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Métodos de Elementos Finitos Mixtos para el Problema Acoplado de Stokes-Darcy

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RESUMEN

El objetivo principal de esta tesis es aproximar un problema de acoplamiento de fluido con medio poroso utilizando Métodos de Elementos Finitos Mixtos. El modelo acoplado está determinado por las ecuaciones de Stokes y Darcy, respectivamente, y las condiciones de interfase correspondientes están dadas por conservación de masa, balance de fuerzas normales y la ley de Beavers-Joseph-Saffman.

Primero se desarrolla un análisis a priori de una formulación primal en el fluido y mixta en el medio poroso, y se demuestra que cualquier par de espacios de elementos finitos estables para Stokes y Darcy implican la estabilidad del esquema de Galerkin correspondiente. Lo anterior extiende resultados previos que demuestran existencia y unicidad de un esquema de Galerkin definido por elementos de Bernardi-Raugel y de Raviart-Thomas de bajo orden.

Posteriormente, se realiza un análisis a priori y a posteriori de una formulación variacional mixta en ambos dominios, del problema acoplado de Stokes-Darcy. Las incógnitas principales consideradas son el pseudo-esfuerzo y la velocidad en el fluido, junto con la velocidad y la presión en el medio poroso. Además, las condiciones de transmisión se convierten en esenciales, lo cual induce la introducción de los valores de la presión del medio poroso y de la velocidad del fluido en la interfase como incógnitas adicionales que cumplen el rol de multiplicadores de Lagrange. Se demuestra existencia y unicidad a nivel continuo, y a nivel discreto se introducen condiciones suficientes para que el esquema de Galerkin asociado sea estable. En particular se pueden utilizar elementos de Raviart-Thomas de bajo orden y elementos constantes a trozo para las velocidades y presiones en ambos dominios, junto con elementos continuos lineales a trozo para los multiplicadores de Lagrange. Además, se obtiene un estimador de error a posteriori, confiable y eficiente para el problema acoplado.

Finalmente, se generalizan los resultados anteriores y se estudia un acoplamiento de un fluido viscoso incompresible con un medio poroso matemáticamente determinado por una ley no lineal. El modelo acoplado no lineal está definido por la ecuación de Stokes y un sistema de Darcy no lineal. En este último la permeabilidad está representada por un operador no lineal, fuertemente monótono y Lipschitz continuo. Se introduce un esquema mixto en ambos dominios y se demuestra existencia y unicidad de solución a nivel continuo y discreto, con su estimación a priori correspondiente. Además se obtiene un estimador de error a posteriori eficiente y confiable para el problema acoplado no lineal.

Para todas las situaciones descritas anteriormente se presentan ensayos numéricos que confirman los resultados teóricos obtenidos.

ABSTRACT

The main purpose of this thesis is to approximate a coupling of fluid flow with porous medium flow by using Mixed Finite Element Methods. Flows are governed by the Stokes and Darcy equations, respectively, and the corresponding interface conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law.

First, we analyze the well-posedness of a mixed formulation, primal in the Stokes domain and dual-mixed in the Darcy region, and we show that use of any pair of stable Stokes and Darcy elements implies the well-posedness of the corresponding Stokes-Darcy Galerkin scheme. This extends previous results showing well-posedness only for Bernardi-Raugel and Raviart-Thomas elements of the lowest order.

Afterwards, we develop the a priori and a posteriori error analysis of a new fully mixed finite element method for the coupled problem. We consider dual-mixed formulations in both domains, which yields the pseudostress and the velocity in the fluid, as well as the velocity and the pressure in the porous medium, as the main unknowns. In addition, since the transmission conditions become essential, we impose them weakly and introduce the values of the porous medium pressure and the fluid velocity on the interface as new unknowns, that play the role of Lagrange multipliers. We prove the unique solvability of the continuous formulation and derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme is well posed. A practicable choice of subspaces is given by the Raviart-Thomas elements of lowest order and piecewise constants for the velocities and pressures, respectively, in both domains, together with continuous piecewise linear elements for the Lagrange multipliers. We also derive a reliable and efficient residual-based a posteriori error estimator for the coupled problem.

Finally, we generalize the above results and we analyze a mixed finite element method for the coupling of viscous incompressible fluid flow with a state law mathematically corresponding to a nonlinear porous medium flow. Flows are governed by the Stokes and nonlinear Darcy equations respetively. In the latter permeability is given by a strongly monotone and Lipschitz-continuous nonlinear operator. We consider dual-mixed approaches in both the Stokes and Darcy regions. This yields a twofold saddle point operator equation as the resulting variational formulation. A well known generalization of the classical Babuška-Brezzi theory is applied to show the well-posedness of the continuous and discrete formulations and to derive the corresponding a priori error estimate. Furthermore, a reliable and efficient residual based a posteriori error estimator is provided.

For all the situations described above, several numerical results illustrating the correct performance of the method, and confirming the theoretical results, are reported.

Contents

1 Introducción **2**

1.1 El problema acoplado de Stokes-Darcy . . . 6

1.2 Formulación primal-mixta . . . 7

1.3 Una formulación de Galerkin conforme . . . 9

2 Convergence of a family of Galerkin discretizations for the Stokes–Darcy coupled problem **12**

2.1 Introduction . . . 12

2.2 The Stokes-Darcy coupled problem . . . 14

2.2.1 The model problem . . . 14

2.2.2 The variational formulation . . . 16

2.2.3 Remarks on the geometry . . . 17

2.3 A compact perturbation of the variational formulation . . . 19

2.4 A class of Galerkin schemes and their convergence . . . 22

2.4.1 Preliminaries . . . 23

2.4.2 The main result . . . 24

2.4.3 Examples of subspaces satisfying the hypotheses . . . 26

2.5 An alternative approach . . . 28

2.6 Numerical results . . . 31

3 Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem **43**

3.1 Introduction . . . 43

3.2 The continuous problem . . . 46

3.2.1 Statement of the model problem . . . 46

3.2.2 The weak formulation . . . 49

3.2.3 The variational system . . . 51

3.3 Analysis of the continuous problem 55
3.3.1 Preliminaries 55
3.3.2 The main results 56
3.4 The Galerkin scheme 61
3.4.1 Preliminaries 61
3.4.2 The main result 65
3.4.3 Stable discrete liftings 66
3.5 A particular choice of discrete spaces 67
3.5.1 Discretization of the domains 67
3.5.2 The discrete liftings 68
3.5.3 Discretization on the interface 72
3.5.4 The main results 75
3.6 Numerical results 77

4 A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes-Darcy coupled problem 84
4.1 Introduction 84
4.2 The Stokes-Darcy coupled problem 87
4.2.1 The model problem 88
4.2.2 The fully-mixed variational formulation 89
4.2.3 A Galerkin method 91
4.3 A residual-based a posteriori error estimator 93
4.3.1 Reliability of the a posteriori error estimator 94
4.3.2 Efficiency of the a posteriori error estimator 102
4.4 Numerical results 108

5 A twofold saddle point approach for the coupling of fluid flow with nonlinear porous media flow 118
5.1 Introduction 118
5.2 The continuous problem 121
5.2.1 Statement of the model problem 121
5.2.2 The dual-mixed formulation 123
5.2.3 Abstract theory for twofold saddle point operator equations 126
5.2.4 An equivalent twofold saddle point formulation 128
5.3 Analysis of the continuous problem 130
5.3.1 Preliminaries 130

5.3.2 The main results 131
5.4 The mixed finite element scheme 135
5.4.1 Preliminaries 135
5.4.2 The main results 136
5.5 A particular mixed finite element scheme 140
5.5.1 The finite element subspaces 140
5.5.2 The discrete inf-sup conditions 142
5.5.3 The main results 144
5.6 The a-posteriori error analysis 146
5.6.1 Reliability of the a posteriori error estimator 148
5.6.2 Efficiency of the a posteriori error estimator 150
5.7 Numerical results 153

6 Conclusiones y trabajo futuro **163**
6.1 Conclusiones 163
6.2 Trabajo futuro 164

Bibliography **165**

Chapter 1

Introducción

En la naturaleza existe una gran cantidad de fenómenos que obedecen al acoplamiento físico entre sólidos y fluidos, sea por la interacción entre desplazamientos de un medio sólido continuo con un medio fluido o bien por la existencia de un flujo que transcurre, en parte libremente y en parte dentro de los poros de un sólido. Fenómenos fisiológicos como el movimiento de la sangre en los vasos sanguíneos y la penetración del aire en los pulmones, y fenómenos hidrológicos como la filtración de aguas superficiales a través de rocas y arena, encajan en esta amplia gama de fenómenos, modelados y estudiados por distintas ramas de la ciencia.

En general las ecuaciones que describen estos modelos son difíciles de resolver analíticamente, por lo cual, la resolución y simulación computacional se hace indispensable. Es por esto que una parte importante de la comunidad científica dedicada al área del análisis numérico se ha centrado a desarrollar nuevas herramientas que permitan modelar eficientemente sistemas de interacción donde se combinan sólidos y fluidos.

En particular, el análisis numérico para el acoplamiento de fluidos viscosos incompresibles (modelados por la ecuación de Stokes) con flujo en medio poroso (modelado por la ecuación de Darcy) se ha convertido en un área de investigación muy activa durante las últimas dos decadas. Esto se debe, por una parte, a la creciente necesidad de resolver este problema de la forma más eficiente y precisa posible, y por otra parte, a la amplia variedad de aplicaciones que emplean este modelo, como por ejemplo los diversos procesos industriales que involucran filtración. En la literatura se puede encontrar una cantidad importante de trabajos relacionados a este tema (ver [2], [24], [20], [21], [26], [34], [32], [33], [35], [39], [45], [48], [61], [63], [67], [71], [74], [75], [78] y sus referencias). Los últimos resultados disponibles además incluyen medios porosos con grietas, problemas no lineales, y la incorporación de la ecuación de Brinkman en el modelo (ver [17], [37] y [83]).

Uno de los primeros trabajos relacionado a este tema es [63], en el cual se desarrolla la teoría matemática y el análisis numérico de una formulación variacional mixta para el problema acoplado. En ella se emplea un método primal en el fluido y un método dual-mixto en el medio poroso, esto es, la velocidad y la presión se consideran como las incógnitas en la región gobernada por la ecuación de Stokes, mientras que la velocidad se introduce como una incógnita adicional en la región porosa. Las condiciones de interfase estan dadas por la conservación de masa, el balance de fuerzas normales y por la ley de Beavers-Joseph-Saffman. Dado que una de estas se transforma en una condición esencial, es necesario introducir la traza de la presión del medio poroso como un multiplicador de Lagrange para mantener las discretizaciones por separado. Además, en este trabajo se prueba la existencia y unicidad de solución de la formulación continua correspondiente y se proporciona un detallado análisis de un método de elementos finitos mixtos no conforme. Es importante mencionar que dicha no conformidad se debe a que el multiplicador de Lagrange es aproximado por funciones constantes a trozos, las cuales no están contenidas en el espacio de Sobolev de las trazas en la interfase.

Recientemente, en [45], se ha introducido y analizado una nueva discretización por elementos finitos para la formulación mixta propuesta en [63], la primera en ser conforme, cuya estabilidad se demuestra utilizando un esquema de Galerkin específico. Este esquema se define usando elementos de Bernardi-Raugel para la velocidad en la región del fluido, elementos de Raviart-Thomas de bajo orden para la velocidad de filtración en el medio poroso, elementos constantes a trozos para las presiones, y elementos continuos lineales a trozos para el multiplicador de Lagrange en la interfase. Esta discretización resulta ser el primer método de elementos finitos mixtos conforme para la formulación primal/dual-mixta introducida en [63].

El propósito de esta tesis es ampliar la gama de discretizaciones existentes para el problema acoplado. Lo anterior se realiza, por una parte, generalizando los resultados obtenidos en [45], permitiendo una libre elección de elementos finitos para el esquema de Galerkin asociado a la formulación variacional del problema acoplado. Por otra parte, se proponen nuevos métodos numéricos que permiten la introducción de incógnitas adicionales de interés físico y la utilización de la misma familia de elementos finitos en ambos medios. Además, extendemos el estudio a casos más generales, desarrollando un análisis a priori y a posteriori para un acoplamiento de Stokes-Darcy no lineal.

Este trabajo se organiza de la siguiente manera. En el **Capítulo 2**, se modifican los resultados obtenidos en [45], proporcionando condiciones generales suficientes sobre los subespacios de elementos finitos, para garantizar unicidad, estabilidad y convergencia del método de elementos finitos mixtos asociado. Más precisamente, se mejoran los resultados obtenidos en [45] y se demuestra que la utilización de cualquier par de elementos finitos estables para los sistemas de Stokes y Darcy implican la estabilidad del esquema de Galerkin para el problema acoplado. En particular, para la ecuación de Stokes se pueden utilizar los elementos de Taylor-Hood, Bernardi-Raugel y el elemento MINI, mientras que en el dominio de Darcy se pueden utilizar elementos de Raviart-Thomas de cualquier orden. El análisis se fundamenta en el hecho de que el operador que define la formulación variacional continua está dado por una perturbación compacta de un operador que mantiene los dominios desacoplados. Este capítulo esta constituido por el siguiente artículo:

G.N. Gatica, R. Oyarzúa and F.-J. Sayas, *Convergence of a family of Galerkin discretizations for the Stokes-Darcy coupled problem.* Numerical Methods for Partial Differential Equations DOI 10.1002/num, to appear.

En el **Capítulo 3** se introduce una nueva formulación variacional, dual-mixta en ambos dominios, para el problema acoplado de Stokes-Darcy, cuya estructura se obtiene introduciendo el pseudo-esfuerzo y la velocidad en el fluido, junto con la velocidad y la presión en el medio poroso, como incógnitas principales del modelo. Lo anterior hace que las ecuaciones de transmisión se conviertan en condiciones esenciales, lo que nos induce a imponerlas de forma débil introduciendo, la traza de la velocidad del fluido y de la presión del medio poroso como incógnitas adicionales, las cuales también tienen interés físico. Entonces, se ordenan las ecuaciones variacionales resultantes de forma tal que se obtiene una estructura mixta doble con formas bilineales diagonales, cuyas condicciones inf-sup sean fácilmente verificables, y se aplican las teorías de Fredholm y Babuška-Brezzi para demostrar existencia y unicidad de solución del esquema propuesto. Es importante mencionar que, sin ningún tipo de error adicional, es posible recuperar la presión y el gradiente de velocidad en el fluido, realizando un postproceso simple de las incógnitas involucradas, y sin la utilización de diferenciación numérica. Por otro lado, a nivel discreto se define un esquema de Galerkin y se introducen hipótesis generales sobre los espacios de elementos finitos para asegurar su estabilidad. Este capítulo esta constituido por el siguiente artículo:

G.N. Gatica, R. Oyarzúa and F.-J. Sayas, *Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem.* Mathematics of Computation, to appear.

En el **Capítulo 4** se desarrolla un análisis de error a posteriori para la formulación variacional descrita en el Capítulo 3, en donde se obtiene un estimador de error a posteriori residual, confiable y eficiente, para el problema acoplado. Los elementos finitos considerados son elementos de Raviart-Thomas para el pseudo-esfuerzo en el fluido y la velocidad de filtración en el medio poroso, elementos constantes a trozos para la velocidad del fluido y la presión en el medio poroso, y elementos continuos lineales a trozos y continuos para los multiplicadores de Lagrange definidos en la interfase. En la demostración de confiabilidad del estimador se utilizan descomposiciones de Helmholtz en ambos dominios y propiedades de aproximación local de los interpolantes de Clément y Raviart-Thomas. Por otro lado, algunas de las principales herramientas utilizadas para demostrar la eficiencia del estimador son desigualdades inversas y técnicas de localización basadas en funciones burbuja sobre lados y triángulos. Este capítulo esta constituido por el siguiente artículo:

G.N. Gatica, R. Oyarzúa and F.-J. Sayas, *A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes-Darcy coupled problem.* Preprint 2010-12, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile, (2010).

Finalmente, en el **Capítulo 5** se extienden los resultados obtenidos en los capítulos anteriores, desarrollando un análisis a priori y a posteriori para una formulación variacional de un acoplamiento de Stokes-Darcy no lineal, cuya no linealidad queda definida al considerar la permeabilidad del medio poroso como un operador no lineal, fuertemente monótono y Lipschitz-continuo, que depende de la norma del gradiente de presión. En el modelo se consideran el pseudo-esfuerzo y la velocidad en el fluido, junto a la velocidad, la presión y el gradiente de presión en el medio poroso, como las incógnitas principales del modelo, y se obtiene una estructura dual-mixta en el fluido y dual-dual-mixta en el medio poroso. Al igual que en el Capítulo 3, las condiciones de interfase resultan ser esenciales, lo que nos conduce a imponerlas de forma débil introduciendo las trazas de la velocidad del fluido y de la presión del medio poroso como multiplicadores de Lagrange. Así, aplicando una conocida generalización de la teoría de Babuška-Brezzi, se demuestran existencia y unicidad de solución de las formulaciones continua y discreta y la estimación de error a priori correspondiente. Por otro lado, utilizando argumentos similares a los empleados en el Capítulo 4, se obtiene un estimador de error a posteriori, confiable y eficiente, para el problema no lineal. Este capítulo esta constituido por el siguiente artículo:

G.N. Gatica, R. Oyarzúa and F.-J. Sayas, *A twofold saddle point approach for the coupling of fluid flow with nonlinear porous media flow.* Preprint 2010-19, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile, (2010).

A continuación se introduce de forma detallada cada una de las ecuaciones del problema acoplado Stokes-Darcy, junto con la geometría a considerar. Además, se hace un resumen de los resultados obtenidos en [45], el cual corresponde a la memoria de título de Ingeniero Matemático del autor de esta tesis (ver [68]), a nivel continuo y discreto, describiendo lo mejor posible las razones que motivaron el presente trabajo.

1.1 El problema acoplado de Stokes-Darcy

El problema acoplado de Stokes-Darcy modela el movimiento de un fluido viscoso incompresible que ocupa una región Ω_1, el cual fluye desde y hacia un medio poroso Ω_2, saturado por el mismo fluido, a través de una interfase común Γ_2. Para simplificar el análisis, consideramos un modelo donde la región Ω_2 está rodeada por Ω_1 y por lo tanto $\partial\Omega_2 = \Gamma_2$ (ver Figura 1.1). Como interpretación física de este modelo, se podría pensar en Ω_2 como la sección transversal de un medio poroso tridimensional, por ejemplo un cilindro paralelo al eje x_3, inmerso en un fluido viscoso. En particular, este tipo de modelos tiene aplicaciones en procesos de percolación de diversos materiales que se utilizan en la industria química y farmacéutica (ver [57], [60], [82] y sus referencias).

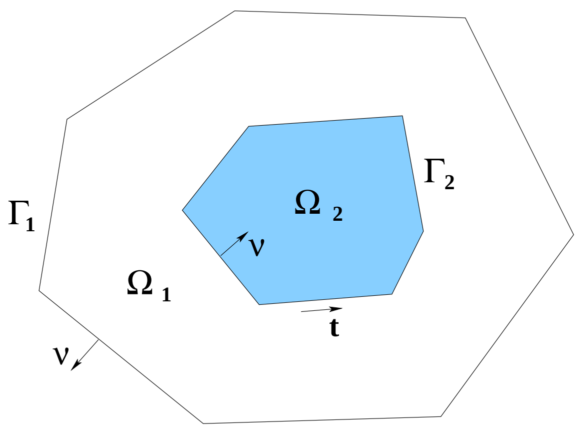

Figura 1.1: Geometría del problema.

Para describir las ecuaciones que gobiernan el problema, comenzamos con algunas definiciones. Sean $\mu > 0$ la viscosidad de fluido y $\mathbf{K}$ un tensor simétrico y uniformemente definido positivo en Ω_2 que representa la permeabilidad del medio poroso, y supongamos que existe $C > 0$ tal que $\|\mathbf{K}(x)\, z\| \,\leq\, C\,\|z\|$ para todo $x \in \Omega_2$, y para todo $z \in \mathbb{R}^2$. Entonces, las ecuaciones constitutivas están dadas por las leyes de Stokes y de Fick, respectivamente, esto es:

$$\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1) \,=\, -\,p_1\,\mathbf{I} \,+\, 2\,\mu\,\mathbf{e}(\mathbf{u}_1) \quad \text{en} \quad \Omega_1\,, \quad \text{y} \quad \mathbf{u}_2 \,=\, -\,\mathbf{K}\,\nabla\,p_2 \quad \text{en} \quad \Omega_2\,,$$

donde $(\mathbf{u}_1, \mathbf{u}_2)$ y (p_1, p_2) denotan las velocidades y presiones en los dominios correspondientes, $\mathbf{I}$ es la matriz identidad de 2×2, $\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)$ es el tensor de esfuerzos, y

$$\mathbf{e}(\mathbf{u}_1) \,:=\, \frac{1}{2}\left(\nabla\mathbf{u}_1 + (\nabla\mathbf{u}_1)^{\mathtt{t}}\right)$$

es el tensor de deformaciones. De aquí en adelante, dado cualquier espacio normado U, U^2 y $U^{2\times 2}$ denotan el espacio de vectores y matrices cuadradas de orden 2 con coeficientes en U, respectivamente. También, el superíndice $^{\mathtt{t}}$ representará la transpuesta de una matriz. Así, dados $\mathbf{f}_1 \in [L^2(\Omega_1)]^2$ y $f_2 \in L^2(\Omega_2)$ tal que $\int_{\Omega_2} f_2 = 0$, el problema acoplado es: Encontrar $(\mathbf{u}_1, \mathbf{u}_2)$ y (p_1, p_2) tales que

$$\left\{\begin{array}{rcllll} -\,\mathbf{div}\,\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1) & = & \mathbf{f}_1 & \text{en} & \Omega_1 & \text{(conservación de momento)}\,, \\ \operatorname{div}\mathbf{u}_1 & = & 0 & \text{en} & \Omega_1 & \text{(conservación de masa)}\,, \\ \mathbf{u}_1 & = & \mathbf{0} & \text{sobre} & \Gamma_1 & \text{(deslizamiento nulo)}\,, \\ \operatorname{div}\mathbf{u}_2 & = & f_2 & \text{en} & \Omega_2 & \text{(conservación de masa)}\,, \\ \mathbf{u}_1\cdot\boldsymbol{\nu} & = & \mathbf{u}_2\cdot\boldsymbol{\nu} & \text{en} & \Gamma_2 & \text{(conservación de masa)}\,, \\ \big(\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)\boldsymbol{\nu}\big)\cdot\boldsymbol{\nu} & = & -\,p_2 & \text{sobre} & \Gamma_2 & \text{(balance de fuerzas normales)}\,, \\ -\dfrac{\kappa}{\mu}\,\big(\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)\,\boldsymbol{\nu}\big)\cdot\mathbf{t} & = & \mathbf{u}_1\cdot\mathbf{t} & \text{sobre} & \Gamma_2 & \text{(ley de B-J-S),} \end{array}\right. \tag{1.1}$$

donde $\boldsymbol{\nu}$ es el vector normal unitario exterior a Ω_1, $\mathbf{t}$ es el vector tangencial a Γ_2, $\kappa := \frac{\sqrt{(\nu K\mathbf{t})\cdot\mathbf{t}}}{\alpha}$ es la constante de fricción y α es un parámetro positivo que se determina experimentalmente. La condición de Beavers-Joseph-Saffman (B-J-S) establece que la velocidad de deslizamiento en Γ_2 es proporcional al esfuerzo cortante en Γ_2, bajo el supuesto experimental que $\mathbf{u}_2\cdot\mathbf{t}$ es despreciable (ver [16], [59], y [76] para mayores detalles de esta condición de interfase).

Para finalizar esta sección, es importante observar que aunque la geometría descrita por la Figura 1.1 es elegida para simplificar el análisis del problema, el caso de un fluido filtrándose a través de una sola parte de la frontera del medio poroso no produce mayores complicaciones para el análisis matemático. Por ejemplo, si consideramos un fluido sobre un medio poroso, obtenemos una nueva frontera no vacía $\Gamma := \partial\Omega_2\backslash\bar{\Gamma}_2$, sobre la cual es necesario incorporar una condición de frontera adicional. Siguiendo las ideas de [39] y [63] (ver también [37]), usualmente se considera la condición de Neumann homogénea:

$$\mathbf{u}_2\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{sobre} \quad \Gamma, \tag{1.2}$$

lo cual significa que no hay flujo a través de Γ. En la Sección 2.2.3 del Capítulo 2 se explica con mayor detalle este tipo de modelos.

1.2 Formulación primal-mixta

Definamos el conjunto $\Omega := \Omega_1 \cup \Gamma_2 \cup \Omega_2$ y los espacios:

$$L^2_0(\Omega) \,:=\, \left\{ q \,\in\, L^2(\Omega) : \quad \int_\Omega q \,=\, 0 \right\},$$

$$[H^1_{\Gamma_1}(\Omega_1)]^2 := \left\{ \mathbf{v}_1 \in [H^1(\Omega_1)]^2 : \ \mathbf{v}_1 = \mathbf{0} \ \text{ sobre } \ \Gamma_1 \right\}$$

y

$$H(\mathrm{div}\,;\Omega_2) := \left\{ \mathbf{v}_2 \in [L^2(\Omega_2)]^2 : \ \mathrm{div}\ \mathbf{v}_2 \in L^2(\Omega_2) \right\}.$$

A su vez, introduzcamos los espacios producto

$$\mathbf{H} := [H^1_{\Gamma_1}(\Omega_1)]^2 \times H(\mathrm{div}\,;\Omega_2) \quad \text{y} \quad \mathbf{Q} := L^2_0(\Omega) \times H^{1/2}(\Gamma_2),$$

dotados con las normas $\|\mathbf{v}\|_{\mathbf{H}} := \|\mathbf{v}_1\|^2_{[H^1(\Omega_1)]^2} + \|\mathbf{v}_2\|^2_{H(\mathrm{div}\,;\Omega_2)}$ para todo $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}$, y $\|(q,\xi)\|_{\mathbf{Q}} := \|q\|^2_{L^2(\Omega)} + \|\xi\|^2_{H^{1/2}(\Gamma_2)}$ para todo $(q,\xi) \in \mathbf{Q}$. Entonces, definiendo las incógnitas globales $\mathbf{u} := (\mathbf{u}_1, \mathbf{u}_2)$, $p := p_1\chi|_{\Omega_1} + p_2\chi|_{\Omega_2}$, y el multiplicador de Lagrange

$$\lambda := p_2 = -\big(\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)\boldsymbol{\nu}\big)\cdot\boldsymbol{\nu} = p_1 - 2\mu\,\boldsymbol{\nu}\cdot\mathbf{e}(\mathbf{u}_1)\boldsymbol{\nu} \quad \text{sobre} \quad \Gamma_2\,,$$

procedemos como en [63] y obtenemos la formulación variacional mixta: Encontrar $(\mathbf{u},(p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$ tal que

$$\begin{aligned} \mathbf{a}(\mathbf{u},\mathbf{v}) + \mathbf{b}(\mathbf{v},(p,\lambda)) &= \mathbf{F}(\mathbf{v}) \qquad && \forall\, \mathbf{v} := (\mathbf{v}_1,\mathbf{v}_2) \in \mathbf{H}\,, \\ \mathbf{b}(\mathbf{u},(q,\xi)) &= \mathbf{G}(q,\xi) && \forall\,(q,\xi) \in \mathbf{Q}\,, \end{aligned} \tag{1.3}$$

donde las formas bilineales $\mathbf{a} : \mathbf{H}\times\mathbf{H} \to \mathbb{R}$ y $\mathbf{b} : \mathbf{H}\times\mathbf{Q} \to \mathbb{R}$, y los funcionales $\mathbf{F} : \mathbf{H} \to \mathbb{R}$ y $\mathbf{G} : \mathbf{Q} \to \mathbb{R}$, están definidos por

$$\begin{aligned} \mathbf{a}(\mathbf{u},\mathbf{v}) &:= 2\,\mu \int_{\Omega_1} \mathbf{e}(\mathbf{u}_1) : \mathbf{e}(\mathbf{v}_1) + \frac{\mu}{\kappa}\int_{\Gamma_2} (\mathbf{u}_1\cdot\mathbf{t})\,(\mathbf{v}_1\cdot\mathbf{t}) + \int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}_2\cdot\mathbf{v}_2\,, \\ \mathbf{b}(\mathbf{v},(q,\xi)) &:= -\int_{\Omega_1} q\,\mathrm{div}\,\mathbf{v}_1 - \int_{\Omega_2} q\,\mathrm{div}\,\mathbf{v}_2 + \langle \mathbf{v}_1\cdot\boldsymbol{\nu} - \mathbf{v}_2\cdot\boldsymbol{\nu}, \xi\rangle_{\Gamma_2}\,, \end{aligned} \tag{1.4}$$

$$\mathbf{F}(\mathbf{v}) = \int_{\Omega_1} \mathbf{f}_1\cdot\mathbf{v}_1 \quad \text{y} \quad \mathbf{G}(q,\xi) := -\int_{\Omega_2} f_2\,q,$$

y $\langle\cdot,\cdot\rangle_{\Gamma_2}$ denota la paridad dual de $H^{-1/2}(\Gamma_2)$ y $H^{1/2}(\Gamma_2)$ con respecto al producto interior de $L^2(\Gamma_2)$.

En [45] se utiliza la teoría clásica de Babuška-Brezzi para demostrar existencia y unicidad de solución de (1.3). En efecto, es fácil ver que $\mathbf{a}$ y $\mathbf{b}$ son continuas, lo cual se sigue de la desigualdad de Cauchy-Schwarz y la estimación de trazas en $[H^1_{\Gamma_1}(\Omega_1)]^2$ y $H(\mathrm{div}\,;\Omega_2)$ (para los detalles, ver [63, Lemas 2.1 y 3.1]). Además, no es difícil demostrar que $\mathbf{b}$ satisface la condición inf-sup continua y que $\mathbf{a}$ es fuertemente coerciva en el espacio nulo de $\mathbf{b}$ (ver [45, Lemas 2.1 y 2.2]). Por otro lado, es claro que $\mathbf{F}$ y $\mathbf{G}$ también son acotadas, con

$$\|\mathbf{F}\|_{\mathbf{H}'} \le \|\mathbf{f}_1\|_{0,\Omega_1} \quad \text{y} \quad \|\mathbf{G}\|_{\mathbf{Q}'} \le \|f_2\|_{0,\Omega_2}.$$

Entonces, aplicando [54, Teorema I.4.1] se obtiene la existencia y unicidad del problema.

Teorema 1.2.1 *([45, Teorema 2.3], [63, Teorema 3.1]) El problema (1.3) tiene única solución. Además, existe $C > 0$, que depende de la constante de la condición inf-sup de* **b***, de la constante de coercividad de* **a***, y de las constantes de acotamiento de* **a** *y* **b***, tal que*

$$\|\mathbf{u}\|_{\mathbf{H}} + \|(p,\lambda)\|_{\mathbf{Q}} \le C \left\{ \|\mathbf{F}\|_{\mathbf{H}'} + \|\mathbf{G}\|_{\mathbf{Q}'} \right\}.$$

1.3 Una formulación de Galerkin conforme

A continuación introducimos el esquema de Galerkin propuesto en [45], el cual resulta ser el primer método de elementos finitos conforme para el problema continuo (1.3).

Sean $\mathcal{T}_1$ y $\mathcal{T}_2$ triangulaciones regulares de $\bar{\Omega}_1$ y $\bar{\Omega}_2$, respectivamente, formadas por triángulos T de diámetro h_T. Sea también $h := \max\{h_1, h_2\}$, donde para cada $i \in \{1,2\}$, $h_i := \max\{\, h_T : T \in \mathcal{T}_i \,\}$. Para cada $T \in \mathcal{T}_2$, $\mathrm{RT}_0(T)$ es el espacio local de Raviart-Thomas de orden cero, esto es

$$\mathrm{RT}_0(T) := \left\langle \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} \right\rangle,$$

donde $x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ en un vector genérico de $\mathbb{R}^2$.

En lo que sigue, dado un entero no negativo k y un subconjunto S de $\mathbb{R}^2$, $\mathbf{P}_k(S)$ denotará al espacio de polinomios definidos sobre S, de grado menor o igual que k. Para cada $T \in \mathcal{T}_1$, $\mathrm{BR}(T)$ es el espacio local de Bernardi-Raugel (ver [22], [54]), definido por

$$\mathrm{BR}(T) := [\mathbf{P}_1(T)]^2 \oplus \mathrm{span}\,\{\, \eta_2\,\eta_3\,\mathbf{n}_1, \eta_1\,\eta_3\,\mathbf{n}_2, \eta_1\,\eta_2\,\mathbf{n}_3 \,\}\,,$$

donde $\{\eta_1, \eta_2, \eta_3\}$ son las coordenadas baricéntricas de T y $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ son los vectores normales unitarios exteriores a los lados opuestos de los correspondientes vértices de T.

Definamos así los siguientes subespacios de elementos finitos para las velocidades y la presión:

$$\begin{aligned}
\mathbf{H}_{h_1} &:= \left\{ \mathbf{v} \in [C(\bar{\Omega}_1)]^2 : \quad \mathbf{v}|_T \in \mathrm{BR}(T) \quad \forall\, T \in \mathcal{T}_1\,, \quad \mathbf{v} = \mathbf{0} \ \text{sobre} \ \Gamma_1 \right\}, \\
\mathbf{H}_{h_2} &:= \left\{ \mathbf{v} \in H(\mathrm{div}\,;\Omega_2) : \quad \mathbf{v}|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_2 \right\}, \\
\mathbf{Q}_h &:= \left\{ q \in L^2(\Omega) : \quad q|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_1 \cup \mathcal{T}_2 \right\}, \\
\text{y} \quad \mathbf{Q}_{h,0} &:= \mathbf{Q}_h \cap L^2_0(\Omega)\,.
\end{aligned} \tag{1.5}$$

Para definir el subespacio de elementos finitos para $\lambda \in H^{1/2}(\Gamma_2)$, asumamos que los vértices de $\mathcal{T}_1$ y $\mathcal{T}_2$ coinciden en la interfase Γ_2 y que $\mathcal{T}_2$ es uniformemente regular cerca de Γ_2, es decir, que existe $C > 0$ tal que $|\gamma_i| \geq C h_2$ para cada $i \in \{1, ..., n\}$, donde $\{\gamma_1, ..., \gamma_n\}$ es la partición de Γ_2 heredada de las triangulaciones $\mathcal{T}_1$ y $\mathcal{T}_2$. Introduciendo una segunda partición independiente $\{\tilde{\gamma}_1, \tilde{\gamma}_2, ..., \tilde{\gamma}_m\}$ de Γ_2 casi uniformemente regular, es decir que satisfaga la desigualdad $|\tilde{\gamma}_j| \geq C\tilde{h}$ para cada $j \in \{1, 2, ..., m\}$, con $\tilde{h} := \max\{|\tilde{\gamma}_j| : j \in \{1, 2, ..., m\}\}$ y $C > 0$ independiente de h y $\tilde{h}$, se define el espacio de elementos finitos para λ

$$\mathbf{Q}_{\tilde{h}} := \left\{ \xi \in C(\Gamma_2) : \quad \xi|_{\tilde{\gamma}_j} \in \mathbf{P}_1(\tilde{\gamma}_j) \quad \forall\, j \in \{1, ..., m\} \right\}. \tag{1.6}$$

Bajo las hipótesis anteriores, es posible demostrar la existencia de constantes $C_0 \in (0, 1)$ y $C_1 > 0$, independientes de h_2 y $\tilde{h}$, tales que para cada $h_2 \leq C_0\tilde{h}$ y $\xi \in \mathbf{Q}_{\tilde{h}}$, se satisface la condición inf-sup

$$\sup_{\substack{\psi \in \Psi_0 \\ \psi \neq \mathbf{0}}} \frac{\langle \psi, \xi \rangle_\Sigma}{\|\psi\|_{H^{-1/2}(\Sigma)}} \geq C_1 \|\xi\|_{H^{1/2}(\Sigma)}, \tag{1.7}$$

donde

$$\Psi_0 := \left\{ \xi \in L^2(\Gamma_2) : \quad \xi|_{\gamma_j} \in \mathbf{P}_0(\gamma_j) \quad \forall\, j \in \{1, ..., n\} \right\}.$$

Así, introduciendo los espacios discretos $\mathbf{H}_h := \mathbf{H}_{h_1} \times \mathbf{H}_{h_2}$ y $\mathbf{Q}_{h,\tilde{h}} := \mathbf{Q}_{h,0} \times \mathbf{Q}_{\tilde{h}}$, el esquema de Galerkin conforme para (1.3) queda definido por: Encontrar $(\mathbf{u}_h, (p_h, \lambda_{\tilde{h}})) \in \mathbf{H}_h \times \mathbf{Q}_{h,\tilde{h}}$ tal que

$$\begin{aligned} \mathbf{a}(\mathbf{u}_h, \mathbf{v}) + \mathbf{b}(\mathbf{v}, (p_h, \lambda_{\tilde{h}})) &= \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{v}_1 \qquad \forall\, \mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}_h\,, \\ \mathbf{b}(\mathbf{u}_h, (q, \xi)) &= -\int_{\Omega_2} f_2\, q \qquad \forall\, (q, \xi) \in \mathbf{Q}_{h,\tilde{h}}\,. \end{aligned} \tag{1.8}$$

Utilizando la teoría clásica de Babuška-Brezzi, en [45] se establece la existencia y unicidad de solución y estabilidad del esquema de Galerkin (1.8), junto con su estimación de error a priori correspondiente. A continuación se resumen estos resultados.

Teorema 1.3.1 *Existen constantes $C_0 \in (0, 1)$ y $h_0 > 0$ tales que para cada $h_2 \leq \min\{h_0, C_0\tilde{h}\}$, el problema (1.8) tiene una única solución $(\mathbf{u}_h, (p_h, \lambda_{\tilde{h}})) \in \mathbf{H}_h \times \mathbf{Q}_{h,\tilde{h}}$. Además, existen constantes C, $\bar{C} > 0$, independientes de h y $\tilde{h}$, tales que*

$$\|(\mathbf{u}, (p, \lambda)) - (\mathbf{u}_h, (p_h, \lambda_{\tilde{h}}))\|_{\mathbf{H}\times\mathbf{Q}} \leq C\{\|\mathbf{f}_1\|_{[L^2(\Omega_1)]^2} + \|f_2\|_{L^2(\Omega_2)}\}$$

y

$$\|(\mathbf{u}, (p, \lambda)) - (\mathbf{u}_h, (p_h, \lambda_{\tilde{h}}))\|_{\mathbf{H}\times\mathbf{Q}} \leq \bar{C} \inf_{(\mathbf{v}_h, (q_h, \xi_{\tilde{h}})) \in \mathbf{H}_h \times \mathbf{Q}_{h,\tilde{h}}} \|(\mathbf{u}, (p, \lambda)) - (\mathbf{v}_h, (q_h, \xi_{\tilde{h}}))\|_{\mathbf{H}\times\mathbf{Q}}.$$

Es importante mencionar que la elección de $\mathbf{H}_{h_1}$ como subespacio de elementos finitos para $[H^1_{\Gamma_1}(\Omega_1)]^2$ se debe a que el operador de interpolación de Bernardi-Raugel $\Pi_1 : [H^1_{\Gamma_1}(\Omega_1)]^2 \to \mathbf{H}_{h_1}$ (ver [22] y [54]), satisface la propiedad

$$\int_e \Pi_1(\mathbf{v}_1) \cdot \boldsymbol{\nu} = \int_e \mathbf{v}_1 \cdot \boldsymbol{\nu} \tag{1.9}$$

para cada lado e de $\mathcal{T}_1$ y para cada $\mathbf{v}_1 \in [H^1_{\Gamma_1}(\Omega_1)]^2$. Esto nos permite controlar el término $\langle \mathbf{v}_1 \cdot \boldsymbol{\nu}, \xi \rangle_{\Gamma_2}$ en la demostración de la condición inf-sup discreta de $\mathbf{b}$ (ver [45, Lema 4.2] para los detalles). Este subespacio es el único conjunto estable para el problema de Stokes que satisface una condición de esta naturaleza, por lo que la generalización de los resultados de [45] al caso de otros subespacios estables para Stokes se hace imposible. Ahora bien, debido a la inclusión compacta de $L^2(\Gamma_2)$ en $H^{-1/2}(\Gamma_2)$, podemos observar que el término $\langle \mathbf{v}_1 \cdot \boldsymbol{\nu}, \xi \rangle_{\Gamma_2}$, es compacto. Esto sugiere la eventual eliminación de este término de la formulación vía una perturbación compacta, y así utilizar los resultados de Fredholm discreto para demostrar convergencia de otros esquemas de Galerkin para el problema acoplado. Esta idea da origen a los resultados desarrollados en el Capítulo 2.

Por otro lado, el resultado técnico (1.7) tiene un papel importante en la demostración de la condición inf-sup discreta de $\mathbf{b}$, por lo que la desigualdad $h_2 \leq C_0 \tilde{h}$ se transforma en una condición fundamental para el análisis de existencia y unicidad del problema discreto. No obstante, una de las dificultades técnicas al definir el subespacio de elementos finitos para λ de la forma descrita anteriormente, es que la constante $C_0 \in (0,1)$, que determina la elección de $\tilde{h}$, y por lo tanto la elección de la partición independiente, no está explícitamente determinada. Lo anterior obliga a elegir una costante C_0 de forma arbitraria al momento de elaborar ejemplos numéricos que corroboren el buen funcionamiento del método propuesto. En [45] (ver también [14], [46]) se elegió definir la partición independiente $\{\tilde{\gamma}_1, \tilde{\gamma}_2, ..., \tilde{\gamma}_m\}$ tomando sus vértices cada dos vértices de la partición inducida por la triangulacón $\mathcal{T}_2$ sobre Γ_2, lo cual asegura que $h_2 \leq 0.5\tilde{h}$. Esta elección ha arrojado buenos resultados, pero en su momento no habían resultados teóricos que la avalaran. En el Capítulo 3 daremos una solución satisfactoria a este problema.

Chapter 2

Convergence of a family of Galerkin discretizations for the Stokes–Darcy coupled problem

2.1 Introduction

The development of appropriate numerical methods for the coupling of fluid flow (modelled by the Stokes equation) with porous media flow (modelled by the Darcy equation) has become a very active research area lately (see, e.g. [17], [34], [35], [37], [39], [45], [61], [63], [74], [78], [83] and the references therein). In particular, the analysis in [39] is based on mortar finite element techniques, and Hood-Taylor and lowest order Raviart-Thomas spaces are employed for the Stokes and Darcy regions, respectively. On the other hand, a priori error estimates for the matching case are developed in [63], whereas discontinuous Galerkin schemes for the non-matching case are considered in [21] and [74]. More recently, a conforming mixed finite element discretization has been introduced and analyzed in [45]. The model from [45] consists of a porous medium entirely enclosed within a fluid region, which constitutes a slight simplification of the problem considered in [35] and [63]. The variational formulation in [45] follows the approach from [63] and employs the primal method in the fluid and the dual-mixed one in the porous medium. This means that only the original velocity and pressure unknowns are considered in the Stokes domain, whereas a further unknown (velocity) is added in the Darcy region. Interface conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law, which yields the introduction of the trace of the porous medium pressure as another Lagrange multiplier. Stability of a specific Galerkin scheme is the main result in [45]. This scheme is defined by using Bernardi-Raugel elements for the velocity in the fluid region, Raviart-Thomas elements of lowest order for the filtration velocity in the porous media, piecewise constants with null mean value for the pressures, and continuous piecewise linear elements for the additional Lagrange multiplier. The resulting mixed finite element method is the first one which is conforming for the original variational formulation proposed in [63]. On the other hand, it is important to remark that the interpolation properties of the Raviart-Thomas and Bernardi-Raugel operators, mainly those holding on the edges of the triangulations (see eqs. (3.11), (4.2), and (4.7) in [45]), play a key role in the proof of one of the required discrete inf-sup conditions (see Lemma 4.2 in [45]). However, these particular properties are not satisfied in general, and hence the analysis in [45] can not be extended to arbitrary finite element subspaces. This drawback of the approach in [45] has motivated the present contribution.

In the present paper we modify the approach from [45] and provide general sufficient conditions on the finite element subspaces guaranteeing unique solvability, stability, and convergence of the associated mixed finite element method. More precisely, we improve the results from [45] and show that the use of any pair of stable Stokes and Darcy elements implies the well-posedness of the Stokes-Darcy Galerkin scheme. In particular, this includes Hood-Taylor, Bernardi-Raugel and MINI element for the Stokes region, and Raviart-Thomas of any order for the Darcy domain. Our analysis hinges on the fact that the operator defining the continuous variational formulation is given by a compact perturbation of an invertible mapping. However, we also show that under somewhat more demanding hypotheses, these compactness arguments are not needed. The rest of this work is organized as follows. In Section 2.2 we recall the model problem from [45], the continuous variational formulation, and the theorem establishing its well-posedness (unique solvability and continuous dependence). In addition, we observe here that a different geometry of the problem, namely the fluid over the porous medium, does not yield further difficulties to the analysis. Then, in Section 2.3 we consider a compact perturbation of this formulation and apply the well known Babuška-Brezzi theory to show that it is also well-posed. Next, in Section 2.4 we employ a classical result on projection methods for Fredholm operators of index zero to conclude the well-posedness of the Stokes-Darcy Galerkin scheme for any pair of stable Stokes and Darcy elements. In Section 2.5 we give a slightly different set of hypotheses on the discrete spaces allowing an alternative proof of stability of the Galerkin scheme, without identifying any compact perturbation. Finally, in Section 2.6 we present several numerical results illustrating the good performance of the method for different geometries of the coupled problem when the MINI element and the Raviart-Thomas subspace of order 0 are employed.

First of all, some comments concerning notations. In any product of Hilbert spaces that we find in the sequel we will implicitly understand that we are using the product topology. Given any function space U, U^2 and $U^{2\times 2}$ will denote the spaces of vectors and square matrices of order 2, respectively, with entries in U. Throughout the rest of the paper we use the standard terminology for Sobolev spaces. In particular, if S is an open set, its closure, or a Lipschitz continuous curve, and $r \in \mathbb{R}$, then $|\cdot|_{r,S}$ and $\|\cdot\|_{r,S}$ stand for the seminorm and norm in the Sobolev spaces $H^r(S)$. The norm and seminorms for $[H^r(S)]^2$ and $[H^r(S)]^{2\times 2}$ will be equally denoted. Also, we employ $\mathbf{0}$ as a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different occurrences. Finally, χ_S will denote the characteristic function of the set S.

2.2 The Stokes-Darcy coupled problem

2.2.1 The model problem

Let Ω_2 be a bounded and simply connected domain in $\mathbb{R}^2$ with polygonal boundary Γ_2, and let Ω_1 be the annular region bounded by Γ_2 and another closed polygonal curve Γ_1 whose interior contains $\overline{\Omega}_2$. The unit normal vector field on Γ_2, pointing inwards in Ω_2, is denoted $\boldsymbol{\nu}$. The tangential vector field on the same interface, obtained by a $\pi/2$ clockwise rotation of $\boldsymbol{\nu}$, will be denoted $\mathbf{t}$.

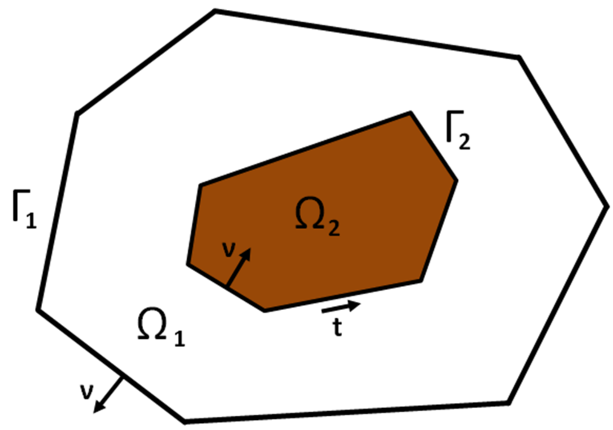

Figure 2.1: A sketch of the geometry of the problem.

