_{1,\Omega} + t \, \|\gamma_h - \hat{\gamma}_h\|_{0,\Omega} \le Ch.$$

Thus, the lemma follows from this estimate and (5.4).

We are now in a position to prove an optimal double-order error estimate for the eigenvalues.

THEOREM 5.8. There exist positive constants h_0 , t_1 , and C such that, $\forall h < h_0$ and $\forall t < t_1$,

$$|\lambda - \lambda_h| \le Ch^2.$$

Proof. We adapt to our case a standard argument for eigenvalue problems (see [2, Lemma 9.1]). Let $(\lambda, \beta, w, \gamma)$ and $(\lambda_h, \beta_h, w_h, \gamma_h)$ be as in Lemma 5.7. We will use the bilinear forms A and B defined in (3.1) and (3.2), respectively, as well as the bilinear form A_h defined in $H_h \times W_h$ as follows:

$$A_h((\beta_h, w_h), (\eta_h, v_h)) := a(\beta_h, \eta_h) + \frac{\kappa}{t^2} \left(\nabla w_h - R\beta_h, \nabla v_h - R\eta_h \right)_{0,\Omega}.$$

With this notation, problems (2.4) and (4.4) can be written as follows:

$$A((\beta, w), (\eta, v)) = \lambda B((\beta, w), (\eta, v)),$$

$$A_h((\beta_h, w_h), (\eta_h, v_h)) = \lambda_h B((\beta_h, w_h), (\eta_h, v_h)),$$

From these equations, straightforward computations lead to (5.5)

$$\begin{aligned} (\lambda_h - \lambda) B((\beta_h, w_h), (\beta_h, w_h)) &= A((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) \\ &- \lambda B((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) \\ &+ [A_h((\beta_h, w_h), (\beta_h, w_h)) - A((\beta_h, w_h), (\beta_h, w_h))]. \end{aligned}$$

Next, we define $\bar{\gamma}_h := \frac{\kappa}{t^2} (\nabla w_h - \beta_h)$. Recalling that $R \nabla w_h = \nabla w_h$ (cf. (4.3)), from the definition of γ_h (cf. (4.4)) we have that $\gamma_h = R \bar{\gamma}_h$. On the other hand, from the definition of A and A_h we write

$$A((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) = a(\beta - \beta_h, \beta - \beta_h) + \kappa^{-1} t^2 \|\gamma - \bar{\gamma}_h\|_{0,\Omega}^2,$$

$$A_h((\beta_h, w_h), (\beta_h, w_h)) - A((\beta_h, w_h), (\beta_h, w_h)) = \kappa^{-1} t^2 \left(\|R\bar{\gamma}_h\|_{0,\Omega}^2 - \|\bar{\gamma}_h\|_{0,\Omega}^2 \right).$$

Therefore,

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$$(\lambda_h - \lambda) B((\beta_h, w_h), (\beta_h, w_h)) = a(\beta - \beta_h, \beta - \beta_h) + \kappa^{-1} t^2 \left(\|\gamma - \bar{\gamma}_h\|_{0,\Omega}^2 + \|R\bar{\gamma}_h\|_{0,\Omega}^2 - \|\bar{\gamma}_h\|_{0,\Omega}^2 \right) - \lambda B((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)).$$

The first and the third term in the right-hand side above are easily bounded by virtue of (5.2) and (5.3). For the second term, we write

(5.6)
$$\|\gamma - \bar{\gamma}_{h}\|_{0,\Omega}^{2} + \|R\bar{\gamma}_{h}\|_{0,\Omega}^{2} - \|\bar{\gamma}_{h}\|_{0,\Omega}^{2} = \|\gamma - R\bar{\gamma}_{h}\|_{0,\Omega}^{2} - 2(\gamma, \bar{\gamma}_{h} - R\bar{\gamma}_{h})_{0,\Omega} = \|\gamma - \gamma_{h}\|_{0,\Omega}^{2} + \frac{2\kappa}{t^{2}}(\gamma, \beta_{h} - R\beta_{h})_{0,\Omega} .$$

