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Analysis of an augmented mixed–primal formulation for the stationary Boussinesq problem*

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Abstract

In this paper we propose and analyze a new mixed variational formulation for the stationary Boussinesq problem. Our method, which employs a technique previously applied to the Navier-Stokes equations, is based first on the introduction of a modified pseudostress tensor depending nonlinearly on the velocity through the respective convective term. Next, the pressure is eliminated, and an augmented approach for the fluid flow, which incorporates Galerkin type terms arising from the constitutive and equilibrium equations, and from the Dirichlet boundary condition, is coupled with a primal-mixed scheme for the main equation modeling the temperature. In this way, the only unknowns of the resulting formulation are given by the aforementined nonlinear pseudostress, the velocity, the temperature, and the normal derivative of the latter on the boundary. An equivalent fixed-point setting is then introduced and the corresponding classical Banach Theorem, combined with the Lax-Milgram Theorem and the Babuška-Brezzi theory, are applied to prove the unique solvability of the continuous problem. In turn, the Brouwer and the Banach fixed point theorems are utilized to establish existence and uniqueness of solution, respectively, of the associated Galerkin scheme. In particular, Raviart-Thomas spaces of order k for the pseudostress, continuous piecewise polynomials of degree $\leq k+1$ for the velocity and the temperature, and piecewise polynomials of degree $\leq k$ for the boundary unknown become feasible choices. Finally, we derive optimal a priori error estimates, and provide several numerical results illustrating the good performance of the augmented mixed-primal finite element method and confirming the theoretical rates of convergence.

Key words: Boussinesq equations, augmented mixed–primal formulation, fixed point theory, finite element methods, a priori error analysis

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