The coupled problem models an incompressible viscous fluid occupying Ω_1, which flows back and forth across Γ_2 into a porous medium living in Ω_2 and saturated with the same fluid. Physically, we may think of Ω_2 as the cross section of a three-dimensional porous medium, given for instance by a long cylinder parallel to the x_3-axis, which is immersed in a viscous fluid. In what follows, $\mu > 0$ denotes the viscosity of the fluid and $\mathbf{K} \in [L^\infty(\Omega_2)]^{2\times 2}$ is a symmetric and uniformly positive definite tensor in Ω_2 representing the permeability of the porous media divided by the viscosity. We assume that there exists $C > 0$ such that

$$\boldsymbol{\xi} \cdot \mathbf{K}(x)\,\boldsymbol{\xi} \geq C\,\|\boldsymbol{\xi}\|^2$$

for almost all $x \in \Omega_2$, and for all $\boldsymbol{\xi} \in \mathbb{R}^2$. The constitutive equations are given by the Stokes and Darcy laws, respectively, that is

$$\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1) \;=\; -\,p_1\,\mathbf{I} \,+\, 2\,\mu\,\mathbf{e}(\mathbf{u}_1) \quad \text{in} \quad \Omega_1\,, \quad \text{and} \quad \mathbf{u}_2 \;=\; -\,\mathbf{K}\,\nabla\,p_2 \quad \text{in} \quad \Omega_2\,,$$

where $(\mathbf{u}_1, \mathbf{u}_2)$ and (p_1, p_2) denote the velocities and pressures in the corresponding domains, $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$, $\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)$ is the stress tensor, and

$$\mathbf{e}(\mathbf{u}_1) \;:=\; \tfrac{1}{2}\left(\nabla\mathbf{u}_1 + (\nabla\mathbf{u}_1)^{\mathtt{t}}\right)$$

is the strain tensor. Hereafter, the superscript $^{\mathtt{t}}$ denotes transposition. Given $\mathbf{f}_1 \in [L^2(\Omega_1)]^2$ and $f_2 \in L^2(\Omega_2)$ such that $\displaystyle\int_{\Omega_2} f_2 = 0$, the Stokes-Darcy coupled problem reads: Find $(\mathbf{u}_1, \mathbf{u}_2)$ and (p_1, p_2) such that

$$\begin{aligned}
-\,\mathbf{div}\,\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1) \;&=\; \mathbf{f}_1 && \text{in} \quad \Omega_1\,,\\
\operatorname{div}\mathbf{u}_1 \;&=\; 0 && \text{in} \quad \Omega_1\,,\\
\mathbf{u}_1 \;&=\; \mathbf{0} && \text{on} \quad \Gamma_1\,,\\
\operatorname{div}\mathbf{u}_2 \;&=\; f_2 && \text{in} \quad \Omega_2\,,\\
\mathbf{u}_1 \cdot \boldsymbol{\nu} \;&=\; \mathbf{u}_2 \cdot \boldsymbol{\nu} && \text{on} \quad \Gamma_2\,,\\
\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)\,\boldsymbol{\nu} \,+\, p_2\,\boldsymbol{\nu} \;&=\; -\,\frac{\mu}{\kappa}\,(\mathbf{u}_1 \cdot \mathbf{t})\,\mathbf{t} && \text{on} \quad \Gamma_2\,,
\end{aligned} \tag{2.1}$$

where $\kappa > 0$ is the friction constant. Note that the second transmission condition on Γ_2 can be decomposed, at least formally, into its normal and tangential components as follows:

$$(\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)\,\boldsymbol{\nu}) \cdot \boldsymbol{\nu} \;=\; -\,p_2 \quad \text{and} \quad (\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)\,\boldsymbol{\nu}) \cdot \mathbf{t} \;=\; -\,\frac{\mu}{\kappa}\,(\mathbf{u}_1 \cdot \mathbf{t}) \quad \text{on} \quad \Gamma_2\,. \tag{2.2}$$

The first equation in (2.2) corresponds to the balance of normal forces, whereas the second one is known as the Beavers-Joseph-Saffman law, which establishes that the slip velocity along Γ_2 is proportional to the shear stress along Γ_2 (assuming also, based on experimental evidences, that $\mathbf{u}_2 \cdot \mathbf{t}$ is negligible). We refer to [16], [59], and [76] for further details on this interface condition.

2.2.2 The variational formulation

We now write $\Omega := \Omega_1 \cup \Gamma_2 \cup \Omega_2$ and define the spaces

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \quad \int_\Omega q = 0 \right\},$$

$$[H^1_{\Gamma_1}(\Omega_1)]^2 := \left\{ \mathbf{v}_1 \in [H^1(\Omega_1)]^2 : \ \mathbf{v}_1 = \mathbf{0} \ \text{on} \ \Gamma_1 \right\},$$

and

$$H(\text{div}\,;\Omega_2) := \left\{ \mathbf{v}_2 \in [L^2(\Omega_2)]^2 : \ \text{div}\ \mathbf{v}_2 \in L^2(\Omega_2) \right\}.$$

In addition, we let

$$\mathbf{H} := [H^1_{\Gamma_1}(\Omega_1)]^2 \times H(\text{div}\,;\Omega_2) \quad \text{and} \quad \mathbf{Q} := L_0^2(\Omega) \times H^{1/2}(\Gamma_2) \tag{2.3}$$

endowed with the product norms. The global unknowns will be $\mathbf{u} := (\mathbf{u}_1, \mathbf{u}_2)$, $\quad p := p_1\chi_{\Omega_1} + p_2\chi_{\Omega_2}$, as well as the Lagrange multiplier $\lambda := p_2 \quad \text{on} \quad \Gamma_2$. We then proceed as in [63] to obtain the weak formulation of this problem. In order to do that, we will need the bilinear forms $\mathbf{a} : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) := 2\,\mu \int_{\Omega_1} \mathbf{e}(\mathbf{u}_1) : \mathbf{e}(\mathbf{v}_1) + \frac{\mu}{\kappa} \int_{\Gamma_2} (\mathbf{u}_1 \cdot \mathbf{t})\,(\mathbf{v}_1 \cdot \mathbf{t}) + \int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}_2 \cdot \mathbf{v}_2, \tag{2.4}$$

and $\mathbf{b} : \mathbf{H} \times \mathbf{Q} \to \mathbb{R}$

$$\mathbf{b}(\mathbf{v}, (q, \xi)) := -\int_{\Omega_1} q \,\text{div}\,\mathbf{v}_1 - \int_{\Omega_2} q \,\text{div}\,\mathbf{v}_2 + \langle \mathbf{v}_1 \cdot \boldsymbol{\nu} - \mathbf{v}_2 \cdot \boldsymbol{\nu}, \xi \rangle_{\Gamma_2}\,, \tag{2.5}$$

$\langle \cdot, \cdot \rangle_{\Gamma_2}$ being the duality pairing of $H^{-1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$ with respect to the $L^2(\Gamma_2)$-inner product. We also consider the functionals $\mathbf{F} : \mathbf{H} \to \mathbb{R}$

$$\mathbf{F}(\mathbf{v}) = \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{v}_1 \qquad \forall \mathbf{v} \in \mathbf{H} \tag{2.6}$$

and $\mathbf{G} : \mathbf{Q} \to \mathbb{R}$

$$\mathbf{G}(q, \xi) = -\int_{\Omega_2} f_2\, q \qquad \forall\,(q, \xi) \in \mathbf{Q}\,. \tag{2.7}$$

The mixed variational formulation of (2.1) is: Find $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, (p, \lambda)) &= \mathbf{F}(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{H}\,, \\ \mathbf{b}(\mathbf{u}, (q, \xi)) &= \mathbf{G}(q, \xi) \qquad \forall\,(q, \xi) \in \mathbf{Q}\,. \end{aligned} \tag{2.8}$$

The classical Babuška-Brezzi theory is applied in [45] to prove that (2.8) is well-posed. In fact, it is easy to see that $\mathbf{a}$ and $\mathbf{b}$ are bounded, which follows from simple applications of the Cauchy-Schwarz inequality and the trace estimates in $[H^1(\Omega_1)]^2$ and $H(\text{div}\,;\Omega_2)$ (see [63, Lemmas 2.1

and 3.1] for details). In addition, it is not difficult to show that **b** satisfies the continuous inf-sup condition and **a** is strongly coercive on the null space of **b** (see [45, Lemmas 2.1 and 2.2]). On the other hand, it is clear that **F** and **G** are also bounded with

$$\|\mathbf{F}\|_{\mathbf{H}'} \,\leq\, \|\mathbf{f}_1\|_{0,\Omega_1} \quad \text{and} \quad \|\mathbf{G}\|_{\mathbf{Q}'} \,\leq\, \|f_2\|_{0,\Omega_2}\,.$$

Consequently, the well-posedness of the continuous formulation (2.8), which follows from a straightforward application of [54, Theorem I.4.1], and which constitutes also one of the main results provided in [63], is established as follows.

Theorem 2.2.1 ([45, Theorem 2.3], [63, Theorem 3.1]) *There exists a unique* $(\mathbf{u},(p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$ *solution to* (2.8). *In addition, there exists* $\tilde{C} > 0$, *depending on the inf-sup constant for* **b**, *the coerciveness constant for* **a**, *and the boundedness constants of* **a** *and* **b**, *such that*

$$\|\mathbf{u}\|_{\mathbf{H}} + \|(p,\lambda)\|_{\mathbf{Q}} \,\leq\, \tilde{C}\left\{\|\mathbf{F}\|_{\mathbf{H}'} + \|\mathbf{G}\|_{\mathbf{Q}'}\right\}.$$

2.2.3 Remarks on the geometry

It is important to remark that, though the geometry described by Figure 2.1 was choosen to simplify the presentation, the case of a fluid flowing only across a part of the boundary of the porous medium does not really yield further complications neither for the analysis in [45] nor for the one in the present paper. In fact, let us assume now a geometry like the one despicted in Figure 2.2 below where $\Gamma_2 := \partial\Omega_1\cap\partial\Omega_2$, $\Gamma_1 := \partial\Omega_1\backslash\bar{\Gamma}_2$, and $\Gamma := \partial\Omega_2\backslash\bar{\Gamma}_2$. In this case, besides the equations given in (2.1) (which hold now with the notations introduced here), one needs to add a boundary condition on Γ. Following [39] and [63] (see also [37]), one usually considers the homogeneous Neumann condition:

$$\mathbf{u}_2\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma\,, \tag{2.9}$$

which constitutes a no flow assumption through Γ. In this way, and having in mind the new geometry, the space **H** becomes now $[H^1_{\Gamma_1}(\Omega_1)]^2\times H_0(\mathrm{div}\,;\Omega_2)$, where

$$H_0(\mathrm{div}\,;\Omega_2) \,:=\, \left\{\mathbf{v}_2\in H(\mathrm{div}\,;\Omega_2): \quad \mathbf{v}_2\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma\right\}, \tag{2.10}$$

and **Q**, **a**, and **b** remain the same as before (cf. (2.3), (2.4), and (2.5)). In particular, the reason for keeping $H^{1/2}(\Gamma_2)$ as the right space for the Lagrange multiplier λ, which differs from the choice of $H^{1/2}_{00}(\Gamma_2)$ adopted in [63], is that λ represents the trace of the porous pressure on Γ_2, and hence there is no physical reason to assume that λ vanishes in Γ. Recall that $H^{1/2}_{00}(\Gamma_2)$ is the subspace of $H^{1/2}(\Gamma_2)$ whose extensions by zero in Γ belong to $H^{1/2}(\partial\Omega_2)$. The present choice of $H^{1/2}(\Gamma_2)$ is also justified in Section 4.1 of [39].

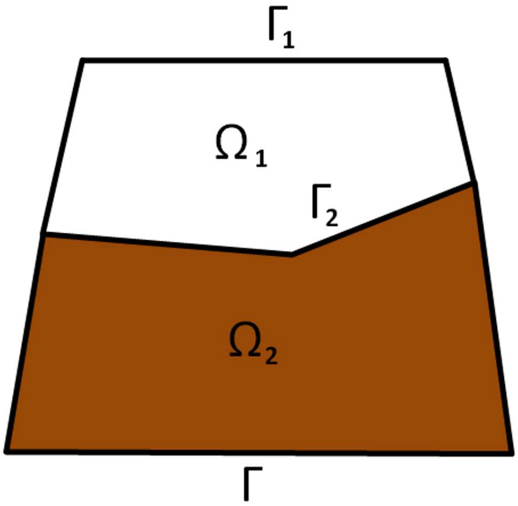

Figure 2.2: A sketch of the geometry of the problem with the fluid over the porous medium.

In connection with the above, we now recall that, given $\mathbf{v}_2 \in H_0(\mathrm{div}\,;\Omega_2)$, the boundary condition $\mathbf{v}_2 \cdot \boldsymbol{\nu} = 0$ on Γ means:

$$\langle\, \mathbf{v}_2 \cdot \boldsymbol{\nu}, E_{00}(\xi)\,\rangle_{\partial\Omega_2} \;=\; 0 \qquad \forall\, \xi \,\in\, H_{00}^{1/2}(\Gamma)\,, \tag{2.11}$$

where $E_{00}(\xi)$ denotes the extension by zero in Γ_2 of each $\xi \in H^{1/2}(\Gamma)$, and $\langle\,\cdot,\cdot\,\rangle_{\partial\Omega_2}$ stands for the duality pairing of $H^{-1/2}(\partial\Omega_2)$ and $H^{1/2}(\partial\Omega_2)$ with respect to the $L^2(\partial\Omega_2)$-inner product. As a consequence, it is not difficult to show (see e.g. Section 2 in [39]) that the restriction of $\mathbf{v}_2 \cdot \boldsymbol{\nu}$ to Γ_2 can be identified with an element of $H^{-1/2}(\Gamma_2)$:

$$\langle\, \mathbf{v}_2 \cdot \boldsymbol{\nu}, \xi\,\rangle_{\Gamma_2} \;:=\; \langle\, \mathbf{v}_2 \cdot \boldsymbol{\nu}, E(\xi)\,\rangle_{\partial\Omega_2} \qquad \forall\, \xi \,\in\, H^{1/2}(\Gamma_2)\,, \tag{2.12}$$

where $E : H^{1/2}(\Gamma_2) \to H^{1/2}(\partial\Omega_2)$ is the bounded linear operator defined by $E(\xi) := \gamma(z)$ for each $\xi \in H^{1/2}(\Gamma_2)$, $\gamma : H^1(\Omega_2) \to H^{1/2}(\partial\Omega_2)$ is the usual trace operator, and $z \in H^1(\Omega_2)$ is the unique solution of:

$$\Delta z \;=\; 0 \quad \text{in} \quad \Omega_2\,, \qquad z = \xi \quad \text{on} \quad \Gamma_2\,, \qquad \nabla z \cdot \boldsymbol{\nu} \;=\; 0 \quad \text{on} \quad \Gamma\,.$$

Moreover, thanks to (2.11) and (2.12), we may also write $\langle\, \mathbf{v}_2 \cdot \boldsymbol{\nu}, \xi\,\rangle_{\Gamma_2} \;:=\; \langle\, \mathbf{v}_2 \cdot \boldsymbol{\nu}, \tilde{\xi}\,\rangle_{\partial\Omega_2}$ with any $\tilde{\xi} \in H^{1/2}(\partial\Omega_2)$ such that $\tilde{\xi} = \xi$ on Γ_2.

From the above analysis, and since $\mathbf{v}_1 \cdot \boldsymbol{\nu}|_{\Gamma_2} \in L^2(\Gamma_2) \subseteq H^{-1/2}(\Gamma_2)$ for each $\mathbf{v}_1 \in [H^1_{\Gamma_1}(\Omega_1)]^2$, it becomes clear that the boundary term in the definition of $\mathbf{b}$ must again be understood as the duality pairing between $H^{-1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$. Consequently, the proofs of the corresponding inf-sup conditions for $\mathbf{a}$ and $\mathbf{b}$ (see Section 4.2 in [39] for details) follow basically the same techniques applied in [45] and [63], thus confirming that no additional difficulties arise. This is also valid for the corresponding discrete analysis, which is illustrated by two numerical examples reported below in Section 2.6.

2.3 A compact perturbation of the variational formulation

In order to define below a suitable compact perturbation of (2.8), we first observe that this formulation is equivalent to: Find $((\mathbf{u},\varphi),(p,\lambda)) \in (\mathbf{H}\times\mathbb{R})\times\mathbf{Q}$ such that

$$\begin{aligned} \widehat{\mathbf{a}}((\mathbf{u},\varphi),(\mathbf{v},\psi)) + \mathbf{b}(\mathbf{v},(p,\lambda)) &= \widehat{\mathbf{F}}(\mathbf{v},\psi) && \forall\,(\mathbf{v},\psi)\in\mathbf{H}\times\mathbb{R}\,,\\ \mathbf{b}(\mathbf{u},(q,\xi)) &= \widehat{\mathbf{G}}(q,\xi) && \forall\,(q,\xi)\in\mathbf{Q}\,, \end{aligned} \tag{2.13}$$

where $\widehat{\mathbf{F}}(\mathbf{v},\psi) = \mathbf{F}(\mathbf{v})$, $\widehat{\mathbf{G}} = \mathbf{G}$, and $\widehat{\mathbf{a}} : (\mathbf{H}\times\mathbb{R})\times(\mathbf{H}\times\mathbb{R})\to\mathbb{R}$ is the bounded bilinear form defined by

$$\widehat{\mathbf{a}}((\mathbf{u},\varphi),(\mathbf{v},\psi)) := \mathbf{a}(\mathbf{u},\mathbf{v}) + \varphi\,\psi\,. \tag{2.14}$$

In fact, it is easy to see that $((\mathbf{u},\varphi),(p,\lambda)) \in (\mathbf{H}\times\mathbb{R})\times\mathbf{Q}$ is a solution of (2.13) if and only if $\varphi = 0$ and $(\mathbf{u},(p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$ is a solution of (2.8). In other words, $\varphi\in\mathbb{R}$ is an artificial unknown, known a priori to vanish, which is introduced here only for convenience.

We now consider the following variational problem: Find $((\mathbf{u},\varphi),(p,\lambda)) \in (\mathbf{H}\times\mathbb{R})\times\mathbf{Q}$ such that

$$\begin{aligned} \widehat{\mathbf{a}}((\mathbf{u},\varphi),(\mathbf{v},\psi)) + \widehat{\mathbf{b}}((\mathbf{v},\psi),(p,\lambda)) &= \widehat{\mathbf{F}}(\mathbf{v},\psi) && \forall\,(\mathbf{v},\psi)\in\mathbf{H}\times\mathbb{R}\,,\\ \widehat{\mathbf{b}}((\mathbf{u},\varphi),(q,\xi)) &= \widehat{\mathbf{G}}(q,\xi) && \forall\,(q,\xi)\in\mathbf{Q}\,, \end{aligned} \tag{2.15}$$

where $\widehat{\mathbf{b}} : (\mathbf{H}\times\mathbb{R})\times\mathbf{Q}\to\mathbb{R}$ is the bounded bilinear form defined by

$$\begin{aligned} \widehat{\mathbf{b}}((\mathbf{v},\psi),(q,\xi)) :=&\ \mathbf{b}(\mathbf{v},(q,\xi)) - \langle \mathbf{v}_1\cdot\boldsymbol{\nu},\xi\rangle_{\Gamma_2} + \psi\int_{\Omega_2} q\\ =&\ -\int_{\Omega_1} q\,\mathrm{div}\,\mathbf{v}_1 - \int_{\Omega_2} q\,\mathrm{div}\,\mathbf{v}_2 - \langle \mathbf{v}_2\cdot\boldsymbol{\nu},\xi\rangle_{\Gamma_2} + \psi\int_{\Omega_2} q\,, \end{aligned} \tag{2.16}$$

for each $((\mathbf{v},\psi),(q,\xi)) \in (\mathbf{H}\times\mathbb{R})\times\mathbf{Q}$. Since $\mathbf{v}_1\cdot\boldsymbol{\nu} \in L^2(\Gamma_2)$ for each $\mathbf{v} := (\mathbf{v}_1,\mathbf{v}_2)\in\mathbf{H}$, and $L^2(\Gamma_2)$ is compactly imbedded into $H^{-1/2}(\Gamma_2)$, we easily verify that the operator induced by the left-hand side of (2.15) is a compact perturbation of the corresponding operator from (2.13). We now proceed to apply the Babuška-Brezzi theory to prove that (2.15) is well-posed, as well. We begin with the inf-sup condition for $\widehat{\mathbf{b}}$.

Lemma 2.3.1 *There exists $\widehat{\beta} > 0$ such that*

$$\sup_{\substack{(\mathbf{v},\psi)\in\mathbf{H}\times\mathbb{R}\\ (\mathbf{v},\psi)\neq\mathbf{0}}} \frac{\widehat{\mathbf{b}}((\mathbf{v},\psi),(q,\xi))}{\|(\mathbf{v},\psi)\|_{\mathbf{H}\times\mathbb{R}}} \geq \widehat{\beta}\,\|(q,\xi)\|_{\mathbf{Q}} \qquad \forall\,(q,\xi)\in\mathbf{Q}\,. \tag{2.17}$$

Proof. It reduces to a slight modification of the proof for the inf-sup condition of $\mathbf{b}$ (cf. [45, Lemma 2.1]). We begin with an arbitrary $\eta \in H^{-1/2}(\Gamma_2)$ and let $z \in H^1(\Omega_2)$ be the unique weak solution of the boundary value problem:

$$-\,\Delta z \,=\, \frac{1}{|\Omega_2|}\,\langle \eta, 1\rangle_{\Gamma_2} \quad \text{in} \quad \Omega_2\,, \qquad \frac{\partial z}{\partial \boldsymbol{\nu}} \,=\, -\,\eta \quad \text{on} \quad \Gamma_2\,, \qquad \int_{\Omega_2} z \,=\, 0\,.$$

The continuous dependence estimate for the above problem yields

$$\|z\|_{1,\Omega_2} \,\leq\, c\,\|\eta\|_{-1/2,\Gamma_2}\,.$$

Then, we define $\widehat{\mathbf{v}}_2 \,:=\, \nabla z$ and $\widehat{\psi} \,:=\, -\frac{1}{|\Omega_2|}\langle \eta, 1\rangle_{\Gamma_2} \,\in\, \mathbb{R}$, so that we have

$$\text{div}\ \widehat{\mathbf{v}}_2 - \widehat{\psi} = 0, \qquad \text{in } \Omega_2, \qquad \widehat{\mathbf{v}}_2\cdot\boldsymbol{\nu} \,=\, -\,\eta, \quad \text{on } \Gamma_2,$$

as well as the bound

$$\|\widehat{\mathbf{v}}_2\|_{H(\text{div}\,;\Omega_2)} + |\widehat{\psi}| \,\leq\, C\,\|\eta\|_{-1/2,\Gamma_2}\,.$$

Next, we set $\widehat{\mathbf{v}} \,:=\, (\mathbf{0}, \widehat{\mathbf{v}}_2) \,\in\, \mathbf{H}$ and notice that

$$\sup_{\substack{(\mathbf{v},\psi)\in\mathbf{H}\times\mathbb{R}\\ (\mathbf{v},\psi)\neq\mathbf{0}}} \frac{\widehat{\mathbf{b}}((\mathbf{v},\psi),(q,\xi))}{\|(\mathbf{v},\psi)\|_{\mathbf{H}\times\mathbb{R}}} \,\geq\, \frac{|\,\widehat{\mathbf{b}}((\widehat{\mathbf{v}},\widehat{\psi}),(q,\xi))\,|}{\|(\widehat{\mathbf{v}},\widehat{\psi})\|_{\mathbf{H}\times\mathbb{R}}} \,=\, \frac{|\,\langle \eta,\xi\rangle_{\Gamma_2}\,|}{\|(\widehat{\mathbf{v}},\widehat{\psi})\|_{\mathbf{H}\times\mathbb{R}}} \,\geq\, c_1\,\frac{|\,\langle \eta,\xi\rangle_{\Gamma_2}\,|}{\|\eta\|_{-1/2,\Gamma_2}}\,,$$

which, using that $\eta \in H^{-1/2}(\Gamma_2)$ is arbitrary, yields

$$\sup_{\substack{(\mathbf{v},\psi)\in\mathbf{H}\times\mathbb{R}\\ (\mathbf{v},\psi)\neq\mathbf{0}}} \frac{\widehat{\mathbf{b}}((\mathbf{v},\psi),(q,\xi))}{\|(\mathbf{v},\psi)\|_{\mathbf{H}\times\mathbb{R}}} \,\geq\, c_1\,\|\xi\|_{1/2,\Gamma_2}\,. \tag{2.18}$$

On the other hand, since $q \in L^2_0(\Omega)$, a well known result on the surjectivity of the divergence operator (see for instance [54, Corollary I.2.4]) yields the existence of $\mathbf{z} \in [H^1_0(\Omega)]^2$ such that $\text{div}\ \mathbf{z} \,=\, -\,q$ in Ω and $\|\mathbf{z}\|_{1,\Omega} \,\leq\, c\,\|q\|_{0,\Omega}$. Thus, defining $\widehat{\mathbf{v}} \,:=\, (\mathbf{z}|_{\Omega_1}, \mathbf{z}|_{\Omega_2}) \,\in\, \mathbf{H}$, we find that $\text{div}\ \widehat{\mathbf{v}}_i \,=\, -q$ in Ω_i, and $\|\widehat{\mathbf{v}}\|_{\mathbf{H}} \,\leq\, C\,\|q\|_{0,\Omega}$, whence

$$\begin{aligned} \sup_{\substack{(\mathbf{v},\psi)\in\mathbf{H}\times\mathbb{R}\\ (\mathbf{v},\psi)\neq\mathbf{0}}} \frac{\widehat{\mathbf{b}}((\mathbf{v},\psi),(q,\xi))}{\|(\mathbf{v},\psi)\|_{\mathbf{H}\times\mathbb{R}}} &\geq \frac{|\,\widehat{\mathbf{b}}((\widehat{\mathbf{v}},0),(q,\xi))\,|}{\|(\widehat{\mathbf{v}},0)\|_{\mathbf{H}\times\mathbb{R}}} \\ = \frac{|\,\|q\|^2_{0,\Omega} \,-\, \langle \widehat{\mathbf{v}}_2\cdot\boldsymbol{\nu},\xi\rangle_{\Gamma_2}\,|}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} &\geq c_2\,\|q\|_{0,\Omega} \,-\, c_3\,\|\xi\|_{1/2,\Gamma_2}\,. \end{aligned} \tag{2.19}$$

Finally, it is easy to see that (2.17) follows from (2.18) and (2.19). □

The strong coerciveness of $\widehat{\mathbf{a}}$ on the null space of $\widehat{\mathbf{b}}$,

$$\widehat{\mathbf{V}} \,:=\, \Big\{\,(\mathbf{v},\psi)\in\mathbf{H}\times\mathbb{R}: \quad \widehat{\mathbf{b}}((\mathbf{v},\psi),(q,\xi)) \,=\, 0 \quad \forall\,(q,\xi)\in\mathbf{Q}\,\Big\},$$

is shown next as a consequence of some properties concerning this set.

Lemma 2.3.2 *Given* $(\mathbf{v}, \psi) \in \mathbf{H} \times \mathbb{R}$*,*

$$\int_{\Omega_1} q \operatorname{div} \mathbf{v}_1 + \int_{\Omega_2} q \operatorname{div} \mathbf{v}_2 - \psi \int_{\Omega_2} q = 0 \qquad \forall q \in L^2_0(\Omega), \tag{2.20}$$

if and only if

$$\begin{aligned} \operatorname{div} \mathbf{v}_1 &= \frac{1}{|\Omega_1|} \langle \mathbf{v}_1 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2} && \text{in } \Omega_1, \\ \operatorname{div} \mathbf{v}_2 &= -\frac{1}{|\Omega_2|} \langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2} && \text{in } \Omega_2, \\ \psi &= -\frac{1}{|\Omega_1|} \langle \mathbf{v}_1 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2} - \frac{1}{|\Omega_2|} \langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2}. \end{aligned} \tag{2.21}$$

Proof. It is simple to see, using that $q - \left(\frac{1}{|\Omega_2|} \int_\Omega q \right) \chi_{\Omega_2}$ belongs to $L^2_0(\Omega)$ for each $q \in L^2(\Omega)$, that (2.20) is equivalent to

$$\int_{\Omega_1} q \operatorname{div} \mathbf{v}_1 + \int_{\Omega_2} q \operatorname{div} \mathbf{v}_2 - \psi \int_{\Omega_2} q + \left(\frac{1}{|\Omega_2|} \langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2} + \psi \right) \int_\Omega q = 0 \qquad \forall q \in L^2(\Omega),$$

which can be easily broken as the pair of conditions

$$\int_{\Omega_1} q \left(\operatorname{div} \mathbf{v}_1 + \frac{1}{|\Omega_2|} \langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2} + \psi \right) = 0 \qquad \forall q \in L^2(\Omega_1),$$

$$\int_{\Omega_2} q \left(\operatorname{div} \mathbf{v}_2 + \frac{1}{|\Omega_2|} \langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2} \right) = 0 \qquad \forall q \in L^2(\Omega_2).$$

That these two conditions are equivalent to (2.21) is straightforward to verify. □

Lemma 2.3.3 *If* $(\mathbf{v}, \psi) \in \widehat{\mathbf{V}}$*, then* $\operatorname{div} \mathbf{v}_2 = 0$ *in* Ω_2*. Therefore there exists* $\widehat{\alpha} > 0$ *such that*

$$\widehat{\mathbf{a}}((\mathbf{v}, \psi), (\mathbf{v}, \psi)) \geq \widehat{\alpha} \, \|(\mathbf{v}, \psi)\|^2_{\mathbf{H} \times \mathbb{R}} \qquad \forall \, (\mathbf{v}, \psi) \in \widehat{\mathbf{V}} \, . \tag{2.22}$$

Proof. It is clear that $(\mathbf{v}, \psi) \in \widehat{\mathbf{V}}$ is equivalent to (2.20) together with the boundary condition

$$\mathbf{v}_2 \cdot \boldsymbol{\nu} = 0 \qquad \text{on} \quad \Gamma_2, \tag{2.23}$$

which thanks to the characterization (2.21) proves the first part of the result. Hence, Korn's and Poincaré's inequalities and the fact that $\mathbf{K}^{-1}$ is symmetric and positive definite yield (2.22), completing the proof. □

In this way, applying again [54, Theorem I.4.1], we conclude the well-posedness of the variational formulation (2.15).

2.4 A class of Galerkin schemes and their convergence

In this section we provide sufficient conditions guaranteeing unique solvability, stability, and Céa's estimate for the Galerkin scheme of (2.8). To this end, we make use of the equivalence between the Galerkin schemes of (2.8) and (2.13), and apply the following classical result on projection methods for Fredholm operators of index zero.

Theorem 2.4.1 *Let $(X, \langle \cdot, \cdot \rangle_X)$ be a Hilbert space, let A, $K : X \to X$ be bounded linear operators, and let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of finite dimensional subspaces of X. Assume that:*

i) *A is bijective, K is compact, and $A + K$ is injective,*

ii) *for each $x \in X$:* $\lim\limits_{n\to+\infty} \left\{ \inf\limits_{z_n\in X_n} \|x - z_n\|_X \right\} = 0,$

iii) *the Galerkin scheme associated to the pair (A, X_n) is convergent, that is*

- *there exists $N \in \mathbb{N}$ such that for each $x \in X$ and for each $n \geq N$ there exists a unique $x_n \in X_n$ satisfying*

$$\langle A(x_n), z_n\rangle_X = \langle A(x), z_n\rangle_X \qquad \forall\, z_n \in X_n\,, \tag{2.24}$$

- *for each $x \in X$ there holds* $\lim\limits_{n\to+\infty} x_n = x.$

Then, the Galerkin scheme associated to the pair $(A + K, X_n)$ is also convergent.

Proof. See [62, Theorem 13.7]). □

We complement Theorem 2.4.1 with some useful remarks. We first recall that the operator $\mathcal{G}_n : X \to X_n$ mapping x into the unique solution x_n of (2.24) is called the Galerkin projector associated to the pair (A, X_n). The Galerkin scheme (2.24) is said to be stable if the projectors $\{\mathcal{G}_n\}_{n\geq N}$ are uniformly bounded. A simple application of the Banach-Steinhaus Theorem shows that the convergence of x_n to x implies the stability of (2.24). Conversely, the stability of (2.24) together with the assumption ii) yield convergence. In fact, to see the latter let $M > 0$ such that $\|\mathcal{G}_n\| \leq M \quad \forall\, n \geq N$. Then, since $\mathcal{G}_n$ coincides with the identity operator $\mathcal{I}$ on X_n, it follows that $\|x - x_n\|_X = \|(\mathcal{I} - \mathcal{G}_n)(x - z_n)\|_X \leq (1 + M)\, \|x - z_n\|_X \quad \forall\, z_n \in X_n$, and hence $\|x - x_n\|_X \leq (1 + M) \inf\limits_{z_n\in X_n} \|x - z_n\|_X \to 0$ as $n \to +\infty$.

Now, since the operators induced by the left-hand sides of (2.13) and (2.15) are bijective, and the difference between them is a compact operator, a straightforward application of Theorem 2.4.1 establishes that the convergence of a Galerkin method applied to (2.13) is equivalent to the convergence of the Galerkin scheme of (2.15), when the same subspaces are employed. According to this, it suffices to analyze the Galerkin scheme of (2.15) (see Section 2.4.2 below).

2.4.1 Preliminaries

We first let $H_h(\Omega_1)$, $H_h(\Omega_2)$, $Q_h(\Omega_1)$, $Q_h(\Omega_2)$, and $Q_h(\Gamma_2)$ be finite dimensional subspaces of $[H^1_{\Gamma_1}(\Omega_1)]^2$, $\mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)$, $L^2(\Omega_1)$, $L^2(\Omega_2)$, and $H^{1/2}(\Gamma_2)$, respectively. Then, we define

$$Q_{h,0}(\Omega_1) := Q_h(\Omega_1) \cap L^2_0(\Omega_1)\,, \qquad Q_{h,0}(\Omega_2) := Q_h(\Omega_2) \cap L^2_0(\Omega_2)\,,$$

$$Q_h(\Omega) := \{q \in L^2(\Omega) : \quad q|_{\Omega_i} \in Q_h(\Omega_i) \quad \forall\, i \in \{1,2\}\,\}\,, \quad Q_{h,0}(\Omega) := Q_h(\Omega) \cap L^2_0(\Omega)\,,$$

$$\mathbf{H}_h := H_h(\Omega_1) \times H_h(\Omega_2)\,, \quad \text{and} \quad \mathbf{Q}_h := Q_{h,0}(\Omega) \times Q_h(\Gamma_2)\,. \tag{2.25}$$

In this way, the Galerkin schemes of (2.8), (2.13), and (2.15) are given, respectively, by:

Find $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$\begin{aligned} \mathbf{a}(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{b}(\mathbf{v}_h, (p_h, \lambda_h)) &= \mathbf{F}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{H}_h\,, \\ \mathbf{b}(\mathbf{u}_h, (q_h, \xi_h)) &= \mathbf{G}(q_h, \xi_h) && \forall\, (q_h, \xi_h) \in \mathbf{Q}_h\,, \end{aligned} \tag{2.26}$$

Find $((\mathbf{u}_h, \varphi_h), (p_h, \lambda_h)) \in (\mathbf{H}_h \times \mathbb{R}) \times \mathbf{Q}_h$ such that

$$\begin{aligned} \widehat{\mathbf{a}}((\mathbf{u}_h, \varphi_h), (\mathbf{v}_h, \psi_h)) + \mathbf{b}(\mathbf{v}_h, (p_h, \lambda_h)) &= \widehat{\mathbf{F}}(\mathbf{v}_h, \psi_h) && \forall\, (\mathbf{v}_h, \psi_h) \in \mathbf{H}_h \times \mathbb{R}\,, \\ \mathbf{b}(\mathbf{u}_h, (q_h, \xi_h)) &= \widehat{\mathbf{G}}(q_h, \xi_h) && \forall\, (q_h, \xi_h) \in \mathbf{Q}_h\,, \end{aligned} \tag{2.27}$$

and

Find $((\mathbf{u}_h, \varphi_h), (p_h, \lambda_h)) \in (\mathbf{H}_h \times \mathbb{R}) \times \mathbf{Q}_h$ such that

$$\begin{aligned} \widehat{\mathbf{a}}((\mathbf{u}_h, \varphi_h), (\mathbf{v}_h, \psi_h)) + \widehat{\mathbf{b}}((\mathbf{v}_h, \psi_h), (p_h, \lambda_h)) &= \widehat{\mathbf{F}}(\mathbf{v}_h, \psi_h) && \forall\, (\mathbf{v}_h, \psi_h) \in \mathbf{H}_h \times \mathbb{R}\,, \\ \widehat{\mathbf{b}}((\mathbf{u}_h, \varphi_h), (q_h, \xi_h)) &= \widehat{\mathbf{G}}(q_h, \xi_h) && \forall\, (q_h, \xi_h) \in \mathbf{Q}_h\,. \end{aligned} \tag{2.28}$$

Similarly as for the continuous case, it is easy to see that (2.26) and (2.27) are equivalent. More precisely, $((\mathbf{u}_h, \varphi_h), (p_h, \lambda_h)) \in (\mathbf{H}_h \times \mathbb{R}) \times \mathbf{Q}_h$ is a solution of (2.27) if and only if $\varphi_h = 0$ and $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ is a solution of (2.26).

Throughout the rest of Section 2.4 we assume the following hypotheses on the subspaces:

(H.1) The pair $(H_h(\Omega_1), Q_h(\Omega_1))$ is stable for the Stokes problem, that is, there exists $\beta_1 > 0$, independent of h, such that for each $q_{1,h} \in Q_{h,0}(\Omega_1)$ there holds

$$\sup_{\substack{\mathbf{v}_{1,h} \in H_h(\Omega_1) \\ \mathbf{v}_{1,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_1} q_{1,h}\, \mathrm{div}\, \mathbf{v}_{1,h}}{\|\mathbf{v}_{1,h}\|_{1,\Omega_1}} \geq \beta_1\, \|q_{1,h}\|_{0,\Omega_1}\,. \tag{2.29}$$

In addition, the space of constant functions on Ω_1 is contained in $Q_h(\Omega_1)$.

(H.2) The triple $(H_h(\Omega_2), Q_h(\Omega_2), Q_h(\Gamma_2))$ is stable for the Darcy problem, that is, there exists $\beta_2 > 0$, independent of h, such that for each $(q_{2,h}, \xi_h) \in Q_{h,0}(\Omega_2) \times Q_h(\Gamma_2)$ there holds

$$\sup_{\substack{\mathbf{v}_{2,h} \in H_h(\Omega_2) \\ \mathbf{v}_{2,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_2} q_{2,h} \, \mathrm{div} \, \mathbf{v}_{2,h} \; + \; \langle \mathbf{v}_{2,h} \cdot \boldsymbol{\nu}, \xi_h \rangle_{\Gamma_2}}{\|\mathbf{v}_{2,h}\|_{\mathrm{div},\Omega_2}} \;\geq\; \beta_2 \left\{ \|q_{2,h}\|_{0,\Omega_2} + \|\xi_h\|_{1/2,\Gamma_2} \right\}. \tag{2.30}$$

Here we have used the symbol $\|\cdot\|_{\mathrm{div},\Omega_2}$ for the norm of $\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$. In addition, $\mathrm{div}\, \mathbf{v}_{2,h} \in Q_h(\Omega_2)$, for all $\mathbf{v}_{2,h} \in H_h(\Omega_2)$, and the spaces of constant functions on Ω_2 and Γ_2 are contained in $Q_h(\Omega_2)$ and $Q_h(\Gamma_2)$, respectively.

(H.3) The discrete spaces satisfy the approximation properties

$$\lim_{h \to 0} \left\{ \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{H}} \right\} \; = \; 0 \qquad \forall \, \mathbf{v} \, \in \, \mathbf{H}\,, \tag{2.31}$$

$$\lim_{h \to 0} \left\{ \inf_{(q_h,\xi_h) \in \mathbf{Q}_h} \|(q,\xi) - (q_h,\xi_h)\|_{\mathbf{Q}} \right\} \; = \; 0 \qquad \forall \, (q,\xi) \, \in \, \mathbf{Q}\,. \tag{2.32}$$

It is important to remark here that, in the case of the geometry described in Subsection 2.2.3, $H_h(\Omega_2)$ is a finite dimensional subspace of $H_0(\mathrm{div}\,;\Omega_2)$ (cf. (2.10)) throughout the whole Section 2.4.

2.4.2 The main result

As already announced, we now analyze the Galerkin scheme (2.28). We begin with the following lemma establishing the discrete inf-sup condition for $\widehat{\mathbf{b}}$.

Lemma 2.4.1 *There exists $\beta > 0$, independent of h, such that*

$$S_h(q_h,\xi_h) := \sup_{\substack{(\mathbf{v}_h,\psi_h) \in \mathbf{H}_h \times \mathbb{R} \\ (\mathbf{v}_h,\psi_h) \neq \mathbf{0}}} \frac{\widehat{\mathbf{b}}((\mathbf{v}_h,\psi_h),(q_h,\xi_h))}{\|(\mathbf{v}_h,\psi_h)\|_{\mathbf{H}\times\mathbb{R}}} \geq \beta \, \|(q_h,\xi_h)\|_{\mathbf{Q}} \qquad \forall \, (q_h,\xi_h) \in \mathbf{Q}_h\,. \tag{2.33}$$

Proof. Limiting the set where the supremum is taken to elements of the form $(\mathbf{0},\psi_h)$, we obtain

$$S_h(q_h,\xi_h) \geq \sup_{\substack{\psi_h \in \mathbb{R} \\ \psi_h \neq 0}} \frac{\psi_h \displaystyle\int_{\Omega_2} q_h}{|\psi_h|} = \left| \int_{\Omega_2} q_h \right| = \left| \int_{\Omega_1} q_h \right|, \qquad \forall (q_h,\xi_h) \in \mathbf{Q}_h. \tag{2.34}$$

Restricting now the supremum to the set of elements $((-\mathbf{v}_{1,h},\mathbf{0}),0)$ and using (2.29), we obtain

$$\begin{aligned} S_h(q_h,\xi_h) \geq & \sup_{\substack{\mathbf{v}_{1,h} \in H_h(\Omega_1) \\ \mathbf{v}_{1,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_1} (\mathrm{div}\, \mathbf{v}_{1,h})\, q_h}{\|\mathbf{v}_{1,h}\|_{1,\Omega_1}} \geq \beta_1 \left\| q_h - \frac{1}{|\Omega_1|} \int_{\Omega_1} q_h \right\|_{0,\Omega_1} - c_1 \left| \int_{\Omega_1} q_h \right| \\ \geq & \;\; \beta_1 \|q_h\|_{0,\Omega_1} - c_2 \left| \int_{\Omega_1} q_h \right|, \qquad \forall (q_h,\xi_h) \in \mathbf{Q}_h. \end{aligned} \tag{2.35}$$

Finally, considering now elements $((\mathbf{0}, -\mathbf{v}_{2,h}), 0)$ and using (2.30), we obtain

$$
\begin{aligned}
S_h(q_h, \xi_h) \geq & \sup_{\substack{\mathbf{v}_{2,h} \in H_h(\Omega_2) \\ \mathbf{v}_{2,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_2} (\mathrm{div}\, \mathbf{v}_{2,h})\, q_h + \langle \mathbf{v}_{2,h} \cdot \boldsymbol{\nu}, \xi_h \rangle_{\Gamma_2}}{\|\mathbf{v}_{2,h}\|_{\mathrm{div},\Omega_2}} \\
\geq & \;\beta_2 \Big(\Big\| q_h - \frac{1}{|\Omega_2|} \int_{\Omega_2} q_h \Big\|_{0,\Omega_2} + \|\xi_h\|_{1/2,\Gamma_2} \Big) - c_3 \Big| \int_{\Omega_2} q_h \Big| \\
\geq & \;\beta_2 \Big(\|q_h\|_{0,\Omega_2} + \|\xi_h\|_{1/2,\Gamma_2} \Big) - c_4 \Big| \int_{\Omega_2} q_h \Big|, \qquad \forall (q_h, \xi_h) \in \mathbf{Q}_h.
\end{aligned}
\tag{2.36}
$$

Adding inequalities (2.35) and (2.36) to $(c_2 + c_4)$ times (2.34), we obtain an inequality that is equivalent to the one in the statement of the lemma. □

The strong coerciveness of $\widehat{\mathbf{a}}$ on the discrete null space of $\widehat{\mathbf{b}}$

$$
\widehat{\mathbf{V}}_h \;:=\; \Big\{ (\mathbf{v}_h, \psi_h) \in \mathbf{H}_h \times \mathbb{R} : \quad \widehat{\mathbf{b}}((\mathbf{v}_h, \psi_h), (q_h, \xi_h)) \;=\; 0 \quad \forall\, (q_h, \xi_h) \in \mathbf{Q}_h \Big\},
$$

is shown next.

Lemma 2.4.2 *If* $(\mathbf{v}_h, \psi_h) \in \widehat{\mathbf{V}}_h$*, then* $\mathrm{div}\, \mathbf{v}_{2,h} = 0$ *in* Ω_2 *and*

$$
\widehat{\mathbf{a}}((\mathbf{v}_h, \psi_h), (\mathbf{v}_h, \psi_h)) \;\geq\; \widehat{\alpha}\, \|(\mathbf{v}_h, \psi_h)\|^2_{\mathbf{H}_h \times \mathbb{R}} \qquad \forall\, (\mathbf{v}_h, \psi_h) \in \widehat{\mathbf{V}}_h\,, \tag{2.37}
$$

with the same constant as in Lemma 2.3.3.

Proof. We first observe, according to the definition of $\widehat{\mathbf{b}}$ (cf. (2.16)), that

$$
-\int_{\Omega_1} q_h\, \mathrm{div}\, \mathbf{v}_{1,h} \;-\; \int_{\Omega_2} q_h\, \mathrm{div}\, \mathbf{v}_{2,h} \;+\; \psi_h \int_{\Omega_2} q_h \;=\; 0 \quad \forall\, q_h \in Q_{h,0}(\Omega) \tag{2.38}
$$

and

$$
\langle \mathbf{v}_{2,h} \cdot \boldsymbol{\nu}, \xi_h \rangle_{\Gamma_2} \;=\; 0 \qquad \forall\, \xi_h \in Q_h(\Gamma_2)\,. \tag{2.39}
$$

Since by Hypotheses (H.1) and (H.2), we have constant functions in $Q_h(\Omega_1)$, $Q_h(\Omega_2)$ and $Q_h(\Gamma_2)$, we can test (2.38) with $q_h := -\frac{1}{|\Omega_1|}\chi_{\Omega_1} + \frac{1}{|\Omega_2|}\chi_{\Omega_2}$, and (2.39) with $\xi_h = 1$ to deduce that

$$
\psi_h \;=\; -\frac{1}{|\Omega_1|} \int_{\Gamma_2} \mathbf{v}_{1,h} \cdot \boldsymbol{\nu}\,. \tag{2.40}
$$

Next, given an arbitrary $q_{2,h} \in Q_h(\Omega_2)$, we define $\tilde{q}_h := -\Big(\frac{1}{|\Omega_1|}\int_{\Omega_2} q_{2,h}\Big)\chi_{\Omega_1} + q_{2,h}\chi_{\Omega_2} \in Q_{h,0}(\Omega)$ and test (2.38) with this function. Using (2.40), we find that

$$
\int_{\Omega_2} q_{2,h}\, \mathrm{div}\; \mathbf{v}_{2,h} \;=\; 0 \qquad \forall\, q_{2,h} \in Q_h(\Omega_2)\,,
$$

which, using that by (H.2) $\mathrm{div}\, \mathbf{v}_{2,h} \in Q_h(\Omega_2)$, proves the first statement of the result. The second one follows as in Lemma 2.3.3. □

Consequently, stability of the Galerkin scheme (2.28) follows from the previous lemmas and the discrete Babuška-Brezzi theory (see, [54, Theorem II.1.1] for instance). Hence, as shown right after Theorem 2.4.1, stability and the approximation hypothesis (H.3) imply convergence of the method. Having established this, Theorem 2.4.1 implies that the Galerkin scheme (2.27) is also convergent and therefore stable, which means that we have the Céa quasioptimality estimate for h small enough. In this way, since (2.27) and (2.26) are equivalent, we have shown the following main result.