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For $\beta \in \mathrm{H}^2(\Omega)^2 \cap \mathrm{H}^1_0(\Omega)$, we denote by $\beta^{\mathrm{I}} \in \mathrm{H}_h$ the standard Clément interpolant of β , which satisfies

(5.7)
$$\left\|\beta^{\mathrm{I}}\right\|_{1,\Omega} \leq C \left\|\beta\right\|_{1,\Omega}$$
 and $\left\|\beta - \beta^{\mathrm{I}}\right\|_{1,\Omega} \leq Ch \left\|\beta\right\|_{2,\Omega}$

It follows that

$$(\gamma, \beta_h - R\beta_h)_{0,\Omega} = (\gamma, (\beta_h - \beta^{\mathrm{I}}) - R(\beta_h - \beta^{\mathrm{I}}))_{0,\Omega} + (\gamma, \beta^{\mathrm{I}} - R\beta^{\mathrm{I}})_{0,\Omega}$$
$$\leq \|\gamma\|_{0,\Omega} \|(\beta_h - \beta^{\mathrm{I}}) - R(\beta_h - \beta^{\mathrm{I}})\|_{0,\Omega} + (\gamma, \beta^{\mathrm{I}} - R\beta^{\mathrm{I}})_{0,\Omega}.$$

Thus, using (4.2) and Lemma 3.3 from [10], we obtain

$$\begin{aligned} (\gamma, \beta_h - R\beta_h)_{0,\Omega} &\leq Ch \left\|\gamma\right\|_{0,\Omega} \left\|\beta_h - \beta^{\mathrm{I}}\right\|_{1,\Omega} + Ch^2 \left\|\operatorname{div}\gamma\right\|_{0,\Omega} \left\|\beta\right\|_{1,\Omega} \\ &\leq Ch \left\|\gamma\right\|_{0,\Omega} \left(\left\|\beta - \beta_h\right\|_{1,\Omega} + \left\|\beta - \beta^{\mathrm{I}}\right\|_{1,\Omega}\right) + Ch^2 \left\|\operatorname{div}\gamma\right\|_{0,\Omega} \left\|\beta\right\|_{1,\Omega}, \end{aligned}$$

and from Lemma 5.7, (5.7), and Proposition 3.10, we have

$$(\gamma, \beta_h - R\beta_h)_{0,\Omega} \le Ch \|\gamma\|_{0,\Omega} \left(Ch + Ch \|\beta\|_{2,\Omega}\right) + Ch^2 |\lambda| \|w\|_{2,\Omega} \|\beta\|_{1,\Omega} \le Ch^2 |\lambda|.$$

Finally, we use this estimate, (5.5), (5.6), and the fact that $B((\beta_h, w_h), (\beta_h, w_h)) = (\boldsymbol{\sigma} \nabla w_h, \nabla w_h)_{0,\Omega} \neq 0$ (cf. Remark 4.2) to obtain

$$|\lambda - \lambda_h| \le C \frac{\|\beta - \beta_h\|_{1,\Omega}^2 + \|w - w_h\|_{1,\Omega}^2 + \kappa^{-1}t^2 \|\gamma - \gamma_h\|_{0,\Omega}^2 + Ch^2 |\lambda|}{|B((\beta_h, w_h), (\beta_h, w_h))|}$$

Consequently, from Lemma 5.7,

$$|\lambda - \lambda_h| \le Ch^2$$

and we conclude the proof. \Box

6. Numerical results. In this section we report some numerical experiments carried out with our method applied to problem (2.4).

For all the computations we have taken $\Omega := (0, 6) \times (0, 4)$ and typical parameters of steel: the Young modulus has been chosen $E = 1.44 \times 10^{11}$ Pa and the Poisson ratio $\nu = 0.30$. All the lengths are measured in meters and the shear correction factor has been taken k = 5/6. We recall that the buckling coefficients λ_b may be easily computed from the eigenvalues of (2.4) (cf. Remark 2.1).

We have used uniform meshes as shown in Fig. 6.1; the meaning of the refinement parameter N can be easily deduced from this figure. Notice that $h \sim N^{-1}$.