Theorem 2.4.2 *Assume that the hypotheses* (H.1)*,* (H.2)*, and* (H.3) *hold. Then, there exists* $h_0 > 0$ *such that for each* $h \leq h_0$ *the Galerkin scheme* (2.26) *has a unique solution* $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$*. Moreover, there exist positive constants* C_1 *and* C_2*, independent of* h*, such that*

$$\|u_h\|_{\mathbf{H}} + \|(p_h, \lambda_h))\|_{\mathbf{Q}} \leq C_1 \left\{ \|\mathbf{F}\|_{\mathbf{H}_h'} + \|\mathbf{G}\|_{\mathbf{Q}_h'} \right\} \qquad \forall\, h \leq h_0\,, \tag{2.41}$$

and

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}} + \|(p, \lambda) - (p_h, \lambda_h)\|_{\mathbf{Q}} \\ &\quad \leq C_2 \left\{ \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}} + \inf_{(q_h, \xi_h) \in \mathbf{Q}_h} \|(p, \lambda) - (q_h, \xi_h)\|_{\mathbf{Q}} \right\} \qquad \forall\, h \leq h_0\,. \end{aligned} \tag{2.42}$$

We end this subsection by remarking that, as usual, the approximation properties of the finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$ together with the Céa estimate (2.42) yield the corresponding rates of convergence of the Galerkin scheme (2.26).

2.4.3 Examples of subspaces satisfying the hypotheses

There is a large variety of stable Stokes elements available in the literature: the MINI element, Bernardi-Raugel, Hood-Taylor, conforming Crouzeix-Raviart, and others (see, e.g. [19] and [54]). Similarly, stable Darcy elements are usually defined in terms of Raviart-Thomas, Brezzi-Douglas-Marini, and related subspaces (see, e.g [19]). In order to specify some subspaces satisfying (H.1), (H.2) and (H.3), we now let $\mathcal{T}_1$ and $\mathcal{T}_2$ be members of regular families of triangulations, satisfying the angle condition, of $\bar{\Omega}_1$ and $\bar{\Omega}_2$, respectively, by triangles T of diameter h_T, and let $h := \max\{h_1, h_2\}$, where $h_i := \max\{\, h_T : \; T \in \mathcal{T}_i \,\}$ for each $i \in \{1, 2\}$. Also, given S, an open set, its closure, or a Lipschitz continuous curve of $\mathbb{R}^2$, and a non-negative integer k, we denote by $\mathbf{P}_k(S)$ the space of polynomials defined on S of degree $\leq k$.

The Bernardi-Raugel space

For each $T \in \mathcal{T}_1$ we let $\mathcal{BR}(T)$ be the local Bernardi-Raugel space (see [22], [54]), that is

$$\mathcal{BR}(T) := [\mathbf{P}_1(T)]^2 \oplus \text{span}\,\{\, \theta_2\, \theta_3\, \boldsymbol{\nu}_1, \theta_1\, \theta_3\, \boldsymbol{\nu}_2, \theta_1\, \theta_2\, \boldsymbol{\nu}_3 \,\}\ , \tag{2.43}$$

where $\{\theta_1, \theta_2, \theta_3\}$ are the barycentric coordinates of T, and $\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3\}$ are the unit outward normals to the opposite sides of the corresponding vertices of T. Then, the Bernardi-Raugel element is the pair $(H_h(\Omega_1), Q_h(\Omega_1))$, where

$$H_h(\Omega_1) := \Big\{ \mathbf{v}_{1,h} \in [C(\bar{\Omega}_1)]^2 : \quad \mathbf{v}_{1,h}|_T \in \mathcal{BR}(T) \quad \forall\, T \in \mathcal{T}_1 \,, \quad \mathbf{v}_{1,h} = \mathbf{0} \text{ on } \Gamma_1 \Big\} \tag{2.44}$$

and

$$Q_h(\Omega_1) := \Big\{ q_{1,h} \in L^2(\Omega_1) : \quad q_{1,h}|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_1 \Big\} . \tag{2.45}$$

The proof of the corresponding inf-sup condition (2.29) follows straighforwardly from the analysis in Section 2.1 of Chapter II of [54]. In addition, it is clear that $\mathbf{P}_0(\Omega_1)$ is contained in $Q_h(\Omega_1)$, and hence $(H_h(\Omega_1), Q_h(\Omega_1))$ satisfies the hypothesis (H.1).

The MINI element

For each $T \in \mathcal{T}_1$ we let $\mathcal{M}(T)$ be the space (see [6], [54])

$$\mathcal{M}(T) := [\mathbf{P}_1(T) \oplus \text{span}\,\{b_T\}]^2 \,, \tag{2.46}$$

where $b_T := \theta_1\, \theta_2\, \theta_3$ is a $\mathbb{P}_3$ bubble function in T. Then, the MINI element subspace is the pair $(H_h(\Omega_1), Q_h(\Omega_1))$, where

$$H_h(\Omega_1) := \Big\{ \mathbf{v}_{1,h} \in [C(\bar{\Omega}_1)]^2 : \quad \mathbf{v}_{1,h}|_T \in \mathcal{M}(T) \quad \forall\, T \in \mathcal{T}_1 \,, \quad \mathbf{v}_{1,h} = \mathbf{0} \text{ on } \Gamma_1 \Big\} \tag{2.47}$$

and

$$Q_h(\Omega_1) := \Big\{ q_{1,h} \in C(\bar{\Omega}_1) : \quad q_{1,h}|_T \in \mathbf{P}_1(T) \quad \forall\, T \in \mathcal{T}_1 \Big\} . \tag{2.48}$$

In this case, the proof of the corresponding inf-sup condition (2.29) follows from the analysis in Section 4.1 of Chapter II of [54]. In particular, we refer to Lemma 4.1 in there. In addition, it is also clear that $\mathbf{P}_0(\Omega_1)$ is contained in $Q_h(\Omega_1)$, and hence $(H_h(\Omega_1), Q_h(\Omega_1))$ satisfies the hypothesis (H.1), as well.

The Raviart-Thomas element

For each $T \in \mathcal{T}_2$ we let $\mathcal{RT}_0(T)$ be the local Raviart-Thomas space of lowest order, that is

$$\mathcal{RT}_0(T) := [\mathbf{P}_0(T)]^2 \oplus \mathbf{P}_0(T)\, m \,,$$

where $m(\mathbf{x}) := (x_1, x_2)^{\mathsf{t}}$. Then, we define the pair $(H_h(\Omega_2), Q_h(\Omega_2))$ as

$$H_h(\Omega_2) := \Big\{ \mathbf{v}_{2,h} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\text{div}; \Omega_{\mathrm{D}}) : \quad \mathbf{v}_{2,h}|_T \in \mathcal{RT}_0(T) \quad \forall\, T \in \mathcal{T}_2 \Big\} \tag{2.49}$$

and

$$Q_h(\Omega_2) := \Big\{ q_{2,h} \in L^2(\Omega_2) : \quad q_{2,h}|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_2 \Big\} . \tag{2.50}$$

It is easy to see that div $\mathbf{v}_{2,h} \in Q_h(\Omega_2) \quad \forall \mathbf{v}_{2,h} \in H_h(\Omega_2)$, and $\mathbb{P}_0(\Omega_2) \subseteq Q_h(\Omega_2)$. Next, in order to define the subspace $Q_h(\Gamma_2)$, we let $\{\gamma_1, \gamma_2, ..., \gamma_n\}$ be the partition of Γ_2 inherited from the triangulation $\mathcal{T}_2$, and introduce a second partition $\{\tilde{\gamma}_1, \tilde{\gamma}_2, ..., \tilde{\gamma}_m\}$ of Γ_2, also made of line segments, such that $\{\tilde{\gamma}_1, \tilde{\gamma}_2, ..., \tilde{\gamma}_m\}$ is a derefinement of $\{\gamma_1, \gamma_2, ..., \gamma_n\}$. In other words, for each $j \in \{1, 2, ..., n\}$ there exists $i \in \{1, 2, ..., m\}$ such that $\gamma_j \subseteq \tilde{\gamma}_i$. Then, we set $\tilde{h} := \max\{|\tilde{\gamma}_j| : j \in \{1, ..., m\}\}$, redefine $h := \max\{h_1, h_2, \tilde{h}\}$, and introduce

$$Q_h(\Gamma_2) := \Big\{ \xi_h \in C(\Gamma_2) : \quad \xi_h|_{\tilde{\gamma}_j} \in \mathbf{P}_1(\tilde{\gamma}_j) \quad \forall\, j \in \{1, ..., m\} \Big\}. \tag{2.51}$$

It is clear that $\mathbf{P}_0(\Gamma_2) \subseteq Q_h(\Gamma_2)$. In addition, from the analysis provided in [45] we deduce the existence of a constant $C_0 \in (0, 1]$ such that (2.30) holds for each $h_2 \le C_0\, \tilde{h}$. This technical requirement explains the need of defining $Q_h(\Gamma_2)$ on the second partition $\{\tilde{\gamma}_1, \tilde{\gamma}_2, ..., \tilde{\gamma}_m\}$ instead of $\{\gamma_1, \gamma_2, ..., \gamma_n\}$. We omit further details here and refer to Sections 3 and 4 in [45]. In particular, it is not difficult to see that the same arguments apply for the case of the geometry described in Subsection 2.2.3 with $H_h(\Omega_2)$ redefined as a subspace of $H_0(\mathrm{div}\,; \Omega_2)$, that is

$$H_h(\Omega_2) := \Big\{ \mathbf{v}_{2,h} \in H_0(\mathrm{div}\,; \Omega_2) : \quad \mathbf{v}_{2,h}|_T \in \mathcal{RT}_0(T) \quad \forall\, T \in \mathcal{T}_2 \Big\}. \tag{2.52}$$

According to the above, we conclude that the triple $(H_h(\Omega_2), Q_h(\Omega_2), Q_h(\Gamma_2))$ satisfies the hypothesis (H.2).

Finally, the satisfaction of the hypothesis (H.3) follows in each case from standard density arguments and the approximation properties of the subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$ involved (see, e.g. [19], [54], and [73]).

2.5 An alternative approach

In this section we present a slightly different (and somewhat more demanding) set of hypotheses of the discrete spaces allowing a proof of stability of a Galerkin scheme for all $h > 0$, without having to recur to the asymptotic limit (cf. constant h_0 in Theorem 2.4.2). This means in particular that we will not eliminate compact terms from the equation. However, we will still impose conditions on the discrete spaces separately for the Stokes and Darcy domains, with no coupling hypothesis whatsoever. For technical reasons, we will have to change the determination of the pressure, requesting now that

$$\int_{\Omega_2} p = 0\,.$$

Hence, throughout this section we replace the original definition of the space $\mathbf{Q}$ (cf. (2.3)) by $\mathbf{Q} := [L^2(\Omega_1) \times L^2_0(\Omega_2)] \times H^{1/2}(\Gamma_2)$, keep $\mathbf{H}$ as in (2.3), and consider the bilinear and linear forms of formulas (2.4), (2.5), (2.6), and (2.7) defined on the present space $\mathbf{H} \times \mathbf{Q}$. Having this in mind, our aim is the numerical solution of the following well posed problem: Find $(\mathbf{u},(p,\lambda)) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\mathbf{u},\mathbf{v}) + \mathbf{b}(\mathbf{v},(p,\lambda)) &= \mathbf{F}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}\,, \\ \mathbf{b}(\mathbf{u},(q,\xi)) &= \mathbf{G}(q,\xi) \quad \forall (q,\xi) \in \mathbf{Q}\,. \end{aligned} \tag{2.53}$$

For this purpose, we take $\mathbf{H}_h$ as in (2.25), let $\mathbf{Q}_h := [Q_h(\Omega_1) \times Q_{h,0}(\Omega_2)] \times Q_h(\Gamma_2)$, and define the corresponding Galerkin equations as in (2.26).

Next, we let $H_{h,0}(\Omega_1) := H_h(\Omega_1) \cap [H_0^1(\Omega_1)]^2$ and instead of (H.1) consider the following more stringent assumptions:

(H.4) There exist β_3, $\beta_4 > 0$, independent of h, and $\mathbf{u}_h^0 \in H_h(\Omega_1)$, $\mathbf{u}_h^0 \neq \mathbf{0}$, such that

$$\sup_{\substack{\mathbf{v}_{1,h} \in H_{h,0}(\Omega_1) \\ \mathbf{v}_{1,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_1} q_{1,h} \operatorname{div} \mathbf{v}_{1,h}}{\|\mathbf{v}_{1,h}\|_{1,\Omega_1}} \geq \beta_3 \|q_{1,h}\|_{0,\Omega_1} \qquad \forall\, q_{1,h} \in Q_{h,0}(\Omega_1)\,, \tag{2.54}$$

and

$$\int_{\Gamma_2} \mathbf{u}_h^0 \cdot \boldsymbol{\nu} \geq \beta_4 \|\mathbf{u}_h^0\|_{1,\Omega_1}\,. \tag{2.55}$$

In addition, the space of constant functions on Ω_1 is contained in $Q_h(\Omega_1)$.

(H.5) There exists $\beta_5 > 0$, independent of h, such that

$$\sup_{\substack{\mathbf{v}_{1,h} \in H_h(\Omega_1) \\ \mathbf{v}_{1,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_1} q_{1,h} \operatorname{div} \mathbf{v}_{1,h}}{\|\mathbf{v}_{1,h}\|_{1,\Omega_1}} \geq \beta_5 \|q_{1,h}\|_{0,\Omega_1} \qquad \forall\, q_{1,h} \in Q_h(\Omega_1)\,. \tag{2.56}$$

In addition, the space of constant functions on Ω_1 is contained in $Q_h(\Omega_1)$.

It is not difficult to see that (2.29) in (H.1) follows straightforwardly from (2.54) as well as from (2.56), and that (2.56) implies (2.55). Also, note that (2.54) is the necessary inf-sup condition that the pair of spaces has to satisfy to be applicable to the Stokes problem with homogeneous boundary conditions on the whole boundary, whereas (2.56) is related to using the space for the Stokes problem with Dirichlet boundary conditions only on Γ_1. In what follows we show that (H.4) is sufficient for (H.5) and that (H.2) and (H.5) yield the discrete inf-sup condition for $\mathbf{b}$.

Lemma 2.5.1 (H.4) $\Longrightarrow$ (H.5).

Proof. Given $q_{1,h} \in Q_h(\Omega_1)$, we let $q_0 \in Q_{h,0}(\Omega_1)$ and $d \in \mathbb{R}$ such that $q_{1,h} = q_0 + d$. Then, using (2.54) it follows that

$$S_h(q_{1,h}) := \sup_{\substack{\mathbf{v}_{1,h} \in H_h(\Omega_1) \\ \mathbf{v}_{1,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_1} q_{1,h} \operatorname{div} \mathbf{v}_{1,h}}{\|\mathbf{v}_{1,h}\|_{1,\Omega_1}} \geq \sup_{\substack{\mathbf{v}_{1,h} \in H_{h,0}(\Omega_1) \\ \mathbf{v}_{1,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_1} q_0 \operatorname{div} \mathbf{v}_{1,h}}{\|\mathbf{v}_{1,h}\|_{1,\Omega_1}} \geq \beta_3 \|q_0\|_{0,\Omega_1}\,. \tag{2.57}$$

Next, bounding $S_h(q_{1,h})$ from below with $\mathbf{v}_{1,h} := d\,\mathbf{u}_h^0 \in H_h(\Omega_1)$ and using (2.55), we find that

$$S_h(q_{1,h}) \,\geq\, |d| \,\frac{\displaystyle\int_{\Gamma_2} \mathbf{u}_h^0 \cdot \boldsymbol{\nu}}{\|\mathbf{u}_h^0\|_{1,h}} \,-\, C\,\|q_0\|_{0,\Omega_1} \,\geq\, \beta_4\,|d| \,-\, C\,\|q_0\|_{0,\Omega_1}\,. \tag{2.58}$$

In this way, adding (2.57) to $\frac{\beta_3}{2C}$ times (2.58) we obtain the required estimate (2.56). □

Lemma 2.5.2 *Assume that* (H.2) *and* (H.5) *hold. Then there exists $\beta > 0$, independent of h, such that*

$$S_h(q_h, \xi_h) \,:=\, \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\mathbf{v}_h, (q_h, \xi_h))}{\|\mathbf{v}_h\|_{\mathbf{H}}} \,\geq\, \beta\,\|(q_h, \xi_h)\|_{\mathbf{Q}} \qquad \forall\,(q_h, \xi_h) \,\in\, \mathbf{Q}_h\,.$$

Proof. We proceed similarly as in the proof of Lemma 2.4.1. Restricting the supremum to the set of elements $(-\mathbf{v}_{1,h}, \mathbf{0})$ and using (2.56) (cf. (H.5)), we obtain

$$S_h(q_h, \xi_h) \geq \sup_{\substack{\mathbf{v}_{1,h} \in H_h(\Omega_1) \\ \mathbf{v}_{1,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_1} q_{1,h} \,\mathrm{div}\,\mathbf{v}_{1,h} - \langle \mathbf{v}_{1,h} \cdot \boldsymbol{\nu}, \xi_h \rangle_{\Gamma_2}}{\|\mathbf{v}_{1,h}\|_{1,\Omega_1}} \geq \beta_5\,\|q_{1,h}\|_{0,\Omega_1} - C\,\|\xi_h\|_{1/2,\Gamma_2}\,. \tag{2.59}$$

Now, considering elements $(\mathbf{0}, -\mathbf{v}_{2,h})$ and using (2.30) (cf. (H.2)), we find that

$$S_h(q_h, \xi_h) \geq \sup_{\substack{\mathbf{v}_{2,h} \in H_h(\Omega_2) \\ \mathbf{v}_{2,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_2} q_{2,h} \,\mathrm{div}\;\mathbf{v}_{2,h} \,+\, \langle \mathbf{v}_{2,h} \cdot \boldsymbol{\nu}, \xi_h \rangle_{\Gamma_2}}{\|\mathbf{v}_{2,h}\|_{\mathrm{div},\Omega_2}} \,\geq\, \beta_2 \left\{ \|q_{2,h}\|_{0,\Omega_2} + \|\xi_h\|_{1/2,\Gamma_2} \right\}. \tag{2.60}$$

Finally, adding (2.60) to $\frac{\beta_2}{2C}$ times (2.59) we conclude the discrete inf-sup condition for $\mathbf{b}$. □

The ellipticity of $\mathbf{a}$ in the discrete kernel of $\mathbf{b}$ is established next.

Lemma 2.5.3 *Let*

$$\mathbf{V}_h \,:=\, \left\{ \mathbf{v}_h \in \mathbf{H}_h : \quad \mathbf{b}(\mathbf{v}_h, (q_h, \xi_h)) \,=\, 0 \quad \forall\,(q_h, \xi_h) \,\in\, \mathbf{Q}_h \right\}.$$

Then there exists $\alpha > 0$, independent of h, such that

$$\mathbf{a}(\mathbf{v}_h, \mathbf{v}_h) \,\geq\, \alpha\,\|\mathbf{v}_h\|_{\mathbf{H}}^2 \qquad \forall\,\mathbf{v}_h \,\in\, \mathbf{V}_h\,.$$

Proof. It suffices to observe that

$$\mathbf{V}_h \,\subseteq\, \tilde{\mathbf{V}}_h \,:=\, \left\{ \mathbf{v}_h \in \mathbf{H}_h : \quad \mathrm{div}\,\mathbf{v}_{2,h} \,\in\, \mathbb{P}_0(\Omega_2) \right\},$$

and that $\mathbf{a}$ is elliptic in $\tilde{\mathbf{V}}_h$. □

As a consequence of Lemmas 2.5.2 and 2.5.3 we have the following theorem (recall the definitions of $\mathbf{Q}$ and $\mathbf{Q}_h$ in this section).

Theorem 2.5.1 *Assume that the hypotheses* (H.2) *and* (H.5) *hold. Then for each* $h > 0$ *the Galerkin scheme of* (2.53) *has a unique solution* $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$. *Moreover, there exist positive constants* C_1 *and* C_2, *independent of* h, *such that*

$$\|u_h\|_{\mathbf{H}} + \|(p_h, \lambda_h))\|_{\mathbf{Q}} \le C_1 \left\{ \|\mathbf{F}\|_{\mathbf{H}_h'} + \|\mathbf{G}\|_{\mathbf{Q}_h'} \right\}, \tag{2.61}$$

and

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}} + \|(p, \lambda) - (p_h, \lambda_h)\|_{\mathbf{Q}} \\ &\le C_2 \left\{ \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}} + \inf_{(q_h, \xi_h) \in \mathbf{Q}_h} \|(p, \lambda) - (q_h, \xi_h)\|_{\mathbf{Q}} \right\}. \end{aligned} \tag{2.62}$$

Note that, as announced at the beginning of this section, Theorem 2.5.1 does not recur to the asymptotic limit. However, according to the logic connectivity of the inequalities in (H.1), (H.4), and (H.5) (see Figure 5.1 below), the discrete inf-sup condition (2.29) in (H.1) is the less restrictive of all.

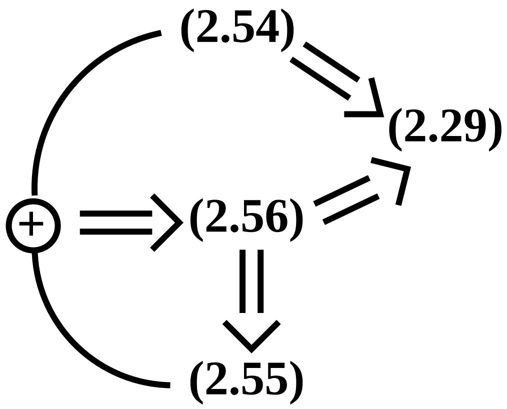

Figure 5.1: Illustration of the logic connectivity of the main assumptions.

2.6 Numerical results

Numerical results showing the good performance of the mixed finite element scheme (2.26) with the Bernardi-Raugel and Raviart-Thomas subspaces (cf. (2.44), (2.45), (2.49), (2.50), and (2.51)) were provided in [45]. In order to confirm the same behaviour with other stable Stokes elements, in this section we present three examples illustrating the performance of (2.26) with the MINI element and Raviart-Thomas subspaces (cf. (2.47), (2.48), (2.49), (2.50), and (2.51)) for two different geometries of the coupled problem. In this case, similarly as established by [45, Theorem 4.3], one can prove that, for sufficiently smooth continuous solutions, there also holds a rate of convergence of $O(h)$. In particular, for the approximation properties of the MINI element subspace we refer again to [54, Chapter II, Section 4.1].

We now introduce additional notations. The variable N stands for the number of degrees of freedom defining $\mathbf{H}_h$ and $\mathbf{Q}_h$, and the individual errors are denoted by:

$$\mathsf{e}(\mathbf{u}_1) \,:=\, \|\mathbf{u}_1 - \mathbf{u}_{1,h}\|_{1,\Omega_1}\,, \quad \mathsf{e}(\mathbf{u}_2) \,:=\, \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{\mathrm{div},\Omega_2}\,,$$

$$\mathsf{e}(p) \,:=\, \|p_1 - p_{1,h}\|_{0,\Omega_1} \,+\, \|p_2 - p_{2,h}\|_{0,\Omega_2}\,, \quad \text{and} \quad \mathsf{e}(\lambda) \,:=\, \|\lambda - \lambda_h\|_{1/2,\Gamma_2}\,,$$

where $\mathbf{u}_h \,:=\, (\mathbf{u}_{1,h}, \mathbf{u}_{2,h}) \in \mathbf{H}_h$, $p_{1,h} \,=\, p_h|_{\Omega_1}$, and $p_{2,h} \,=\, p_h|_{\Omega_2}$. Also, we let $r(\mathbf{u}_1)$, $r(\mathbf{u}_2)$, $r(p)$, and $r(\lambda)$ be the experimental rates of convergence given by

$$r(\mathbf{u}_1) \,:=\, \frac{\log(\mathsf{e}(\mathbf{u}_1)/\mathsf{e}'(\mathbf{u}_1))}{\log(h/h')}\,, \quad r(\mathbf{u}_2) \,:=\, \frac{\log(\mathsf{e}(\mathbf{u}_2)/\mathsf{e}'(\mathbf{u}_2))}{\log(h/h')}\,,$$

$$r(p) \,:=\, \frac{\log(\mathsf{e}(p)/\mathsf{e}'(p))}{\log(h/h')}\,, \quad \text{and} \quad r(\lambda) \,:=\, \frac{\log(\mathsf{e}(\lambda)/\mathsf{e}'(\lambda))}{\log(\tilde{h}/\tilde{h}')}\,,$$

where h and h' (resp. $\tilde{h}$ and $\tilde{h}'$) denote two consecutive meshsizes with errors e and e'.

In what follows we describe the data of the examples. In all cases we choose $\mu = 1$, $\kappa = 1$, and $\mathbf{K} \,=\, \mathbf{I}$, the identity matrix of $\mathbb{R}^{2\times 2}$.

In Example 1 we take $\Omega_2 \,:=\, (-1/2, 1/2) \,\times\, (-1/2, 1/2)$ and $\Omega_1 \,:=\, (-1, 1) \,\times\, (-1, 1)\setminus\Omega_2$, which represents a porous medium completely surrounded by a fluid. Then we choose the data $\mathbf{f}_1$ and f_2 so that the exact solution is given by

$$\mathbf{u}_1(x_1, x_2) \,=\, \frac{1}{100}\begin{pmatrix} \sin(\pi x_1)^2\cos(\pi x_1)(x_2^2 - 1)(\pi x_2^2 \sin(\pi x_2) - 4x_2\cos(\pi x_2) - \pi\sin(\pi x_2)) \\ \\ \pi\cos(\pi x_2)\sin(\pi x_1)(3\cos(\pi x_1)^2 - 1)(x_2 - 1)^2(x_2 + 1)^2 \end{pmatrix},$$

$$\mathbf{u}_2(x_1, x_2) \,=\, -\frac{1}{4}\begin{pmatrix} (54x_1^2 - \frac{27}{2})\sin(\pi x_2)^3 \\ \\ 18\pi(x_1^3 - \frac{3}{4}x_1)\sin(\pi x_2)^2\cos(\pi x_2) \end{pmatrix},$$

and

$$p(x_1, x_2) \,=\, \frac{1}{4}(18x_1^3 - \frac{27}{2}x_1)\sin(\pi x_2)^3\,.$$

In Example 2 we consider $\Omega_1 \,:=\, (-1, 1) \,\times\, (0, 1)$ and $\Omega_2 \,:=\, (-1, 1) \,\times\, (-1, 0)$, which constitutes a particular case of the geometry analyzed in Subsection 2.2.3, and choose the data

$\mathbf{f}_1$ and f_2 so that the exact solution is given by

$$\mathbf{u}_1(x_1,x_2) \;=\; \left(\begin{array}{c} -2\sin(\pi x_1)^2(x_2-1) \\ \\ 2\pi\sin(\pi x_1)(x_2-1)^2\cos(\pi x_1) \end{array}\right),$$

$$\mathbf{u}_2(x_1,x_2) \;=\; \left(\begin{array}{c} -3\pi\sin(\pi x_1)^2\cos(\pi x_2)\cos(\pi x_1) \\ \\ \pi\sin(\pi x_1)^3\sin(\pi x_2) \end{array}\right),$$

and

$$p(x_1,x_2) \;:=\; \begin{cases} (x_1^5+x_1^3)e^{2x_2} \quad \text{in} \quad \Omega_1\,, \\ \\ \sin(\pi x_1)^3\cos(\pi x_2) \quad \text{in} \quad \Omega_2\,. \end{cases}$$

Finally, in Example 3 we consider the same geometry of Example 2, and take the data $\mathbf{f}_1$ and f_2 given by

$$\mathbf{f}_1(x_1,x_2) \;=\; \left(\begin{array}{c} -4\sin(x_1x_2)\,x_1 \;+\; \exp(x_2^3) \\ \\ 4\,\exp(3x_1) \;+\; 4\,x_2 \end{array}\right)$$

and

$$f_2(x_1,x_2) \;=\; x_1^3\left(\exp(x_2^2)-0.5\right).$$

We observe that the solutions of Examples 1 and 2 show very oscillating behavior, and that Example 3 corresponds to a more realistic situation in which the exact solution is unknown.

The numerical results shown below were obtained using a MATLAB implementation. According to the technical requirement established by the inf-sup condition (2.30) for the Raviart-Thomas subspace, namely $h_2 \,\leq\, C_0\,\tilde{h}$, and since the constant $C_0 \in (0,1)$ is not explicitly known, we simply put a vertex of the independent partition $\{\tilde{\gamma}_1,\tilde{\gamma}_2,\cdots,\tilde{\gamma}_m\}$ every two vertices of $\mathcal{T}_2$ on Γ_2, thus insuring that, locally on Γ_2, $h_2 \,\leq\, \frac{1}{2}\,\tilde{h}$. As we will see below, this choice works out well in both examples. In addition, there is no need of taking sufficiently small values of h (as technically suggested by the inequality $h \,\leq\, h_0$ in Theorem 2.4.2) since the resulting discrete schemes become all well posed for the degrees of freedom employed in the present examples.

In Tables 6.1 and 6.2 we present the convergence history of Examples 1 and 2, respectively, for a set of uniform triangulations of the computational domain $\bar{\Omega}$. We see there that the dominant error in both examples is given by $\mathtt{e}(\mathbf{u}_2)$, though this is more evident in Example 1. In addition, we observe that the rate of convergence $O(h)$ is attained by all the unknowns. Furthermore, the rates of convergence of $\mathtt{e}(p)$ and $\mathtt{e}(\lambda)$ are a bit higher than $O(h)$ in Example 2, which, however, is just a special behavior of this particular solution. The experimental rates of convergence and the dominant components of the error can also be checked from Figures 6.1 and 6.9 below where we display the meshsize h and the errors $\mathsf{e}(\mathbf{u}_1)$, $\mathsf{e}(\mathbf{u}_2)$, and $\mathsf{e}(p)$ vs. the degrees of freedom N. Next, from Figure 6.2 throughout Figure 6.8 (resp. Figure 6.10 throughout Figure 6.16) we display the approximate and exact solutions of Example 1 for N=106881 (resp. Example 2 for N = 98371). It is clear from these figures that the MINI element subspace provides very accurate approximations of the velocity and pressure in the fluid Ω_1. In particular, the quality of these approximations are not affected at all by the strong oscillations of some solutions. Similarly, the Raviart-Thomas subspace reconstructs quite accurately the velocity and pressure in the porous media Ω_2, and the trace λ of the pressure on the interface Γ_2.

Next, in Table 6.3 we present the convergence history of Example 3 for a set of uniform triangulations of the computational domain $\bar{\Omega}$. The errors and experimental rates of convergence shown there are computed by considering the discrete solution obtained with the finest mesh (N = 786563) as the *exact solution*. Similarly as for Examples 1 and 2 we observe that the rate of convergence $O(h)$ is attained by all the unknowns, and in this case the dominant error is given by $\mathsf{e}(p)$. The experimental rates of convergence and the dominance of $\mathsf{e}(p)$ can also be checked from Figure 6.17 where we display the meshsize h and the errors $\mathsf{e}(\mathbf{u}_1)$, $\mathsf{e}(\mathbf{u}_2)$, and $\mathsf{e}(p)$ vs. the degrees of freedom N. Next, from Figure 6.18 throughout Figure 6.21 we show the approximate solutions obtained for N=98371. Note that in this example the normal on the interface $\Gamma_2 := (-1,1)\times\{0\}$ is given by $\boldsymbol{\nu} = (0,-1)^{\mathtt{t}}$, and hence the first transmission condition becomes equality of the second components of $\mathbf{u}_1$ and $\mathbf{u}_2$. This can be verified at the discrete level in Figure 6.20 where we display 3D and 2D joint pictures of the second components of $\mathbf{u}_{1,h}$ and $\mathbf{u}_{2,h}$.

Summarizing, the numerical results reported here confirm the good performance of the mixed finite element scheme (2.26) for different geometries of the coupled problem and with any pair of stable Stokes and Darcy subspaces.

N	h	$\mathsf{e}(\mathbf{u}_1)$	$r(\mathbf{u}_1)$	$\mathsf{e}(\mathbf{u}_2)$	$r(\mathbf{u}_2)$	$\mathsf{e}(p)$	$r(p)$	$\tilde{h}$	$\mathsf{e}(\lambda)$	$r(\lambda)$
441	0.354	0.1006	–	8.0254	–	0.1317	–	0.707	1.3451	–
1713	0.177	0.0530	0.944	4.1266	0.980	0.0497	1.436	0.354	0.9517	0.510
6753	0.088	0.0254	1.076	2.0873	0.994	0.0200	1.327	0.177	0.5030	0.930
26817	0.044	0.0125	1.029	1.0467	1.001	0.0091	1.136	0.088	0.2213	1.191
106881	0.022	0.0062	1.010	0.5237	1.002	0.0044	1.059	0.044	0.0971	1.191

Table 6.1: degrees of freedom, meshsizes, errors, and rates of convergence (EXAMPLE 1)

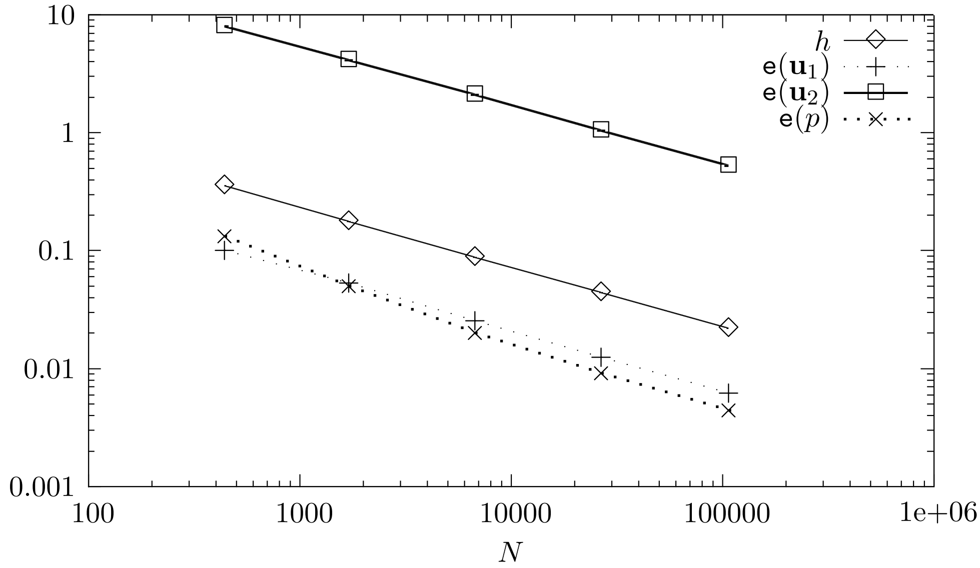

Figure 6.1: meshsize h and errors vs. degrees of freedom N (Example 1)

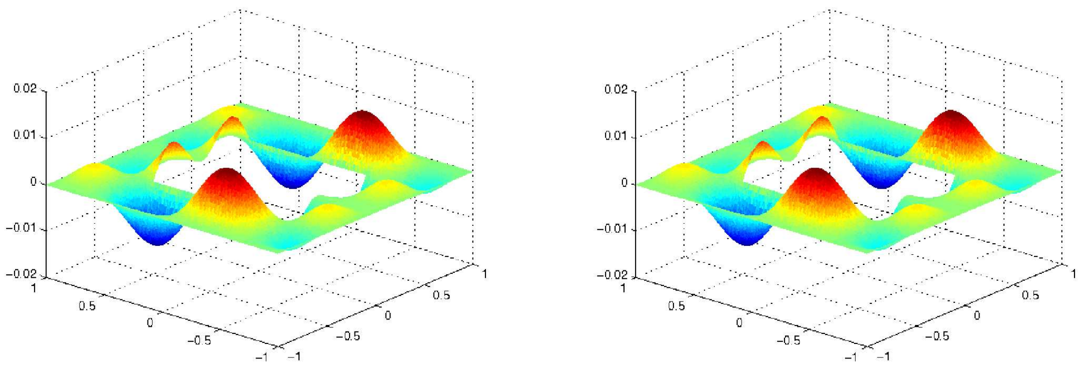

Figure 6.2: first components of $\mathbf{u}_{1,h}$ and $\mathbf{u}_1$ (Example 1)

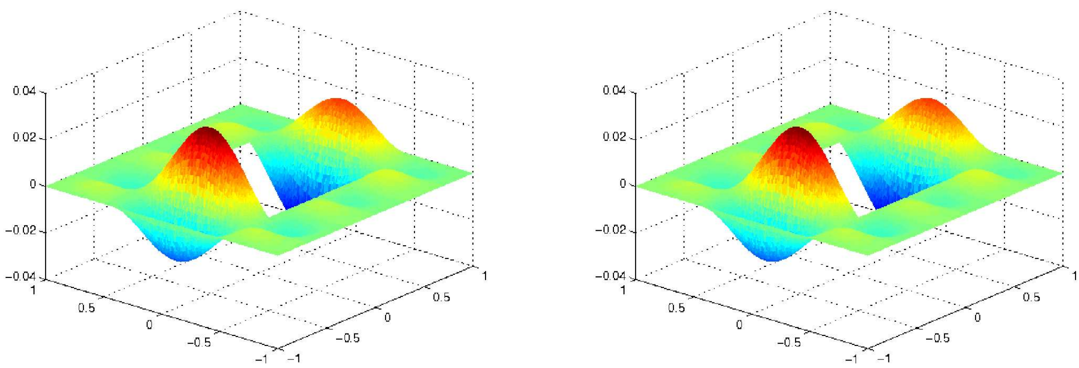

Figure 6.3: second components of $\mathbf{u}_{1,h}$ and $\mathbf{u}_1$ (Example 1)

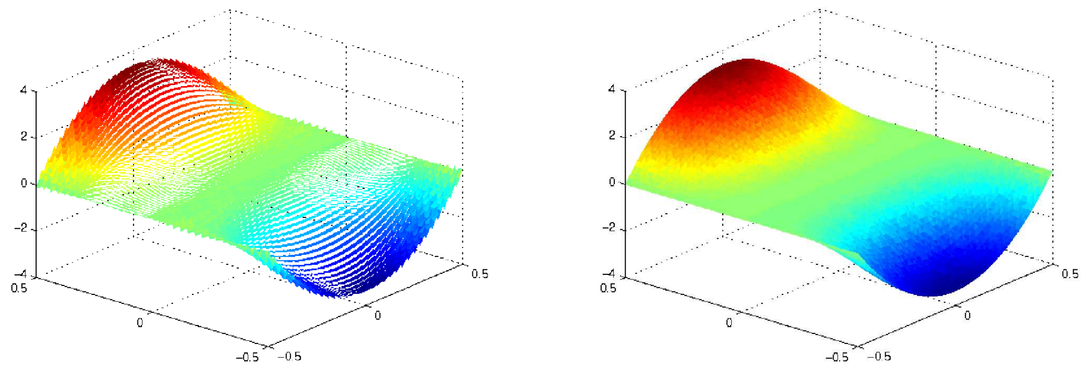

Figure 6.4: first components of $\mathbf{u}_{2,h}$ and $\mathbf{u}_2$ (Example 1)

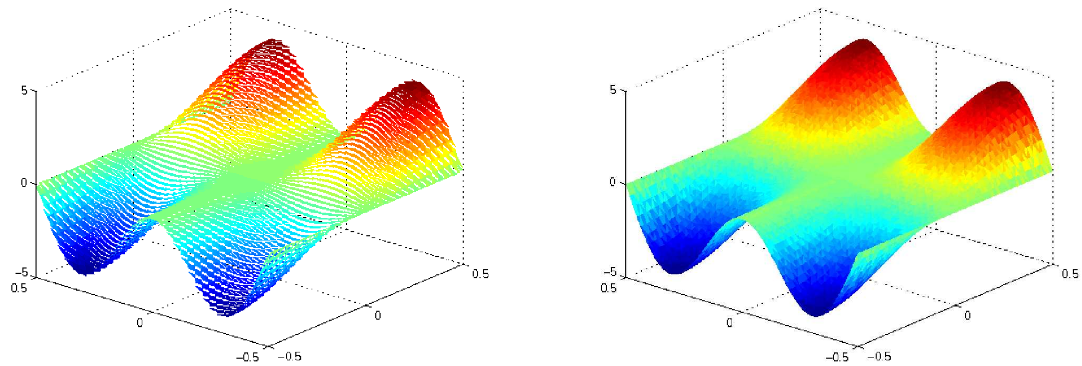

Figure 6.5: second components of $\mathbf{u}_{2,h}$ and $\mathbf{u}_2$ (Example 1)

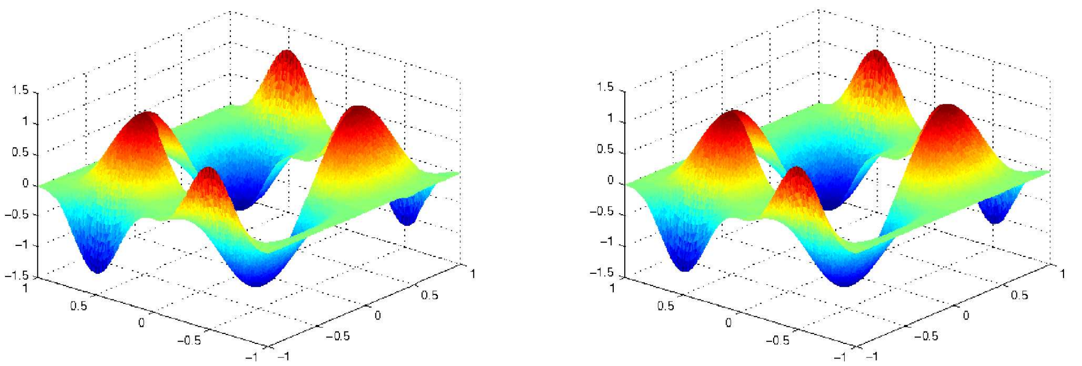

Figure 6.6: $p_{1,h}$ and p_1 (Example 1)

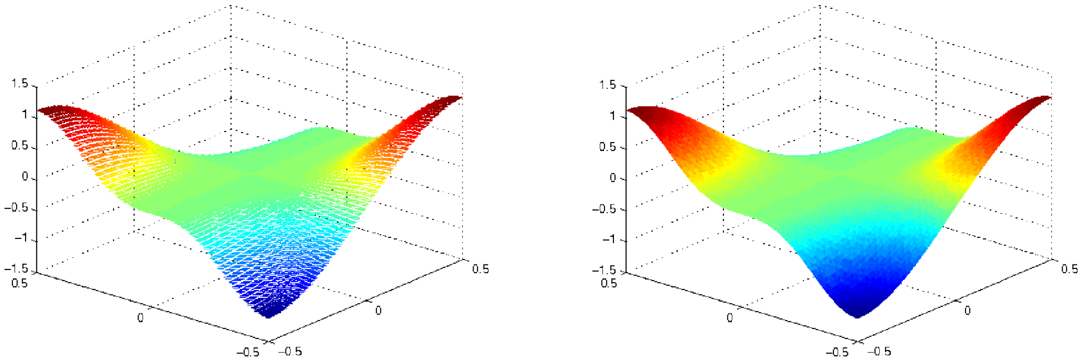

Figure 6.7: $p_{2,h}$ and p_2 (Example 1)

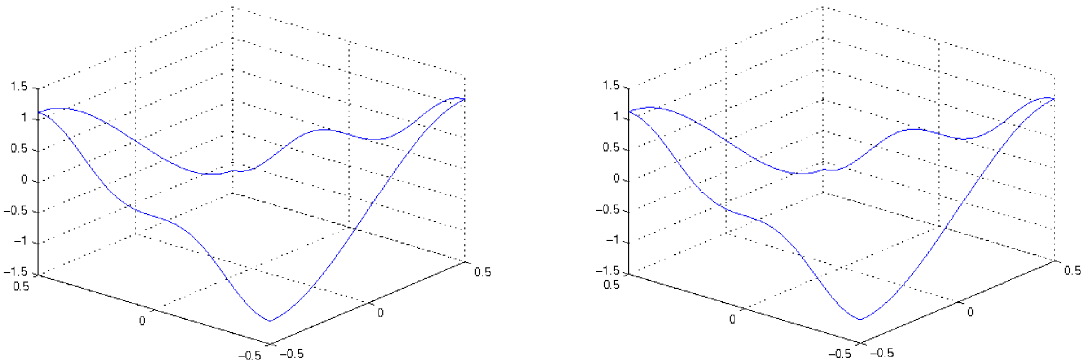

Figure 6.8: λ_h and λ (Example 1)

N	h	$\mathsf{e}(\mathbf{u}_1)$	$r(\mathbf{u}_1)$	$\mathsf{e}(\mathbf{u}_2)$	$r(\mathbf{u}_2)$	$\mathsf{e}(p)$	$r(p)$	$\tilde{h}$	$\mathsf{e}(\lambda)$	$r(\lambda)$
101	0.707	10.3690	–	16.6390	–	5.4791	–	1.000	1.4761	–
391	0.353	5.9337	0.824	9.9808	0.731	2.3571	1.246	0.500	0.9826	0.601
1547	0.176	2.8182	1.082	5.2415	0.936	0.9442	1.330	0.250	0.4868	1.021
6163	0.088	1.4010	1.011	2.6517	0.985	0.3532	1.422	0.125	0.1584	1.624
24611	0.044	0.6959	1.010	1.3298	0.996	0.1298	1.445	0.062	0.0535	1.567
98371	0.022	0.3465	1.006	0.6654	0.999	0.0467	1.476	0.031	0.0187	1.517

Table 6.2: degrees of freedom, meshsizes, errors, and rates of convergence (Example 2)

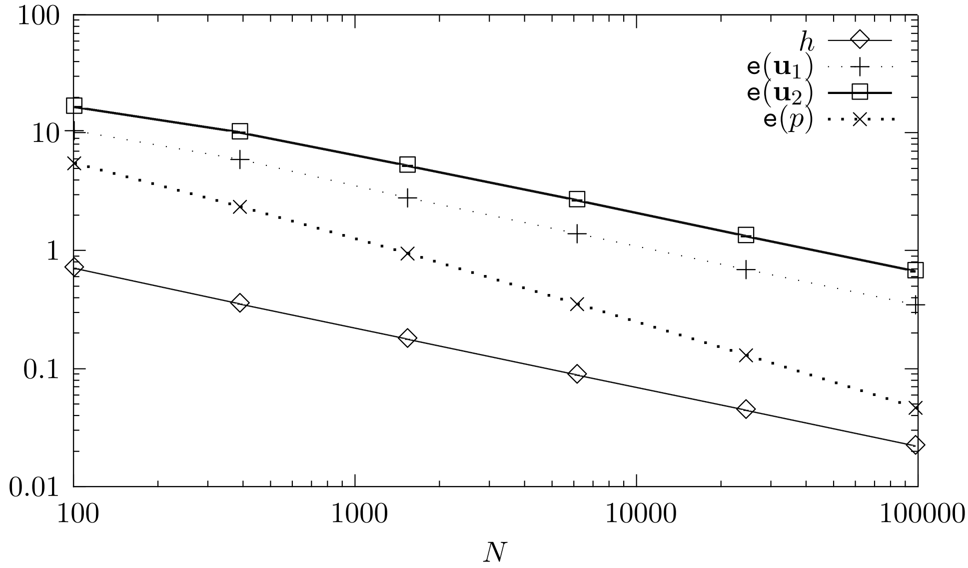

Figure 6.9: meshsize h and errors vs. degrees of freedom N (EXAMPLE 2)

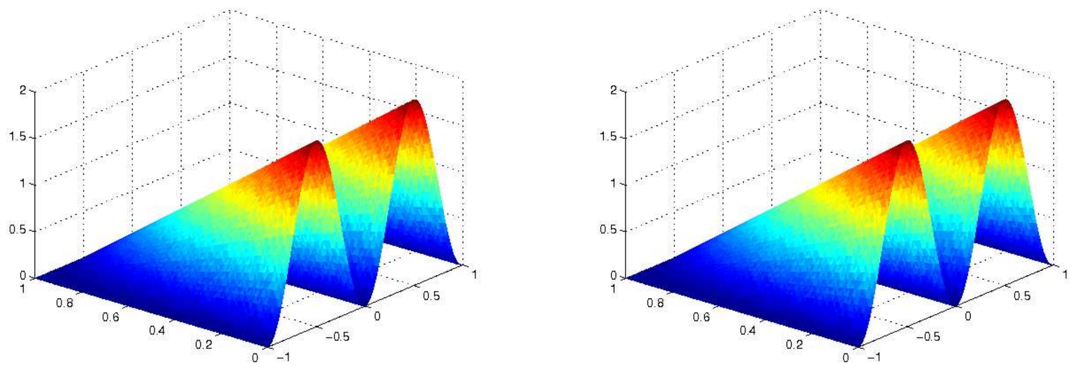

Figure 6.10: first components of $\mathbf{u}_{1,h}$ and $\mathbf{u}_1$ (EXAMPLE 2)

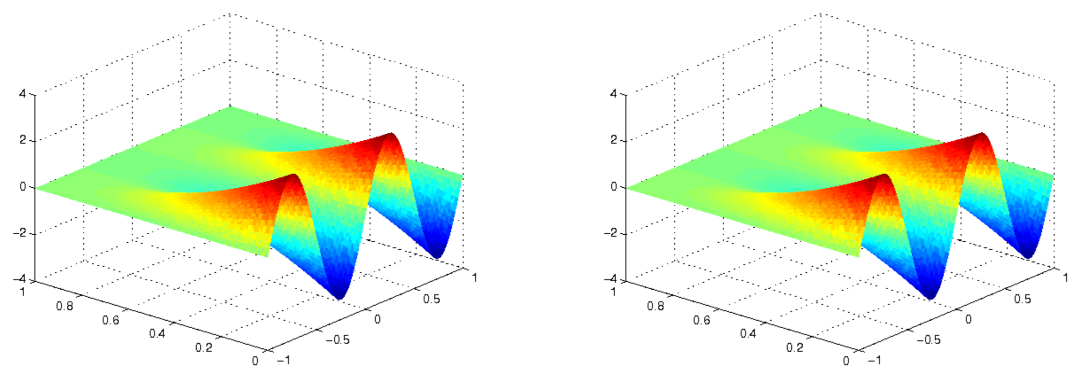

Figure 6.11: second components of $\mathbf{u}_{1,h}$ and $\mathbf{u}_1$ (EXAMPLE 2)

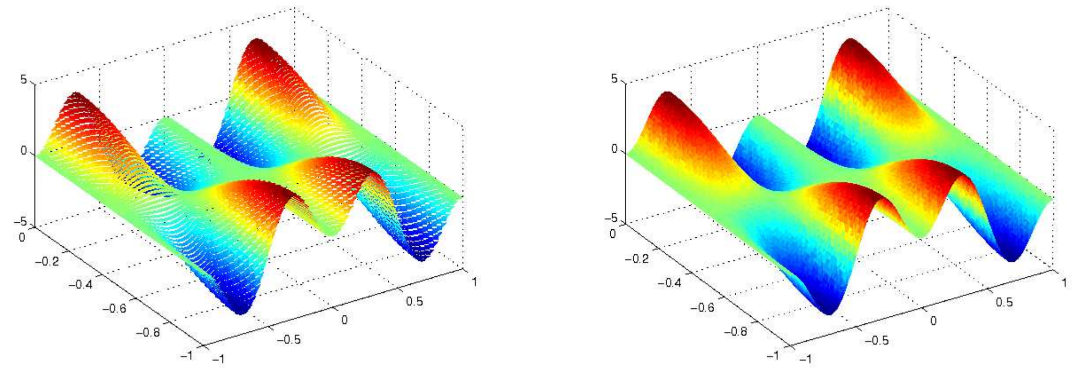

Figure 6.12: first components of $\mathbf{u}_{2,h}$ and $\mathbf{u}_2$ (EXAMPLE 2)

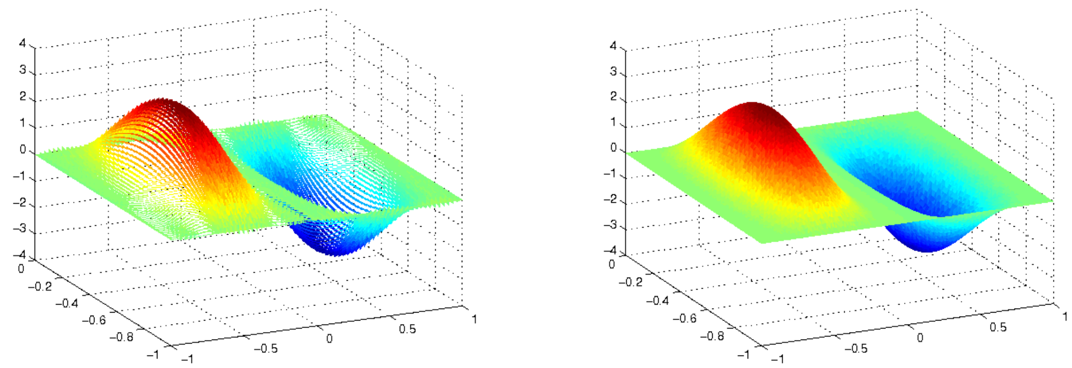

Figure 6.13: second components of $\mathbf{u}_{2,h}$ and $\mathbf{u}_2$ (EXAMPLE 2)

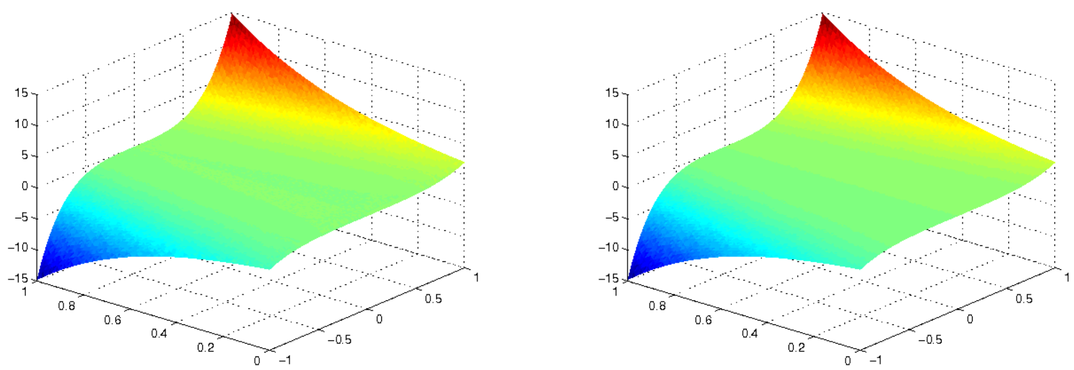

Figure 6.14: $p_{1,h}$ and p_1 (EXAMPLE 2)

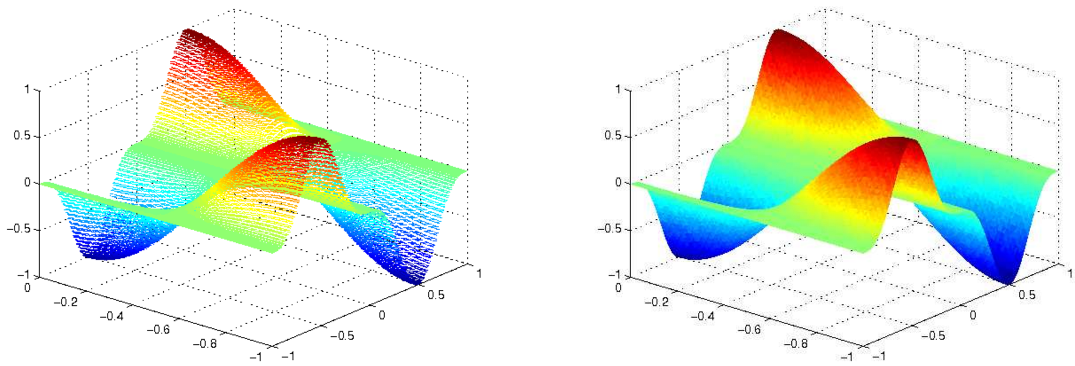

Figure 6.15: $p_{2,h}$ and p_2 (EXAMPLE 2)

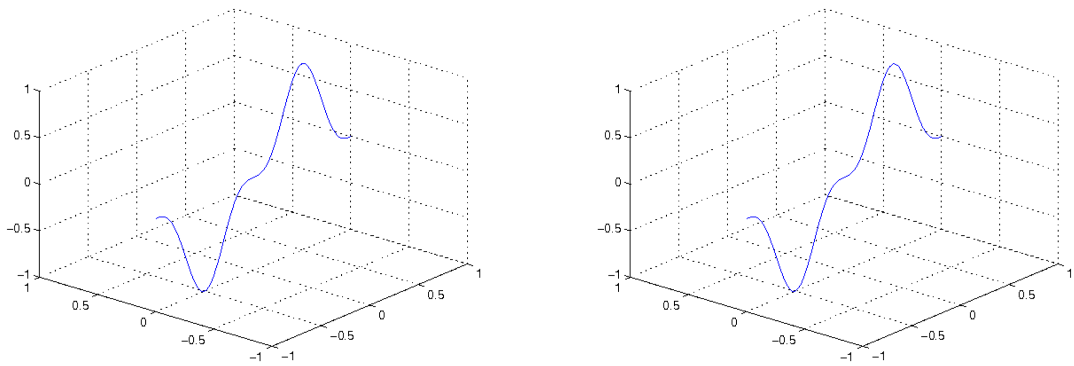

Figure 6.16: λ_h and λ (EXAMPLE 2)

N	h	$e(\mathbf{u}_1)$	$r(\mathbf{u}_1)$	$e(\mathbf{u}_2)$	$r(\mathbf{u}_2)$	$e(p)$	$r(p)$	$\tilde{h}$	$e(\lambda)$	$r(\lambda)$
101	0.707	1.8331	–	0.3848	–	6.3700	–	1.000	0.3869	–
391	0.353	1.1978	0.628	0.1890	1.050	3.3163	0.964	0.500	0.1501	1.399
1547	0.176	0.6544	0.879	0.0901	1.077	1.5455	1.110	0.250	0.0781	0.950
6163	0.088	0.3455	0.924	0.0445	1.020	0.6902	1.166	0.125	0.0372	1.073
24611	0.044	0.1806	0.937	0.0221	1.011	0.3242	1.091	0.062	0.0169	1.139
98371	0.022	0.0953	0.922	0.0110	1.007	0.1701	0.931	0.031	0.0074	1.192

Table 6.3: degrees of freedom, meshsizes, errors, and rates of convergence (EXAMPLE 3)

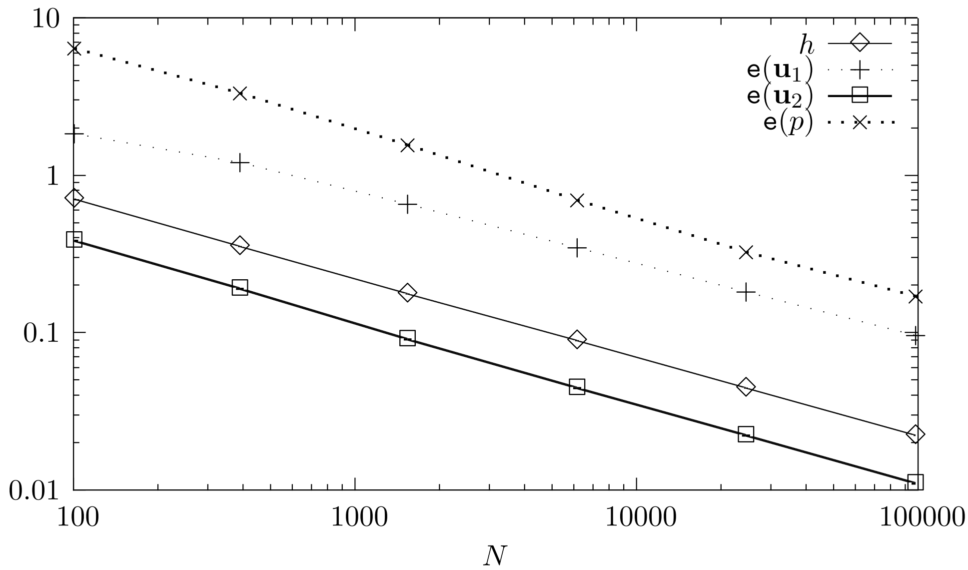

Figure 6.17: meshsize h and errors vs. degrees of freedom N (Example 3)

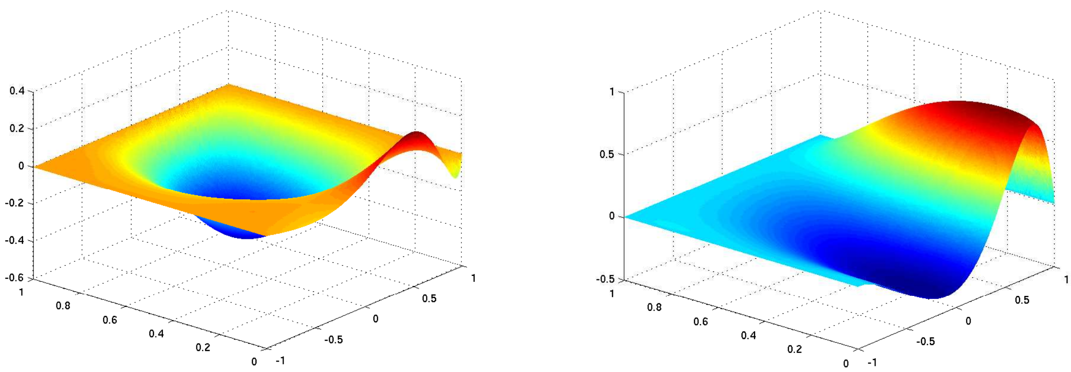

Figure 6.18: first and second components of $\mathbf{u}_{1,h}$ (Example 3)

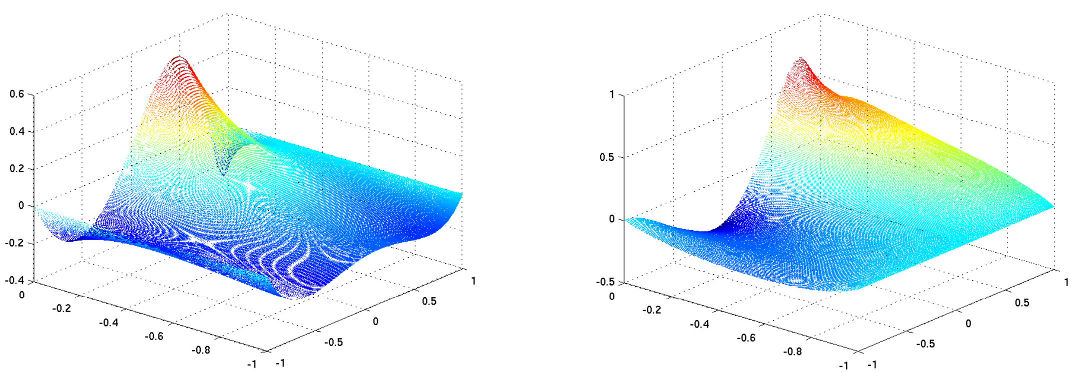

Figure 6.19: first and second components of $\mathbf{u}_{2,h}$ (Example 3)

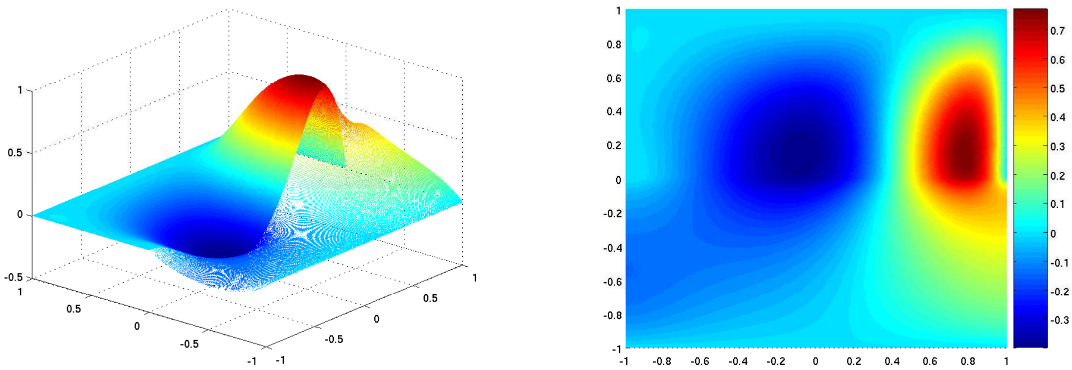

Figure 6.20: second components of $\mathbf{u}_{1,h}$ and $\mathbf{u}_{2,h}$ (Example 3)

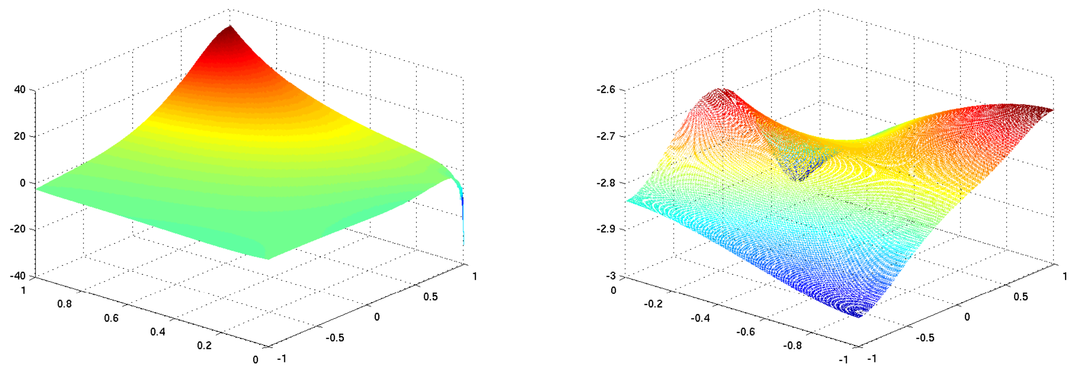

Figure 6.21: $p_{1,h}$ and $p_{2,h}$ (Example 3)

Chapter 3

Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem

3.1 Introduction

The derivation of suitable numerical methods for the coupling of fluid flow (modelled by the Stokes equations) with porous media flow (modelled by the Darcy equation) has become a very active research area during the last decade (see, e.g. [2], [20], [21], [26], [34], [35], [39], [45], [48], [61], [63], [67], [71], [74], [75], [78], and the references therein). This fact has been motivated by the diverse applications of this coupled model (in petroleum engineering, hydrology, and environmental sciences, to name a few), and also by the increasing need of simpler, more accurate, and more efficient procedures to solve it. Moreover, the latest results available in the literature also include porous media with cracks, nonlinear problems, and the incorporation of the Brinkman equation in the model (see, e.g. [17], [37], and [83]).

In general, most of the finite element formulations developed for the Stokes-Darcy coupled problem are based on appropriate combinations of stable elements for the free fluid flow and for the porous medium flow. The first theoretical results in this direction go back to [35] and [63]. An iterative subdomain method employing the primal variational formulation and standard finite element subspaces in both domains is proposed in [35]. Alternatively, the approach from [63] applies the primal method in the fluid and the dual-mixed one in the porous medium, which means that only the original velocity and pressure unknowns are considered in the Stokes domain, whereas a further unknown (velocity) is added in the Darcy region. The corresponding interface conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-

Saffman law. Since one of them becomes essential, the trace of the porous medium pressure needs to be incorporated as an additional Lagrange multiplier.

More recently, new mixed finite element discretizations of the variational formulation from [63] have been introduced and analyzed in [45] and [48]. The stability of a specific Galerkin method is the main result in [45]. This scheme is defined by using Bernardi-Raugel elements for the velocity in the fluid region, Raviart-Thomas elements of lowest order for the filtration velocity in the porous media, piecewise constants with null mean value for the pressures, and continuous piecewise linear elements for the Lagrange multiplier on the interface. The resulting mixed finite element method is the first one which is conforming for the primal/dual-mixed formulation proposed in [63]. The results from [45] are improved in [48] where it is shown that the use of any pair of stable Stokes and Darcy elements implies the stability of the corresponding Stokes-Darcy Galerkin scheme. In particular, this includes Hood-Taylor, Bernardi-Raugel and MINI element for the Stokes region, and Raviart-Thomas of any order for the Darcy domain. The analysis in [48] hinges on the fact that the operator defining the continuous variational formulation is given by a compact perturbation of an invertible mapping.

On the other hand, mortar finite element techniques, discontinuous Galerkin (DG) schemes, and stabilized formulations have also been applied to solve the Stokes-Darcy coupled problem. We first refer to [39] where a non-matching approach is combined with Hood-Taylor and lowest order Raviart-Thomas spaces in the Stokes and Darcy regions, respectively. Also, stabilized formulations for the free fluid flow combined with stable elements for the Darcy equation are considered in [2] and [74], while stabilized formulations for the porous medium flow combined with stable elements for the Stokes equations are provided in [31] and [78]. Similarly, stabilized formulations in the whole domain are presented in [26] and [71]. It is important to notice here that the formulations in [2] and [26] are able to approximate the Stokes and Darcy flows with the same finite element subpaces. Other stabilized formulations with this characteristic are developed in [20], [21], [67], and [75]. In particular, a stabilized piecewise linear/piecewise constant method with an added penalization of pressure jumps over the edges is proposed in [21]. In addition, Crouzeix-Raviart elements for the velocities and piecewise constants for the pressures in both regions, combined with a stabilization term penalizing the jumps of the discontinuous velocities over the edges, are employed in [75]. This approach differs from the one in [20] where the stabilization term depends on the normal vectors of the interior edges. In connection with these references we remark that different finite element subspaces in each flow region may lead to different approximation properties for each subproblem. For instance, one could obtain a more accurate velocity field in the Stokes domain than in the Darcy region. On the contrary, employing the same spaces guarantees the same accurateness along the entire domain and leads to simpler and more efficient computational codes.

The purpose of the present work is to contribute in the development of new numerical methods for the 2D Stokes-Darcy coupled problem, allowing on one hand the introduction of further unknowns of physical interest, and on the other hand, the utilization of the same family of finite element subspaces in both media, without requiring any stabilization term. To reach this aim we consider dual-mixed formulations in both domains, which yields the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium, as the main unknowns. The pressure and the gradient of the velocity in the fluid can then be computed as a very simple postprocess of the above unknowns, in which no numerical differentiation is applied, and hence no further sources of error arise. In addition, since the transmission conditions become essential, we impose them weakly and introduce the traces of the porous media pressure and the fluid velocity, which are also variables of importance from a physical point of view, as the corresponding Lagrange multipliers. Then, we apply the well known Fredholm and Babuška-Brezzi theories to prove the unique solvability of a suitably chosen continuous formulation and derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme becomes well posed. In particular, among the several different ways in which the equations and unknowns can be ordered, we choose the one yielding a doubly mixed structure for which the inf-sup conditions of the off-diagonal bilinear forms follow straightforwardly. In this way, the arguments of the continuous analysis can be easily adapted to the discrete case.

The rest of this paper is organized as follows. In Section 3.2 we introduce the main aspects of the continuous problem, which includes the coupled model, its weak formulation, and the corresponding variational system. The Fredholm theorems and the classical Babuška-Brezzi theory are applied in Section 3.3 to analyze the continuous problem. Then, in Section 3.4 we define the Galerkin scheme and derive general hypotheses on the finite element subspaces ensuring that the discrete scheme becomes well posed. In addition, we show that the assumption of existence of uniformly bounded discrete liftings of the normal traces on the interface simplifies the statement of one of the hypotheses. Next, in Section 3.5 we describe a specific choice of finite element subspaces, namely Raviart-Thomas of lowest order and piecewise constants on both domains, and piecewise linears on the interface, and show that they satisfy all the required assumptions. In particular, we prove that a quasiuniformity condition in a neighborhood of the interface implies the existence of the above mentioned discrete liftings. Finally, several numerical examples employing these spaces, illustrating the good performance of the method, and confirming the theoretical order of convergence, are reported in Section 3.6.

We end this section by summarizing in advance, and according to the already mentioned purpose of the paper, the main advantages of the present fully-mixed approach: it provides either direct finite element approximations or very simple postprocess formulae for several additional quantities of physical interest; it yields, under a special ordering of the resulting equations and unknowns, a unified and straightforward analysis of the continuous and discrete formulations; it leads to independent but analogously structured stability assumptions on the finite element subspaces for the Stokes and Darcy regions; and it allows the utilization of the same kind of finite elements in both media, with the consequent simplification of the respective code.

3.2 The continuous problem

3.2.1 Statement of the model problem

The Stokes-Darcy coupled problem consists of an incompressible viscous fluid occupying a region Ω_S, which flows back and forth across the common interface into a porous medium living in another region Ω_D and saturated with the same fluid. Physically, we consider a simplified 2D model where Ω_D is surrounded by a bounded region Ω_S (see Figure 3.1 below). Their common interface is supposed to be a Lipschitz curve Σ and we assume that $\partial\Omega_D = \Sigma$. The remaining part of the boundary of Ω_S is also assumed to be a Lipschitz curve Γ_S. For practical purposes, we can assume that both Γ_S and Σ are polygons, although this fact will not be used in the general considerations about the formulation of the problem. The unit normal vector field on the boundaries $\mathbf{n}$ is chosen pointing outwards from Ω_S (and therefore inwards to Ω_D when seen on Σ). On Σ we also consider a unit tangent vector field $\mathbf{t}$ in any fixed orientation of this closed curve.

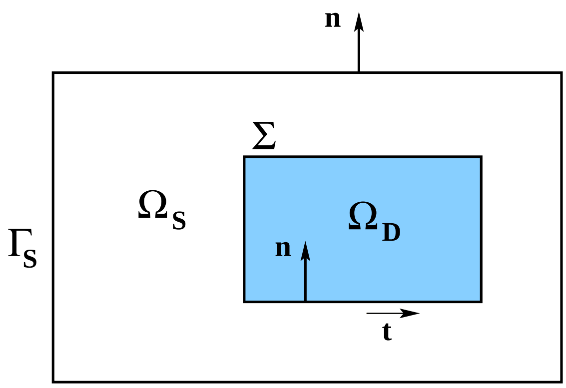

Figure 3.1: The domains for our simplified 2D Stokes–Darcy model

The mathematical model is defined by two separate groups of equations and a set of coupling terms. In Ω_S, the governing equations are those of the Stokes problem, which are written in the following non-standard velocity-pressure-pseudostress formulation:

$$\begin{aligned} \boldsymbol{\sigma}_{\mathrm{S}} &= -p_{\mathrm{S}}\,\mathrm{I} + \nu\,\nabla\mathbf{u}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} + \mathbf{f}_{\mathrm{S}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \\ \operatorname{div}\mathbf{u}_{\mathrm{S}} &= 0 \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}\,, \end{aligned} \tag{3.1}$$

where $\nu > 0$ is the viscosity of the fluid, $\mathbf{u}_{\mathrm{S}}$ is the fluid velocity, p_{S} is the pressure, $\boldsymbol{\sigma}_{\mathrm{S}}$ is the pseudostress tensor, I is the 2×2 identity matrix, and $\mathbf{f}_{\mathrm{S}}$ are known source terms. Here, div is the usual divergence operator acting on vector fields,

$$\nabla\mathbf{u} = \left(\frac{\partial u_i}{\partial x_j}\right)\,, \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\sigma} = \Big(\operatorname{div}(\sigma_{i1},\sigma_{i2})\Big)\,,$$

i.e., the divergence operator applied to a matrix valued function (a tensor) is taken row-wise. On the other hand, the flow equations in Ω_{D} are those of the linearized Darcy model:

$$\mathbf{u}_{\mathrm{D}} = -\,\mathrm{K}\,\nabla p_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \qquad \operatorname{div}\mathbf{u}_{\mathrm{D}} = f_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \tag{3.2}$$

where the unknowns are the pressure p_{D} and the flow $\mathbf{u}_{\mathrm{D}}$. The matrix valued function K, describing permeability of Ω_{D} divided by the viscosity ν, satisfies $\mathrm{K}^t = \mathrm{K}$, has $L^\infty(\Omega_{\mathrm{D}})$ components and is uniformly elliptic. Finally, f_{D} are source terms. We will see that a necessary and sufficient condition for well posedness of the model equations is

$$\int_{\Omega_{\mathrm{D}}} f_{\mathrm{D}} = 0. \tag{3.3}$$

Finally, the transmission conditions on Σ are given by

$$\begin{aligned} \mathbf{u}_{\mathrm{S}}\cdot\mathbf{n} &= \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \qquad \text{on} \quad \Sigma\,, \\ \boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n} + \nu\,\kappa^{-1}\,(\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t})\,\mathbf{t} &= -\,p_{\mathrm{D}}\,\mathbf{n} \qquad \text{on} \quad \Sigma\,, \end{aligned} \tag{3.4}$$

where $\kappa := \dfrac{\sqrt{(\nu\,K\,\mathbf{t})\cdot\mathbf{t}}}{\alpha}$ is the friction coefficient, and α is a positive parameter to be determined experimentally. The first equation in (3.4) corresponds to mass conservation on Σ, whereas the second one can be decomposed into its normal and tangential components as follows:

$$\big(\boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n}\big)\cdot\mathbf{n} = -\,p_{\mathrm{D}} \quad \text{and} \quad \big(\boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n}\big)\cdot\mathbf{t} = -\,\nu\,\kappa^{-1}\,(\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t}) \quad \text{on} \quad \Sigma\,,$$

which constitute the balance of normal forces and the Beavers-Joseph-Saffman law, respectively. The latter establishes that the slip velocity along Σ is proportional to the shear stress along Σ (assuming also, based on experimental evidences, that $\mathbf{u}_{\mathrm{D}}\cdot\mathbf{t}$ is negligible). We refer to [16], [59], and [76] for further details on this interface condition. Throughout the rest of the paper we assume, without loss of generality, that κ is a positive constant.

The description of our model problem is completed by observing that the equations in the Stokes domain (cf. (3.1)) can be rewritten equivalently as

$$\begin{aligned} \nu^{-1}\,\boldsymbol{\sigma}_{\mathrm{S}}^{d} \,=\, \nabla\mathbf{u}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \,+\, \mathbf{f}_{\mathrm{S}} \,=\, \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \\ p_{\mathrm{S}} \,=\, -\,\tfrac{1}{2}\,\mathrm{tr}\boldsymbol{\sigma}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{u}_{\mathrm{S}} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}\,, \end{aligned} \tag{3.5}$$

where tr stands for the usual trace of tensors, that is $\mathrm{tr}\boldsymbol{\tau} \,:=\, \tau_{11} \,+\, \tau_{22}$, and

$$\boldsymbol{\tau}^{d} \,:=\, \boldsymbol{\tau} \,-\, \tfrac{1}{2}\,(\mathrm{tr}\boldsymbol{\tau})\,\mathrm{I}$$

is the deviatoric part of the tensor $\boldsymbol{\tau}$. The third equation in (3.5) allows us to eliminate p_{S} from the system and compute it as a simple postprocess of the solution. Similarly, the first equation in (3.5) yields a straightforward postprocess formula for the gradient of the velocity in the fluid. Note that a constant c added to both p_{S} and p_{D} is not perceived by the system: its only effect is a correction in $\boldsymbol{\sigma}_{\mathrm{S}}$ that has to be subtracted c times the identity matrix.

We end this section by remarking that, though the geometry described by Figure 3.1 was choosen to simplify the presentation, the case of a fluid flowing only across a part of the boundary of the porous medium does not really yield further complications for the analysis in the present paper. For instance, if we consider a fluid over the porous medium, $\partial\Omega_{\mathrm{S}}$ stays given by $\Gamma_{\mathrm{S}} \cup \Sigma$, but now with both curves meeting at their end points, whereas a new piece of $\partial\Omega_{\mathrm{D}}$, say Γ, such that $\partial\Omega_{\mathrm{D}} \,=\, \Sigma \cup \Gamma$, needs to be identified. In this case, besides the equations given in the present section (which hold now with the notations introduced here), a boundary condition on Γ needs to be added. Following [39] and [63] (see also [37]), one usually considers the homogeneous Neumann condition:

$$\mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma\,, \tag{3.6}$$

which constitutes a no flow assumption through Γ. We refer to [39] for further details and emphasize that only minor modifications will need to be incorporated into the forthcoming analysis. In particular, this is certainly valid for the discrete analysis, which is illustrated by two numerical examples reported below in Section 3.6. Alternatively, instead of (3.6) one can consider the homogeneous Dirichlet condition:

$$p_{\mathrm{D}} \,=\, 0 \quad \text{on} \quad \Gamma\,, \tag{3.7}$$

which, as will be explained at the end of Section 3.3 below, becomes a unique solvability condition for the resulting variational formulation.

3.2.2 The weak formulation

Let us first introduce some general functional spaces. If $\mathcal{O}$ is a domain, Γ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}, \quad \text{and} \quad \mathbf{H}^r(\Gamma) := [H^r(\Gamma)]^2.$$

In the particular case $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\Gamma}$ (for $H^r(\Gamma)$ and $\mathbf{H}^r(\Gamma)$).

Also, the Hilbert space

$$\mathbf{H}(\mathrm{div}\,;\mathcal{O}) := \left\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\ \mathbf{w} \in L^2(\mathcal{O})\right\},$$

is standard in the realm of mixed problems (see [19] or [54] for instance). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ and $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div}\,;\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div}\,;\mathcal{O}}$, respectively. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\mathcal{O})$, then $\mathbf{div}\ \boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O})$. Note also that $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$ can be characterized as the space of matrix valued functions $\boldsymbol{\tau}$ such that $\mathbf{c}^t\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}\,;\mathcal{O})$ for any constant column vector $\mathbf{c}$. In addition, it is easy to see that there holds:

$$\mathbb{H}(\mathbf{div}\,;\mathcal{O}) = \mathbb{H}_0(\mathbf{div}\,;\mathcal{O}) \oplus \mathbb{P}_0(\mathcal{O})\,\mathrm{I}, \tag{3.8}$$

where

$$\mathbb{H}_0(\mathbf{div}\,;\mathcal{O}) := \left\{\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}\,;\mathcal{O}) : \quad \int_{\mathcal{O}} \mathrm{tr}\boldsymbol{\sigma} = 0\right\} \tag{3.9}$$

and $\mathbb{P}_0(\mathcal{O})$ is the space of constant polynomials on $\mathcal{O}$. More precisely, each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\mathcal{O})$ can be decomposed uniquely as:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\,\mathrm{I}, \quad \text{with} \quad \boldsymbol{\tau}_0 \in \mathbb{H}_0(\mathbf{div}\,;\mathcal{O}) \quad \text{and} \quad c := \frac{1}{2\,|\mathcal{O}|}\int_{\mathcal{O}} \mathrm{tr}\,\boldsymbol{\tau} \in \mathbb{R}. \tag{3.10}$$

This decomposition will be utilized below to analyze the weak formulation of our problem.

On the other hand, for simplicity of notations we will also denote, with $\star \in \{\mathrm{S}, \mathrm{D}\}$

$$(u, v)_\star := \int_{\Omega_\star} u\,v, \qquad (\mathbf{u}, \mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u}\cdot\mathbf{v}, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau},$$

where $\boldsymbol{\sigma} : \boldsymbol{\tau} = \mathrm{tr}(\boldsymbol{\sigma}^t\boldsymbol{\tau}) = \sum_{ij=1}^{2} \sigma_{ij}\tau_{ij}$. Note the following simple and useful identity

$$\boldsymbol{\sigma}^d : \boldsymbol{\tau}^d = \boldsymbol{\sigma}^d : \boldsymbol{\tau} = \boldsymbol{\sigma} : \boldsymbol{\tau} - \frac{1}{2}(\mathrm{tr}\boldsymbol{\sigma})\,(\mathrm{tr}\boldsymbol{\tau}).$$

The symbols for the $L^2(\Sigma)$ and $\mathbf{L}^2(\Sigma)$ inner products

$$\langle \xi, \lambda \rangle_\Sigma := \int_\Sigma \xi\,\lambda, \qquad \langle \boldsymbol{\xi}, \boldsymbol{\lambda} \rangle_\Sigma := \int_\Sigma \boldsymbol{\xi} \cdot \boldsymbol{\lambda},$$

will also be employed for their extensions as the duality products $H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)$ and $\mathbf{H}^{-1/2}(\Sigma) \times \mathbf{H}^{1/2}(\Sigma)$, respectively.

The unknowns in the weak (mixed) formulation will be the two unknowns in (3.2) and the unknowns of (3.5) without the pressure p_S. The corresponding spaces will be:

$$\boldsymbol{\sigma}_\mathrm{S} \in \mathbb{H}(\mathbf{div}\,;\Omega_\mathrm{S}), \qquad \mathbf{u}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S}), \qquad \mathbf{u}_\mathrm{D} \in \mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{D}), \qquad p_\mathrm{D} \in L^2(\Omega_\mathrm{D}). \tag{3.11}$$

In addition, we will need to define two unknowns on the coupling boundary

$$\boldsymbol{\varphi} := -\mathbf{u}_\mathrm{S} \in \mathbf{H}^{1/2}(\Sigma), \qquad \lambda := p_\mathrm{D} \in H^{1/2}(\Sigma). \tag{3.12}$$

Note that in principle the spaces for $\mathbf{u}_\mathrm{S}$ and p_D do not allow enough regularity for the traces above to exist. However, solutions of (3.2) and (3.5) have these unknowns in $\mathbf{H}^1(\Omega_\mathrm{S})$ and $H^1(\Omega_\mathrm{D})$ respectively.

In order to obtain the weak formulation of (3.2)–(3.4)–(3.5), we apply the divergence theorem to the first equation in both (3.2) and (3.5), that is to those equations relating $\boldsymbol{\sigma}_\mathrm{S}$ and $\mathbf{u}_\mathrm{D}$ to other magnitudes. Then, due to the mixed nature of the model, the Dirichlet condition in (3.5) and the traces of p_D and $\mathbf{u}_\mathrm{S}$ on Σ become natural and hence they are incorporated directly in the weak formulation. On the contrary, both transmission conditions in (3.4) become essential, whence they have to be imposed independently, thus yielding the introduction of the auxiliary unknowns (3.12) as the corresponding Lagrange multipliers. According to the above, the weak equations can be written as follows: we look for the unknowns

$$\begin{aligned} (\boldsymbol{\sigma}_\mathrm{S}, \mathbf{u}_\mathrm{S}, \boldsymbol{\varphi}) &\in \mathbb{H}(\mathbf{div}\,;\Omega_\mathrm{S}) \times \mathbf{L}^2(\Omega_\mathrm{S}) \times \mathbf{H}^{1/2}(\Sigma), \\ (\mathbf{u}_\mathrm{D}, p_\mathrm{D}, \lambda) &\in \mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{D}) \times L^2(\Omega_\mathrm{D}) \times H^{1/2}(\Sigma) \end{aligned} \tag{3.13}$$

satisfying two variational equations

$$\nu^{-1}\,(\boldsymbol{\sigma}_\mathrm{S}^d, \boldsymbol{\tau}_\mathrm{S}^d)_\mathrm{S} + (\mathbf{div}\ \boldsymbol{\tau}_\mathrm{S}, \mathbf{u}_\mathrm{S})_\mathrm{S} + \langle \boldsymbol{\tau}_\mathrm{S}\,\mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma = 0 \qquad \forall\, \boldsymbol{\tau}_\mathrm{S} \in \mathbb{H}(\mathbf{div}\,;\Omega_\mathrm{S}), \tag{3.14}$$

$$(\mathrm{K}^{-1}\,\mathbf{u}_\mathrm{D}, \mathbf{v}_\mathrm{D})_\mathrm{D} - (\mathrm{div}\ \mathbf{v}_\mathrm{D}, p_\mathrm{D})_\mathrm{D} - \langle \mathbf{v}_\mathrm{D} \cdot \mathbf{n}, \lambda \rangle_\Sigma = 0 \qquad \forall\, \mathbf{v}_\mathrm{D} \in \mathbf{H}(\mathbf{div}\,;\Omega_\mathrm{D}), \tag{3.15}$$

two differential equations

$$\begin{aligned} \mathbf{div}\ \boldsymbol{\sigma}_\mathrm{S} + \mathbf{f}_\mathrm{S} &= \mathbf{0} \qquad \text{in} \quad \Omega_\mathrm{S}, \\ \mathrm{div}\ \mathbf{u}_\mathrm{D} &= f_\mathrm{D} \qquad \text{in} \quad \Omega_\mathrm{D}, \end{aligned} \tag{3.16}$$

with source terms $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ and $f_D \in L^2(\Omega_D)$, and two restrictions on the boundary

$$\begin{aligned} \boldsymbol{\varphi} \cdot \mathbf{n} + \mathbf{u}_D \cdot \mathbf{n} &= 0 \qquad \text{in} \quad H^{-1/2}(\Sigma), \\ \boldsymbol{\sigma}_S \mathbf{n} + \lambda \mathbf{n} - \nu \kappa^{-1} (\boldsymbol{\varphi} \cdot \mathbf{t}) \mathbf{t} &= 0 \qquad \text{in} \quad \mathbf{H}^{-1/2}(\Sigma). \end{aligned} \tag{3.17}$$

The apparently wrong sign in the term where λ appears in the second equation of (3.17) is due to the fact that the normal on Σ points inwards from the point of view of Ω_D.

Different orderings of the equations and unknown will emphasize different structural properties of the system. We will show three possibilities shortly.

Theorem 3.2.1 *Assume that we have a solution* (3.13) *of the system* (3.14)–(3.15)–(3.16)–(3.17) *and that we define* $p_S := -\frac{1}{2} tr \boldsymbol{\sigma}_S$. *Then* $\mathbf{u}_S \in \mathbf{H}^1(\Omega_S)$, $p_D \in H^1(\Omega_D)$, $\boldsymbol{\varphi} = -\mathbf{u}_S$ *on* Σ, $\lambda = p_D$ *on* Σ *and we have a solution of the system* (3.1)–(3.2)–(3.4).

Proof. It is a simple application of well known results on distribution theory and Sobolev spaces of $H^1(\mathcal{O})$ and $\mathbf{H}(\text{div}\,; \mathcal{O})$ type. □

3.2.3 The variational system

The weak system (3.14)–(3.15)–(3.16)–(3.17) can be described in purely variational form. To do that, we now test the equations (3.16) and the first equation of (3.17) with arbitrary $\mathbf{v} \in \mathbf{L}^2(\Omega_S)$, $q \in L^2(\Omega_D)$, and $\xi \in H^{1/2}(\Sigma)$, respectively, which give

$$(\mathbf{div}\ \boldsymbol{\sigma}_S, \mathbf{v}_S)_S = -(\mathbf{f}_S, \mathbf{v}_S)_S \qquad \forall \mathbf{v}_S \in \mathbf{L}^2(\Omega_S), \tag{3.18}$$

$$(\text{div}\ \mathbf{u}_D, q_D)_D = (f_D, q_D)_D \qquad \forall q_D \in L^2(\Omega_D), \tag{3.19}$$

and

$$\langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_\Sigma + \langle \mathbf{u}_D \cdot \mathbf{n}, \xi \rangle_\Sigma = 0 \qquad \forall \xi \in H^{1/2}(\Sigma). \tag{3.20}$$

In addition, for convenience of the subsequent analysis we consider the decomposition (3.8)–(3.9) with $\mathcal{O} = \Omega_S$, and from now on redefine the fluid pseudostress as

$$\boldsymbol{\sigma}_S + \mu \mathrm{I} \qquad \text{with the new unknowns} \qquad \boldsymbol{\sigma}_S \in \mathbb{H}_0(\mathbf{div}\,; \Omega_S) \quad \text{and} \quad \mu \in \mathbb{R}. \tag{3.21}$$

In this way, the variational formulation of the second transmission condition in (3.17) becomes

$$\langle \boldsymbol{\sigma}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma - \nu \kappa^{-1} \langle \boldsymbol{\varphi} \cdot \mathbf{t}, \boldsymbol{\psi} \cdot \mathbf{t} \rangle_\Sigma + \mu \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma = 0 \qquad \forall \boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Sigma), \tag{3.22}$$

and the equation (3.14) is rewritten, equivalently, as

$$\nu^{-1} (\boldsymbol{\sigma}_S^d, \boldsymbol{\tau}_S^d)_S + (\mathbf{div}\ \boldsymbol{\tau}_S, \mathbf{u}_S)_S + \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma = 0 \qquad \forall \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}\,; \Omega_S), \tag{3.23}$$

and

$$\eta \, \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_\Sigma \, = \, 0 \qquad \forall \, \eta \, \in \, \mathbb{R} \, . \tag{3.24}$$

As a consequence of the above, we find that the resulting variational formulation reduces to a system of seven equations ((3.15), (3.18) – (3.20), (3.22) – (3.24)) and seven unknowns, which can be written in terms of the following nine bilinear forms:

$$\begin{array}{lllllllll} \mathrm{A} & : & \nu^{-1} \, (\boldsymbol{\sigma}_\mathrm{S}^d, \boldsymbol{\tau}_\mathrm{S}^d)_\mathrm{S} & \mathrm{D} & : & \nu \, \kappa^{-1} \, \langle \boldsymbol{\varphi} \cdot \mathbf{t}, \boldsymbol{\psi} \cdot \mathbf{t} \rangle_\Sigma & \mathrm{G} & : & -(\mathrm{div} \, \mathbf{u}_\mathrm{D}, q_\mathrm{D})_\mathrm{D} \\ \mathrm{B} & : & (\mathbf{div} \, \boldsymbol{\sigma}_\mathrm{S}, \mathbf{v}_\mathrm{S})_\mathrm{S} & \mathrm{E} & : & \langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_\Sigma & \mathrm{H} & : & - \langle \mathbf{u}_\mathrm{D} \cdot \mathbf{n}, \xi \rangle_\Sigma \\ \mathrm{C} & : & \langle \boldsymbol{\sigma}_\mathrm{S} \, \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma & \mathrm{F} & : & (\mathrm{K}^{-1} \, \mathbf{u}_\mathrm{D}, \mathbf{v}_\mathrm{D})_\mathrm{D} & \mathrm{J} & : & \eta \, \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_\Sigma \end{array} \tag{3.25}$$

On the left of each column of (3.25) we have added a key letter for the nine different bilinear forms (or related operators). It is easy to see that all these bilinear forms are bounded. Also, those with both arguments in the same space

$$\mathrm{A} : \quad \nu^{-1} \, (\boldsymbol{\sigma}_\mathrm{S}^d, \boldsymbol{\tau}_\mathrm{S}^d)_\mathrm{S} \, , \qquad \mathrm{D} : \quad \nu \, \kappa^{-1} \, \langle \boldsymbol{\varphi} \cdot \mathbf{t}, \boldsymbol{\psi} \cdot \mathbf{t} \rangle_\Sigma \, , \qquad \mathrm{F} : \quad (\mathrm{K}^{-1} \, \mathbf{u}_\mathrm{D}, \mathbf{v}_\mathrm{D})_\mathrm{D}$$

are symmetric and positive semidefinite. In addition, the bilinear forms

$$\mathrm{D} : \quad \nu \, \kappa^{-1} \, \langle \boldsymbol{\varphi} \cdot \mathbf{t}, \boldsymbol{\psi} \cdot \mathbf{t} \rangle_\Sigma \, , \qquad \mathrm{E} : \quad \langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_\Sigma$$

are compact by the compact inclusion of $\mathbf{H}^{1/2}(\Sigma)$ in $\mathbf{L}^2(\Sigma)$.

Now, it is quite clear that there are many different ways of ordering the variational system. In order to illustrate this fact and identify a suitable form, in Table 3.1 below we show three options, emphasizing different structural properties of them. On the left of each row we indicate the corresponding equation. Asides the row and the column involving the unknown μ, we observe in ((1)) that the remaining equations show two blocks on the diagonal: the Stokes block in mixed form with a penalization term and the Darcy block in mixed form. The coupling is limited to E and E^t. Changing the sign of the fourth equation we obtain a symmetric system, whereas changing the sign of the second and third equations we see the sign of the underlying quadratic form: off–diagonal terms compose a skew–symmetric matrix and diagonal terms are positive semidefinite. Similarly, asides again the row and the column involving μ, we observe in ((2)) that the variables are grouped by character and a different mixed structure, with a non–symmetric and negative semidefinite penalization term, is recovered. Nevertheless, a good feature of this system is the fact that D and E are compact, so taking away the penalization term, the remaining system consists of a purely mixed problem, which can be decoupled in two mixed problems. On the other hand, ((3)) shows a particular overlapping of the Stokes and Darcy blocks, which, at first sight, seems to mix everything in an unconvenient way. However, a closer look to this ordering allows to identify a doubly-mixed structure in which the interior mixed formulation contains the same penalization term observed in ((2)). Moreover, all the block bilinear forms, except the one defining the penalization term, show a diagonal structure, which constitutes an advantageous feature when proving the corresponding inf-sup conditions.