FIG. 6.1. Rectangular plate. Uniform meshes.

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6.1. Uniformly compressed rectangular plate. For this test we have used $\sigma = I$, which corresponds to a uniformly compressed plate.

6.1.1. Simply supported plate. First, we have considered a simply supported plate, since analytical solutions are available in that case (see [19, 20]). Even though our theoretical analysis has been developed only for clamped plates, we think that the results of Sections 4 and 5 should hold true for more general boundary conditions, as well. The results that follow give some numerical evidence of this.

In Table 6.1 we report the four lowest eigenvalues $(\lambda_i, i = 1, 2, 3, 4)$ for a thickness t = 0.001, computed by our method with four different meshes (N = 2, 4, 8, 16). The table includes computed orders of convergence, as well as more accurate values extrapolated by means of a least-squares fitting. Furthermore, the last column shows the exact eigenvalues.

TABLE 6.1

Lowest eigenvalues λ_i (multiplied by 10^{-10}) of a uniformly compressed simply supported plate with thickness t = 0.001.

Eigenvalue	N = 2	N = 4	N = 8	N = 16	Order	Extrapolated	Exact
λ_1	1.1793	1.1759	1.1752	1.1750	2.14	1.1750	1.1749
λ_2	2.2638	2.2602	2.2596	2.2595	2.68	2.2595	2.2595
λ_3	3.7293	3.6441	3.6224	3.6170	1.98	3.6151	3.6152
λ_4	4.1573	4.0892	4.0726	4.0685	2.03	4.0672	4.0671

It can be seen from Table 6.1 that the method converges to the exact values with an optimal quadratic order.

Figure 6.2 shows the transverse displacements for the principal buckling mode computed with the finest mesh (N = 16).



FIG. 6.2. Uniformly compressed simply supported plate; principal buckling mode.

6.1.2. Clamped plate. In Table 6.2 we present the results for the lowest eigenvalue of a uniformly compressed clamped rectangular plate, with varying thickness. We have used the same meshes as in the previous test. Again, we have computed the orders of convergence, and more accurate values obtained by a least-squares procedure. Furthermore, in the last row we also report for each mesh the limit values as t goes to zero obtained by extrapolation.

TABLE	6.	.2
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Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of uniformly compressed clamped plates with varying thickness.

Thickness	N = 2	N = 4	N = 8	N = 16	Order	Extrapolated
t = 0.1	3.4031	3.3440	3.3293	3.3258	2.02	3.3246
t = 0.01	3.4324	3.3723	3.3571	3.3533	1.99	3.3520
t = 0.001	3.4327	3.3726	3.3574	3.3536	1.99	3.3522
t = 0.0001	3.4327	3.3726	3.3574	3.3536	1.98	3.3522
t = 0 (extrap.)	3.4327	3.3726	3.3574	3.3536	1.99	3.3523

Figure 6.3 shows the transverse displacements for the principal buckling mode, for t = 0.1, and the finest mesh (N = 16).



FIG. 6.3. Uniformly compressed clamped plate; principal buckling mode.

According to Lemma 3.7, the values on the last row of Table 6.2 should correspond to the lowest eigenvalues of a Kirchhoff-Love uniformly compressed clamped plate with thickness t = 1. As a further test, we have also computed the latter, by using the methods analyzed in [6] and [17]. We show the obtained results in Table 6.3, where an excellent agreement with the last row of Table 6.2 can be appreciated.

TABLE 6.3

Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of a uniformly compressed clamped thin plate (Kirchhoff-Love model) computed with the methods from [6] and [17].

Method	N = 8	N = 12	N = 16	N = 20	Order	Extrapolated
[6]	3.3718	3.3611	3.3573	3.3555	1.97	3.3523
[17]	3.3514	3.3519	3.3521	3.3522	1.95	3.3523

It is clear that the results from the Reissner-Mindlin model do not deteriorate as the plate thickness become smaller, which confirms that our method is *locking-free*.

6.2. Clamped plate uniformly compressed in one direction. We have used for this test

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which corresponds to a plate uniformly compressed in one direction. Notice that in this test σ is only positive semi-definite. Table 6.4 shows the same quantities as Table 6.2 in this case.

TABLE 6.4 Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of clamped plates with varying thickness, uniformly

compressed in one direction.

Thickness	N = 2	N = 4	N = 8	N = 16	Order	Extrapolated
t = 0.1	6.7969	6.7274	6.7104	6.7066	2.05	6.7052
t = 0.01	6.8825	6.8143	6.7971	6.7930	2.00	6.7915
t = 0.001	6.8834	6.8151	6.7980	6.7939	2.00	6.7924
t = 0.0001	6.8834	6.8152	6.7980	6.7939	2.00	6.7924
t = 0 (extrap.)	6.8834	6.8152	6.7980	6.7939	2.00	6.7924

Figure 6.4 shows the principal buckling mode for t = 0.1 and the finest mesh (N = 16).



FIG. 6.4. Clamped plate uniformly compressed in one direction; principal buckling mode.

Finally, Table 6.5 shows the same quantities as Table 6.3 in this case. Once more, an excellent agreement with the values extrapolated from the Reissner-Mindlin model (last row of Table 6.4) can be clearly appreciated.

TABLE 6.5

Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of a clamped thin plate (Kirchhoff-Love model) uniformly compressed in one direction, computed with the methods from [6] and [17].

Method	N = 8	N = 12	N = 16	N = 20	Order	Extrapolated
[6]	6.8450	6.8158	6.8056	6.8009	2.00	6.7925
[17]	6.7904	6.7913	6.7917	6.7920	1.92	6.7926

6.3. Shear loaded clamped plate. In this case we have used

$$oldsymbol{\sigma} = egin{bmatrix} 0 & 1 \ 1 & 0 \end{bmatrix},$$

which corresponds to a uniform shear load. Notice that σ is indefinite in this test. The numerical results are reported in Table 6.6, Figure 6.5, and Table 6.7, using the same pattern as the previous tests.

TABLE 6.6 Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of shear loaded clamped plates with varying thickness.

Thickness	N = 4	N = 8	N = 12	N = 16	Order	Extrapolated
t = 0.1	9.4306	9.2179	9.1783	9.1645	1.99	9.1464
t = 0.01	9.6098	9.3923	9.3514	9.3371	1.98	9.3184
t = 0.001	9.6116	9.3942	9.3533	9.3389	1.98	9.3202
t = 0.0001	9.6117	9.3942	9.3533	9.3389	1.98	9.3202
t = 0 (extrap.)	9.6117	9.3942	9.3533	9.3389	1.98	9.3202



FIG. 6.5. Shear loaded clamped plate; principal buckling mode.

TABLE 6.7

Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of a shear loaded clamped thin plate (Kirchhoff-Love model) computed with the methods from [6] and [17].

Method	N = 8	N = 12	N = 16	N = 20	Order	Extrapolated
[6]	9.4625	9.3840	9.3563	9.3435	1.98	9.3203
[17]	9.3660	9.3408	9.3319	9.3278	1.99	9.3204

In all cases, an excellent agreement between the numerical experiments and the theoretical results detailed in Section 5 can be noticed and the method appears thoroughly locking-free.

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REFERENCES

- D. N. ARNOLD AND R. S. FALK, A uniformly accurate finite element method for the Reissner-Mindlin plate, SIAM J. Numer. Anal., 26 (1989), pp. 1276–1290.
- [2] I. BABUŠKA AND J. OSBORN, Eigenvalue Problems, in Handbook of Numerical Analysis, Vol. II, P. G. Ciarlet and J. L. Lions, eds., North-Holland, Amsterdam, 1991, pp. 641–787.