Throughout the rest of the paper we adopt the structure ((3)) for our analysis. This means that we group unknowns and spaces as follows:

$$\begin{gathered}\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda) \in \mathbb{X}_0 := \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbf{H}^{1/2}(\Sigma) \times H^{1/2}(\Sigma)\,,\\ \underline{\mathbf{u}} := (\mathbf{u}_{\mathrm{S}}, p_{\mathrm{D}}, \mu) \in \mathbb{M} := \mathbf{L}^2(\Omega_{\mathrm{S}}) \times L^2(\Omega_{\mathrm{D}}) \times \mathbb{R}\,.\end{gathered} \tag{3.26}$$

In this way, the variational system of our problem reads: Find $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}$ such that

$$\begin{aligned}\mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}}) &= \mathcal{F}(\underline{\boldsymbol{\tau}}) \qquad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi) \in \mathbb{X}_0\,,\\ \mathcal{B}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{v}}) &= \mathcal{G}(\underline{\mathbf{v}}) \qquad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}, \eta) \in \mathbb{M}\,,\end{aligned} \tag{3.27}$$

where

$$\mathcal{F}(\underline{\boldsymbol{\tau}}) := 0, \qquad \mathcal{G}(\underline{\mathbf{v}}) = \mathcal{G}((\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}, \eta)) := -(\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} - (f_{\mathrm{D}}, q_{\mathrm{D}})\,, \tag{3.28}$$

and $\mathcal{A}$ and $\mathcal{B}$ are the bounded bilinear forms defined by

$$\begin{aligned}\mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) &= \mathbf{a}((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}})) + \mathbf{b}((\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}), (\boldsymbol{\varphi}, \lambda))\\ &\quad + \mathbf{b}((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (\boldsymbol{\psi}, \xi)) - \mathbf{c}((\boldsymbol{\varphi}, \lambda), (\boldsymbol{\psi}, \xi))\,,\end{aligned} \tag{3.29}$$

with

$$\mathbf{a}((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}})) := \nu^{-1}(\boldsymbol{\sigma}_{\mathrm{S}}^d, \boldsymbol{\tau}_{\mathrm{S}}^d)_{\mathrm{S}} + (\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}}\,,$$

$$[\mathrm{A} + \mathrm{F}]$$

$$\mathbf{b}((\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}), (\boldsymbol{\psi}, \xi)) := \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma - \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \xi\rangle_\Sigma\,,$$

$$[\mathrm{C} + \mathrm{H}]$$

$$\mathbf{c}((\boldsymbol{\varphi}, \lambda), (\boldsymbol{\psi}, \xi)) := \nu\,\kappa^{-1}\langle \boldsymbol{\varphi}\cdot\mathbf{t}, \boldsymbol{\psi}\cdot\mathbf{t}\rangle_\Sigma + \langle \boldsymbol{\varphi}\cdot\mathbf{n}, \xi\rangle_\Sigma - \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda\rangle_\Sigma$$

$$[\mathrm{D} + \mathrm{E} - \mathrm{E}^{\mathrm{t}}]\,,$$

and

$$\mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) := (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} - (\mathrm{div}\,\mathbf{v}_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}} + \eta\,\langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma \qquad [\mathrm{B} + \mathrm{G} + \mathrm{J}]\,. \tag{3.30}$$

It is quite evident from (3.29) that $\mathcal{A}$ has a mixed structure with penalization term given by $-\,\mathbf{c}$, which confirms the doubly-mixed character of (3.27). Note also that $\mathbf{c}$ is non–symmetric and positive semidefinite (this fact will be emphasized and utilized in Section 3.3). In addition, we remark again that the diagonal character of the bilinear forms $\mathbf{a}$, $\mathbf{b}$, and $\mathcal{B}$ will yield simpler and more straightforward proofs of the corresponding inf-sup conditions.

((1))	$\boldsymbol{\sigma}_{\mathrm{S}}$	$\mathbf{u}_{\mathrm{S}}$	$\boldsymbol{\varphi}$	$\mathbf{u}_{\mathrm{D}}$	p_{D}	λ	μ
(3.23)	A	B^t	C^t				
(3.18)	B						
(3.22)	C		$-\mathrm{D}$			E^t	J^t
(3.15)				F	G^t	H^t	
(3.19)				$-\mathrm{G}$			
(3.20)			E	$-\mathrm{H}$			
(3.24)			J				

((2))	$\boldsymbol{\sigma}_{\mathrm{S}}$	$\mathbf{u}_{\mathrm{D}}$	$\mathbf{u}_{\mathrm{S}}$	p_{D}	$\boldsymbol{\varphi}$	λ	μ
(3.23)	A		B^t		C^t		
(3.15)		F		G^t		H^t	
(3.18)	B						
(3.19)		G					
(3.22)	C				$-\mathrm{D}$	E^t	J^t
(3.20)		H			$-\mathrm{E}$		
(3.24)					$-\mathrm{J}$		

((3))	$\boldsymbol{\sigma}_{\mathrm{S}}$	$\mathbf{u}_{\mathrm{D}}$	$\boldsymbol{\varphi}$	λ	$\mathbf{u}_{\mathrm{S}}$	p_{D}	μ
(3.23)	A		C^t		B^t		
(3.15)		F		H^t		G^t	
(3.22)	C		$-\mathrm{D}$	E^t			J^t
(3.20)		H	$-\mathrm{E}$				
(3.18)	B						
(3.19)		G					
(3.24)			J				

Table 3.1: Three different forms of structuring the variational system.

3.3 Analysis of the continuous problem

The approach that we will follow for the analysis of the continuous problem (3.27) is the one of Fredholm theorems and Babuška-Brezzi theory for mixed problems.

3.3.1 Preliminaries

We group here some merely technical results and further notations that will serve for the forthcoming analysis. For elementary results on Hilbert space theory, we refer to [38] for example. The first of them is an abstract result on Hilbert spaces that can be read as follows: a symmetric positive definite bilinear form in a Hilbert space that can be made elliptic by the addition of a compact bilinear form, is necessarily elliptic.

Lemma 3.3.1 *Let X be a Hilbert space, and let $a : X \times X \to \mathbb{R}$ and $k : X \times X \to \mathbb{R}$ be bounded bilinear forms. Assume that a is symmetric and positive definite, k is compact, and there exists $\alpha > 0$ such that*

$$a(x,x) \,+\, k(x,x) \,\geq\, \alpha\,\|x\|^2 \qquad \forall\, x \,\in\, X\,.$$

Then there exists $\beta > 0$ such that

$$a(x,x) \,\geq\, \beta\,\|x\|^2 \qquad \forall\, x \,\in\, X\,.$$

Proof. Let $A : X \to X'$ and $K : X \to X'$ be the linear and bounded operators induced by a and k, respectively, that is $A(x) = a(x,\cdot)$ and $K(x) = k(x,\cdot)$ for each $x \in X$. The hypotheses on a and k imply that A is selfadjoint and injective, K is compact, and $A+K$ is invertible, whence A is Fredholm of index zero. It follows that A is an invertible selfadjoint positive definite operator, and hence, by elementary spectral properties of bounded selfadjoint operators, A is necessarily elliptic. □

Lemma 3.3.2 *There exists $c > 0$ such that*

$$\|\mathbf{v}_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \,\geq\, c\,\|\mathbf{v}_\mathrm{D}\|_{\mathrm{div}\,,\Omega_\mathrm{D}} \qquad \forall\,\mathbf{v}_\mathrm{D} \,\in\, \mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{D}) \quad \textit{such that} \quad \mathrm{div}\;\mathbf{v}_\mathrm{D} \,\in\, \mathbb{P}_0(\Omega_\mathrm{D})\,.$$

Proof. Let $\mathbf{v}_\mathrm{D} \in \mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{D})$ such that $\mathrm{div}\;\mathbf{v}_\mathrm{D} \in \mathbb{P}_0(\Omega_\mathrm{D})$. It is simple to see that

$$\|\mathbf{v}_\mathrm{D}\|^2_{\mathrm{div}\,,\Omega_\mathrm{D}} \,=\, \|\mathbf{v}_\mathrm{D}\|^2_{0,\Omega_\mathrm{D}} \,+\, k(\mathbf{v}_\mathrm{D},\mathbf{v}_\mathrm{D})\,,$$

where $k : \mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{D}) \times \mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{D}) \to \mathbb{R}$ is the bounded bilinear form defined by

$$k(\mathbf{w}_\mathrm{D},\mathbf{v}_\mathrm{D}) \,:=\, \frac{1}{|\Omega_\mathrm{D}|}\left\{\int_{\Omega_\mathrm{D}} \mathrm{div}\;\mathbf{w}_\mathrm{D}\right\}\left\{\int_{\Omega_\mathrm{D}} \mathrm{div}\;\mathbf{v}_\mathrm{D}\right\} \qquad \forall\,\mathbf{w}_\mathrm{D},\,\mathbf{v}_\mathrm{D} \,\in\, \mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{D})\,.$$

Since k is clearly compact, a direct application of Lemma 3.3.1 ends the proof. □

Lemma 3.3.3 *There exists $c_1 > 0$ such that*

$$\|\boldsymbol{\tau}_{\mathrm{S}}^{d}\|_{0,\Omega_{\mathrm{S}}}^{2} + \|\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}^{2} \geq c_1 \|\boldsymbol{\tau}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}^{2} \qquad \forall\, \boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}})\,. \tag{3.31}$$

Proof. See [12, Lemma 3.1] or [19, Proposition 3.1, Chapter IV]. □

Lemma 3.3.4 *Let $(X, \langle\cdot,\cdot\rangle_X)$ and $Y, \langle\cdot,\cdot\rangle_Y)$ be Hilbert spaces and let $A : X \to X$, $B : X \to Y$, and $C : Y \to Y$ be bounded linear operators. Assume that A is elliptic, B is surjective, and C is positive semidefinite, that is, respectively*

i) *there exists $\alpha > 0$ such that $\langle A(x), x\rangle_X \geq \alpha\,\|x\|_X^2 \quad \forall\, x \in X$,*

ii) *there exists $\beta > 0$ such that $\|B^*(y)\|_X \geq \beta\,\|y\|_Y \quad \forall\, y \in Y$,*

iii) $\langle C(y), y\rangle_Y \geq 0 \qquad \forall\, y \in Y$.

Then the matrix operator $T := \begin{bmatrix} A & B^* \\ B & -C \end{bmatrix} : X \times Y \to X \times Y$ is bijective.

Proof. It suffices to observe that, being A invertible thanks to i), T is bijective if and only if $S := B\,A^{-1}\,B^* + C : Y \to Y$ is bijective, which follows from the fact that S becomes elliptic. We omit further details and refer to [41, Lemma 2.1] for a nonlinear version of this result. □

We end this section with some notations concerning our product spaces. In fact, we now let

$$\mathbb{X} := \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbf{H}^{1/2}(\Sigma) \times H^{1/2}(\Sigma)\,,$$

recall that $\mathbb{M} := \mathbf{L}^2(\Omega_{\mathrm{S}}) \times L^2(\Omega_{\mathrm{D}}) \times \mathbb{R}$ (cf. (3.26)), and define

$$\|\underline{\boldsymbol{\tau}}\|_{\mathbb{X}} := \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}} + \|\boldsymbol{\psi}\|_{1/2,\Sigma} + \|\xi\|_{1/2,\Sigma} \quad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi) \in \mathbb{X}\,,$$

and

$$\|\underline{\mathbf{v}}\|_{\mathbb{M}} := \|\mathbf{v}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + |\eta| \qquad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}, \eta) \in \mathbb{M}\,.$$

Note that $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{M}}$ are equivalent to the product norms that make $\mathbb{X}$ and $\mathbb{M}$ (and hence $\mathbb{X}_0$ and $\mathbb{M}_0$) Hilbert spaces. We will use them for all forthcoming estimates.

3.3.2 The main results

We begin by showing that (3.27) has a one dimensional kernel. More precisely, we have the following result.

Lemma 3.3.5 *Let $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) := ((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda), (\mathbf{u}_{\mathrm{S}}, p_{\mathrm{D}}, \mu)) \in \mathbb{X}_0 \times \mathbb{M}$ be a solution of (3.27) with homogeneous right hand side. Then there exists $c \in \mathbb{R}$ such that*

$$\underline{\boldsymbol{\sigma}} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, c) \quad \text{and} \quad \underline{\mathbf{u}} = (\mathbf{0}, c, -c)$$

Proof. Testing the equations (3.27) with $\underline{\boldsymbol{\tau}} = (\boldsymbol{\sigma}_{\rm S}, \mathbf{u}_{\rm D}, -\boldsymbol{\varphi}, -\lambda)$ and $\underline{\mathbf{v}} = (-\mathbf{u}_{\rm S}, -p_{\rm D}, \mu)$, and then adding them, we find that

$$0 = \nu^{-1} (\boldsymbol{\sigma}_{\rm S}^d, \boldsymbol{\sigma}_{\rm S}^d)_{\rm S} + ({\rm K}^{-1}\mathbf{u}_{\rm D}, \mathbf{u}_{\rm D})_{\rm D} + \nu\, \kappa^{-1} \langle \boldsymbol{\varphi} \cdot \mathbf{t}, \boldsymbol{\varphi} \cdot \mathbf{t} \rangle_{\Sigma} .$$

Note that this is equivalent to changing the sign of either the second and third rows in ((1)) or all the rows but the first two in ((2)) or all the rows but the first two and the last one in ((3)) (see Table 3.1), and then adding all them. It is clear from the above equation that

$$\boldsymbol{\sigma}_{\rm S}^d = \mathbf{0} \quad \text{in} \quad \Omega_{\rm S}, \qquad \mathbf{u}_{\rm D} = \mathbf{0} \quad \text{in} \quad \Omega_{\rm D}, \quad \text{and} \quad \boldsymbol{\varphi} \cdot \mathbf{t} = 0 \quad \text{on} \quad \Sigma .$$

Using Theorem 3.2.1 it follows that $\nabla \mathbf{u}_{\rm S} = \nu^{-1} \boldsymbol{\sigma}_{\rm S}^d = \mathbf{0}$ (cf . (3.5)) and $-\mathbf{u}_{\rm S} \cdot \mathbf{t} = 0$, which implies that $\mathbf{u}_{\rm S} = \mathbf{0}$ in $\Omega_{\rm S}$. Hence, again by Theorem 3.2.1 we have that $\boldsymbol{\varphi} = \mathbf{0}$ and $\mathbf{div}\, \boldsymbol{\sigma}_{\rm S} = \mathbf{0}$, which, together with the fact that $\boldsymbol{\sigma}_{\rm S} \in \mathbb{H}_0(\mathbf{div}\,; \Omega_{\rm S})$ and $\boldsymbol{\sigma}_{\rm S}^d = \mathbf{0}$, yields $\boldsymbol{\sigma}_{\rm S} = \mathbf{0}$ in $\Omega_{\rm S}$. Next, since $\nabla p_{\rm D} = {\rm K}^{-1} \mathbf{u}_{\rm D} = \mathbf{0}$, we deduce the existence of $c \in \mathbb{R}$ such that $p_{\rm D} = c$ in $\Omega_{\rm D}$, whence $\lambda = c$ on Σ. According to the above, the equation (3.22) reduces now to $\mu\, \mathbf{n} + c\, \mathbf{n} = \mathbf{0}$ on Σ, which gives $\mu = -c$. □

Our next goal is to demonstrate that a simple restriction on the pressure in the Darcy domain solves the indetermination generated by the non-null kernel of (3.27). To this end, we now let

$$\mathbb{M}_0 := \mathbf{L}^2(\Omega_{\rm S}) \times L_0^2(\Omega_{\rm D}) \times \mathbb{R} ,$$

where

$$L_0^2(\Omega_{\rm D}) := \left\{ q \in L^2(\Omega_{\rm D}) : \quad \int_{\Omega_{\rm D}} q = 0 \right\} ,$$

and consider the reduced problem: Find $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ such that

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}}) &= \mathcal{F}(\underline{\boldsymbol{\tau}}) \qquad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\rm S}, \mathbf{v}_{\rm D}, \boldsymbol{\psi}, \xi) \in \mathbb{X}_0 , \\ \mathcal{B}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{v}}) &= \mathcal{G}(\underline{\mathbf{v}}) \qquad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_{\rm S}, q_{\rm D}, \eta) \in \mathbb{M}_0 . \end{aligned} \tag{3.32}$$

Throughout the rest of the section we follow the analysis suggested by the Babuška-Brezzi theory to conclude finally that (3.32) is well posed. This requires the inf-sup condition for $\mathcal{B}$ and the invertibility of the operator induced by $\mathcal{A}$ in the kernel of $\mathcal{B}$. We begin with the first.

Lemma 3.3.6 *There exists $\beta > 0$ such that*

$$\sup_{\underline{\boldsymbol{\tau}} \in \mathbb{X}_0 \backslash \mathbf{0}} \frac{\mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbb{X}}} \geq \beta\, \|\underline{\mathbf{v}}\|_{\mathbb{M}} \qquad \forall\, \underline{\mathbf{v}} \in \mathbb{M}_0 . \tag{3.33}$$

Proof. We first observe that the diagonal character of $\mathcal{B}$ (cf. (3.30)) guarantees that (3.33) is equivalent to the following three independent inf-sup conditions:

$$\sup_{\boldsymbol{\tau}_{\rm S} \in \mathbb{H}_0(\mathbf{div};\Omega_{\rm S}) \backslash \mathbf{0}} \frac{(\mathbf{div}\, \boldsymbol{\tau}_{\rm S}, \mathbf{v}_{\rm S})_{\rm S}}{\|\boldsymbol{\tau}_{\rm S}\|_{\mathbf{div},\Omega_{\rm S}}} \geq \beta_{\rm S}\, \|\mathbf{v}_{\rm S}\|_{0,\Omega_{\rm S}} \qquad \forall\, \mathbf{v}_{\rm S} \in \mathbf{L}^2(\Omega_{\rm S}) , \tag{3.34}$$

$$\sup_{\mathbf{v}_{\rm D}\,\in\,\mathbf{H}({\rm div};\Omega_{\rm D})\backslash\mathbf{0}} \frac{({\rm div}\,\mathbf{v}_{\rm D}, q_{\rm D})_{\rm D}}{\|\mathbf{v}_{\rm D}\|_{{\rm div},\Omega_{\rm D}}} \,\geq\, \beta_{\rm D}\,\|q_{\rm D}\|_{0,\Omega_{\rm D}} \qquad \forall\, q_{\rm D} \,\in\, L^2_0(\Omega_{\rm D})\,, \tag{3.35}$$

$$\sup_{\boldsymbol{\psi}\,\in\,\mathbf{H}^{1/2}(\Sigma)\backslash\mathbf{0}} \frac{\eta\,\langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma}{\|\boldsymbol{\psi}\|_{1/2,\Sigma}} \,\geq\, \beta_\Sigma\,|\eta| \qquad \forall\,\eta\,\in\,\mathbb{R}\,, \tag{3.36}$$

with $\beta_{\rm S}$, $\beta_{\rm D}$, $\beta_\Sigma > 0$. For instance, the above statement follows from a direct application of the characterization result for the inf-sup condition on product spaces provided in [52, Theorem 5].

Now, given $\mathbf{v}_{\rm S} \in \mathbf{L}^2(\Omega_{\rm S})$ we define $\boldsymbol{\tau}$ as the $\mathbb{H}_0(\mathbf{div}\,;\Omega_{\rm S})$–component of $\nabla\mathbf{z} \in \mathbb{H}(\mathbf{div}\,;\Omega_{\rm S})$, where $\mathbf{z} \in \mathbf{H}^1(\Omega_{\rm S})$ is the unique solution of the boundary value problem:

$$\Delta\mathbf{z} \,=\, \mathbf{v}_{\rm S} \quad \text{in} \quad \Omega_{\rm S}\,, \qquad \mathbf{z} \,=\, \mathbf{0} \quad \text{on} \quad \partial\Omega_{\rm S}\,.$$

This proves the surjectivity of the operator $\mathbf{div}\,:\,\mathbb{H}_0(\mathbf{div}\,;\Omega_{\rm S}) \,\rightarrow\, \mathbf{L}^2(\Omega_{\rm S})$, which is (3.34). Similarly, it is easy to see that ${\rm div}\,:\,\mathbf{H}({\rm div}\,;\Omega_{\rm D}) \,\rightarrow\, L^2(\Omega_{\rm D})$ is also surjective, which yields (3.35).

On the other hand, the inf-sup condition (3.36) is equivalent to the surjectivity of the operator $\boldsymbol{\psi} \,\rightarrow\, \langle\boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma$ from $\mathbf{H}^{1/2}(\Sigma)$ to $\mathbb{R}$, which in turn is equivalent to showing the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}(\Sigma)$ such that $\langle\boldsymbol{\psi}_0\cdot\mathbf{n}, 1\rangle_\Sigma \neq 0$. In fact, we pick one corner point of Σ and define a function v that is continuous, linear on each side of Σ, equal to one in the chosen vertex and zero on all other ones. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors on the two sides of Σ that meet at the corner point, then $\boldsymbol{\psi}_0 := v\,(\mathbf{n}_1 + \mathbf{n}_2)$ satisfies the required property. □

We now let $\mathbb{V}$ be the kernel of $\mathcal{B}$, that is

$$\mathbb{V} \,:=\, \big\{\, \underline{\boldsymbol{\tau}} \,\in\, \mathbb{X}_0 : \quad \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) \,=\, 0 \quad \forall\,\underline{\mathbf{v}} \,\in\, \mathbb{M}_0 \,\big\}\,.$$

It is easy to see from the definition of $\mathcal{B}$ (cf. (3.30)) that $\mathbb{V} \,=\, \mathbb{V}_1 \,\times\, \mathbb{V}_2$, where

$$\mathbb{V}_1 \,=\, \tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\rm S}) \times \tilde{\mathbf{H}}({\rm div};\Omega_{\rm D}) \quad \text{and} \quad \mathbb{V}_2 \,=\, \tilde{\mathbf{H}}^{1/2}(\Sigma) \times H^{1/2}(\Sigma)\,,$$

with

$$\tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\rm S}) \,:=\, \Big\{ \boldsymbol{\tau}_{\rm S} \,\in\, \mathbb{H}_0(\mathbf{div};\Omega_{\rm S}) : \quad \mathbf{div}\,\boldsymbol{\tau}_{\rm S} \,=\, \mathbf{0} \Big\}\,,$$

$$\tilde{\mathbf{H}}({\rm div};\Omega_{\rm D}) \,:=\, \Big\{ \mathbf{v}_{\rm D} \,\in\, \mathbf{H}({\rm div};\Omega_{\rm D}) : \quad {\rm div}\,\mathbf{v}_{\rm D} \,\in\, \mathbb{P}_0(\Omega_{\rm D}) \Big\}\,,$$

and

$$\tilde{\mathbf{H}}^{1/2}(\Sigma) \,:=\, \Big\{ \boldsymbol{\psi} \,\in\, \mathbf{H}^{1/2}(\Sigma) : \quad \langle\boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma \,=\, 0 \Big\}\,.$$

Then, in what follows we apply Lemma 3.3.4 to prove that the operator induced by $\mathcal{A}$ (cf. (3.29)) is invertible in $\mathbb{V}$. This means showing that $\mathbf{a}$ is elliptic on $\mathbb{V}_1$, $\mathbf{b}$ satisfies the inf-sup condition on $\mathbb{V}_1 \times \mathbb{V}_2$, and $\mathbf{c}$ is positive semidefinite on $\mathbb{V}_2$.

As remarked in Section 3.2, the condition on $\mathbf{c}$ is pretty straightforward since

$$\mathbf{c}((\boldsymbol{\varphi},\lambda),(\boldsymbol{\varphi},\lambda)) \,=\, \nu\,\kappa^{-1}\,\|\boldsymbol{\varphi}\cdot\mathbf{t}\|^2_{0,\Sigma} \,\geq\, 0 \quad \forall\,(\boldsymbol{\varphi},\lambda)\in\mathbf{H}^{1/2}(\Sigma)\times H^{1/2}(\Sigma)\,. \tag{3.37}$$

The remaining conditions for $\mathbf{a}$ and $\mathbf{b}$ are established in the following lemmas.

Lemma 3.3.7 *There exists* $\alpha_1 > 0$ *such that for each* $(\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}) \in \mathbb{V}_1$ *there holds*

$$\mathbf{a}((\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}),(\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}})) \,\geq\, \alpha_1\left\{\|\boldsymbol{\tau}_{\mathrm{S}}\|^2_{\mathbf{div},\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|^2_{\mathrm{div},\Omega_{\mathrm{D}}}\right\}.$$

Proof. It suffices to observe that

$$\begin{aligned}\mathbf{a}((\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}),(\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}})) &= \nu^{-1}\,\|\boldsymbol{\tau}^d_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + (\mathbf{K}^{-1}\,\mathbf{v}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}}\\ &\geq c\left\{\|\boldsymbol{\tau}^d_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}}\right\},\end{aligned}$$

and then apply Lemmas 3.3.3 and 3.3.2. □

Lemma 3.3.8 *There exists* $\tilde{\beta} > 0$ *such that*

$$\sup_{(\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}})\in\mathbb{V}_1\backslash\mathbf{0}} \frac{\mathbf{b}((\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}),(\boldsymbol{\psi},\xi))}{\|(\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}})\|} \geq \tilde{\beta}\,\|(\boldsymbol{\psi},\xi)\| \qquad \forall\,(\boldsymbol{\psi},\xi)\in\mathbb{V}_2\,. \tag{3.38}$$

Proof. Analogously to the proof of Lemma 3.3.6, and thanks to the diagonal character of $\mathbf{b}$, we find that (3.38) is equivalent to the following two independent inequalities:

$$\sup_{\boldsymbol{\tau}_{\mathrm{S}}\in\tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}})\backslash\mathbf{0}} \frac{\langle\boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \,\geq\, \tilde{\beta}_{\mathrm{S}}\,\|\boldsymbol{\psi}\|_{1/2,\Sigma} \quad \forall\,\boldsymbol{\psi}\,\in\,\tilde{\mathbf{H}}^{1/2}(\Sigma)\,, \tag{3.39}$$

$$\sup_{\mathbf{v}_{\mathrm{D}}\in\tilde{\mathbf{H}}(\mathrm{div};\Omega_{\mathrm{D}})\backslash\mathbf{0}} \frac{\langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div},\Omega_{\mathrm{D}}}} \,\geq\, \tilde{\beta}_{\mathrm{D}}\,\|\xi\|_{1/2,\Sigma} \qquad \forall\,\xi\,\in\,H^{1/2}(\Sigma)\,, \tag{3.40}$$

with $\tilde{\beta}_{\mathrm{S}},\,\tilde{\beta}_{\mathrm{D}} > 0$.

Now, given $\boldsymbol{\chi}\in\mathbf{H}^{-1/2}(\Sigma)$ we let $\boldsymbol{\tau}$ be the $\mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}})$–component of $\nabla\,\mathbf{z}\in\mathbb{H}(\mathbf{div}\,;\Omega_{\mathrm{S}})$, where $\mathbf{z}\in\mathbf{H}^1(\Omega_{\mathrm{S}})$ is the unique solution of the boundary value problem:

$$\Delta\mathbf{z} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \quad \mathbf{z} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}\,, \quad \nabla\mathbf{z}\,\mathbf{n} = \boldsymbol{\chi} \quad \text{on} \quad \Sigma\,. \tag{3.41}$$

In other words, $\boldsymbol{\tau} := \nabla\,\mathbf{z} - c\,\mathbf{I}$, where $c := \dfrac{1}{2\,|\Omega_{\mathrm{S}}|}\displaystyle\int_{\Omega_{\mathrm{S}}}\mathrm{tr}\,\nabla\,\mathbf{z}$ (cf. (3.10)), which implies that $\boldsymbol{\tau}\in\tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}})$ and $\boldsymbol{\tau}\,\mathbf{n} = \boldsymbol{\chi} - c\,\mathbf{n}$ on Σ. It follows that $\langle\boldsymbol{\tau}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma = \langle\boldsymbol{\chi},\boldsymbol{\psi}\rangle_\Sigma$ for each $\boldsymbol{\psi}\in\tilde{\mathbf{H}}^{1/2}(\Sigma)$, which proves the surjectivity of the operator $\boldsymbol{\tau}\to\boldsymbol{\tau}\,\mathbf{n}$ from $\tilde{\mathbb{H}}_0(\mathbf{div};\Omega_{\mathrm{S}})$ to $\big(\tilde{\mathbf{H}}^{1/2}(\Sigma)\big)'$, that is (3.39).

Similarly, given $\chi\in\mathrm{H}^{-1/2}(\Sigma)$ we define $\mathbf{v} := \nabla z\in\mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{D}})$, where $z\in H^1(\Omega_{\mathrm{D}})$ is the unique solution of the boundary value problem:

$$\Delta z = \frac{1}{|\Omega_{\mathrm{D}}|}\,\langle\chi,1\rangle_\Sigma \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \quad \nabla z\cdot\mathbf{n} = \chi \quad \text{on} \quad \Sigma\,, \quad \int_{\Omega_{\mathrm{D}}} z = 0\,. \tag{3.42}$$

It follows that $\mathbf{v} \in \tilde{\mathbf{H}}(\mathrm{div};\Omega_{\mathrm{D}})$ and $\mathbf{v}\cdot\mathbf{n} = \chi$ on Σ, which proves the surjectivity of the operator $\mathbf{v} \to \mathbf{v}\cdot\mathbf{n}$ from $\tilde{\mathbf{H}}(\mathrm{div};\Omega_{\mathrm{D}})$ to $\mathrm{H}^{-1/2}(\Sigma)$, that is (3.40). □

As a consequence of the previous analysis we conclude that $\mathcal{A}$ is invertible in the kernel of $\mathcal{B}$. This result and the inf-sup condition for $\mathcal{B}$ (cf. Lemma 3.3.6) allow to establish the following theorem.

Theorem 3.3.1 *For each pair* $(\mathcal{F},\mathcal{G}) \in \mathbb{X}_0' \times \mathbb{M}_0'$ *there exists a unique* $(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ *solution to* (3.32)*, and there exists a constant* $C > 0$*, independent of the solution, such that*

$$\|(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})\|_{\mathbb{X}\times\mathbb{M}} \;\leq\; C\left\{ \|\mathcal{F}\|_{\mathbb{X}_0'} + \|\mathcal{G}\|_{\mathbb{M}_0'} \right\}.$$

In particular, if $(\mathcal{F},\mathcal{G})$ *is given by* (3.28) *and there holds* $\int_{\Omega_{\mathrm{D}}} f_{\mathrm{D}} = 0$ (*cf.* (3.3))*, then the solution of* (3.32) *is also a solution of the original variational formulation* (3.27)*.*

Proof. The well posedness of (3.32) follows from a straightforward application of the classical Babuška-Brezzi theory for mixed problems (see, e.g. [54, Theorem I.4.1] or [19, Chapter II]). Now, let $(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ be the solution of (3.32) with $(\mathcal{F},\mathcal{G})$ given by (3.28). Since the first equations of (3.27) and (3.32) coincide, it only remains to show that $\underline{\boldsymbol{\sigma}}$ verifies the second equation of (3.27) to conclude that $(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})$ also solves that problem. In fact, taking $\underline{\boldsymbol{\tau}} = (\mathbf{0},\mathbf{0},\mathbf{0},\xi)$ in the first equation of (3.32) we deduce that $\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} + \boldsymbol{\varphi}\cdot\mathbf{n} = 0$ on Σ, and hence, according to the definition of $\mathcal{B}$ (cf. (3.30)) and the second equation of (3.32), we obtain that

$$\begin{aligned}\mathcal{B}(\underline{\boldsymbol{\sigma}},(\mathbf{0},1,0)) &= -(\mathrm{div}\,\mathbf{u}_{\mathrm{D}},1)_{\mathrm{D}} = \langle \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n},1\rangle_{\Sigma} = -\langle \boldsymbol{\varphi}\cdot\mathbf{n},1\rangle_{\Sigma} \\ &= \mathcal{B}(\underline{\boldsymbol{\sigma}},(\mathbf{0},0,-1)) = \mathcal{G}((\mathbf{0},0,-1)) = 0.\end{aligned}$$

Then, given $\underline{\mathbf{v}} = (\mathbf{v}_{\mathrm{S}},q_{\mathrm{D}},\eta) \in \mathbb{M}$, where $q_{\mathrm{D}} = q_0 + c$, with $(q_0,c) \in L_0^2(\Omega_{\mathrm{D}}) \times \mathbb{R}$, we use the above identity and again the second equation of (3.32), to find that

$$\begin{aligned}\mathcal{B}(\underline{\boldsymbol{\sigma}},\underline{\mathbf{v}}) &= \mathcal{B}(\underline{\boldsymbol{\sigma}},(\mathbf{v}_{\mathrm{S}},q_0,\eta)) = \mathcal{G}((\mathbf{v}_{\mathrm{S}},q_0,\eta)) = -(\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} - (f_{\mathrm{D}},q_0)_{\mathrm{D}} \\ &= -(\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} - \left(f_{\mathrm{D}} - \frac{1}{|\Omega_{\mathrm{D}}|}\int_{\Omega_{\mathrm{D}}} f_{\mathrm{D}}, q_{\mathrm{D}}\right)_{\mathrm{D}},\end{aligned}$$

which, thanks to the assumption (3.3), becomes $\mathcal{B}(\underline{\boldsymbol{\sigma}},\underline{\mathbf{v}}) = \mathcal{G}(\underline{\mathbf{v}}) \quad \forall\, \underline{\mathbf{v}} \in \mathbb{M}$. □

Note from the last identity in the previous proof that if we solve (3.32) with $(\mathcal{F},\mathcal{G})$ given by (3.28) but (3.3) is not satisfied, then we are finding a solution of (3.27) for a slightly modified right hand side, with $\mathbf{f}_{\mathrm{S}}$ unchanged but with $f_{\mathrm{D}} - \frac{1}{|\Omega_{\mathrm{D}}|}\int_{\Omega_{\mathrm{D}}} f_{\mathrm{D}}$ instead of f_{D}. Moreover, we can actually prove the following result characterizing the solvability of (3.27).

Theorem 3.3.2 *Problem* (3.27) *with* $(\mathcal{F},\mathcal{G})$ *given by* (3.28) *is solvable if and only if* (3.3) *holds. In that case, the solution is defined up to a multiple of the vector* $((\mathbf{0},\mathbf{0},\mathbf{0},1),(\mathbf{0},1,-1))$.

Proof. It suffices to observe that the operator induced by the left hand side of (3.27), say $\mathcal{L}$, is Fredholm of index zero. In fact, using that $L^2(\Omega_{\rm D}) = L^2_0(\Omega_{\rm D}) \oplus \mathbb{P}_0(\Omega_{\rm D})$, we decompose the pressure unknown $p_{\rm D}$ in (3.27) as $p_{\rm D} = p_0 + c$ with $p_0 \in L^2_0(\Omega_{\rm D})$ and $c \in \mathbb{P}_0(\Omega_{\rm D})$, and similarly for the corresponding test functions $q_{\rm D} \in L^2(\Omega_{\rm D})$. In this way, it is easy to realize that (3.27) is equivalent to a compact perturbation of a problem equivalent to (3.32). Since the latter is well posed, this proves the announced property of $\mathcal{L}$. Now, the kernel of the adjoint operator $\mathcal{L}^*$ is the same as $\mathcal{L}$ because this operator is symmetric up to some sign changes of its rows (see Table 3.1). Therefore, by the Fredholm alternative, the system (3.27) is solvable if and only if the right hand side vanishes when applied to an element of the kernel of the adjoint. With the right hand side (3.28) and the kernel given in Lemma 3.3.5 this is just condition (3.3).
□

At this point we remark that the above analysis also applies when the fluid lies over the porous medium and the additional Neumann boundary condition (3.6) is incorporated into the model (as described at the end of Section 3.2.1). In particular, it is easy to see that (3.3) and its equivalence with the solvability of the original formulation (3.27) remain unchanged in this case. On the other hand, if we assume (3.7) instead of (3.6), the condition (3.3) does not hold any more and the solvability analysis of (3.27) becomes simpler. Indeed, following the same arguments of the proof of Lemma 3.3.5, we find now, thanks to the fact that $\nabla p_{\rm D} = \mathbf{0}$ in $\Omega_{\rm D}$ and $p_{\rm D} = 0$ on Γ, that $p_{\rm D} = 0$ in $\Omega_{\rm D}$, which leads to a trivial kernel for (3.27). In other words, there is no need of incorporating any further restriction on the pressure $p_{\rm D}$ and the subsequent reduced problem (3.32) since the homogeneous Dirichlet boundary condition (3.7) already insures the uniqueness of solution. Consequently, up to minor modifications, the solvability analysis of (3.27) becomes very similar to the corresponding analysis of the present formulation (3.32).

3.4 The Galerkin scheme

In this section we introduce and analyze the Galerkin scheme of the reduced problem (3.32).

3.4.1 Preliminaries

Here we define the discrete system and establish suitable assumptions on the finite element subspaces ensuring later on that it becomes well posed. For this purpose, we first select two collections of discrete spaces:

$$\begin{aligned} &\mathbf{H}_h(\Omega_{\rm D}) \subseteq \mathbf{H}(\mathrm{div}\,;\Omega_{\rm D})\,, \qquad L_h(\Omega_{\rm D}) \subseteq L^2(\Omega_{\rm D})\,, \qquad \Lambda_h^{\rm D}(\Sigma) \subseteq H^{1/2}(\Sigma)\,, \\ &\mathbf{H}_h(\Omega_{\rm S}) \subseteq \mathbf{H}(\mathrm{div}\,;\Omega_{\rm S})\,, \qquad L_h(\Omega_{\rm S}) \subseteq L^2(\Omega_{\rm S})\,, \qquad \Lambda_h^{\rm S}(\Sigma) \subseteq H^{1/2}(\Sigma)\,. \end{aligned} \tag{3.43}$$

However, the spaces for the Stokes domain will have to be doubled. In particular, in the case of the matrix valued unknown $\boldsymbol{\sigma}_{\rm S}$ we will consider the space of matrix valued functions whose rows belong to $\mathbf{H}_h(\Omega_{\rm S})$. According to this we now define

$$\mathbf{L}_h(\Omega_{\rm S}) := L_h(\Omega_{\rm S}) \times L_h(\Omega_{\rm S}), \qquad \boldsymbol{\Lambda}_h^{\rm S}(\Sigma) := \Lambda_h^{\rm S}(\Sigma) \times \Lambda_h^{\rm S}(\Sigma), \tag{3.44}$$

$$\mathbb{H}_h(\Omega_{\rm S}) := \{ \boldsymbol{\tau} : \Omega_{\rm S} \to \mathbb{R}^{2\times 2} : \quad \mathbf{c}^t \boldsymbol{\tau} \in \mathbf{H}_h(\Omega_{\rm S}) \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \} \subseteq \mathbb{H}(\mathbf{div}\,; \Omega_{\rm S}), \tag{3.45}$$

and

$$\mathbb{H}_{h,0}(\Omega_{\rm S}) := \mathbb{H}_h(\Omega_{\rm S}) \cap \mathbb{H}_0(\mathbf{div}\,; \Omega_{\rm S}). \tag{3.46}$$

In addition, in order to deal with the mean value condition of the Darcy pressure we define

$$L_{h,0}(\Omega_{\rm D}) := L_h(\Omega_{\rm D}) \cap L_0^2(\Omega_{\rm D}). \tag{3.47}$$

In this way, we define the global finite element subspaces as:

$$\begin{aligned} \mathbb{X}_{h,0} &:= \mathbb{H}_{h,0}(\Omega_{\rm S}) \times \mathbf{H}_h(\Omega_{\rm D}) \times \boldsymbol{\Lambda}_h^{\rm S}(\Sigma) \times \Lambda_h^{\rm D}(\Sigma), \\ \mathbb{M}_{h,0} &:= \mathbf{L}_h(\Omega_{\rm S}) \times L_{h,0}(\Omega_{\rm D}) \times \mathbb{R}, \end{aligned} \tag{3.48}$$

and consider the following Galerkin scheme for (3.32): Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$ such that

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) + \mathcal{B}(\underline{\boldsymbol{\tau}}_h, \underline{\mathbf{u}}_h) &= \mathcal{F}(\underline{\boldsymbol{\tau}}_h) \qquad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbb{X}_{h,0}, \\ \mathcal{B}(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{v}}_h) &= \mathcal{G}(\underline{\mathbf{v}}_h) \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbb{M}_{h,0}. \end{aligned} \tag{3.49}$$

Note that the different structures shown in Table 3.1 are inherited by the linear system associated to (3.49) once we have chosen bases for all the discrete spaces.

In what follows we derive general hypotheses on the spaces (3.43) that will allow us to show in Section 3.4.2 below that (3.49) is well posed. Our approach consists of adapting to the present discrete case the arguments employed in the analysis of the continuous problem, mainly those from the proofs of Lemmas 3.3.6, 3.3.7, and 3.3.8. We begin by observing that in order to have meaningful spaces $\mathbb{H}_{h,0}(\Omega_{\rm S})$ and $L_{h,0}(\Omega_{\rm D})$ (cf. (3.46) and (3.47)), we need to be able to eliminate multiples of the identity matrix from $\mathbb{H}_h(\Omega_{\rm S})$ and constants polynomials from $L_h(\Omega_{\rm D})$. This request is certainly satisfied if we assume that:

(H.0) $[\mathbb{P}_0(\Omega_{\rm S})]^2 \subseteq \mathbf{H}_h(\Omega_{\rm S})$ *and* $\mathbb{P}_0(\Omega_{\rm D}) \subseteq L_h(\Omega_{\rm D})$.

We remark that the above hypothesis is only related to the ability of the spaces to deal with problems inherent to the kernel of (3.27). In particular, it follows that $\mathrm{I} \in \mathbb{H}_h(\Omega_{\rm S})$ for all h, and hence there holds the decomposition:

$$\mathbb{H}_h(\Omega_{\rm S}) = \mathbb{H}_{h,0}(\Omega_{\rm S}) \oplus \mathbb{P}_0(\Omega_{\rm S})\,\mathrm{I}. \tag{3.50}$$

Next, following the same diagonal argument utilized in the proof of Lemma 3.3.6, we deduce that $\mathcal{B}$ satisfies the discrete inf-sup condition uniformly on $\mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$ if and only if there exist $\beta_{\rm S}\,,\ \beta_{\rm D}\,,\ \beta_\Sigma\ >\ 0$, independent of h, such that

$$\sup_{\boldsymbol{\tau}_h\,\in\,\mathbb{H}_{h,0}(\Omega_{\rm S})\backslash\mathbf{0}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_h, \mathbf{v}_h)_{\rm S}}{\|\boldsymbol{\tau}_h\|_{\mathbf{div},\Omega_{\rm S}}} \ \geq\ \beta_{\rm S}\,\|\mathbf{v}_h\|_{0,\Omega_{\rm S}} \qquad \forall\,\mathbf{v}_h\,\in\,\mathbf{L}_h(\Omega_{\rm S})\,, \tag{3.51}$$

$$\sup_{\mathbf{v}_h\,\in\,\mathbf{H}_h(\Omega_{\rm D})\backslash\mathbf{0}} \frac{(\mathrm{div}\,\mathbf{v}_h, q_h)_{\rm D}}{\|\mathbf{v}_h\|_{\mathrm{div},\Omega_{\rm D}}} \ \geq\ \beta_{\rm D}\,\|q_h\|_{0,\Omega_{\rm D}} \qquad \forall\,q_h\,\in\,L_{h,0}(\Omega_{\rm D})\,, \tag{3.52}$$

$$\sup_{\boldsymbol{\psi}_h\,\in\,\boldsymbol{\Lambda}_h^{\rm S}(\Sigma)\backslash\mathbf{0}} \frac{\eta\,\langle\boldsymbol{\psi}_h\cdot\mathbf{n}, 1\rangle_\Sigma}{\|\boldsymbol{\psi}_h\|_{1/2,\Sigma}} \ \geq\ \beta_\Sigma\,|\eta| \qquad \forall\,\eta\,\in\,\mathbb{R}\,. \tag{3.53}$$

However, since $\mathbf{div}\ \mathbb{H}_h(\Omega_{\rm S})\ =\ \mathbf{div}\ \mathbb{H}_{h,0}(\Omega_{\rm S})$ (cf. (3.50)), the supremum in (3.51) remains the same if taken on $\mathbb{H}_h(\Omega_{\rm S})$ instead of $\mathbb{H}_{h,0}(\Omega_{\rm S})$, and hence this inequality turns out to be equivalent to the following inf-sup condition:

$$\sup_{\tau_h\,\in\,\mathbf{H}_h(\Omega_{\rm S})\backslash\mathbf{0}} \frac{(\mathrm{div}\ \tau_h, v_h)_{\rm S}}{\|\tau_h\|_{\mathrm{div}\,,\Omega_{\rm S}}} \ \geq\ \beta_{\rm S}\,\|v_h\|_{0,\Omega_{\rm S}} \qquad \forall\,v_h\,\in\,L_h(\Omega_{\rm S})\,.$$

Notice also that a sufficient condition for (3.53) is the existence of $\boldsymbol{\psi}_0\,\in\,\mathbf{H}^{1/2}(\Sigma)$ such that $\boldsymbol{\psi}_0\,\in\,\boldsymbol{\Lambda}_h^{\rm S}(\Sigma)\quad\forall\,h$ and $\langle\boldsymbol{\psi}_0\cdot\mathbf{n}, 1\rangle_\Sigma \neq 0$. Consequently, we now introduce the following hypothesis summarizing the above analysis:

(H.1) *There exist* $\beta_{\rm S}\,,\ \beta_{\rm D}\ >\ 0$, *independent of* h, *and there exists* $\boldsymbol{\psi}_0\,\in\,\mathbf{H}^{1/2}(\Sigma)$, *such that*

$$\sup_{\tau_h\,\in\,\mathbf{H}_h(\Omega_{\rm S})\backslash\mathbf{0}} \frac{(\mathrm{div}\ \tau_h, v_h)_{\rm S}}{\|\tau_h\|_{\mathrm{div}\,,\Omega_{\rm S}}} \ \geq\ \beta_{\rm S}\,\|v_h\|_{0,\Omega_{\rm S}} \qquad \forall\,v_h\,\in\,L_h(\Omega_{\rm S})\,, \tag{3.54}$$

$$\sup_{\mathbf{v}_h\,\in\,\mathbf{H}_h(\Omega_{\rm D})\backslash\mathbf{0}} \frac{(\mathrm{div}\,\mathbf{v}_h, q_h)_{\rm D}}{\|\mathbf{v}_h\|_{\mathrm{div},\Omega_{\rm D}}} \ \geq\ \beta_{\rm D}\,\|q_h\|_{0,\Omega_{\rm D}} \qquad \forall\,q_h\,\in\,L_{h,0}(\Omega_{\rm D})\,, \tag{3.55}$$

$$\boldsymbol{\psi}_0\,\in\,\boldsymbol{\Lambda}_h^{\rm S}(\Sigma) \quad \forall\,h \quad \textit{and} \quad \langle\boldsymbol{\psi}_0\cdot\mathbf{n}, 1\rangle_\Sigma \neq 0\,. \tag{3.56}$$

On the other hand, we now look at the discrete kernel of $\mathcal{B}$, which is defined by

$$\mathbb{V}_h\ :=\ \left\{\,\underline{\boldsymbol{\tau}}_h\,\in\,\mathbb{X}_{h,0}:\quad \mathcal{B}(\underline{\boldsymbol{\tau}}_h, \underline{\mathbf{v}}_h)\ =\ 0 \quad \forall\,\underline{\mathbf{v}}_h\,\in\,\mathbb{M}_{h,0}\,\right\}.$$

In order to have a more explicit definition of $\mathbb{V}_h$ we introduce the following assumption:

(H.2) $\mathrm{div}\ \mathbf{H}_h(\Omega_{\rm S})\ \subseteq\ L_h(\Omega_{\rm S})$ *and* $\mathrm{div}\ \mathbf{H}_h(\Omega_{\rm D})\ \subseteq\ L_h(\Omega_{\rm D})$.