- [3] A. BERMÚDEZ, R. DURÁN, M. A. MUSCHIETTI, R. RODRÍGUEZ, AND J. SOLOMIN, Finite element vibration analysis of fluid-solid systems without spurious modes, SIAM J. Numer. Anal., 32 (1995), pp. 1280–1295.
- [4] F. BREZZI AND M. FORTIN, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.
- [5] C. CANUTO, Eigenvalue approximations by mixed methods, RAIRO Anal. Numér., 12 (1978), pp. 27–50.
- [6] P. G. CIARLET AND P. A. RAVIART, A mixed finite element method for the biharmonic equation, in Mathematical Aspects of Finite Elements in Partial Differential Equations, C. de Boor, ed., Academic Press, New York, 1974, pp. 125–145.
- [7] M. DAUGE AND M. SURI, Numerical approximation of the spectra of non-compact operators arising in buckling problems, J. Numer. Math., 10 (2002), pp. 193–219.
- [8] J. DESCLOUX, N. NASSIF, AND J. RAPPAZ, On spectral approximation. Part 1: The problem of convergence, RAIRO Anal. Numér., 12 (1978), pp. 97–112.
- [9] —, On spectral approximation. Part 2: Error estimates for the Galerkin method, RAIRO Anal. Numér., 12 (1978), pp. 113–119.
- [10] R. DURÁN, L. HERVELLA-NIETO, E. LIBERMAN, R. RODRÍGUEZ, AND J. SOLOMIN, Approximation of the vibration modes of a plate by Reissner-Mindlin equations, Math. Comp., 68 (1999), pp. 1447–1463.
- [11] R. DURÁN AND E. LIBERMAN, On mixed finite elements methods for the Reissner-Mindlin plate model, Math. Comp., 58 (1992), pp. 561–573.
- [12] R. FALK, Finite Elements for the Reissner-Mindlin Plate, in Mixed Finite Elements, Compatibility Conditions, and Applications, D. Boffi and L. Gastaldi, eds., Springer-Verlag, Berlin, 2008, pp. 195–230.
- [13] K. ISHIHARA, On the mixed finite element approximation for the buckling of plates, Numer. Math., 33 (1979), pp. 195–210.
- [14] T. KATO, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1966.
- [15] S. KITIPORNCHAI, Y. XIANG, C. M. WANG, AND K. M. LIEW, Buckling of thick skew plates, Internat. J. Numer. Methods Engrg., 36, (1993), pp. 1299–1310.
- [16] B. MERCIER, J. OSBORN, J. RAPPAZ, AND P. A. RAVIART, Eigenvalue approximation by mixed and hybrid methods, Math. Comp., 36 (1981), pp. 427–453.
- [17] D. MORA AND R. RODRÍGUEZ, A piecewise linear finite element method for the buckling and the vibration problems of thin plates, Math. Comp., (in press).
- [18] R. RANNACHER, Nonconforming finite element methods for eigenvalue problems in linear plate theory, Numer. Math., 33 (1979), pp. 23–42.
- [19] J. N. REDDY, Energy and Variational Methods in Applied Mechanics, Wiley, New York, 1984.
- [20] —, Mechanics of Laminated Composite Plates Theory and Analysis, CRC Press, Boca Raton, 1997.
- [21] G. SAVARÉ, Regularity results for elliptic equations in Lipschitz domains, J. Funct. Anal., 152 (1998), pp. 176–201.
- [22] J. P. SMITH, Buckling of shear deformable plates using the p-version of the finite element method, Comput. & Structures, 57 (1995), pp. 527–532.
- [23] H. ZHONG AND C. GU, Buckling of simply supported rectangular Reissner-Mindlin plates subjected to linearly varying in-plane loading, J. Engrg. Mech.-ASCE, 132 (2006), pp. 578–581.

Appendix. Uniformly compressed plates.

The aim of this appendix is to show that the results of Sections 3, 4, and 5 can be refined when $\boldsymbol{\sigma} = \boldsymbol{I}$, which corresponds to a uniformly compressed plate. In this case, we are able to give a better characterization of the spectrum of T_t and to prove the spectral approximation without assuming that the family of meshes is *quasi-uniform*.

A.1. Spectral characterization. We have the following counterpart of Theorem 3.1, showing that the spectrum of T_t is simply a shift of the spectrum of a compact operator.

THEOREM A.1. Suppose that $\sigma = I$. For all t > 0, the spectrum of T_t satisfies

$$\operatorname{Sp}(T_t) = \operatorname{Sp}(G) + \kappa^{-1} t^2,$$

where G is the compact operator defined in (3.5).

Proof. The first equation of (2.7) leads in this case to $\psi = f$, due to the fact that $\sigma = I$. Therefore, (2.7) reduces to

$$(A.1) \qquad \begin{cases} a(\beta,\eta) - (\operatorname{curl} p,\eta)_{0,\Omega} = (\nabla f,\eta)_{0,\Omega} & \forall \eta \in \mathrm{H}^1_0(\Omega)^2, \\ -(\beta,\operatorname{curl} q)_{0,\Omega} - \kappa^{-1}t^2 (\operatorname{curl} p,\operatorname{curl} q)_{0,\Omega} = 0 & \forall q \in \mathrm{H}^1(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} + \kappa^{-1}t^2 (\nabla f, \nabla \xi)_{0,\Omega} & \forall \xi \in \mathrm{H}^1_0(\Omega). \end{cases}$$

Next, recall that G is defined in (3.5) as the operator mapping $f \mapsto u$, with $u \in \mathrm{H}^{1}_{0}(\Omega)$ such that

$$(\nabla u, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}^{1}_{0}(\Omega),$$

where $\beta \in \mathrm{H}_0^1(\Omega)^2$ is determined in this case by the first two equations from (A.1). Therefore, the third equation from (A.1) yields $T_t = G + \kappa^{-1} t^2 I$. Since G has been already shown to be compact, this allows us to conclude the theorem. \Box

As a consequence of this theorem, $\operatorname{Sp}(T_t) = \{\kappa^{-1}t^2\} \cup \{\mu_n : n \in \mathbb{N}\}$, with μ_n being a sequence of finite-multiplicity eigenvalues converging to $\kappa^{-1}t^2$. Therefore, in this particular case, the *essential spectrum* of T_t reduces to a unique point: $\kappa^{-1}t^2$.

A.2. Spectral approximation. In this particular case, we will improve the error estimate shown in Section 4 in that we will not need to assume quasi-uniformity of the meshes. Indeed, this property was used above only to prove Proposition 4.3. Instead, we have now the following result.

PROPOSITION A.2. Suppose that $\boldsymbol{\sigma} = \boldsymbol{I}$. Then for any regular family of triangular meshes $\{\mathcal{T}_h\}_{h>0}$, there exists C > 0 such that, for all t > 0,

$$\|T_t - T_{th}\|_h \le Ch.$$

Proof. We will simply sketch the proof, since it follows exactly the same steps as that of Proposition 4.3. First, we notice that in the decomposition (4.12) we have $\psi = f_h \in W_h$ (cf. problem (4.14) with $\boldsymbol{\sigma} = \boldsymbol{I}$).

As a consequence, we infer that the term $\|\nabla \psi - \nabla \psi_{\mathrm{I}}\|_{0,\Omega}$ in (4.15) vanishes. Hence, the last estimate in the proof of Lemma 4.5 changes into

$$\|\beta - \beta_h\|_{1,\Omega} + t \,\|\gamma - \gamma_h\|_{0,\Omega} \le C \left(h \,\|\beta\|_{2,\Omega} + th \,\|p\|_{2,\Omega} + h \,\|\gamma\|_{0,\Omega}\right) \le Ch \,\|f_h\|_{1,\Omega}.$$

By using the above estimate in the proof of Proposition 4.3 (in particular in (4.16)), we obtain

$$\|(T_t - T_{th}) f_h\|_{1,\Omega} = \|w - w_h\|_{1,\Omega} \le Ch \|f_h\|_{1,\Omega}$$

from which we conclude the proof. \Box

As a consequence of Proposition A.2, we can improve Lemma 4.6. In fact, now for t small enough there holds directly

$$\|T_t - T_{th}\|_h \le Ch,$$

with a constant C independent of h and t. By using this instead of property P1, we could give somewhat simpler proofs for the error estimates from Section 5. However the final results, Theorems 5.1, 5.5, and 5.8 are the same, although now valid for any regular family of triangular meshes, without the need of being quasi-uniform.

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