It follows from the definition of $\mathcal{B}$ (cf. (3.30)) and **(H.2)** that $\mathbb{V}_h = \mathbb{V}_{1,h}\times\mathbb{V}_{2,h}$, where

$$\mathbb{V}_{1,h}\ =\ \tilde{\mathbb{H}}_{h,0}(\Omega_{\rm S})\times\tilde{\mathbf{H}}_h(\Omega_{\rm D}) \quad\text{and}\quad \mathbb{V}_{2,h}\ =\ \tilde{\boldsymbol{\Lambda}}_h^{\rm S}(\Sigma)\times\Lambda_h^{\rm D}(\Sigma)\,,$$

with

$$\tilde{\mathbb{H}}_{h,0}(\Omega_S) := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_{h,0}(\Omega_S) : \quad \mathbf{div}\, \boldsymbol{\tau}_h = \mathbf{0} \Big\},$$

$$\tilde{\mathbf{H}}_h(\Omega_D) := \Big\{ \mathbf{v}_h \in \mathbf{H}_h(\Omega_D) : \quad \operatorname{div} \mathbf{v}_h \in \mathbb{P}_0(\Omega_D) \Big\},$$

and

$$\tilde{\boldsymbol{\Lambda}}_h^{S}(\Sigma) := \Big\{ \boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h^{S}(\Sigma) : \quad \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1 \rangle_\Sigma = 0 \Big\}.$$

Note that $\mathbb{V}_h \subseteq \mathbb{V}$, which yields in particular $\mathbb{V}_{1,h} \subseteq \mathbb{V}_1$.

Then, applying the same diagonal argument employed in the proof of Lemma 3.3.8, we find that $\mathbf{b}$ satisfies the discrete inf-sup condition uniformly on $\mathbb{V}_{1,h} \times \mathbb{V}_{2,h}$ if and only if there exist $\tilde{\beta}_S$, $\tilde{\beta}_D > 0$, independent of h, such that

$$\sup_{\boldsymbol{\tau}_h \in \tilde{\mathbb{H}}_{h,0}(\Omega_S)\setminus \mathbf{0}} \frac{\langle \boldsymbol{\tau}_h \, \mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\tau}_h\|_{\mathbf{div},\Omega_S}} \geq \tilde{\beta}_S \, \|\boldsymbol{\psi}_h\|_{1/2,\Sigma} \quad \forall\, \boldsymbol{\psi}_h \in \tilde{\boldsymbol{\Lambda}}_h^{S}(\Sigma), \tag{3.57}$$

$$\sup_{\mathbf{v}_h \in \tilde{\mathbf{H}}_h(\Omega_D)\setminus \mathbf{0}} \frac{\langle \mathbf{v}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma}{\|\mathbf{v}_h\|_{\operatorname{div},\Omega_D}} \geq \tilde{\beta}_D \, \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \in \Lambda_h^{D}(\Sigma). \tag{3.58}$$

In addition, the characterization of the elements of $\tilde{\boldsymbol{\Lambda}}_h^{S}(\Sigma)$ yields the supremum in (3.57) to remain unchanged if taken on $\tilde{\mathbb{H}}_h(\Omega_S) := \big\{ \boldsymbol{\tau}_h \in \mathbb{H}_h(\Omega_S) : \quad \mathbf{div}\ \boldsymbol{\tau}_h = \mathbf{0} \big\}$ instead of $\tilde{\mathbb{H}}_{h,0}(\Omega_S)$, and therefore it is not difficult to see that a sufficient condition for (3.57) is given by:

$$\sup_{\tau_h \in \tilde{\mathbf{H}}_h(\Omega_S)\setminus \mathbf{0}} \frac{\langle \tau_h \cdot \mathbf{n}, \psi_h \rangle_\Sigma}{\|\tau_h\|_{\operatorname{div},\Omega_S}} \geq \tilde{\beta}_S \, \|\psi_h\|_{1/2,\Sigma} \quad \forall\, \psi_h \in \Lambda_h^{S}(\Sigma),$$

where

$$\tilde{\mathbf{H}}_h(\Omega_S) := \big\{ \tau_h \in \mathbf{H}_h(\Omega_S) : \quad \operatorname{div}\ \tau_h = 0 \big\}.$$

In this way, we now add the following hypothesis:

(H.3) *There exist* $\tilde{\beta}_S$, $\tilde{\beta}_D > 0$, *independent of* h, *such that*

$$\sup_{\tau_h \in \tilde{\mathbf{H}}_h(\Omega_S)\setminus \mathbf{0}} \frac{\langle \tau_h \cdot \mathbf{n}, \psi_h \rangle_\Sigma}{\|\tau_h\|_{\operatorname{div},\Omega_S}} \geq \tilde{\beta}_S \, \|\psi_h\|_{1/2,\Sigma} \quad \forall\, \psi_h \in \Lambda_h^{S}(\Sigma), \tag{3.59}$$

$$\sup_{\mathbf{v}_h \in \tilde{\mathbf{H}}_h(\Omega_D)\setminus \mathbf{0}} \frac{\langle \mathbf{v}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma}{\|\mathbf{v}_h\|_{\operatorname{div},\Omega_D}} \geq \tilde{\beta}_D \, \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \in \Lambda_h^{D}(\Sigma). \tag{3.60}$$

We end this section by mentioning that for computational purposes we replace the Galerkin scheme (3.49) by the equivalent one arising from the utilization of the decomposition (3.50). In other words, we drop the explicit unknown approximating $\mu \in \mathbb{R}$ and keep it implicitly by redefining the approximation of the pseudostress $\boldsymbol{\sigma}_S \in \mathbb{H}(\mathbf{div}\,;\Omega_S)$ as an unknown in $\mathbb{H}_h(\Omega_S)$. This can also be seen as a discrete version of the reconstruction of $\boldsymbol{\sigma}_S$ from the decomposition

(3.21). In this way, the equivalent Galerkin scheme reduces to: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ such that

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) + \mathcal{B}(\underline{\boldsymbol{\tau}}_h, \underline{\mathbf{u}}_h) &= \mathcal{F}(\underline{\boldsymbol{\tau}}_h) \qquad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbb{X}_h \,, \\ \mathcal{B}(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{v}}_h) &= \mathcal{G}(\underline{\mathbf{v}}_h) \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbb{M}_h \,, \end{aligned} \tag{3.61}$$

where

$$\begin{aligned} \mathbb{X}_h &:= \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}_h(\Omega_{\mathrm{D}}) \times \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) \times \Lambda_h^{\mathrm{D}}(\Sigma) \,, \\ \mathbb{M}_h &:= \mathbf{L}_h(\Omega_{\mathrm{S}}) \times L_{h,0}(\Omega_{\mathrm{D}}) \,, \end{aligned} \tag{3.62}$$

and $\mathcal{B}$ is redefined by suppressing the third term on the right hand side of (3.30). The numerical results shown below in Section 3.6 consider precisely this scheme in which the mean value condition of $L_{h,0}(\Omega_{\mathrm{D}})$ is imposed through a Lagrange multiplier.

3.4.2 The main result

The following theorem establishes the well posedness of (3.49) and the associated Cea estimate.

Theorem 3.4.1 *Assume that the hypotheses* **(H.0)**, **(H.1)**, **(H.2)**, *and* **(H.3)** *hold. Then the Galerkin scheme* (3.49) *has a unique solution* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$ *and there exists* $C_1 > 0$, *independent of* h, *such that*

$$\|(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \le C_1 \left\{ \|\mathcal{F}|_{\mathbb{X}_{h,0}}\|_{\mathbb{X}'_{h,0}} + \|\mathcal{G}|_{\mathbb{M}_{h,0}}\|_{\mathbb{M}'_{h,0}} \right\}.$$

In addition, there exists $C_2 > 0$, *independent of* h, *such that*

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \le C_2 \left\{ \inf_{\underline{\boldsymbol{\tau}}_h \in \mathbb{X}_{h,0}} \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h\|_{\mathbb{X}} + \inf_{\underline{\mathbf{v}}_h \in \mathbb{M}_{h,0}} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|_{\mathbb{M}} \right\}, \tag{3.63}$$

where $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ *is the unique solution of* (3.32).

Proof. It is clear from the analysis in Section 3.4.1 that **(H.1)** (resp. **(H.3)**) implies the discrete inf-sup condition for $\mathcal{B}$ (resp. for $\mathbf{b}$) uniformly on $\mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$ (resp. on $\mathbb{V}_{1,h} \times \mathbb{V}_{2,h}$). In addition, the fact that $\mathbb{V}_{1,h} \subseteq \mathbb{V}_1$ and Lemma 3.3.7 imply that $\mathbf{a}$ is uniformly elliptic in $\mathbb{V}_{1,h}$, whereas $\mathbf{c}$ is trivially positive semidefinite on $\mathbb{V}_{2,h} \subseteq \mathbb{V}_2 \subseteq \mathbf{H}^{1/2}(\Sigma) \times H^{1/2}(\Sigma)$ (cf. (3.37)). In this way, applying the discrete version of Lemma 3.3.4 we conclude that the discrete operator induced by $\mathcal{A}$ is invertible in $\mathbb{V}_h$ with uniformly bounded inverse. Therefore, the rest of the proof reduces to a straightforward application of the discrete Babuška-Brezzi theory (see, e.g. [54, Theorem II.1.1], [19, Chapter II]). □

It is important to remark here that the second and third terms defining the bilinear form $\mathbf{c}$ are the only ones in the whole variational system where the Darcy and Stokes discrete spaces meet. However, it is also clear from the previous proof that these terms do not play any role in the stability analysis of the Galerkin scheme since $\mathbf{c}$ is already positive semidefinite in the whole space $\mathbf{H}^{1/2}(\Sigma) \times H^{1/2}(\Sigma)$. This fact also explains why each one of the hypotheses **(H.0)**, **(H.1)**, **(H.2)**, and **(H.3)**, is formed by independent conditions concerning the subspaces for the Stokes and Darcy domains separately. Nevertheless, we notice that these independent assumptions show analogue structures, particularly with respect to the kind of operators and continuous spaces involved: compare for instance (3.54) with (3.55) in **(H.1)** and (3.59) with (3.60) in **(H.3)**. This fact confirms the strong possibility of deriving stable finite element subspaces of the same kind in both domains. A specific example in this direction employing the well-known Raviart-Thomas subspaces is given precisely in Section 3.5 below.

Meanwhile, we prove next that the existence of uniformly bounded discrete liftings for the normal traces on Σ coming from both regions simplifies the statement of **(H.3)**.

3.4.3 Stable discrete liftings

The aim of this section is to give sufficient conditions for the inf-sup inequalities (3.59) and (3.60) in hypothesis **(H.3)**. These new conditions have to do with the eventual existence of stable discrete liftings of the normal traces on Σ, and they will be working hypotheses that can be more easily verified for each set of discrete spaces. In particular, these will be the conditions that we will verify for the example with Raviart–Thomas elements in Section 3.5.

We notice first that conditions (3.59) and (3.60) are hypotheses that deal with how the normal components of elements of $\tilde{\mathbf{H}}_h(\Omega_\mathrm{S})$ and $\tilde{\mathbf{H}}_h(\Omega_\mathrm{D})$ are tested with $\Lambda_h^\mathrm{S}(\Sigma)$ and $\Lambda_h^\mathrm{D}(\Sigma)$, respectively. Because of the already mentioned analogue structure of these assumptions, we summarize them as follows with $\star \in \{\mathrm{S}, \mathrm{D}\}$:

$$\sup_{\mathbf{v}_h \in \tilde{\mathbf{H}}_h(\Omega_\star) \setminus \mathbf{0}} \frac{\langle \mathbf{v}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma}{\|\mathbf{v}_h\|_{\mathrm{div},\Omega_\star}} \;\geq\; \tilde{\beta}_\star \, \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \,\in\, \Lambda_h^\star(\Sigma)\,. \tag{3.64}$$

This kind of condition can be broken into two pieces. Let us recall from Section 3.4.1 that

$$\begin{aligned} \tilde{\mathbf{H}}_h(\Omega_\mathrm{S}) \;&:=\; \Big\{ \mathbf{v}_h \,\in\, \mathbf{H}_h(\Omega_\mathrm{S}) : \quad \mathrm{div}\, \mathbf{v}_h \;=\; 0 \Big\}\,, \\ \tilde{\mathbf{H}}_h(\Omega_\mathrm{D}) \;&:=\; \Big\{ \mathbf{v}_h \,\in\, \mathbf{H}_h(\Omega_\mathrm{D}) : \quad \mathrm{div}\, \mathbf{v}_h \,\in\, \mathbb{P}_0(\Omega_\mathrm{D}) \Big\}\,, \end{aligned} \tag{3.65}$$

and for $\star \in \{\mathrm{S}, \mathrm{D}\}$ define

$$\Phi_h^\star(\Sigma) \;:=\; \{\, \mathbf{v}_h \cdot \mathbf{n}|_\Sigma : \quad \mathbf{v}_h \,\in\, \tilde{\mathbf{H}}_h(\Omega_\star)\,\}\,. \tag{3.66}$$

Assume that the linear operator $\mathbf{v}_h \mapsto \mathbf{v}_h \cdot \mathbf{n}$ from $\tilde{\mathbf{H}}_h(\Omega_\star)$ to $\Phi_h^\star(\Sigma)$ has a uniformly bounded right inverse, i.e., there exist a linear operator $\mathcal{L}_h^\star : \Phi_h^*(\Sigma) \to \tilde{\mathbf{H}}_h(\Omega_\mathrm{S})$ and $c_\star > 0$, independent of h, such that

$$\|\mathcal{L}_h^\star(\phi_h)\|_{\mathrm{div}\,,\Omega_\star} \;\leq\; c_\star \, \|\phi_h\|_{-1/2,\Sigma} \quad \text{and} \quad \mathcal{L}_h^\star(\phi_h) \cdot \mathbf{n} \;=\; \phi_h \quad \text{on} \quad \Sigma \quad \forall\, \phi_h \,\in\, \Phi_h^\star(\Sigma)\,. \tag{3.67}$$

Such a uniformly bounded right inverse of the normal trace will henceforth be referred to as a **stable discrete lifting** to $\Omega_\star$ ($\star \in \{\mathrm{S}, \mathrm{D}\}$). Note that by [36], existence of $\mathcal{L}_h^\star$ satisfying (3.67) is equivalent to existence of a Scott–Zhang type operator $\boldsymbol{\pi}_h^\star : \mathbf{H}(\mathrm{div}\,; \Omega_\star) \to \tilde{\mathbf{H}}_h(\Omega_\star)$, linear and uniformly bounded, such that

$$\boldsymbol{\pi}_h^*(\mathbf{v}_h) = \mathbf{v}_h \quad \forall\, \mathbf{v}_h \in \tilde{\mathbf{H}}_h(\Omega_\star)\,, \quad \text{and} \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Sigma$$

$$\Longrightarrow \; \big(\boldsymbol{\pi}_h^\star(\mathbf{v})\big)\cdot\mathbf{n} = 0 \quad \text{on} \quad \Sigma\,.$$

The following lemma reduces the inf-sup condition (3.64) to the inherited interaction between the elements of $\Phi_h^\star(\Sigma)$ and $\Lambda_h^\star(\Sigma)$.

Lemma 3.4.1 *Assume that there exists a stable discrete lifting to $\Omega_\star$. Then* (3.64) *is equivalent to the existence of $\tilde{\beta}_\star > 0$, independent of h, such that*

$$\sup_{\phi_h \in \Phi_h^\star(\Sigma)\backslash \mathbf{0}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \;\geq\; \tilde{\beta}_\star\, \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \in \Lambda_h^\star(\Sigma)\,. \tag{3.68}$$

Proof. It suffices to show that there exist C_1, $C_2 > 0$, independent of h, such that for each $\xi_h \in \Lambda_h^\star(\Sigma)$ there holds

$$C_1 \sup_{\phi_h \in \Phi_h^\star(\Sigma)\backslash \mathbf{0}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \leq \sup_{\mathbf{v}_h \in \tilde{\mathbf{H}}_h(\Omega_\star)\backslash \mathbf{0}} \frac{\langle \mathbf{v}_h\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_h\|_{\mathrm{div},\Omega_\star}} \leq C_2 \sup_{\phi_h \in \Phi_h^\star(\Sigma)\backslash \mathbf{0}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}}. \tag{3.69}$$

In fact, on the one hand

$$\frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \leq c_\star \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\mathcal{L}_h^\star(\phi_h)\|_{\mathrm{div}\,,\Omega_\star}} \leq c_\star \sup_{\mathbf{v}_h \in \tilde{\mathbf{H}}_h(\Omega_\star)\backslash \mathbf{0}} \frac{\langle \mathbf{v}_h\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_h\|_{\mathrm{div}\,,\Omega_\star}} \quad \forall\, \phi_h \in \Phi_h^\star(\Sigma)\,,$$

and on the other hand

$$\frac{\langle \mathbf{v}_h\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_h\|_{\mathrm{div}\,,\Omega_\star}} \leq C \frac{\langle \mathbf{v}_h\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_h\cdot\mathbf{n}\|_{-1/2,\Sigma}} \leq C \sup_{\phi_h \in \Phi_h^\star(\Sigma)\backslash \mathbf{0}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \quad \forall\, \mathbf{v}_h \in \tilde{\mathbf{H}}_h(\Omega_\star)\,,$$

which yield (3.69) with $C_1 = 1/c_\star$ and $C_2 = C$. □

We have thus proved that the existence of stable discrete liftings to Ω_S and Ω_D together with the inf-sup condition (3.68) constitute sufficient conditions for **(H.3)** to hold. To this respect, we find it important to emphasize that (3.68) deals exclusively with spaces of functions defined on the interface Σ.

3.5 A particular choice of discrete spaces

3.5.1 Discretization of the domains

Let $\mathcal{T}_h^\mathrm{S}$ and $\mathcal{T}_h^\mathrm{D}$ be respective triangulations of the domains Ω_S and Ω_D formed by shape-regular triangles in the usual conditions of the finite element literature. Assume that these triangulations match in Σ, so that $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. Let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$). Then, given a triangle T we consider the local Raviart–Thomas space of the lowest order

$$\mathrm{RT}_0(T) := \mathrm{span}\Big\{(1,0),\,(0,1),\,(x_1,x_2)\Big\}.$$

We then define one Raviart–Thomas space on each subdomain and their usual companion spaces of piecewise constant functions: for $\star \in \{\mathrm{S},\mathrm{D}\}$

$$\begin{aligned} \mathbf{H}_h(\Omega_\star) &:= \Big\{\mathbf{v}_h \in \mathbf{H}(\mathrm{div}\,;\Omega_\star):\quad \mathbf{v}_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star\Big\}, \\ L_h(\Omega_\star) &:= \Big\{q_h : \Omega_\star \to \mathbb{R}:\quad q_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star\Big\}. \end{aligned} \tag{3.70}$$

It is clear that **(H.0)** and **(H.2)** are satisfied and it is well known that so are the discrete inf-sup conditions (3.54) and (3.55) in **(H.1)** (see, e.g. [19, Chapter IV] or [69, Chapter 7]). Moreover, the spaces $\Phi_h^{\mathrm{S}}(\Sigma)$ and $\Phi_h^{\mathrm{D}}(\Sigma)$ of discrete normal traces on Σ (cf. (3.66)) are, for the time being, contained in

$$\Phi_h(\Sigma) := \Big\{\xi_h : \Sigma \to \mathbb{R}:\quad \xi_h|_e \in \mathbb{P}_0(e) \qquad \forall \text{ edge } e \in \Sigma_h\Big\}. \tag{3.71}$$

We will see later on, as a corollary of Lemma 3.5.1 below, that actually $\Phi_h^{\mathrm{S}}(\Sigma) = \Phi_h^{\mathrm{D}}(\Sigma) = \Phi_h(\Sigma)$.

Now, although we could keep our options open for the remaining spaces $\Lambda_h^{\mathrm{S}}(\Sigma)$ and $\Lambda_h^{\mathrm{D}}(\Sigma)$, we simplify the situation by taking

$$\Lambda_h^{\mathrm{S}}(\Sigma) = \Lambda_h^{\mathrm{D}}(\Sigma) = \Lambda_h(\Sigma).$$

Gathering Theorem 3.4.1 and Lemma 3.4.1 we are left with the following tasks:

i) prove the existence of stable discrete liftings (or give conditions on the grid that ensure their existence).

ii) choose $\Lambda_h(\Sigma)$ such that we can find $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}(\Sigma)$ satisfying $\boldsymbol{\psi}_0 \in \boldsymbol{\Lambda}_h(\Sigma)\ \forall\, h$ and $\langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, 1\rangle_\Sigma \neq 0$ (cf. (3.56) in **(H.1)**), and such that the inf–sup condition (3.68) holds.

In Sections 3.5.2 and 3.5.3 below we deal precisely with **i)** and **ii)**, respectively.

3.5.2 The discrete liftings

We are going to be able to construct discrete liftings to Ω_{S} and Ω_{D} by demanding some additional conditions on the triangulations. Namely, we ask for $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ to be quasiuniform in a neighborhood of Σ. More precisely, we assume that there is an open neighborhood of Σ, say Ω_Σ, with Lipschitz boundary, and such that the elements intersecting that region are roughly of the same size. In other words, for $\star \in \{\mathrm{S}, \mathrm{D}\}$ we let $\Omega_{\star,\Sigma} := \Omega_\star \cap \Omega_\Sigma$, define

$$\mathcal{T}_{h,\Sigma}^\star := \Big\{ T \in \mathcal{T}_h^\star : \quad T \cap \Omega_{\star,\Sigma} \neq \emptyset \Big\}, \qquad \mathcal{T}_{h,\Sigma} := \mathcal{T}_{h,\Sigma}^{\mathrm{S}} \cup \mathcal{T}_{h,\Sigma}^{\mathrm{D}},$$

and assume that there exists $c > 0$, independent of h, such that

$$\max_{T \in \mathcal{T}_{h,\Sigma}} h_T \leq c \min_{T \in \mathcal{T}_{h,\Sigma}} h_T .$$

Because of the shape-regularity property, this implies that Σ_h is quasiuniform, which means that there exists $C > 0$, independent of h, such that

$$h_\Sigma := \max\Big\{ |e| : \quad e \in \Sigma_h \Big\} \leq C \min\Big\{ |e| : \quad e \in \Sigma_h \Big\}.$$

Moreover, the quasiuniformity of Σ_h implies the inverse inequality in $\Phi_h(\Sigma)$, that is

$$\|\phi_h\|_{-1/2+\delta,\Sigma} \leq C\, h_\Sigma^{-\delta} \|\phi_h\|_{-1/2,\Sigma} \qquad \forall\, \phi_h \in \Phi_h(\Sigma), \qquad \forall\, \delta \in [0, 1/2]. \tag{3.72}$$

Next, in order to define the discrete liftings we need to introduce the Raviart–Thomas interpolation operator. For the forthcoming definitions and arguments $\star$ is a mute symbol taken in $\{\mathrm{S}, \mathrm{D}\}$. Hence, given a sufficiently smooth vector field $\mathbf{v} : \Omega_\star \to \mathbb{R}^2$, we define $\Pi_h^\star(\mathbf{v})$ as the only element of $\mathbf{H}_h(\Omega_\star)$ such that

$$\int_e \Pi_h^\star(\mathbf{v}) \cdot \mathbf{n} = \int_e \mathbf{v} \cdot \mathbf{n} \qquad \forall\, e \in \mathcal{E}_h^\star, \tag{3.73}$$

where $\mathcal{E}_h^\star$ is the set of the edges of the triangulation $\mathcal{T}_h^\star$. Let us review some properties of this operator that we will use in the sequel:

a) The interpolation operator $\Pi_h^\star$ is well defined in $\mathbf{H}^\delta(\Omega_\star) \cap \mathbf{H}(\mathrm{div}\,;\Omega_\star)$ for any $\delta > 0$ (see, e.g. [7, Theorem 3.1]).

b) For all $\mathbf{v}$ there holds $\mathrm{div}\, \Pi_h^\star(\mathbf{v}) = \mathcal{P}_h^\star(\mathrm{div}\, \mathbf{v})$, where $\mathcal{P}_h^\star : L^2(\Omega_\star) \to L_h(\Omega_\star)$ is the orthogonal projector. Equivalently

$$(\mathrm{div}\, \Pi_h^\star(\mathbf{v}), q_h)_\star = (\mathrm{div}\, \mathbf{v}, q_h)_\star \qquad \forall\, q_h \in L_h(\Omega_\star).$$

This property is a simple consequence of the divergence theorem and the interpolation property (3.73) defining $\Pi_h^\star$. In particular, if $\mathrm{div}\, \mathbf{v} \equiv c$, it follows that $\mathrm{div}\, \Pi_h^\star(\mathbf{v}) \equiv c$.

c) If $\mathbf{v} \cdot \mathbf{n} \in \Phi_h(\Sigma)$ then $\Pi_h^\star(\mathbf{v}) \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$. This property also follows from (3.73).

d) There exists $C > 0$, independent of h, such that for each $\mathbf{v} \in \mathbf{H}^\delta(\Omega_\star) \cap \mathbf{H}(\mathrm{div}\,;\Omega_\star)$, with $\delta \in (0,1]$, and for all $T \in \mathcal{T}_h^\star$, there holds (see, e.g. [58, Theorem 3.16])

$$\|\mathbf{v} - \Pi_h^*(\mathbf{v})\|_{0,T} \;\le\; C\, h_T^\delta \left\{ |\mathbf{v}|_{\delta,T} + \|\mathrm{div}\;\mathbf{v}\|_{0,T} \right\}. \tag{3.74}$$

We are now in a position to establish the existence of stable discrete liftings.

Lemma 3.5.1 *Assume that* $\mathcal{T}_h^{\mathrm{S}}$ *and* $\mathcal{T}_h^{\mathrm{D}}$ *are quasiuniform in a neighborhood* Ω_Σ *of* Σ *as explained in the present section. Then there exist uniformly bounded linear operators* $\mathcal{L}_h^\star : \Phi_h(\Sigma) \to \tilde{\mathbf{H}}_h(\Omega_\star)$ *(cf.* (3.65)) *such that* $\mathcal{L}_h^\star(\phi_h) \cdot \mathbf{n} = \phi_h$ *on* Σ *for each* $\phi_h \in \Phi_h(\Sigma)$.

Proof. We start with the lifting to the Stokes domain Ω_{S}. First of all we increase this region across the external boundary Γ_{S} to a new domain $\Omega_{\mathrm{S}}^{\mathrm{ext}}$ with Lipschitz boundary $\Sigma \cup \Gamma_{\mathrm{S}}^{\mathrm{ext}}$. Then we recall that $\Omega_{\mathrm{S},\Sigma} := \Omega_{\mathrm{S}} \cap \Omega_\Sigma$ and remark that $\Omega_{\mathrm{S}} \setminus \Omega_{\mathrm{S},\Sigma}$ is interior to $\Omega_{\mathrm{S}}^{\mathrm{ext}}$, since both parts of its boundary lie at a nonzero distance of $\partial\Omega_{\mathrm{S}}^{\mathrm{ext}}$. We refer to Figure 3.2 for the geometry. The thick lines enclose the extended Stokes domain $\Omega_{\mathrm{S}}^{\mathrm{ext}}$, whereas the shaded area corresponds to the neighborhood Ω_Σ.

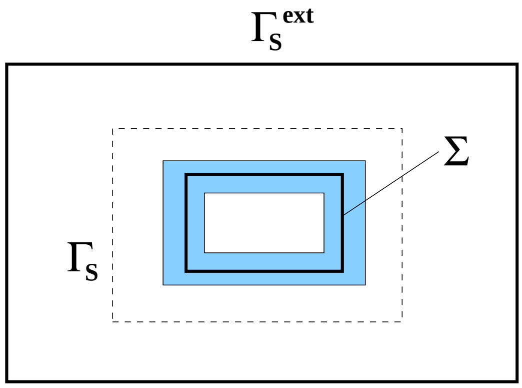

Figure 3.2: The domains in the proof of Lemma 3.5.1.

We now begin the construction of our operator. Given $\phi_h \in \Phi_h(\Sigma)$, we let $v \in H^1(\Omega_{\mathrm{S}}^{\mathrm{ext}})$ be the unique solution of the boundary value problem

$$\Delta v = 0 \quad \text{in} \quad \Omega_{\mathrm{S}}^{\mathrm{ext}}, \quad v = 0 \quad \text{on} \quad \Gamma_{\mathrm{S}}^{\mathrm{ext}}, \quad \partial_{\mathbf{n}} v = \phi_h \quad \text{on} \quad \Sigma,$$

which can be seen as a discrete version of (3.41). Then, as a consequence of the Lax–Milgram lemma and the classical regularity result on polygonal domains (see, e.g. [55]), we obtain, respectively, the following continuity bounds (we write them in the domains where they will be used):

$$\|v\|_{1,\Omega_{\mathrm{S}}} \;\le\; C_1\, \|\phi_h\|_{-1/2,\Sigma}\,, \tag{3.75}$$

$$\|v\|_{5/4,\Omega_{\mathrm{S}}} \;\le\; C_2\, \|\phi_h\|_{-1/4,\Sigma}\,. \tag{3.76}$$

In addition, since $\Omega_{\rm S}\setminus\Omega_{{\rm S},\Sigma}$ is an interior region of $\Omega_{\rm S}^{\rm ext}$, the interior elliptic regularity estimate (see, e.g. [66, Theorem 4.16]) yields

$$\|v\|_{2,\Omega_{\rm S}\setminus\Omega_{{\rm S},\Sigma}} \,\le\, C_3\,\|\phi_h\|_{-1/2,\Sigma}\,, \tag{3.77}$$

Note that in particular $\nabla v\,\in\,\mathbf{H}^{1/4}(\Omega_{\rm S})\,\cap\,\mathbf{H}(\mathrm{div}\,;\Omega_{\rm S})$, and hence, thanks to **a)**, we can define

$$\mathcal{L}_h^{\rm S}(\phi_h)\,:=\,\Pi_h^{\rm S}(\nabla v)\,\in\,\mathbf{H}_h(\Omega_{\rm S})\,.$$

Since $\mathrm{div}\,\nabla v\,=\,\Delta v\,=\,0\quad\text{in}\quad\Omega_{\rm S}$ and $\nabla v\cdot\mathbf{n}\,=\,\partial_{\mathbf{n}}v\,=\,\phi_h\,\in\,\Phi_h(\Sigma)\quad\text{on}\quad\Sigma$, we deduce from **b)** and **c)**, respectively, that

$$\mathrm{div}\,\mathcal{L}_h^{\rm S}(\phi_h)\,=\,0\quad\text{in}\quad\Omega_{\rm S}\quad\text{and}\quad\mathcal{L}_h^{\rm S}(\phi_h)\cdot\mathbf{n}\,=\,\phi_h\quad\text{on}\quad\Sigma\,,$$

which proves that $\mathcal{L}_h^{\rm S}$ is a lifting satisfying $\mathcal{L}_h^{\rm S}(\phi_h)\,\in\,\tilde{\mathbf{H}}_h(\Omega_{\rm S})\quad\forall\,\phi_h\,\in\,\Phi_h(\Sigma)$.

It remains to show that $\mathcal{L}_h^{\rm S}$ is uniformly bounded. To this end, we divide $\Omega_{\rm S}$ into two regions

$$\Omega_{{\rm S},h}^1\,:=\,\cup\Big\{\,T\,\in\,\mathcal{T}_h^{\rm S}:\quad T\,\notin\,\mathcal{T}_{h,\Sigma}^{\rm S}\,\Big\}\,\subseteq\,\Omega_{\rm S}\setminus\Omega_{{\rm S},\Sigma}\quad\text{and}\quad\Omega_{{\rm S},h}^2\,:=\,\Omega_{\rm S}\setminus\Omega_{{\rm S},h}^1\,,$$

where we recall that $\mathcal{T}_{h,\Sigma}^{\rm S}\,:=\,\Big\{T\,\in\,\mathcal{T}_h^{\rm S}:\quad T\,\cap\,\Omega_{{\rm S},\Sigma}\,\neq\,\emptyset\Big\}$. Then, using (3.75), (3.77), and the stability of the Raviart–Thomas projection when applied to functions in $\mathbf{H}^1(\Omega_{{\rm S},h}^1)$, we can bound:

$$\begin{aligned}
\|\mathcal{L}_h^{\rm S}(\phi_h)\|_{\mathrm{div}\,,\Omega_{\rm S}} &= \|\mathcal{L}_h^{\rm S}(\phi_h)\|_{0,\Omega_{\rm S}}\,\le\,\|\mathcal{L}_h^{\rm S}(\phi_h)\|_{0,\Omega_{{\rm S},h}^1}\,+\,\|\mathcal{L}_h^{\rm S}(\phi_h)\|_{0,\Omega_{{\rm S},h}^2}\\
&\le \|\Pi_h^{\rm S}(\nabla v)\|_{0,\Omega_{{\rm S},h}^1}\,+\,\|\nabla v\|_{0,\Omega_{{\rm S},h}^2}\,+\,\|\nabla v\,-\,\Pi_h^{\rm S}(\nabla v)\|_{0,\Omega_{{\rm S},h}^2}\\
&\le C\Big\{\,\|\nabla v\|_{1,\Omega_{\rm S}\setminus\Omega_{{\rm S},\Sigma}}\,+\,\|\phi_h\|_{-1/2,\Sigma}\,+\,\|\nabla v-\Pi_h^{\rm S}(\nabla v)\|_{0,\Omega_{{\rm S},h}^2}\,\Big\}\\
&\le C\Big\{\,\|\phi_h\|_{-1/2,\Sigma}\,+\,\|\nabla v\,-\,\Pi_h^{\rm S}(\nabla v)\|_{0,\Omega_{{\rm S},h}^2}\,\Big\}\,.
\end{aligned}$$

At the same time, applying (3.74) in **d)** to $\nabla v\,\in\,\mathbf{H}^{1/4}(\Omega_{\rm S})\,\cap\,\mathbf{H}(\mathrm{div}\,;\Omega_{\rm S})$, and employing the bound (3.76) and the inverse inequality (3.72) with $\delta\,=\,1/4$, we find that

$$\begin{aligned}
\|\nabla v\,-\,\Pi_h^{\rm S}(\nabla v)\|_{0,\Omega_{{\rm S},h}^2}^2 &\le C\sum_{T\,\in\,\mathcal{T}_{h,\Sigma}^{\rm S}} h_T^{1/2}\,\|\nabla v\|_{1/4,T}^2\,\le\,C\,h_\Sigma^{1/2}\,\|v\|_{5/4,\Omega_{\rm S}}^2\\
&\le C\,h_\Sigma^{1/2}\,\|\phi_h\|_{-1/4,\Sigma}^2\,\le\,C\,\|\phi_h\|_{-1/2,\Sigma}^2\,.
\end{aligned}$$

This estimate and the preceeding inequality give the result.

On the other hand, in the case of the Darcy domain $\Omega_{\rm D}$, the interface Σ constitutes the whole boundary, which implies that $\Omega_{\rm D}\setminus\Omega_{{\rm D},\Sigma}$ is interior to $\Omega_{\rm D}$, and hence there is no need to extend the domain to deal with regularity issues in the (non existent) remaining part of the boundary. According to this, given $\phi_h\,\in\,\Phi_h(\Sigma)$, we now define

$$\mathcal{L}_h^{\rm D}(\phi_h)\,:=\,\Pi_h^{\rm D}(\nabla v)\,\in\,\mathbf{H}_h(\Omega_{\rm D})\,,$$

where $v \in H^1(\Omega_D)$ is the unique solution of the bounday value problem

$$\Delta v = \frac{1}{|\Omega_D|}\int_\Sigma \phi_h \quad \text{in} \quad \Omega_D\,, \quad \partial_{\mathbf{n}} v = \phi_h \quad \text{on} \quad \Sigma\,, \quad \int_{\Omega_D} v = 0\,,$$

which can be seen as a discrete version of (3.42). Since

$$\text{div}\,\nabla v = \Delta v = \frac{1}{|\Omega_D|}\int_\Sigma \phi_h \quad \text{in} \quad \Omega_D \quad \text{and} \quad \nabla v\cdot\mathbf{n} = \partial_{\mathbf{n}} v = \phi_h \in \Phi_h(\Sigma) \quad \text{on} \quad \Sigma\,,$$

we use again **b)** and **c)** to deduce, respectively, that

$$\text{div}\,\mathcal{L}_h^{D}(\phi_h) = \frac{1}{|\Omega_D|}\int_\Sigma \phi_h \in \mathbb{R} \quad \text{in} \quad \Omega_S \quad \text{and} \quad \mathcal{L}_h^{D}(\phi_h)\cdot\mathbf{n} = \phi_h \quad \text{on} \quad \Sigma\,,$$

which proves that $\mathcal{L}_h^D$ is a lifting satisfying $\mathcal{L}_h^D(\phi_h) \in \tilde{\mathbf{H}}_h(\Omega_D) \quad \forall\,\phi_h \in \Phi_h(\Sigma)$. The uniform boundedness of $\mathcal{L}_h^D$ proceeds as in the previous case. We omit further details. □

As a consequence of this lemma, and as already announced in Section 3.5.1, we now notice that $\Phi_h^S(\Sigma)$ and $\Phi_h^D(\Sigma)$ coincide with $\Phi_h(\Sigma)$ (cf. (3.71)), and therefore the inf-sup condition (3.68) reduces simply to the existence of $\tilde{\beta} > 0$, independent of h, such that

$$\sup_{\phi_h \in \Phi_h(\Sigma)\backslash \mathbf{0}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \geq \tilde{\beta}\,\|\xi_h\|_{1/2,\Sigma} \qquad \forall\,\xi_h \in \Lambda_h(\Sigma)\,. \tag{3.78}$$

3.5.3 Discretization on the interface

In this section we discuss on how to choose $\Lambda_h(\Sigma)$ so that **ii)** be satisfied. In fact, there are many possible choices of $\Lambda_h(\Sigma)$ such that (3.78) holds, while the condition requiring the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}(\Sigma)$ such that $\boldsymbol{\psi}_0 \in \Lambda_h(\Sigma) \quad \forall\,h$ and $\langle \boldsymbol{\psi}_0\cdot\mathbf{n}, 1\rangle_\Sigma \neq 0$, is easy to verify if the sequence of subspaces is nested or if we are able to find a coarser space where the hypotheses hold.

<u>OPTION 1</u>. If the partition Σ_h inherited from the interior triangulations is *uniform*, which is feasible only on very simple geometries Σ, we can take $\Lambda_h(\Sigma)$ to be the space of continuous linear elements of the dual grid, that is, on the grid whose nodes are the midpoints of Σ_h. Note that $\dim \Lambda_h(\Sigma) = \dim \Phi_h(\Sigma)$, and that on each corner of Σ there is an element of the dual grid with half of its length on each of the edges that meet in that corner. The inf-sup condition (3.78) for these spaces is verified in [70, Lemma 6.4].

<u>OPTION 2</u>. Let $\widetilde{\Sigma}_h$ be another partition of Σ, completely independent from Σ_h, and take now

$$\Lambda_h(\Sigma) := \mathbb{P}_1(\widetilde{\Sigma}_h) \cap \mathcal{C}(\Sigma), \quad \text{with} \quad \mathbb{P}_1(\widetilde{\Sigma}_h) := \prod_{e\in\widetilde{\Sigma}_h} \mathbb{P}_1(e)\,.$$

If both Σ_h and $\widetilde{\Sigma}_h$ are quasiuniform, then there exists a constant $C_0 \in (0,1]$ such that whenever

$$h_\Sigma \leq C_0\,\widetilde{h}_\Sigma\,, \qquad \widetilde{h}_\Sigma := \max\{\,|\widetilde{e}| : \quad \widetilde{e} \in \widetilde{\Sigma}_h\,\}\,,$$

then (3.78) holds [14, Lemma 3.3]. In this case, if we assume that elements of $\widetilde{\Sigma}_h$ are segments (no element crosses a corner point), then $\boldsymbol{\psi}_0$ can be constructed exactly as explained at the end of the proof of Lemma 3.3.6.

OPTION 3. A very flexible (from the geometric point of view) construction of $\Lambda_h(\Sigma)$ can be done using a coarsened grid. Let us first assume that the number of edges of Σ_h is an even number (we will show a simple strategy in case this number is odd at the end). Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent elements and define

$$\Lambda_h(\Sigma) := \mathbb{P}_1(\Sigma_{2h}) \cap \mathcal{C}(\Sigma). \tag{3.79}$$

Note that because Σ_h is inherited from the interior triangulation, it is automatically of bounded variation (that is, the ratio of lengths of adjacent elements is bounded) and, therefore, so is Σ_{2h}.

Lemma 3.5.2 *The inf-sup condition* (3.78) *holds for the space* (3.79).

Proof. We will actually prove an inequality that is more demanding than (3.78) (see (3.81) below). The structure of the proof (but not the result itself) follows closely [70, Section 7]. Let $\Sigma_{2h} = \{e_i \,|\, i = 1, \ldots, N\}$ be a numbering of the elements of the coarsened grid, where adjacent elements are numbered consecutively and where, in case of need $e_0 = e_N$ and $e_1 = e_{N+1}$. Let also $h_i := |e_i|$. To each pair (e_i, e_{i+1}) we assign a hat function $\eta_i \in \Lambda_h(\Sigma)$, supported in this pair and equal to one in the interior node $\overline{e}_i \cap \overline{e_{i+1}}$. Note that $\{\eta_1, \eta_2, ..., \eta_N\}$ is the usual basis of $\Lambda_h(\Sigma)$.

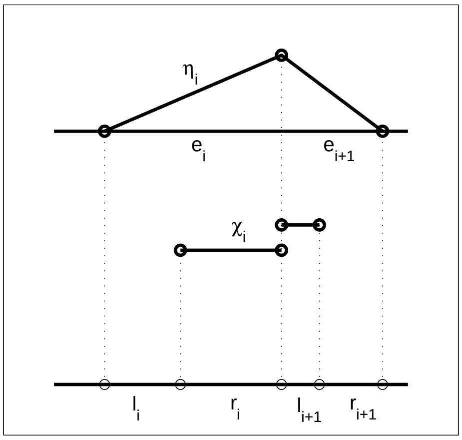

Figure 3.3: Construction of the basis functions of Φ_h°.

For each $e_i \in \Sigma_{2h}$ there are two elements $l_i, r_i \in \Sigma_h$, whose union is e_i. They are tagged as left and right in the numbering direction of Σ_{2h}, so that r_i is adjacent to l_{i+1} (see Figure 3.3). As a consequence of the bounded variation property

$$0 < C_1 \le c_i := \frac{|r_i|}{h_i} \le C_2 < 1 \qquad \text{and} \qquad 0 < C_3 \le \frac{h_i}{h_{i+1}} \le C_4 \qquad \forall i. \tag{3.80}$$

We now define the piecewise constant function $\chi_i \in \Phi_h(\Sigma)$ given by

$$\chi_i = \begin{cases} c_i^{-1} & \text{in } r_i, \\ (1-c_{i+1})^{-1} & \text{in } l_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

The functions χ_i are mutually orthogonal in $L^2(\Sigma)$. We define

$$\Phi_h^\circ := \operatorname{span}\{\chi_1, \ldots, \chi_N\} \subset \Phi_h(\Sigma).$$

The aim of what follows is showing that there exists C (that depends only on the four constants in (3.80)) such that

$$\sup_{\phi_h \in \Phi_h^\circ \setminus \mathbf{0}} \frac{\langle \xi_h, \phi_h \rangle_\Sigma}{\|\phi_h\|_{-s,\Sigma}} \ge C \|\xi_h\|_{s,\Sigma} \qquad \forall \xi \in \Lambda_h(\Sigma), \qquad s \in [0,1]. \tag{3.81}$$

We will prove the result for $s = 0$ and $s = 1$. Given the fact that the dimensions of $\Lambda_h(\Sigma)$ and Φ_h° coincide, an interpolation argument proves the result for the remaining cases. The case $s = 1/2$ implies (3.78), since the supremum in this last inequality is taken over a larger space.

1st step. We first prove (3.81) for $s = 0$. Here we follow [70, Proposition 7.1]. Let us define the operator $T_h : \Lambda_h(\Sigma) \to \Phi_h^\circ$ by

$$T_h \xi_h = T_h\Big(\sum_{i=1}^N \xi_i \eta_i\Big) := \sum_{i=1}^N \xi_i \chi_i.$$

Simple computations can be used to show that for all $\xi_h \in \Lambda_h(\Sigma)$

$$\begin{aligned} \|\xi_h\|_{0,\Sigma}^2 &\le \frac{1}{2} \sum_{i=1}^N \xi_i^2 (h_i + h_{i+1}), \\ \|T_h \xi_h\|_{0,\Sigma}^2 &= \sum_{i=1}^N \xi_i^2 \left(c_i^{-1} h_i + (1-c_{i+1})^{-1} h_{i+1}\right) \le C \sum_{i=1}^N \xi_i^2 (h_i + h_{i+1}) \\ \langle \xi_h, T_h \xi_h \rangle_\Sigma &\ge \frac{1}{2} \sum_{i=1}^N \xi_i^2 \left(h_i(\tfrac{3}{2} - c_i) + h_{i+1}(\tfrac{1}{2} + c_{i+1})\right) \ge C \sum_{i=1}^N \xi_i^2 (h_i + h_{i+1}). \end{aligned}$$

These three inequalities can be used to prove (3.81) when $s = 0$. Note that only the constants C_1 and C_2 of (3.80) are involved in these bounds.

2nd step. An intermediate step requires proving the following inequality:

$$\sum_{i=1}^{N} h_i^2 \int_{e_i} |\phi_h|^2 \leq C \sum_{i=1}^{N} \left(\frac{\langle \phi_h, \eta_i \rangle_\Sigma}{\|\eta_i\|_{1,\Sigma}} \right)^2 \qquad \forall \phi_h \in \Phi_h^\circ. \tag{3.82}$$

The proof retraces the steps of [70, Lemma 7.2]. For integrals of Σ we can use the arc parameterization $\mathbf{x} : [0, |\Sigma|] \to \Sigma$, where $|\Sigma|$ is the length of Σ, and identify

$$\|\eta\|_{1,\Sigma}^2 := \int_0^{|\Sigma|} \Big(|(\eta \circ \mathbf{x})(t)|^2 + |(\eta \circ \mathbf{x})'(t)|^2 \Big) \, \mathrm{d}t.$$

Each of the following inequalities, valid for each η_i and for arbitrary $\phi_h = \sum_{i=1}^N \phi_i \chi_i \in \Phi_h^\circ$, is easy to prove:

$$\begin{aligned}
\|\eta_i\|_{1,\Sigma}^2 &= \frac{1}{3}(h_i + h_{i+1}) + h_i^{-1} + h_{i+1}^{-1} \leq C h_i^{-1}, \\
\sum_{i=1}^{N} h_i^2 \int_{e_i} |\phi_h|^2 &= \sum_{i=1}^{N} h_i^3 \left(c_i^{-1} \phi_i^2 + (1 - c_i)^{-1} \phi_{i-1}^2 \right) \leq C \sum_{i=1}^{N} \phi_i^2 h_i^3, \\
\sum_{i=1}^{N} \phi_i^2 h_i^3 &\leq C \sum_{i=1}^{N} \langle \phi_h, \eta_i \rangle_\Sigma \, h_i^2 \phi_i \\
&\leq C \left(\sum_{i=1}^{N} \left(\frac{\langle \phi_h, \eta_i \rangle_\Sigma}{\|\eta_i\|_{1,\Sigma}} \right)^2 \right)^{1/2} \left(\sum_{i=1}^{N} \|\eta_i\|_{1,\Sigma}^2 h_i^4 \phi_i^2 \right)^{1/2}.
\end{aligned}$$

In particular, note that the second estimate uses that $|r_i| = c_i h_i$ and $|l_i| = (1 - c_i) h_i$. From these inequalities the result follows readily.

3rd step. Once (3.82) has been proved, inequality (3.81) for $s = 1$ can be proved following step by step the proof of [70, Proposition 7.3]. This finishes the proof of the Lemma. □

If the number of elements in Σ_h is odd we simply reduce it to the even case. Indeed, in this case we can prove (3.78) for the subspace of $\Phi_h(\Sigma)$ consisting of elements such that the value of ϕ_h in a fixed set of two adjacent elements coincides. This fixed double element is considered as a single element and hence $\Lambda_h(\Sigma)$ is built as in (3.79) on the resulting even number of elements covering Σ.

3.5.4 The main results

As a consequence of the results and analyses in Sections 3.5.1, 3.5.2, and 3.5.3, we can establish the following theorems.

Theorem 3.5.1 *Let $\mathbf{H}_h(\Omega_S)$, $\mathbf{H}_h(\Omega_D)$, $L_h(\Omega_S)$, and $L_h(\Omega_D)$ be the Raviart-Thomas finite element subspaces given in* (3.70) *and define*

$$\begin{aligned}
\mathbb{H}_h(\Omega_S) &:= \{ \boldsymbol{\tau} : \Omega_S \to \mathbb{R}^{2\times 2} : \quad \mathbf{c}^t \boldsymbol{\tau} \in \mathbf{H}_h(\Omega_S) \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \}\,,\\
\mathbb{H}_{h,0}(\Omega_S) &:= \mathbb{H}_h(\Omega_S) \cap \mathbb{H}_0(\mathbf{div}\,;\Omega_S)\,,\\
\mathbf{L}_h(\Omega_S) &:= L_h(\Omega_S) \times L_h(\Omega_S)\,,\\
L_{h,0}(\Omega_D) &:= L_h(\Omega_D) \cap L^2_0(\Omega_D)\,.
\end{aligned}$$

Assume that $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ are quasiuniform in a neighborhood of Σ and that $\Lambda_h(\Sigma)$ (and hence $\boldsymbol{\Lambda}_h(\Sigma) := \Lambda_h(\Sigma)\times\Lambda_h(\Sigma)$) is given by any of the three options described above. Then the Galerkin scheme (3.49) *with the discrete spaces $\mathbb{X}_{h,0} := \mathbb{H}_{h,0}(\Omega_S)\times\mathbf{H}_h(\Omega_D)\times\boldsymbol{\Lambda}_h(\Sigma)\times\Lambda_h(\Sigma)$ and $\mathbb{M}_{h,0} := \mathbf{L}_h(\Omega_S) \times L_{h,0}(\Omega_D) \times \mathbb{R}$, has a unique solution $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$, which satisfies the corresponding stability and Cea estimates.*

Proof. It follows by gathering the results from Sections 3.4 and 3.5. □

Theorem 3.5.2 *Assume the same hypotheses of Theorem* 3.5.1*. Then the Galerkin scheme* (3.61) *with the spaces $\mathbb{X}_h := \mathbb{H}_h(\Omega_S)\times\mathbf{H}_h(\Omega_D)\times\boldsymbol{\Lambda}_h(\Sigma)\times\Lambda_h(\Sigma)$ and $\mathbb{M}_h := \mathbf{L}_h(\Omega_S)\times L_{h,0}(\Omega_D)$, has a unique solution $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$, which satisfies the corresponding stability and Cea estimates.*

Proof. It follows from Theorem 3.5.1 and the equivalence between (3.49) and (3.61). □

In order to provide the rate of convergence of the Galerkin scheme (3.49), we now recall the approximation properties of the subspaces involved (see, e.g. [13], [19], [58]):

(AP1) For $\star \in \{S, D\}$, for each $\delta \in (0,1]$, and for each $\tau \in \mathbf{H}^\delta(\Omega_\star)$ with $\mathrm{div}\,\tau \in H^\delta(\Omega_\star)$, there exists $\tau_h \in \mathbf{H}_h(\Omega_\star)$ such that

$$\|\tau - \tau_h\|_{\mathrm{div},\Omega_\star} \le C\,h^\delta \left\{ \|\tau\|_{\delta,\Omega_\star} + \|\mathrm{div}\;\tau\|_{\delta,\Omega_\star} \right\}.$$

(AP2) For $\star \in \{S, D\}$, for each $\delta \in [0,1]$, and for each $q \in L^2(\Omega_\star)$, there exists $q_h \in L_h(\Omega_\star)$ such that

$$\|q - q_h\|_{0,\Omega_\star} \le C\,h^\delta\,\|q\|_{\delta,\Omega_\star}\,.$$

(AP3) For each $\delta \in [0,1]$ and for each $\xi \in H^{1/2+\delta}(\Sigma)$, there exists $\xi_h \in \Lambda_h(\Sigma)$ such that

$$\|\xi - \xi_h\|_{1/2,\Sigma} \le C\,h^\delta\,\|\xi\|_{1/2+\delta,\Sigma}\,.$$

The following theorem provides the theoretical rate of convergence of the Galerkin scheme (3.49) (equivalently (3.61)), under suitable regularity assumptions on the exact solution.

Theorem 3.5.3 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}_0 \times \mathbb{M}_0$ *and* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_{h,0} \times \mathbb{M}_{h,0}$ *be the unique solutions of the continuous and discrete formulations* (3.32) *and* (3.49)*, respectively. Assume that there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_{\rm S} \in \mathbb{H}^\delta(\Omega_{\rm S})$, $\mathbf{div}\ \boldsymbol{\sigma}_{\rm S} \in \mathbf{H}^\delta(\Omega_{\rm S})$, $\mathbf{u}_{\rm D} \in \mathbf{H}^\delta(\Omega_{\rm D})$, *and* $\mathrm{div}\ \mathbf{u}_{\rm D} \in H^\delta(\Omega_{\rm D})$*. Then,* $\mathbf{u}_{\rm S} \in \mathbf{H}^{1+\delta}(\Omega_{\rm S})$, $p_{\rm D} \in H^{1+\delta}(\Omega_{\rm D})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\lambda \in H^{1/2+\delta}(\Sigma)$, *and there exists* $C > 0$*, independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned} \|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) - (\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \le C\, h^\delta \Big\{ \|\boldsymbol{\sigma}_{\rm S}\|_{\delta,\Omega_{\rm S}} + \|\mathbf{div}\ \boldsymbol{\sigma}_{\rm S}\|_{\delta,\Omega_{\rm S}} \\ + \|\mathbf{u}_{\rm D}\|_{\delta,\Omega_{\rm D}} + \|\mathrm{div}\ \mathbf{u}_{\rm D}\|_{\delta,\Omega_{\rm D}} + \|\mathbf{u}_{\rm S}\|_{1+\delta,\Omega_{\rm S}} + \|p_{\rm D}\|_{1+\delta,\Omega_{\rm D}} \Big\}. \end{aligned} \tag{3.83}$$

Proof. We first recall from Theorem 3.2.1 that $\nabla \mathbf{u}_{\rm S} = \nu^{-1}\, \boldsymbol{\sigma}_{\rm S}^d$ and $\nabla p_{\rm D} = -\mathbf{K}^{-1}\, \mathbf{u}_{\rm D}$, which implies that $\mathbf{u}_{\rm S} \in \mathbf{H}^{1+\delta}(\Omega_{\rm S})$ and $p_{\rm D} \in H^{1+\delta}(\Omega_{\rm D})$, whence $\boldsymbol{\varphi} = -\mathbf{u}_{\rm S}|_\Sigma \in \mathbf{H}^{1/2+\delta}(\Sigma)$ and $\lambda = p_{\rm D}|_\Sigma \in H^{1/2+\delta}(\Sigma)$. The rest of the proof follows from the corresponding Cea estimate, the above approximation properties, and the fact that, thanks to the trace theorem in $\Omega_{\rm S}$ and $\Omega_{\rm D}$, respectively, there holds

$$\|\boldsymbol{\varphi}\|_{1/2+\delta,\Sigma} \le c\, \|\mathbf{u}_{\rm S}\|_{1+\delta,\Omega_{\rm S}} \quad \text{and} \quad \|\lambda\|_{1/2+\delta,\Sigma} \le c\, \|p_{\rm D}\|_{1+\delta,\Omega_{\rm D}}\,.$$

□

We end this section by commenting that one should be able to extend the analysis of Section 3.5, without difficulties, to the case of Raviart-Thomas finite element subspaces of higher order. In this case, given $k \ge 1$, $\mathrm{RT}_0(T)$ is replaced by $\mathrm{RT}_k(T) := [\mathbb{P}_k(T)]^2 \oplus \mathbb{P}_k(T) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $\Lambda_h(\Sigma)$ is defined in terms of piecewise polynomials of degre $2\,k+1$.

3.6 Numerical results

In this section we present three examples illustrating the performance of the Galerkin scheme (3.61) (equivalently (3.49)) with the subspaces $\mathbb{X}_h := \mathbb{H}_h(\Omega_{\rm S}) \times \mathbf{H}_h(\Omega_{\rm D}) \times \boldsymbol{\Lambda}_h(\Sigma) \times \Lambda_h(\Sigma)$ and $\mathbb{M}_h := \mathbf{L}_h(\Omega_{\rm S}) \times L_{h,0}(\Omega_{\rm D})$ defined in Section 3.5. In particular, we adopt the third option from Section 3.5.3 to choose the space $\Lambda_h(\Sigma)$ of continuous piecewise linear functions on Σ.

We now introduce additional notations. The variable N stands for the number of degrees of freedom defining $\mathbb{X}_h$ and $\mathbb{M}_h$, and the individual errors are denoted by:

$$\mathsf{e}(\boldsymbol{\sigma}_{\rm S}) := \|\boldsymbol{\sigma}_{\rm S} - \boldsymbol{\sigma}_{{\rm S},h}\|_{\mathbf{div},\Omega_{\rm S}}\,, \qquad \mathsf{e}(\mathbf{u}_{\rm S}) := \|\mathbf{u}_{\rm S} - \mathbf{u}_{{\rm S},h}\|_{\mathrm{div},\Omega_{\rm S}}\,,$$

$$\mathsf{e}(\mathbf{u}_{\rm D}) := \|\mathbf{u}_{\rm D} - \mathbf{u}_{{\rm D},h}\|_{\mathrm{div},\Omega_{\rm D}}\,, \qquad \mathsf{e}(p_{\rm D}) := \|p_{\rm D} - p_{{\rm D},h}\|_{0,\Omega_{\rm D}}\,,$$

$$\mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Sigma}\,, \qquad \mathsf{e}(\lambda) := \|\lambda - \lambda_h\|_{1/2,\Sigma}\,,$$

where $\underline{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{S,h}, \mathbf{u}_{D,h}, \boldsymbol{\varphi}_h, \lambda_h) \in \mathbb{X}_h$ and $\underline{\mathbf{u}}_h := (\mathbf{u}_{S,h}, p_{D,h}) \in \mathbb{M}_h$ constitute the unique solution of (3.61).

Also, we let $r(\boldsymbol{\sigma}_S)$, $r(\mathbf{u}_S)$, $r(\mathbf{u}_D)$, $r(p_D)$, $r(\boldsymbol{\varphi})$, and $r(\lambda)$ be the experimental rates of convergence given by

$$r(\%) := \frac{\log(\mathsf{e}(\%)/\mathsf{e}'(\%))}{\log(h/h')} \qquad \text{for each} \quad \% \in \left\{\boldsymbol{\sigma}_S, \mathbf{u}_S, \mathbf{u}_D, p_D, \boldsymbol{\varphi}, \lambda\right\},$$

where h and h' denote two consecutive meshsizes with errors e and e'.

In what follows we describe the data of the examples. In all cases we choose for simplicity $\nu = 1$, $\mathbf{K} = \mathrm{I}$, the identity matrix of $\mathbb{R}^{2\times 2}$, and $\kappa = 1$.

In Example 1 we consider the regions $\Omega_D :=]-1/2, 1/2[\, \times\,]-1/2, 1/2[$ and $\Omega_S :=]-1, 1[\, \times\,]-1, 1[\, \setminus \Omega_D$, which represents a porous medium completely surrounded by a fluid. Then we choose the data $\mathbf{f}_S$ and f_D so that the exact solution is given by

$$\mathbf{u}_S(x_1, x_2) = \begin{pmatrix} -4\,(x_1^2 - 1)^2\,(x_2^2 - 1)\,x_2 \\ \\ 4\,(x_1^2 - 1)\,(x_2^2 - 1)^2\,x_1 \end{pmatrix} \quad \text{in} \quad \Omega_S\,,$$

$$p_S(x_1, x_2) = -\sin(x_1)\,e^{x_2} \quad \text{in} \quad \Omega_S\,,$$

and

$$p_D(x_1, x_2) = -\sin(x_1)\,e^{x_2} \quad \text{in} \quad \Omega_D\,.$$

In Example 2 we let Ω_S and Ω_D be the polygonal domains delimited by the set of points $\{(-1,0), (1,0), (1,1), (-1/2,1)\}$ and $\{(-1/2,-1), (1/2,-1), (1,0), (-1,0)\}$, respectively, which constitutes a particular case of a fluid over a porous medium, and choose the data $\mathbf{f}_S$ and f_D so that the exact solution is given by

$$\mathbf{u}_S(x_1, x_2) = \begin{pmatrix} 2\,(x_2 - 1)\,(x_1 - 1)^2\,(2x_1 - x_2 + 2)\,(2x_1 - 2x_2 + 3) \\ \\ -2\,(x_2 - 1)^2\,(x_1 - 1)\,(4x_1 - x_2)\,(2x_1 - x_2 + 2) \end{pmatrix} \quad \text{in} \quad \Omega_S\,,$$

$$p_S(x_1, x_2) = e^{x_1}\,\sin(x_2) \quad \text{in} \quad \Omega_S\,,$$

and

$$p_D(x_1, x_2) = \sin(x_1)\,(4x_1^2 - (x_2 + 2)^2)^2\,(x_2 + 1)^2 \quad \text{in} \quad \Omega_D\,.$$

Finally, in Example 3 we consider the domains $\Omega_S :=]-1, 1[\, \times\,]0, 1[$ and $\Omega_D :=]-1, 1[\, \times\,]-1, 0[$, which constitutes another case of a fluid over a porous medium, and take the data $\mathbf{f}_S$ and f_D given by

$$\mathbf{f}_S(x_1, x_2) = \begin{pmatrix} -4\sin(x_1 x_2)\,x_1 \;+\; \exp(x_2^3) \\ \\ 4\,\exp(3x_1) \;+\; 4\,x_2 \end{pmatrix}$$

and

$$f_D(x_1, x_2) = x_1^3 \left(\exp(x_2^2) - 0.5 \right).$$

This example corresponds to a more realistic situation in which the exact solution is unknown.

The numerical results shown below were obtained using a MATLAB implementation. In Tables 3.2 and 3.3 we present the convergence history of Examples 1 and 2, respectively, for a set of shape-regular triangulations of the computational domain $\bar{\Omega}_S \cup \bar{\Omega}_D$. We see there that the dominant error in both examples is given by $\mathsf{e}(\boldsymbol{\sigma}_S)$, though this is more evident in Example 1. In addition, we observe that the rate of convergence $O(h)$ provided by Theorem 3.5.3 for $\delta = 1$ is attained by all the unknowns. Next, in Figures 3.4 and 3.5 (resp. Figures 3.6 and 3.7) we display the approximate and exact values of some components of the solution of Example 1 for N=144641 (resp. Example 2 for $N = 273071$). It is clear from these figures that the finite element subspaces employed provide very accurate approximations to the unknowns in both domains. In particular, the quality of these approximations is not affected at all by the strong oscillations of some solutions. The shape-regular character of the meshes is ilustrated in Figure 3.8 for Example 2.

Next, in Table 3.4 we present the convergence history of Example 3 for a set of shape-regular triangulations of the computational domain $\bar{\Omega}_S \cup \bar{\Omega}_D$. The errors and experimental rates of convergence shown there are computed by considering the discrete solution obtained with a finer mesh ($N = 984068$) as the *exact solution*. Similarly as for Examples 1 and 2 we observe that the rate of convergence $O(h)$ is attained by all the unknowns, and the dominant error is also given by $\mathsf{e}(\boldsymbol{\sigma}_S)$. Next, in Figures 3.9, 3.10, and 3.11 we show some components of the approximate solutions obtained for N=123396. Note that in this example the normal on the interface $\Sigma := (-1, 1) \times \{0\}$ is given by $\mathbf{n} = (0, -1)^t$, and hence the first transmission condition becomes equality of the second components of $\mathbf{u}_S$ and $\mathbf{u}_D$. This can be verified at the discrete level in Figure 3.10 where we display 3D and 2D joint pictures of the second components of $\mathbf{u}_{S,h}$ and $\mathbf{u}_{D,h}$.

Summarizing, the numerical results reported here confirm the good performance of the mixed finite element scheme (3.61) with Raviart-Thomas finite element subspaces of lowest order in Ω_S and Ω_D, and continuous piecewise linear functions on the interface Σ, for different geometries of the coupled problem.

We end this paper by mentioning that the only reason for restricting here to 2D is the simple fact that in our previous works [45] and [48] we assumed that dimension. We believe, however, that the present results should be extended, with minor modifications, to the three-dimensional case. Indeed, it is easy to see that the sections concerning the model problem and the general analysis of the continuous and discrete formulations, should look more or less the same as the

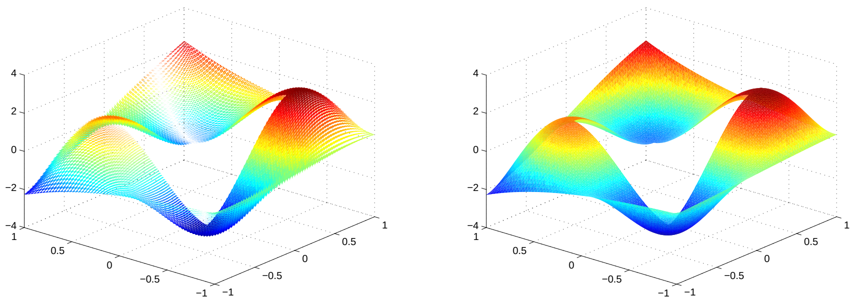

Figure 3.4: components $(1,1)$ of $\boldsymbol{\sigma}_{\mathrm{S},h}$ and $\boldsymbol{\sigma}_{\mathrm{S}}$ (EXAMPLE 1)

ones provided here. Eventual technical difficulties, not too hard to solve, nevertheless, might appear only in the analogue of Section 3.5, probably in the construction of the discrete liftings and the verification of the discrete inf-sup condition (3.78). We hope to address this issue in a separate work.

N	h	$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$
641	0.3536	5.2974		0.3622		0.1204	
2401	0.1768	2.6875	1.0277	0.1802	1.0573	0.0584	1.0957
9281	0.0884	1.3468	1.0219	0.0900	1.0269	0.0289	1.0406
36481	0.0442	0.6737	1.0121	0.0450	1.0128	0.0144	1.0178
144641	0.0221	0.3369	1.0062	0.0225	1.0064	0.0072	1.0064

N	h	$\mathsf{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathsf{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathsf{e}(\lambda)$	$r(\lambda)$
641	0.3536	0.0645		1.0988		0.2572	
2401	0.1768	0.0320	1.0615	0.5390	1.0787	0.1260	1.0807
9281	0.0884	0.0160	1.0253	0.2661	1.0441	0.0619	1.0514
36481	0.0442	0.0080	1.0128	0.1321	1.0232	0.0306	1.0294
144641	0.0221	0.0040	1.0064	0.0658	1.0119	0.0152	1.0159

Table 3.2: degrees of freedom, meshsizes, errors, and rates of convergence (EXAMPLE 1).

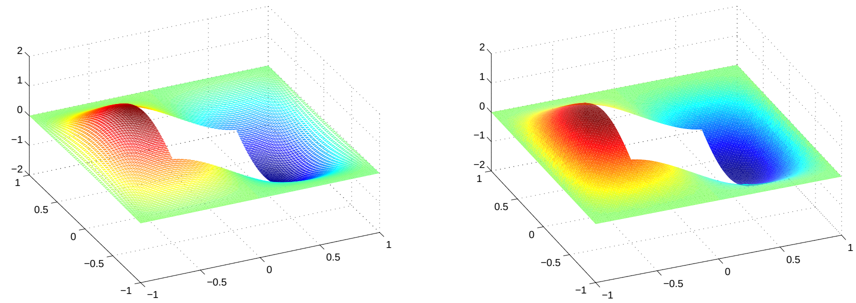

Figure 3.5: second components of $\mathbf{u}_{S,h}$ and $\mathbf{u}_S$ (EXAMPLE 1)

N	h	$\mathsf{e}(\boldsymbol{\sigma}_S)$	$r(\boldsymbol{\sigma}_S)$	$\mathsf{e}(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$\mathsf{e}(\mathbf{u}_D)$	$r(\mathbf{u}_D)$
219	0.2500	15.4093		1.5670		11.2401	
1225	0.0885	6.6580	0.9748	0.7182	0.9063	5.1261	0.9121
7368	0.0347	2.4934	1.0948	0.2789	1.0543	1.6395	1.2707
44595	0.0117	0.9809	1.0363	0.1135	0.9989	0.6796	0.9783
273071	0.0035	0.4041	0.9789	0.0461	0.9931	0.2742	1.0017

N	h	$\mathsf{e}(p_D)$	$r(p_D)$	$\mathsf{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathsf{e}(\lambda)$	$r(\lambda)$
219	0.2500	1.0956		11.6289		6.2759	
1225	0.0885	1.0870	0.0091	4.0529	1.2245	4.2773	0.4454
7368	0.0347	0.1317	2.3529	1.2658	1.2972	1.1155	1.4982
44595	0.0117	0.0434	1.2324	0.5188	0.9907	0.3315	1.3478
273071	0.0035	0.0180	0.9709	0.2010	1.0468	0.1286	1.0452

Table 3.3: degrees of freedom, meshsizes, errors, and rates of convergence (EXAMPLE 2).

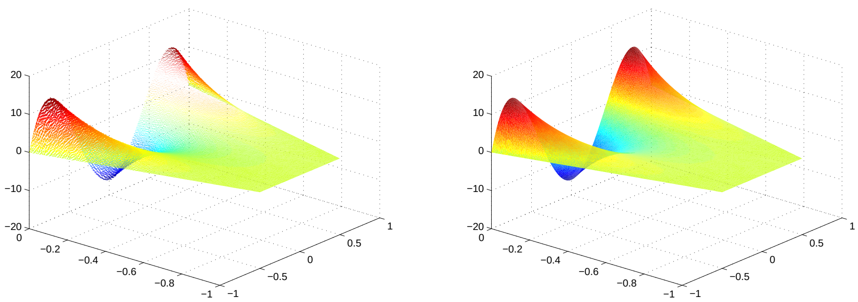

Figure 3.6: first components of $\mathbf{u}_{D,h}$ and $\mathbf{u}_D$ (EXAMPLE 2)

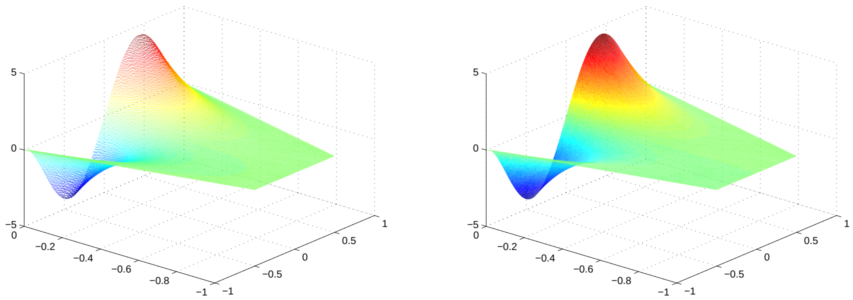

Figure 3.7: $p_{\mathrm{D},h}$ and p_{D} (EXAMPLE 2)

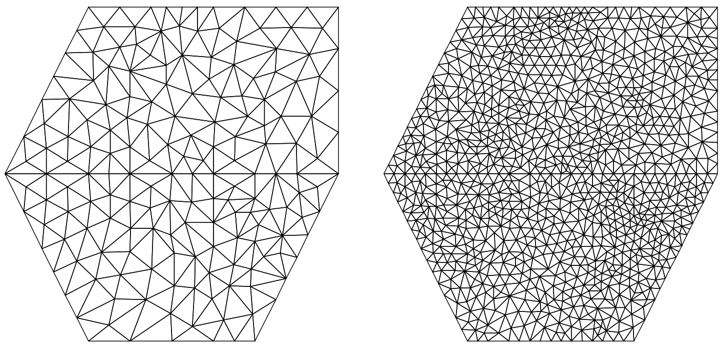

Figure 3.8: meshes for $N = 1225$ and $N = 7368$ (EXAMPLE 2)

N	h	$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$
516	0.3536	6.7467		0.1873		0.1911	
1988	0.1768	3.4022	1.0152	0.0758	1.3414	0.0921	1.0823
7812	0.0884	1.6984	1.0153	0.0340	1.1717	0.0456	1.0273
30980	0.0442	0.8482	1.0080	0.0164	1.0584	0.0227	1.0126
123396	0.0221	0.4243	1.0024	0.0081	1.0208	0.0114	0.9967

N	h	$\mathsf{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathsf{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathsf{e}(\lambda)$	$r(\lambda)$
516	0.3536	0.0301		0.5915		0.1324	
1988	0.1768	0.0112	1.4659	0.2838	1.0890	0.0498	1.4499
7812	0.0884	0.0050	1.1786	0.1449	0.9824	0.0185	1.4472
30980	0.0442	0.0024	1.0655	0.0761	0.9349	0.0062	1.5870
123396	0.0221	0.0012	1.0031	0.0401	0.9271	0.0019	1.7115

Table 3.4: degrees of freedom, meshsizes, errors, and rates of convergence (EXAMPLE 3).

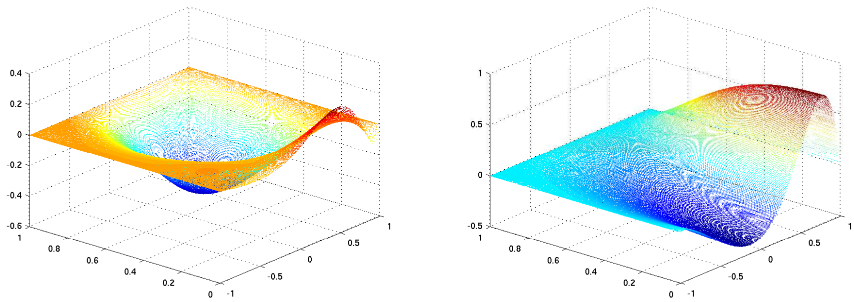

Figure 3.9: first and second components of $\mathbf{u}_{S,h}$ (Example 3)

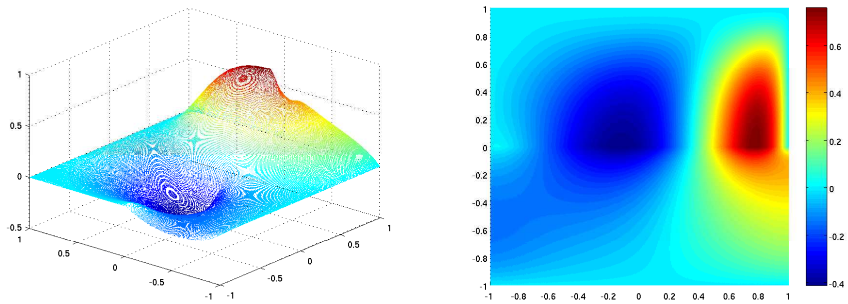

Figure 3.10: second components of $\mathbf{u}_{S,h}$ and $\mathbf{u}_{D,h}$ (Example 3)

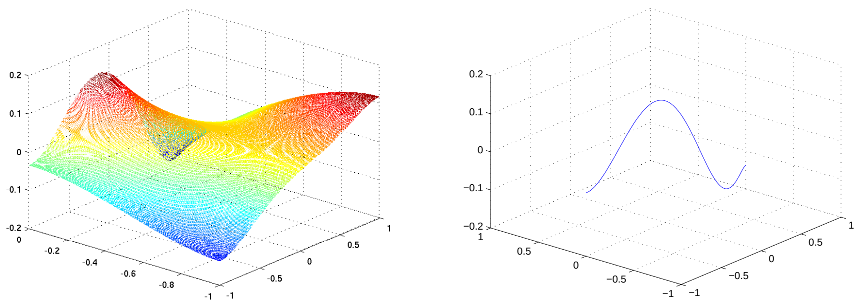

Figure 3.11: $p_{D,h}$ and λ_h (Example 3)

Chapter 4

A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes-Darcy coupled problem

4.1 Introduction

The derivation of new finite element methods for the Stokes-Darcy coupled problem, in which the respective interface conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law, has become a very active research area lately (see, e.g. [2], [17], [20], [21], [26], [35], [37], [39], [45], [48], [63], [67], [71], [74], [75], [78], [83] and the references therein). The above list includes porous media with cracks, nonlinear problems, and the incorporation of the Brinkman equation in the model (see [17], [37], and [83]). In addition, most of the formulations employed are based on appropriate combinations of stable elements for the free fluid flow and for the porous medium flow, and the first theoretical results in this direction go back to [35] and [63]. Indeed, an iterative subdomain method employing the primal variational formulation and standard finite element subspaces in both domains is proposed in [35], whereas the primal method in the fluid and the dual-mixed method in the porous medium are applied in [63]. In this way, the approach from [63] yields the velocity and the pressure in both domains, together with the trace of the porous medium pressure on the interface, as the main unknowns of the coupled problem. This trace unknown is motivated by the fact that one of the transmission conditions becomes essential. Then, new mixed finite element discretizations of the variational formulation from [63] have been introduced and analyzed in [45] and [48]. The stability of a specific Galerkin method is the main result in [45], and the resulting mixed finite element method is the first one that is conforming for the primal/dual-mixed formulation proposed in [63]. The results from [45] are improved in [48] where it is shown that the use of any pair of stable Stokes and Darcy elements implies the stability of the corresponding Stokes-Darcy Galerkin scheme. The analysis in [48] hinges on the fact that the operator defining the continuous variational formulation is given by a compact perturbation of an invertible mapping. Further techniques utilized in the literature include mortar finite element methods, discontinuous Galerkin (DG) schemes, and stabilized formulations (see, e.g. [2], [20], [21], [26], [31], [39], [67], [71], [74], [75], [78]). In particular, the main motivation for employing stabilized formulations either in both domains or in one of them, is the possibility of approximating the Stokes and Darcy flows with the same finite element subpaces. Certainly, different finite element subspaces in each flow region may lead to different approximation properties for each subproblem. On the contrary, using the same spaces guarantees the same accurateness along the entire domain and leads to simpler and more efficient computational codes.

Now, in the recent paper [49] we have developed a new variational approach for the 2D Stokes-Darcy coupled problem, which allows, on one hand, the introduction of further unknowns of physical interest, and on the other hand, the utilization of the same family of finite element subspaces in both media, without requiring any stabilization term. More precisely, in [49] we consider dual-mixed formulations in both domains, which yields the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium, as the main unknowns. The pressure and the gradient of the velocity in the fluid can then be computed as a very simple postprocess of the above unknowns, in which no numerical differentiation is applied, and hence no further sources of error arise. In addition, since the transmission conditions become essential, we impose them weakly and introduce the traces of the porous media pressure and the fluid velocity, which are also variables of importance from a physical point of view, as the corresponding Lagrange multipliers. Then, we apply the well known Fredholm and Babuška-Brezzi theories to prove the unique solvability of the resulting continuous formulation and derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme becomes well posed. Among the several different ways in which the equations and unknowns can be ordered, we choose the one yielding a doubly mixed structure for which the inf-sup conditions of the off-diagonal bilinear forms follow straightforwardly. In this way, the arguments of the continuous analysis can be easily adapted to the discrete case. In particular, a feasible choice of subspaces is given by Raviart-Thomas elements of lowest order and piecewise constants for the velocities and pressures, respectively, in both domains, together with continuous piecewise linear elements for the Lagrange multipliers.

On the other hand, it is well known that in order to guarantee a good convergence behaviour of most finite element solutions, specially under the eventual presence of singularities, one usually needs to apply an adaptive algorithm based on a posteriori error estimates. These are represented by global quantities $\boldsymbol{\eta}$ that are expressed in terms of local indicators η_T defined on each element T of a given triangulation $\mathcal{T}$. The estimator $\boldsymbol{\eta}$ is said to be efficient (resp. reliable) if there exists $C_{\texttt{eff}} > 0$ (resp. $C_{\texttt{rel}} > 0$), independent of the meshsizes, such that

$$C_{\texttt{eff}}\,\boldsymbol{\eta} + \text{h.o.t.} \;\leq\; \|error\| \;\leq\; C_{\texttt{rel}}\,\boldsymbol{\eta} + \text{h.o.t.}\,,$$

where h.o.t. is a generic expression denoting one or several terms of higher order. In particular, the a posteriori error analysis of variational formulations with saddle-point structure has already been widely investigated by many authors (see, e.g. [8], [9], [10], [23], [25], [28], [43], [56], [64], [65], [72], [79], and the references therein). These contributions refer mainly to reliable and efficient a posteriori error estimators based on local and global residuals, local problems, postprocessing, and functional-type error estimates. In addition, the applications include Stokes and Oseen equations, Poisson problem, linear elasticity, and general elliptic partial differential equations of second order. However, up to our knowledge, the first a posteriori error analysis for the Stokes-Darcy coupled problem has been provided recently in [15], where a reliable and efficient residual-based a posteriori error estimator for the variational formulation analyzed in [45] is derived. Partially following known approaches, the proof of reliability makes use of suitable auxiliary problems, diverse continuous inf-sup conditions satisfied by the bilinear forms involved, and local approximation properties of the Clément interpolant and Raviart-Thomas operator. Similarly, Helmholtz decomposition, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions, are the main tools for proving the efficiency of the estimator.

Motivated by the discussion in the above paragraphs, our purpose now is to additionally contribute in the direction of [15] and provide the a posteriori error analysis of the fully-mixed variational approach introduced in [49]. According to this, the rest of this work is organized as follows. In Section 4.2 we recall from [49] the Stokes-Darcy coupled problem and its continuous and discrete fully-mixed variational formulations. The kernel of the present work is given by Section 4.3, where we develop the a posteriori error analysis. In Section 4.3.1 we employ the global continuous inf-sup condition, Helmholtz decompositions in both domains, and the local approximation properties of the Clément and Raviart-Thomas operators, to derive a reliable residual-based a posteriori error estimator. An interesting feature of our proof of reliability is the previous transformation of the global continuous inf-sup condition into an equivalent estimate involving global inf-sup conditions for each one of the components of the product space to which the vector of unknowns belongs. Then, in Section 4.3.2 we apply again Helmholtz decompositions, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions to prove the efficiency of the estimator. This proof benefits partially from the fact that some components of the a posteriori error estimator coincide with those obtained in [15] and the related work [25]. Finally, numerical results confirming the reliability and efficiency of the a posteriori error estimator and showing the good performance of the associated adaptive algorithm, are presented in Section 4.4.

We end this section with some notations to be used below. In particular, in what follows we utilize the standard terminology for Sobolev spaces. In addition, if $\mathcal{O}$ is a domain, Γ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}, \quad \text{and} \quad \mathbf{H}^r(\Gamma) := [H^r(\Gamma)]^2.$$

However, for $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\Gamma}$ (for $H^r(\Gamma)$ and $\mathbf{H}^r(\Gamma)$). Also, the Hilbert space

$$\mathbf{H}(\mathrm{div}\,;\mathcal{O}) := \left\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\ \mathbf{w} \in L^2(\mathcal{O})\right\},$$

is standard in the realm of mixed problems (see, e.g. [19] or [54]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ and $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div}\,;\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div}\,;\mathcal{O}}$, respectively. On the other hand, the symbol for the $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ inner products

$$\langle \xi, \lambda \rangle_\Gamma := \int_\Gamma \xi\,\lambda \quad \forall\, \xi,\, \lambda \in L^2(\Gamma), \qquad \langle \boldsymbol{\xi}, \boldsymbol{\lambda} \rangle_\Gamma := \int_\Gamma \boldsymbol{\xi}\cdot\boldsymbol{\lambda} \quad \forall\, \boldsymbol{\xi},\, \boldsymbol{\lambda} \in \mathbf{L}^2(\Gamma)$$

will also be employed for their respective extensions as the duality products $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{1/2}(\Gamma)$. Finally, we employ $\mathbf{0}$ as a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

4.2 The Stokes-Darcy coupled problem

In this section we follow very closely the presentation from [49] to introduce the model problem and the corresponding continuous and discrete mixed variational formulations.

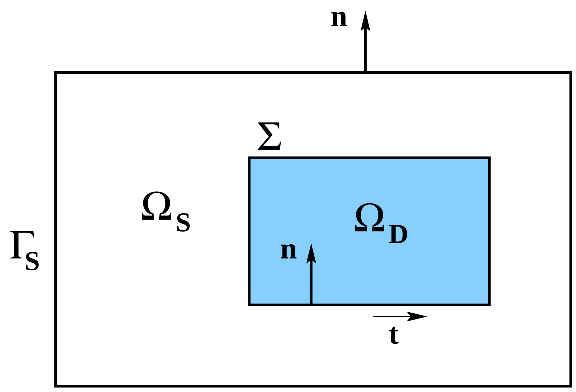

Figure 4.1: Geometry of the problem

4.2.1 The model problem

The Stokes-Darcy coupled problem models the interaction of an incompressible viscous fluid occupying a region Ω_S, which flows back and forth across the common interface into a porous medium living in another region Ω_D and saturated with the same fluid. Physically, we consider a simplified 2D model where Ω_D is surrounded by a bounded region Ω_S (see Figure 4.1 below). Their common interface is supposed to be a Lipschitz curve Σ and we assume that $\partial\Omega_D = \Sigma$. The remaining part of the boundary of Ω_S is also assumed to be a Lipschitz curve Γ_S. For practical purposes, we can assume that both Γ_S and Σ are polygons. The unit normal vector field on the boundaries $\mathbf{n}$ is chosen pointing outwards from Ω_S (and therefore inwards to Ω_D when seen on Σ). On Σ we also consider a unit tangent vector field $\mathbf{t}$ in any fixed orientation of this closed curve.

The governing equations in Ω_S are those of the Stokes problem, which are written in the following non-standard velocity-pressure-pseudostress formulation:

$$\begin{gathered}\boldsymbol{\sigma}_S = -p_S\,\mathbf{I} + \nu\,\nabla\mathbf{u}_S \quad \text{in} \quad \Omega_S\,, \qquad \mathbf{div}\,\boldsymbol{\sigma}_S + \mathbf{f}_S = \mathbf{0} \quad \text{in} \quad \Omega_S\,, \\ \operatorname{div}\,\mathbf{u}_S = 0 \quad \text{in} \quad \Omega_S\,, \qquad \mathbf{u}_S = \mathbf{0} \quad \text{on} \quad \Gamma_S\,,\end{gathered} \tag{4.1}$$

where $\nu > 0$ is the viscosity of the fluid, $\mathbf{u}_S$ is the fluid velocity, p_S is the pressure, $\boldsymbol{\sigma}_S$ is the pseudostress tensor, I is the 2×2 identity matrix and $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ are known source terms. Here, div is the usual divergence operator acting on vector fields, and **div** denotes the action of div to the rows of each tensor. On the other hand, the flow equations in Ω_D are those of the linearized Darcy model:

$$\mathbf{u}_D = -\,\mathbf{K}\,\nabla p_D \quad \text{in} \quad \Omega_D\,, \qquad \operatorname{div}\,\mathbf{u}_D = f_D \quad \text{in} \quad \Omega_D\,, \tag{4.2}$$

where the unknowns are the pressure p_D and the flow $\mathbf{u}_D$, and the source term, given by $f_D \in L^2(\Omega_D)$, satisfies $\int_{\Omega_D} f_D = 0$. The matrix valued function $\mathbf{K}$, describing permeability of Ω_D divided by the viscosity ν, is symmetric, has $L^\infty(\Omega_D)$ components and is uniformly elliptic. Finally, the transmission conditions on Σ are given by

$$\begin{aligned} \mathbf{u}_S \cdot \mathbf{n} &= \mathbf{u}_D \cdot \mathbf{n} \quad \text{on} \quad \Sigma\,, \\ \boldsymbol{\sigma}_S\,\mathbf{n} + \nu\,\kappa^{-1}\,(\mathbf{u}_S \cdot \mathbf{t})\,\mathbf{t} &= -\,p_D\,\mathbf{n} \quad \text{on} \quad \Sigma\,, \end{aligned} \tag{4.3}$$

where $\kappa := \alpha^{-1}\sqrt{(\nu\,\mathbf{K}\,\mathbf{t})\cdot\mathbf{t}}$ is the friction coefficient and α is an experimentally determined positive parameter. The first equation in (4.3) corresponds to mass conservation on Σ, whereas the normal and tangential components of the second one constitute the balance of normal forces and the Beavers-Joseph-Saffman law, respectively. Throughout the rest of the paper we assume, without loss of generality, that κ is a positive constant.

We complete the description of our model problem by observing that the equations in the Stokes domain (cf. (4.1) can be rewritten equivalently as

$$\begin{aligned} \nu^{-1}\,\boldsymbol{\sigma}_S^d &= \nabla \mathbf{u}_S \quad \text{in} \quad \Omega_S\,, \qquad \mathbf{div}\,\boldsymbol{\sigma}_S + \mathbf{f}_S = \mathbf{0} \quad \text{in} \quad \Omega_S\,, \\ p_S &= -\tfrac{1}{2}\,\mathrm{tr}\,\boldsymbol{\sigma}_S \quad \text{in} \quad \Omega_S\,, \qquad \mathbf{u}_S = \mathbf{0} \quad \text{on} \quad \Gamma_S\,, \end{aligned} \tag{4.4}$$

where $\mathrm{tr}\,\boldsymbol{\tau} := \tau_{11} + \tau_{22}$, and

$$\boldsymbol{\tau}^d := \boldsymbol{\tau} - \tfrac{1}{2}\,(\mathrm{tr}\,\boldsymbol{\tau})\,\mathrm{I}$$

is the deviatoric part of the tensor $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$.

We end this section by remarking that, though the geometry described by Figure 4.1 was choosen to simplify the presentation, the case of a fluid flowing only across a part of the boundary of the porous medium does not yield further complications for the a posteriori error analysis of the problem. We already discussed this issue in [49, Section 2.1], in connection with the respective a priori error analysis, and further details can be found in [39].

4.2.2 The fully-mixed variational formulation

We add two new unknowns to the system, namely $\boldsymbol{\varphi} := -\,\mathbf{u}_S|_\Sigma$ and $\lambda := p_D|_\Sigma$. The system will be written in terms of the unknowns $\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_S, \mathbf{u}_D, \boldsymbol{\varphi}, \lambda)$ and $\underline{\mathbf{u}} := (\mathbf{u}_S, p_D)$. Then we recall from [49, Lemma 3.5] that the coupled problem given by (4.2), (4.3), and (4.4) has the one-dimensional kernel defined by

$$\{\,((\boldsymbol{\sigma}_S, \mathbf{u}_D, \boldsymbol{\varphi}, \lambda), (\mathbf{u}_S, p_D)) : \quad \boldsymbol{\sigma}_S = -\,c\,\mathbf{I},\ \mathbf{u}_D = \mathbf{0},\ \boldsymbol{\varphi} = \mathbf{0},\ \lambda = c,\ \mathbf{u}_S = \mathbf{0},\ p_D = c\,;\ c \in \mathbb{R}\}\ .$$

Hence, in order to solve this indetermination, we introduce

$$L_0^2(\Omega_D) := \left\{ q \in L^2(\Omega_D) : \quad \int_{\Omega_D} q = 0 \right\},$$

and define the product spaces

$$\mathbb{X} := \mathbb{H}(\mathbf{div}\,;\Omega_S) \times \mathbf{H}(\mathrm{div}\,;\Omega_D) \times \mathbf{H}^{1/2}(\Sigma) \times H^{1/2}(\Sigma), \quad \mathbb{M} := \mathbf{L}^2(\Omega_S) \times L_0^2(\Omega_D),$$

endowed with the product norms

$$\|\underline{\boldsymbol{\tau}}\|^2_{\mathbb{X}} := \|\boldsymbol{\tau}_S\|^2_{\mathbf{div},\Omega_S} + \|\mathbf{v}_D\|^2_{\mathrm{div}\,;\Omega_D} + \|\boldsymbol{\psi}\|^2_{1/2,\Sigma} + \|\xi\|^2_{1/2,\Sigma} \qquad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \mathbf{v}_D, \boldsymbol{\psi}, \xi) \in \mathbb{X},$$

and

$$\|\underline{\mathbf{v}}\|^2_{\mathbb{M}} := \|\mathbf{v}_S\|^2_{0.\Omega_S} + \|q_D\|^2_{0,\Omega_D} \qquad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_S, q_D) \in \mathbb{M}.$$

In this way, as explained in [49, Sections 2 and 3]), it suffices to consider from now on the following modified variational formulation of (4.2), (4.3), and (4.4): Find $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbb{M}$ such that

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}}) &= \mathcal{F}(\underline{\boldsymbol{\tau}}) \qquad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \mathbf{v}_D, \boldsymbol{\psi}, \xi) \in \mathbb{X}, \\ \mathcal{B}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{v}}) &= \mathcal{G}(\underline{\mathbf{v}}) \qquad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_S, q_D) \in \mathbb{M}, \end{aligned} \tag{4.5}$$

where

$$\mathcal{F}(\underline{\boldsymbol{\tau}}) := 0, \qquad \mathcal{G}(\underline{\mathbf{v}}) = \mathcal{G}((\mathbf{v}_S, q_D)) := -(\mathbf{f}_S, \mathbf{v}_S)_S - (f_D, q_D)_D, \tag{4.6}$$

and $\mathcal{A}$ and $\mathcal{B}$ are the bounded bilinear forms defined by

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) :=\ & \mathbf{a}((\boldsymbol{\sigma}_S, \mathbf{u}_D), (\boldsymbol{\tau}_S, \mathbf{v}_D)) + \mathbf{b}((\boldsymbol{\tau}_S, \mathbf{v}_D), (\boldsymbol{\varphi}, \lambda)) \\ & + \mathbf{b}((\boldsymbol{\sigma}_S, \mathbf{u}_D), (\boldsymbol{\psi}, \xi)) - \mathbf{c}((\boldsymbol{\varphi}, \lambda), (\boldsymbol{\psi}, \xi)), \end{aligned} \tag{4.7}$$

with

$$\begin{aligned} \mathbf{a}((\boldsymbol{\sigma}_S, \mathbf{u}_D), (\boldsymbol{\tau}_S, \mathbf{v}_D)) &:= \nu^{-1} (\boldsymbol{\sigma}_S^d, \boldsymbol{\tau}_S^d)_S + (\mathbf{K}^{-1} \mathbf{u}_D, \mathbf{v}_D)_D, \\ \mathbf{b}((\boldsymbol{\tau}_S, \mathbf{v}_D), (\boldsymbol{\psi}, \xi)) &:= \langle \boldsymbol{\tau}_S \, \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma - \langle \mathbf{v}_D \cdot \mathbf{n}, \xi \rangle_\Sigma, \\ \mathbf{c}((\boldsymbol{\varphi}, \lambda), (\boldsymbol{\psi}, \xi)) &:= \nu\, \kappa^{-1} \langle \boldsymbol{\varphi} \cdot \mathbf{t}, \boldsymbol{\psi} \cdot \mathbf{t} \rangle_\Sigma + \langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_\Sigma - \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma, \end{aligned}$$

and

$$\mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) := (\mathbf{div}\, \boldsymbol{\tau}_S, \mathbf{v}_S)_S - (\mathrm{div}\, \mathbf{v}_D, q_D)_D. \tag{4.8}$$

Hereafter we utilize, for each $\star \in \{S, D\}$, the following notations

$$(u, v)_\star := \int_{\Omega_\star} u\, v, \qquad (\mathbf{u}, \mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u} \cdot \mathbf{v}, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau},$$

for all $u, v \in L^2(\Omega_\star)$, $\mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega_\star)$, and $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$, where $\boldsymbol{\sigma} : \boldsymbol{\tau} := \mathrm{tr}(\boldsymbol{\sigma}^t \boldsymbol{\tau})$.

We find it important to remark that $\boldsymbol{\varphi}$ and λ can be interpreted as Lagrange multipliers associated to the transmission conditions (4.3). In addition, we notice that (4.5) is equivalent to the variational formulation defined in [49, Section 3.2, eq. (3.2)], in which $\boldsymbol{\sigma}_{\mathrm{S}}$ is decomposed into $\boldsymbol{\sigma}_{\mathrm{S}} = \boldsymbol{\sigma} + \mu$, with $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}})$ and $\mu \in \mathbb{R}$, where

$$\mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}}) \,:=\, \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega_{\mathrm{S}}) \,:\, \int_{\Omega_{\mathrm{S}}} \mathrm{tr}\,\boldsymbol{\tau} \,=\, 0 \right\}.$$

The following result taken from [49] establishes, in particular, the well-posedness of (4.5).

Theorem 4.2.1 *For each pair* $(\mathcal{F},\mathcal{G}) \in \mathbb{X}' \times \mathbb{M}'$ *there exists a unique* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbb{M}$ *solution to (4.5), and there exists a constant* $C > 0$*, independent of the solution, such that*

$$\|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})\|_{\mathbb{X}\times\mathbb{M}} \,\leq\, C \left\{ \|\mathcal{F}\|_{\mathbb{X}'} \,+\, \|\mathcal{G}\|_{\mathbb{M}_0'} \right\}. \tag{4.9}$$

Proof. See [49, Theorem 3.9]. □

We end this section with the converse of the derivation of (4.5). More precisely, the following theorem establishes that the unique solution of (4.5), with $\mathcal{F}$ and $\mathcal{G}$ given by (4.6), solves the original transmission problem described in Section 4.2.1. This result will be used later on in Section 4.3.2 to prove the efficiency of our a posteriori error estimator. We remark that no extra regularity assumptions on the data, but only $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ and $f_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}})$, are required here.

Theorem 4.2.2 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ *be the unique solution of the variational formulation* (4.5) *with* $\mathcal{F}$ *and* $\mathcal{G}$ *given by (4.6). Then* $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} = -\,\mathbf{f}_{\mathrm{S}}$ *in* $\mathbf{L}^2(\Omega_{\mathrm{S}})$*,* $\nu^{-1}\,\boldsymbol{\sigma}_{\mathrm{S}}^{d} = \nabla\,\mathbf{u}_{\mathrm{S}}$ *in* $\mathbb{L}^2(\Omega_{\mathrm{S}})$*,* $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$*,* $\mathrm{div}\,\mathbf{u}_{\mathrm{D}} = f_{\mathrm{D}}$ *in* $L^2(\Omega_{\mathrm{D}})$*,* $\mathbf{u}_{\mathrm{D}} = -\,\mathbf{K}\,\nabla\,p_{\mathrm{D}}$ *in* $\mathbf{L}^2(\Omega_{\mathrm{D}})$*,* $p_{\mathrm{D}} \in H^1(\Omega_{\mathrm{D}})$*,* $\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} + \boldsymbol{\varphi}\cdot\mathbf{n} = 0$ *on* $H^{-1/2}(\Sigma)$*,* $\boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n} + \lambda\,\mathbf{n} - \nu\,\kappa^{-1}\,(\boldsymbol{\varphi}\cdot\mathbf{t})\,\mathbf{t} = 0$ *on* $\mathbf{H}^{-1/2}(\Sigma)$*,* $\lambda = p_{\mathrm{D}}$ *on* $H^{1/2}(\Sigma)$*,* $\boldsymbol{\varphi} = -\,\mathbf{u}_{\mathrm{S}}$ *on* $\mathbf{H}^{1/2}(\Sigma)$*, and* $\mathbf{u}_{\mathrm{S}} = 0$ *on* $\mathbf{H}^{1/2}(\Gamma_{\mathrm{S}})$*.*

Proof. It basically follows by applying integration by parts backwardly in (4.5) and using suitable test functions. We omit further details. □

4.2.3 A Galerkin method

Although the analysis in [49] provides general hypotheses for the well-posedness of a Galerkin scheme of (4.5), we will consider here the particular case described in [49, Section 5]. Let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D} formed by shape-regular triangles T of diameter h_T, and assume that $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ match in Σ, so that their union is a triangulation of $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. Then, for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we let $\mathrm{RT}_0(T)$ be the local lowest order Raviart-Thomas space,

$$\mathrm{RT}_0(T) := \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}.$$

For each $\star \in \{S, D\}$ we define the global spaces

$$\mathbf{H}_h(\Omega_\star) := \Big\{ \mathbf{v}_h \in \mathbf{H}(\mathrm{div}\,;\Omega_\star): \quad \mathbf{v}_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \Big\}, \tag{4.10}$$

and

$$L_h(\Omega_\star) := \Big\{ q_h : \Omega_\star \to \mathbb{R}: \quad q_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \Big\}.$$

Hereafter, given a non-negative integer k and a subset S of $\mathbb{R}^2$, $\mathbf{P}_k(S)$ stands for the space of polynomials defined on S of degree $\leq k$. Next, we let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$), and first assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. Note that because Σ_h is inherited from one of the interior triangulations, it is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded) and, therefore, so is Σ_{2h}. If the cardinal of Σ_h is odd, we start by joining to adjacent elements and construct Σ_{2h} from this reduced partition.

Employing the above notations, we now introduce

$$\begin{aligned}
\mathbb{H}_h(\Omega_{\mathrm{S}}) &:= \{ \boldsymbol{\tau} : \Omega_{\mathrm{S}} \to \mathbb{R}^{2\times 2} : \quad \mathbf{c}^t\boldsymbol{\tau} \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \},\\
\mathbf{L}_h(\Omega_{\mathrm{S}}) &:= L_h(\Omega_{\mathrm{S}}) \times L_h(\Omega_{\mathrm{S}}),\\
L_{h,0}(\Omega_{\mathrm{D}}) &:= L_h(\Omega_{\mathrm{D}}) \cap L_0^2(\Omega_{\mathrm{D}}),\\
\Lambda_h(\Sigma) &:= \{ \xi_h \in C(\Sigma): \quad \xi_h|_e \in \mathbb{P}_1(e) \quad \forall\, e \text{ edge of } \Sigma_{2h} \},\\
\boldsymbol{\Lambda}_h(\Sigma) &:= \Lambda_h(\Sigma) \times \Lambda_h(\Sigma),
\end{aligned}$$

and the product spaces

$$\mathbb{X}_h := \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}_h(\Omega_{\mathrm{D}}) \times \boldsymbol{\Lambda}_h(\Sigma) \times \Lambda_h(\Sigma) \quad \text{and} \quad \mathbb{M}_h := \mathbf{L}_h(\Omega_{\mathrm{S}}) \times L_{h,0}(\Omega_{\mathrm{D}}).$$

In this way, the Galerkin scheme of (4.5) becomes: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ such that

$$\begin{aligned}
\mathcal{A}(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}}_h) &= \mathcal{F}(\underline{\boldsymbol{\tau}}) \quad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi) \in \mathbb{X}_h,\\
\mathcal{B}(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{v}}) &= \mathcal{G}(\underline{\mathbf{v}}) \quad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}) \in \mathbb{M}_h,
\end{aligned} \tag{4.11}$$

where $\underline{\boldsymbol{\sigma}}_h = (\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h, \lambda_h)$ and $\underline{\mathbf{u}}_h := (\mathbf{u}_{\mathrm{S},h}, p_{\mathrm{D},h})$.

The following theorem, also taken from [49], provide the well-posedness of (4.11), the associated Céa estimate, and the corresponding theoretical rate of convergence.

Theorem 4.2.3 *The Galerkin scheme (4.11) has a unique solution* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$*. Moreover, there exist* C_1*,* $C_2 > 0$*, independent of* h*, such that*

$$\|(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \,\le\, C_1 \left\{ \|\mathcal{F}|_{\mathbb{X}_h}\|_{\mathbb{X}_h'} \,+\, \|\mathcal{G}|_{\mathbb{M}_h}\|_{\mathbb{M}_h'} \right\},$$

and

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} \,+\, \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \,\le\, C_2 \left\{ \inf_{\underline{\boldsymbol{\tau}}_h \in \mathbb{X}_h} \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h\|_{\mathbb{X}} \,+\, \inf_{\underline{\mathbf{v}}_h \in \mathbb{M}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|_{\mathbb{M}} \right\}.$$

If there exists $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_{\rm S} \in \mathbb{H}^\delta(\Omega_{\rm S})$*,* $\mathbf{div}\,\boldsymbol{\sigma}_{\rm S} \in \mathbf{H}^\delta(\Omega_{\rm S})$*,* $\mathbf{u}_{\rm D} \in \mathbf{H}^\delta(\Omega_{\rm D})$*, and* $\mathrm{div}\,\mathbf{u}_{\rm D} \in H^\delta(\Omega_{\rm D})$*, then,* $\mathbf{u}_{\rm S} \in \mathbf{H}^{1+\delta}(\Omega_{\rm S})$*,* $p_{\rm D} \in H^{1+\delta}(\Omega_{\rm D})$*,* $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$*,* $\lambda \in H^{1/2+\delta}(\Sigma)$*, and there exists* $C > 0$*, independent of* h*, such that*

$$\begin{aligned} \|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \,-\, (\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \,&\le\, C\, h^\delta \Big\{ \|\boldsymbol{\sigma}_{\rm S}\|_{\delta,\Omega_{\rm S}} \,+\, \|\mathbf{div}\,\boldsymbol{\sigma}_{\rm S}\|_{\delta,\Omega_{\rm S}} \\ &+\, \|\mathbf{u}_{\rm D}\|_{\delta,\Omega_{\rm D}} \,+\, \|\mathrm{div}\,\mathbf{u}_{\rm D}\|_{\delta,\Omega_{\rm D}} \,+\, \|\mathbf{u}_{\rm S}\|_{1+\delta,\Omega_{\rm S}} \,+\, \|p_{\rm D}\|_{1+\delta,\Omega_{\rm D}} \Big\}. \end{aligned} \tag{4.12}$$

Proof. See [49, Theorems 5.3, 5.4 and 5.5]. □

Note that the proofs of [49, Theorems 5.3, 5.4 and 5.5] require $\mathcal{T}_h^{\rm S}$ and $\mathcal{T}_h^{\rm D}$ are quasiuniform in a neighborhood of Σ. Based on a recent result on stable discrete liftings of the normal trace of Raviart–Thomas elements in [77], the theorem can be easily generalized to any shape–regular triangulation.

4.3 A residual-based a posteriori error estimator

We first introduce some notations. For each $T \in \mathcal{T}_h^{\rm S} \cup \mathcal{T}_h^{\rm D}$ we let $\mathcal{E}(T)$ be the set of edges of T, and we denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h^{\rm S} \cup \mathcal{T}_h^{\rm D}$, subdivided as follows:

$$\mathcal{E}_h \,=\, \mathcal{E}_h(\Gamma_{\rm S}) \cup \mathcal{E}_h(\Omega_{\rm S}) \cup \mathcal{E}_h(\Omega_{\rm D}) \cup \mathcal{E}_h(\Sigma)\,,$$

where $\mathcal{E}_h(\Gamma_{\rm S}) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Gamma_{\rm S} \,\}$, $\mathcal{E}_h(\Omega_\star) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Omega_\star \,\}$ for each $\star \in \{\mathrm{S},\mathrm{D}\}$, and $\mathcal{E}_h(\Sigma) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Sigma \,\}$. Note that $\mathcal{E}_h(\Sigma)$ is the set of edges defining the partition Σ_h. Analogously, we let $\mathcal{E}_{2h}(\Sigma)$ be the set of *double* edges defining the partition Σ_{2h}. In what follows, h_e stands for the diameter of a given edge $e \in \mathcal{E}_h \cup \mathcal{E}_{2h}(\Sigma)$. Now, let $\star \in \{\mathrm{D},\mathrm{S}\}$ and let $q \in [L^2(\Omega_\star)]^m$, with $m \in \{1,2\}$, such that $q|_T \in [C(T)]^m$ for each $T \in \mathcal{T}_h^\star$. Then, given $e \in \mathcal{E}_h(\Omega_\star)$, we denote by $[q]$ the jump of q across e, that is $[q] := (q|_{T'})|_e - (q|_{T''})|_e$, where T' and T'' are the triangles of $\mathcal{T}_h^\star$ having e as an edge. Also, we fix a unit normal vector $\mathbf{n}_e := (n_1, n_2)^{\tt t}$ to the edge e (its particular orientation is not relevant) and let $\mathbf{t}_e := (-n_2, n_1)^{\tt t}$ be the corresponding fixed unit tangential vector along e. Hence, given $\mathbf{v} \in \mathbf{L}^2(\Omega_\star)$ and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$ such that $\mathbf{v}|_T \in [C(T)]^2$ and $\boldsymbol{\tau}|_T \in [C(T)]^{2\times2}$, respectively, for each $T \in \mathcal{T}_h^\star$, we let $[\mathbf{v}\cdot\mathbf{t}_e]$ and

$[\boldsymbol{\tau}\,\mathbf{t}_e]$ be the tangential jumps of $\mathbf{v}$ and $\boldsymbol{\tau}$, across e, that is $[\mathbf{v}\cdot\mathbf{t}_e] := \{(\mathbf{v}|_{T'})|_e - (\mathbf{v}|_{T''})|_e\}\cdot\mathbf{t}_e$ and $[\boldsymbol{\tau}\,\mathbf{t}_e] := \{(\boldsymbol{\tau}|_{T'})|_e - (\boldsymbol{\tau}|_{T''})|_e\}\,\mathbf{t}_e$, respectively. From now on, when no confusion arises, we will simply write $\mathbf{t}$ and $\mathbf{n}$ instead of $\mathbf{t}_e$ and $\mathbf{n}_e$, respectively. Finally, for suffiently smooth scalar, vector and tensors fields q, $\mathbf{v} := (v_1, v_2)^{\mathtt{t}}$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, we let

$$\mathbf{curl}\,\mathbf{v} := \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix}, \qquad \mathrm{curl}\,q := \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1}\right)^{\mathtt{t}},$$

$$\mathrm{rot}\,\mathbf{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \text{and} \quad \mathbf{rot}\,\boldsymbol{\tau} := \left(\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2}\right)^{\mathtt{t}}.$$

Next, let $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X}\times\mathbb{M}$ and $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) := ((\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h, \lambda_h), (\mathbf{u}_{\mathrm{S},h}, p_{\mathrm{D},h})) \in \mathbb{X}_h\times\mathbb{M}_h$ be the unique solutions of (4.5) and (4.11), respectively. Then, we introduce the global a posteriori error estimator:

$$\Theta := \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \Theta_{\mathrm{S},T}^2 + \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \Theta_{\mathrm{D},T}^2 \right\}^{1/2}, \tag{4.13}$$

where, for each $T \in \mathcal{T}_h^{\mathrm{S}}$:

$$\begin{aligned}\Theta_{\mathrm{S},T}^2 &:= \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2\,\|\mathbf{rot}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 + h_T^2\,\|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e\,\|[\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}]\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e\,\|\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\,\|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\left\|\boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\right\|_{0,e}^2 + h_e\,\left\|\nu^{-1}\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t} + \boldsymbol{\varphi}_h'\right\|_{0,e}^2 \right\},\end{aligned}$$

and for each $T \in \mathcal{T}_h^{\mathrm{D}}$:

$$\begin{aligned}\Theta_{\mathrm{D},T}^2 &:= \|f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\,\|\mathrm{rot}\,(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 + h_T^2\,\|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{D}})} h_e\,\left\|[\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}]\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\,\left\|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} + \lambda_h'\right\|_{0,e}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 + h_e\,\|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \right\}.\end{aligned}$$

The derivatives $\boldsymbol{\varphi}_h'$ and λ_h' have to be understood as tangential derivatives in the direction imposed by the tangential vector field $\mathbf{t}$ on Σ.

4.3.1 Reliability of the a posteriori error estimator

The main result of this section is stated as follows.

Theorem 4.3.1 *There exists $C_{\texttt{rel}} > 0$, independent of h, such that*

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \le C_{\texttt{rel}}\,\Theta\,. \tag{4.14}$$

We begin the derivation of (4.14) by recalling that the continous dependence result given by (4.9) is equivalent to the global inf-sup condition for the continuous formulation (4.5). Then, applying this estimate to the error $(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h) \in \mathbb{X} \times \mathbb{M}$, we obtain

$$\|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \le C \sup_{\substack{(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})\in\mathbb{X}\times\mathbb{M}\\ (\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})\neq\mathbf{0}}} \frac{|R(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})|}{\|(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})\|_{\mathbb{X}\times\mathbb{M}}}\,, \tag{4.15}$$

where $R : \mathbb{X} \times \mathbb{M} \to \mathbb{R}$ is the residual functional

$$R(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}}) := \mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h) + \mathcal{B}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{v}}), \quad \forall\,(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}}) \in \mathbb{X}\times\mathbb{M}\,.$$

More precisely, according to (4.5) and the definitions of $\mathcal{A}$ and $\mathcal{B}$ (cf. (4.7), (4.8), we find that for any $(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}}) := ((\boldsymbol{\tau}_{\rm S}, \mathbf{v}_{\rm D}, \boldsymbol{\psi}, \xi), (\mathbf{v}_{\rm S}, q_{\rm D})) \in \mathbb{X}\times\mathbb{M}$ there holds

$$R(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}}) = R_1(\boldsymbol{\tau}_{\rm S}) + R_2(\mathbf{v}_{\rm D}) + R_3(\boldsymbol{\psi}) + R_4(\xi) + R_5(\mathbf{v}_{\rm S}) + R_6(q_{\rm D})\,,$$

where

$$\begin{aligned}
R_1(\boldsymbol{\tau}_{\rm S}) &:= -\nu^{-1}\int_{\Omega_{\rm S}} \boldsymbol{\sigma}^d_{{\rm S},h} : \boldsymbol{\tau}^d_{\rm S} - \int_{\Omega_{\rm S}} \mathbf{u}_{{\rm S},h}\cdot \mathbf{div}\,\boldsymbol{\tau}_{\rm S} - \langle \boldsymbol{\tau}_{\rm S}\,\mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma\,,\\
R_2(\mathbf{v}_{\rm D}) &:= -\int_{\Omega_{\rm D}} \mathbf{K}^{-1}\mathbf{u}_{{\rm D},h}\cdot\mathbf{v}_{\rm D} + \int_{\Omega_{\rm D}} p_{{\rm D},h}\,{\rm div}\,\mathbf{v}_{\rm D} + \langle \mathbf{v}_{\rm D}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma\,,\\
R_3(\boldsymbol{\psi}) &:= -\langle \boldsymbol{\sigma}_{{\rm S},h}\,\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma - \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma + \nu\,\kappa^{-1}\langle \boldsymbol{\psi}\cdot\mathbf{t}, \boldsymbol{\varphi}_h\cdot\mathbf{t}\rangle_\Sigma\,,\\
R_4(\xi) &:= \langle \mathbf{u}_{{\rm D},h}\cdot\mathbf{n}, \xi\rangle_\Sigma + \langle \boldsymbol{\varphi}_h\cdot\mathbf{n}, \xi\rangle_\Sigma\,,\\
R_5(\mathbf{v}_{\rm S}) &:= -\int_{\Omega_{\rm S}} \mathbf{v}_{\rm S}\cdot(\mathbf{f}_{\rm S} + \mathbf{div}\,\boldsymbol{\sigma}_{{\rm S},h})\,,
\end{aligned}$$

and

$$R_6(q_{\rm D}) := -\int_{\Omega_{\rm D}} q_{\rm D}\,(f_{\rm D} - {\rm div}\,\mathbf{u}_{{\rm D},h})\,.$$

Hence, the supremum in (4.15) can be bounded in terms of R_i, $i \in \{1,...,6\}$, which yields

$$\begin{aligned}
\|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} &\le C\,\Big\{\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_{\rm S})'} + \|R_2\|_{\mathbf{H}_{\Gamma_{\rm D}}({\rm div};\Omega_{\rm D})'}\\
&+\|R_3\|_{\mathbf{H}^{1/2}(\Sigma)'} + \|R_4\|_{H^{1/2}(\Sigma)'} + \|R_5\|_{\mathbf{L}^2(\Omega_{\rm S})'} + \|R_6\|_{L^2_0(\Omega_{\rm D})'}\Big\}\,.
\end{aligned} \tag{4.16}$$

Throughout the rest of this section we provide suitable upper bounds for each one of the terms on the right hand side of (4.16). The following lemma, whose proof follows from straightforward applications of the Cauchy-Schwarz inequality, is stated first.

Lemma 4.3.1 *There hold*

$$\|R_5\|_{\mathbf{L}^2(\Omega_S)'} = \|\mathbf{f}_S + \mathbf{div}\,\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S} = \left\{ \sum_{T\in\mathcal{T}_h^S} \|\mathbf{f}_S + \mathbf{div}\,\boldsymbol{\sigma}_{S,h}\|_{0,T}^2 \right\}^{1/2}, \tag{4.17}$$

and

$$\|R_6\|_{L_0^2(\Omega_D)'} \le \|f_D - \operatorname{div}\mathbf{u}_{D,h}\|_{0,\Omega_D} = \left\{ \sum_{T\in\mathcal{T}_h^D} \|f_D - \operatorname{div}\mathbf{u}_{D,h}\|_{0,T}^2 \right\}^{1/2}. \tag{4.18}$$

The next lemma estimates the supremums on the spaces defined in the interface Σ.

Lemma 4.3.2 *There exist C_3, $C_4 > 0$, independent of h, such that*

$$\|R_3\|_{\mathbf{H}^{1/2}(\Sigma)'} \le C_3 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\|\boldsymbol{\sigma}_{S,h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\right\|_{0,e}^2 \right\}^{1/2}, \tag{4.19}$$

and

$$\|R_4\|_{H^{1/2}(\Sigma)'} \le C_4 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\mathbf{u}_{D,h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 \right\}^{1/2}. \tag{4.20}$$

Proof. It is clear from the definition of R_3 that

$$R_3(\boldsymbol{\psi}) = -\langle \boldsymbol{\sigma}_{S,h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}, \boldsymbol{\psi}\rangle_\Sigma \qquad \forall\boldsymbol{\psi}\in\mathbf{H}^{1/2}(\Sigma),$$

and hence

$$\|R_3\|_{\mathbf{H}^{1/2}(\Sigma)'} = \left\|\boldsymbol{\sigma}_{S,h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\right\|_{-1/2,\Sigma}. \tag{4.21}$$

In order to estimate $\left\|\boldsymbol{\sigma}_{S,h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\right\|_{-1/2,\Sigma}$ in terms of local quantities we now apply a technical result from [27]. Taking $\boldsymbol{\tau}_S = \mathbf{0}$, $\mathbf{v}_D = \mathbf{0}$ and $\xi = 0$ in the first equation of (4.11), we have

$$\langle \boldsymbol{\sigma}_{S,h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}, \boldsymbol{\psi}\rangle_\Sigma = 0 \qquad \forall\,\boldsymbol{\psi} \in \boldsymbol{\Lambda}_h(\Sigma),$$

which says that $\boldsymbol{\sigma}_{S,h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}$ is $\mathbf{L}^2(\Sigma)$-orthogonal to $\boldsymbol{\Lambda}_h(\Sigma)$. Hence, applying [27, Theorem 2], and recalling that Σ_h and Σ_{2h} are of bounded variation, we deduce that

$$\begin{aligned}
&\left\|\boldsymbol{\sigma}_{S,h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\right\|_{-1/2,\Sigma}^2 \\
&\quad\le C \sum_{e\in\mathcal{E}_{2h}(\Sigma)} h_e \left\|\boldsymbol{\sigma}_{S,h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\right\|_{0,e}^2 \\
&\quad\le C \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\|\boldsymbol{\sigma}_{S,h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\right\|_{0,e}^2,
\end{aligned}$$

which, together with (4.21), yields (4.19).

The proof of (4.20) proceeds analogously. Since

$$\|R_4\|_{H^{1/2}(\Sigma)'} = \|\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} + \boldsymbol{\varphi}_h \cdot \mathbf{n}\|_{-1/2,\Sigma},$$

and $\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} + \boldsymbol{\varphi}_h \cdot \mathbf{n}$ is $L^2(\Sigma)$-orthogonal to $\Lambda_h(\Sigma)$ (this is a consequence of the first equation of (4.11)), another straightforward application of [27, Theorem 2] yields the required estimate. □

Our next goal is to bound the remaining terms in right hand side of (4.16), for which we need some preliminary results. We begin with the following lemma showing the existence of stable Helmholtz decompositions for $\mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div};\Omega_\mathrm{D})$ and $\mathbb{H}(\mathbf{div};\Omega_\mathrm{S})$.

Lemma 4.3.3

a) *There exists* $C_\mathrm{D} > 0$ *such that every* $\mathbf{v}_\mathrm{D} \in \mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div};\Omega_\mathrm{D})$ *can be decomposed as* $\mathbf{v}_\mathrm{D} = \mathbf{w} + \mathrm{curl}\,\beta$, *where* $\mathbf{w} \in \mathbf{H}^1(\Omega_\mathrm{D})$, $\beta \in H^1(\Omega_\mathrm{D})$, $\int_{\Omega_\mathrm{S}} \beta = 0$ *and*

$$\|\mathbf{w}\|_{1,\Omega_\mathrm{D}} + \|\beta\|_{1,\Omega_\mathrm{D}} \leq C_\mathrm{D}\,\|\mathbf{v}_\mathrm{D}\|_{\mathrm{div}\,;\Omega_\mathrm{D}}.$$

b) *There exists* $C_\mathrm{S} > 0$ *such that every* $\boldsymbol{\tau}_\mathrm{S} \in \mathbb{H}(\mathbf{div};\Omega_\mathrm{S})$ *can be decomposed as* $\boldsymbol{\tau}_\mathrm{S} = \boldsymbol{\eta} + \mathbf{curl}\,\boldsymbol{\chi}$, *where* $\boldsymbol{\eta} \in \mathbb{H}^1(\Omega_\mathrm{S})$, $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_\mathrm{S})$ *and*

$$\|\boldsymbol{\eta}\|_{1,\Omega_\mathrm{S}} + \|\boldsymbol{\chi}\|_{1,\Omega_\mathrm{S}} \leq C_\mathrm{S}\,\|\boldsymbol{\tau}_\mathrm{S}\|_{\mathbf{div}\,;\Omega_\mathrm{S}}.$$

Proof. Let G be a convex domain with smooth boundary that contains Ω_D. Given $\mathbf{v}_\mathrm{D} \in \mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div};\Omega_\mathrm{D})$, we take $z \in H_0^1(G) \cap H^2(G)$ to be the unique solution of

$$-\Delta z = \begin{Bmatrix} \mathrm{div}\,\mathbf{v}_\mathrm{D} & \text{in} & \Omega_\mathrm{D} \\ 0 \quad \text{in} \quad G \setminus \bar{\Omega}_\mathrm{D} & & \end{Bmatrix} \quad \text{in} \quad G, \quad z = 0 \quad \text{on} \quad \partial G.$$

It follows that

$$\|z\|_{2,G} \leq C\,\|\mathrm{div}\,\mathbf{v}_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \leq C\,\|\mathbf{v}_\mathrm{D}\|_{\mathrm{div}\,;\Omega_\mathrm{D}},$$

and hence, defining $\mathbf{w} := -\nabla z$ in Ω_D, we find that div $\mathbf{w} = \mathrm{div}\,\mathbf{v}_\mathrm{D}$ in Ω_D and

$$\|\mathbf{w}\|_{1,\Omega_\mathrm{D}} \leq \|z\|_{2,\Omega_\mathrm{D}} \leq \|z\|_{2,G} \leq C\,\|\mathbf{v}_\mathrm{D}\|_{\mathrm{div}\,;\Omega_\mathrm{D}}.$$

In addition, since $\mathrm{div}\,(\mathbf{v}_\mathrm{D} - \mathbf{w}) = 0$ and Ω_D is connected, there exists $\beta \in H^1(\Omega_\mathrm{D})$, with $\int_{\Omega_\mathrm{D}} \beta = 0$, such that $\mathbf{v}_\mathrm{D} - \mathbf{w} = \mathrm{curl}\,\beta$ in Ω_D. In this way, using the generalized Poincaré inequality and the above estimate for $\mathbf{w}$, we deduce that

$$\|\beta\|_{1,\Omega_\mathrm{D}} \leq C\,|\beta|_{1,\Omega_\mathrm{D}} = C\,\|\mathrm{curl}\,\beta\|_{0,\Omega_\mathrm{D}} = C\,\|\mathbf{v}_\mathrm{D} - \mathbf{w}\|_{0,\Omega_\mathrm{D}} \leq C\|\mathbf{v}_\mathrm{D}\|_{\mathrm{div}\,;\Omega_\mathrm{D}},$$

which completes the proof of a).

We now let $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div}\,;\Omega_{\mathrm{S}})$. Since Ω_{S} is not necessarily connected, we first perform a suitable extension of $\boldsymbol{\tau}_{\mathrm{S}}$ to the domain $\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$, and then apply a) to each row of the resulting tensor. More precisely, let $\boldsymbol{\tau}_{\mathrm{S},i} \in \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{S}})$ be the i-th row of $\boldsymbol{\tau}_{\mathrm{S}}$, $i \in \{1,2\}$, and let $\phi_i \in H^1(\Omega_{\mathrm{D}})$ be the unique solution of the Neumann problem:

$$\Delta\phi_i \,=\, -\,\frac{\langle \boldsymbol{\tau}_{\mathrm{S},i}\cdot\mathbf{n}, 1\rangle_\Sigma}{|\Omega_{\mathrm{D}}|} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \qquad \frac{\partial\phi_i}{\partial\mathbf{n}} \,=\, \boldsymbol{\tau}_{\mathrm{S},i}\cdot\mathbf{n} \quad \text{on} \quad \Sigma\,, \qquad \int_{\Omega_{\mathrm{D}}} \phi_i \,=\, 0\,.$$

Then we define $\boldsymbol{\tau}_i^{ext} \,=\, \begin{cases} \boldsymbol{\tau}_{\mathrm{S},i} & \text{in} \quad \Omega_{\mathrm{S}} \\ \nabla\phi_i & \text{in} \quad \Omega_{\mathrm{D}} \end{cases}$, and notice that $\boldsymbol{\tau}_i^{ext} \in \mathbf{H}(\mathrm{div}\,;\Omega)$ and

$$\begin{aligned} \|\boldsymbol{\tau}_i^{ext}\|_{\mathrm{div}\,;\Omega} \,&\le\, \|\boldsymbol{\tau}_{\mathrm{S},i}\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}} \,+\, \|\nabla\phi_i\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}} \\ &\le\, \|\boldsymbol{\tau}_{\mathrm{S},i}\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}} \,+\, C\,\|\boldsymbol{\tau}_{\mathrm{S},i}\cdot\mathbf{n}\|_{-1/2,\Sigma} \,\le\, C\,\|\boldsymbol{\tau}_{\mathrm{S},i}\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}}\,. \end{aligned}$$

Proceeding as in the proof of a), but now for $\boldsymbol{\tau}_i^{ext} \in \mathbf{H}(\mathrm{div}\,;\Omega)$, we deduce the existence of $\mathbf{w}_i \in \mathbf{H}^1(\Omega)$ and $\beta_i \in H^1(\Omega)$, with $\displaystyle\int_\Omega \beta_i \,=\, 0$, such that $\boldsymbol{\tau}_i^{ext} \,=\, \mathbf{w}_i \,+\, \mathrm{curl}\,\beta_i$ in Ω, and

$$\|\mathbf{w}_i\|_{1,\Omega} \,+\, \|\beta_i\|_{1,\Omega} \,\le\, C\,\|\boldsymbol{\tau}_i^{ext}\|_{\mathrm{div}\,;\Omega} \,\le\, C\,\|\boldsymbol{\tau}_{\mathrm{S},i}\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}}\,.$$

Hence, the proof of b) follows by defining i-th row of $\boldsymbol{\eta} \,:=\, \mathbf{w}_i|_{\Omega_{\mathrm{S}}}$ and $\boldsymbol{\chi} \,:=\, (\beta_1|_{\Omega_{\mathrm{S}}}, \beta_2|_{\Omega_{\mathrm{S}}})$. □

We next recall two well–known approximation operators: the Raviart-Thomas interpolator (see [19] for example) and the Clément operator onto the space of continuous piecewise linear functions [25].

The Raviart-Thomas interpolation operator $\Pi_h^\star \,:\, \mathbf{H}^1(\Omega_\star) \,\to\, \mathbf{H}_h(\Omega_\star)$ (recall the discrete spaces in (4.10)), $\star \in \{\mathrm{S},\mathrm{D}\}$, is given by the conditions

$$\Pi_h^\star\mathbf{v} \,\in\, \mathbf{H}_h(\Omega_\star) \quad \text{and} \quad \int_e \Pi_h^\star\mathbf{v}\cdot\mathbf{n} \,=\, \int_e \mathbf{v}\cdot\mathbf{n} \qquad \forall \text{ edge } e \text{ of } \mathcal{T}_h^\star\,. \tag{4.22}$$

As a consequence of (4.22), there holds

$$\mathrm{div}\,(\Pi_h^\star\mathbf{v}) \,=\, \mathcal{P}_h^\star(\mathrm{div}\,\mathbf{v})\,, \tag{4.23}$$

where $\mathcal{P}_h^\star$, $\star \in \{\mathrm{S},\mathrm{D}\}$, is the $L^2(\Omega_\star)$-orthogonal projector onto the piecewise constant functions on $\Omega_\star$. A tensor version of $\Pi_h^\star$, say $\boldsymbol{\Pi}_h^\star : \mathbb{H}^1(\Omega_\star) \to \mathbb{H}_h(\Omega_\star)$, which is defined row-wise by $\Pi_h^\star$, and a vector version of $\mathcal{P}_h^\star$, say $\mathbf{P}_h^\star$, which is the $\mathbf{L}^2(\Omega_\star)$-orthogonal projector onto the piecewise constant vectors on $\Omega_\star$, might also be required. The local approximation properties of $\Pi_h^\star$ (and hence of $\boldsymbol{\Pi}_h^\star$) are stated as follows.

Lemma 4.3.4 *For each $\star \in \{\mathrm{S},\mathrm{D}\}$ there exist constants c_1, $c_2 > 0$, independent of h, such that for all $\mathbf{v} \in \mathbf{H}^1(\Omega_\star)$ there hold*

$$\|\mathbf{v} - \Pi_h^\star \mathbf{v}\|_{0,T} \,\le\, c_1\, h_T\, \|\mathbf{v}\|_{1,T} \qquad \forall\, T \,\in\, \mathcal{T}_h^\star\,,$$

and

$$\|\mathbf{v}\cdot\mathbf{n} - \Pi_h^\star \mathbf{v}\cdot\mathbf{n}\|_{0,e} \,\le\, c_2\, h_e^{1/2}\, \|\mathbf{v}\|_{1,T_e} \qquad \forall \text{ edge } e \text{ of } \mathcal{T}_h^\star\,,$$

where T_e is a triangle of $\mathcal{T}_h^\star$ containing e on its boundary.

Proof. See [19]. □

The Clément operators $I_h^\star : H^1(\Omega_\star) \to X_{\star,h}$ approximate optimally non–smooth functions by continuous piecewise linear functions:

$$X_{\star,h} \,:=\, \{v \,\in\, C(\bar{\Omega}_\star):\quad v|_T \,\in\, \mathbf{P}_1(T) \quad \forall\, T \,\in\, \mathcal{T}_h^\star\} \qquad \text{for each } \star \,\in\, \{\mathrm{S},\mathrm{D}\}\,.$$

Of this operator, we will only use its approximation properties (see below). In addition, we will make use of a vector version of $I_h^\star$, say $\mathbf{I}_h^\star : \mathbf{H}^1(\Omega_\star) \to \mathbf{X}_{\star,h} := X_{\star,h} \times X_{\star,h}$, which is defined componentwise by $I_h^\star$. The following lemma establishes the local approximation properties of $I_h^\star$ (and hence of $\mathbf{I}_h^\star$).

Lemma 4.3.5 *For each $\star \in \{\mathrm{S},\mathrm{D}\}$ there exist constants c_3, $c_4 > 0$, independent of h, such that for all $v \in H^1(\Omega_\star)$ there hold*

$$\|v - I_h^\star v\|_{0,T} \,\le\, c_3\, h_T\, \|v\|_{1,\Delta_\star(T)} \qquad \forall\, T \,\in\, \mathcal{T}_h^\star\,,$$

and

$$\|v - I_h^\star v\|_{0,e} \,\le\, c_4\, h_e^{1/2}\, \|v\|_{1,\Delta_\star(e)} \qquad \forall\, e \in \mathcal{E}_h\,,$$

where

$$\Delta_\star(T) \,:=\, \cup\{T' \,\in\, \mathcal{T}_h^\star:\quad T'\cap T \neq \mathbf{0}\} \quad \text{and} \quad \Delta_\star(e) \,:=\, \cup\{T' \,\in\, \mathcal{T}_h^\star:\quad T'\cap e \neq \mathbf{0}\}\,.$$

Proof. See [30]. □

Finally, we require the technical results given by the following two lemmas.

Lemma 4.3.6 *Let $\boldsymbol{\eta} \in \mathbb{H}^1(\Omega_\mathrm{S})$ and $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_\mathrm{S})$. Then there hold*

$$|R_1(\boldsymbol{\eta} - \boldsymbol{\Pi}_h^\mathrm{S}\boldsymbol{\eta})| \,\le\, c_1\,\nu^{-1} \sum_{T\in\mathcal{T}_h^\mathrm{S}} h_T\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}\, \|\boldsymbol{\eta}\|_{1,T} \,+\, c_2 \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2}\, \|\mathbf{u}_{\mathrm{S},h} \,+\, \boldsymbol{\varphi}_h\|_{0,e}\, \|\boldsymbol{\eta}\|_{1,T_e}\,,$$

and

$$
\begin{aligned}
&|R_1(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}))| \\
&\le c_3\,\nu^{-1} \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} h_T\,\|\mathbf{rot}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}\,\|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(T)} + c_4\,\nu^{-1} \sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e^{1/2}\,\|[\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}]\|_{0,e}\,\|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(e)} \\
&+ c_4\,\nu^{-1} \sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e^{1/2}\,\|\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}\|_{0,e}\,\|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(e)} + c_4 \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2}\,\left\|\nu^{-1}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t} + \boldsymbol{\varphi}_h'\right\|_{0,e}\|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(e)}.
\end{aligned}
$$

Proof. We first let $\boldsymbol{\zeta} := \boldsymbol{\eta} - \boldsymbol{\Pi}_h^{\mathrm{S}}\boldsymbol{\eta}$ and observe, according to (4.22) and (4.23), that

$$
\int_e \mathbf{p}\cdot\boldsymbol{\zeta}\,\mathbf{n} = 0 \quad \forall\,\mathbf{p}\in[\mathbb{P}_0(e)]^2\,, \quad \forall \text{ edge } e \text{ of } \mathcal{T}_h^{\mathrm{S}}\,, \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\zeta} = \mathbf{div}\,\boldsymbol{\eta} - \mathbf{P}_h^{\mathrm{S}}(\mathbf{div}\,\boldsymbol{\eta})\,.
$$

Then, since $\boldsymbol{\sigma}_{\mathrm{S},h}^d : \boldsymbol{\zeta}^d = \boldsymbol{\sigma}_{\mathrm{S},h}^d : \boldsymbol{\zeta}$ and $\mathbf{u}_{\mathrm{S},h}$ is constant on each $T \in \mathcal{T}_h^{\mathrm{S}}$, we deduce from the definition of R_1 and the above identities that

$$
\begin{aligned}
R_1(\boldsymbol{\zeta}) = &\; -\nu^{-1} \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \int_T \boldsymbol{\sigma}_{\mathrm{S},h}^d : \boldsymbol{\zeta}^d - \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \int_T \mathbf{u}_{\mathrm{S},h}\cdot\mathbf{div}\,\boldsymbol{\zeta} - \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \boldsymbol{\varphi}_h\cdot\boldsymbol{\zeta}\,\mathbf{n} \\
= &\; -\nu^{-1} \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \int_T \boldsymbol{\sigma}_{\mathrm{S},h}^d : \boldsymbol{\zeta} - \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \boldsymbol{\varphi}_h\cdot\boldsymbol{\zeta}\,\mathbf{n} \\
= &\; -\nu^{-1} \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \int_T \boldsymbol{\sigma}_{\mathrm{S},h}^d : \boldsymbol{\zeta} - \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e (\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h)\cdot\boldsymbol{\zeta}\,\mathbf{n}\,.
\end{aligned}
$$

We next let $\boldsymbol{\rho} := \boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}$. Then, using that $\mathbf{div}\,(\mathbf{curl}\,\boldsymbol{\rho}) = \mathbf{0}$, noting that $(\mathbf{curl}\,\boldsymbol{\rho})\,\mathbf{n} = (\nabla\boldsymbol{\rho})\,\mathbf{t}$ on Σ, integrating by parts on each $T \in \mathcal{T}_h^{\mathrm{S}}$ and on Σ, and observing that $\boldsymbol{\varphi}_h' \in \mathbf{L}^2(\Sigma)$, we obtain

$$
\begin{aligned}
R_1(\mathbf{curl}\,\boldsymbol{\rho}) &= -\nu^{-1} \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}_{\mathrm{S},h}^d : \mathbf{curl}\,\boldsymbol{\rho} - \langle(\mathbf{curl}\,\boldsymbol{\rho})\,\mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma \\
&= \nu^{-1} \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \left(-\int_T \boldsymbol{\rho}\cdot\mathbf{rot}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d + \int_{\partial T} \boldsymbol{\rho}\cdot\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}\right) + \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \boldsymbol{\rho}\cdot\boldsymbol{\varphi}_h' \\
&= -\sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \nu^{-1} \int_T \boldsymbol{\rho}\cdot\mathbf{rot}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d + \sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{S}})} \nu^{-1} \int_e \boldsymbol{\rho}\cdot[\boldsymbol{\sigma}_{\mathrm{S},h}^d\,\mathbf{t}] \\
&\quad + \sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{S}})} \nu^{-1} \int_e \boldsymbol{\rho}\cdot(\boldsymbol{\sigma}_{\mathrm{S},h}^d\,\mathbf{t}) + \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \boldsymbol{\rho}\cdot\left(\nu^{-1}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d\,\mathbf{t} + \boldsymbol{\varphi}_h'\right)\,.
\end{aligned}
$$

Hence, straightforward applications of the Cauchy-Schwarz inequality to the above identities, together with the approximation properties of Lemmas 4.3.4 and 4.3.5, namely,

$$
\|\boldsymbol{\eta} - \boldsymbol{\Pi}_h^{\mathrm{S}}\boldsymbol{\eta}\|_{0,T} \le c_1\,h_T\,\|\boldsymbol{\eta}\|_{1,T}\,, \qquad \|\boldsymbol{\eta}\,\mathbf{n} - \boldsymbol{\Pi}_h^{\mathrm{S}}\boldsymbol{\eta}\,\mathbf{n}\|_{0,e} \le c_2\,h_e^{1/2}\,\|\boldsymbol{\eta}\|_{1,T}
$$

$$
\|\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}\|_{0,T} \le c_3\,h_T\,\|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(T)}\,, \qquad \|\boldsymbol{\chi} - \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}\|_{0,e} \le c_4\,h_e^{1/2}\,\|\boldsymbol{\chi}\|_{1,\Delta_{\mathrm{S}}(e)}\,,
$$

for each $T \in \mathcal{T}_h^{\mathrm{S}}$ and for each $e \in \mathcal{E}(T)$, imply the required estimates and finish the proof. □

Lemma 4.3.7 *Let* $\mathbf{w} \in \mathbf{H}^1(\Omega_D)$ *and* $\beta \in H^1(\Omega_D)$*. Then there hold*

$$|R_2(\mathbf{w} - \Pi_h^{\mathrm{D}}\mathbf{w})| \,\le\, c_1 \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} h_T\, \|\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T}\, \|\mathbf{w}\|_{1,T} \,+\, c_2 \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2}\, \|p_{\mathrm{D},h} \,-\, \lambda_h\|_{0,e}\, \|\mathbf{w}\|_{1,T_e}\,,$$

and

$$\begin{aligned} |R_2(\mathrm{curl}\,(\beta - I_h^{\mathrm{D}}\beta))| \;&\le\; c_3 \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} h_T\, \|\mathrm{rot}\,(\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h})\|_{0,T}\, \|\beta\|_{1,\Delta_{\mathrm{D}}(T)} \\ &+\; c_4 \sum_{e\in\mathcal{E}_h(\Omega_{\mathrm{D}})} h_e^{1/2}\, \|[\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}]\|_{0,e}\, \|\beta\|_{1,\Delta_{\mathrm{D}}(e)} \\ &+\; c_4 \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2}\, \left\|\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} + \lambda_h'\right\|_{0,e}\, \|\beta\|_{1,\Delta_{\mathrm{D}}(e)}\,. \end{aligned}$$

Proof. Since R_1 and R_2 have analogue structures, the proof proceeds similarly as for Lemma 4.3.6. □

We are now in a position to bound the residual functionals R_1 and R_2.

Lemma 4.3.8 *There exists* $C_1 > 0$*, independent of* h*, such that*

$$\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})'} \;\le\; C_1 \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \widehat{\Theta}_{\mathrm{S},T}^2 \right\}^{1/2}, \tag{4.24}$$

where, for each $T \in \mathcal{T}_h^{\mathrm{S}}$:

$$\begin{aligned} \widehat{\Theta}_{\mathrm{S},T}^2 \;:=\; & h_T^2\, \|\mathbf{rot}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \;+\; h_T^2\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e\, \|[\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}]\|_{0,e}^2 \,+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}\|_{0,e}^2 \\ &+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\, \left\|\nu^{-1}\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t} + \boldsymbol{\varphi}_h'\right\|_{0,e}^2 \,+\, h_e\, \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \right\} \end{aligned}$$

Proof. Given $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ we know from Lemma 4.3.3 that there exist $\boldsymbol{\eta} \in \mathbb{H}^1(\Omega_{\mathrm{S}})$ and $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$ such that $\boldsymbol{\tau}_{\mathrm{S}} = \boldsymbol{\eta} + \mathbf{curl}\,\boldsymbol{\chi}$ in Ω_{S} and

$$\|\boldsymbol{\eta}\|_{1,\Omega_{\mathrm{S}}} \,+\, \|\boldsymbol{\chi}\|_{1,\Omega_{\mathrm{S}}} \,\le\, C\, \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div}\,;\Omega_{\mathrm{S}}}\,. \tag{4.25}$$

Then, since $R_1(\boldsymbol{\tau}_{\mathrm{S},h}) = 0 \quad \forall\, \boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_h(\Omega_{\mathrm{S}})$, which follows from the first equation of the Galerkin scheme (4.11) taking $(\mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \boldsymbol{\xi}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$, we obtain

$$R_1(\boldsymbol{\tau}_{\mathrm{S}}) \,=\, R_1(\boldsymbol{\tau}_{\mathrm{S}} \,-\, \boldsymbol{\tau}_{\mathrm{S},h}) \qquad \forall\, \boldsymbol{\tau}_{\mathrm{S},h} \,\in\, \mathbb{H}_h(\Omega_{\mathrm{S}})\,. \tag{4.26}$$

In particular, we let $\boldsymbol{\tau}_{\mathrm{S},h} := \mathbf{\Pi}_h^{\mathrm{S}}\boldsymbol{\eta} + \mathbf{curl}\,(\mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi})$, which can be seen as a discrete Helmholtz decomposition of $\boldsymbol{\tau}_{\mathrm{S},h}$, and obtain

$$R_1(\boldsymbol{\tau}_{\mathrm{S}}) \,=\, R_1(\boldsymbol{\eta} \,-\, \mathbf{\Pi}_h^{\mathrm{S}}\boldsymbol{\eta}) \,+\, R_1(\mathbf{curl}\,(\boldsymbol{\chi} \,-\, \mathbf{I}_h^{\mathrm{S}}\boldsymbol{\chi}))\,. \tag{4.27}$$

Hence, applying Lemma 4.3.6 and noticing that the numbers of triangles in $\#\Delta_{\rm S}(T)$ and $\#\Delta_{\rm S}(e)$ are bounded, and finally using the estimate (4.25), we prove the upper bound (4.24). $\square$

Lemma 4.3.9 *There exists $C_2 > 0$, independent of h, such that*

$$\|R_2\|_{\mathbf{H}_{\Gamma_{\rm D}}(\mathrm{div};\Omega_{\rm D})'} \;\leq\; C_2 \left\{ \sum_{T\in\mathcal{T}_h^{\rm D}} \widehat{\Theta}_{{\rm D},T}^2 \right\}^{1/2}, \tag{4.28}$$

where, for each $T \in \mathcal{T}_h^{\rm D}$:

$$\widehat{\Theta}_{{\rm D},T}^2 \;:=\; h_T^2\,\|\mathrm{rot}\,(\mathbf{K}^{-1}\mathbf{u}_{{\rm D},h})\|_{0,T}^2 \;+\; h_T^2\,\|\mathbf{K}^{-1}\mathbf{u}_{{\rm D},h}\|_{0,T}^2 \;+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\rm D})} h_e\,\left\|[\mathbf{K}^{-1}\mathbf{u}_{{\rm D},h}\cdot\mathbf{t}]\right\|_{0,e}^2$$
$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\left\|\mathbf{K}^{-1}\mathbf{u}_{{\rm D},h}\cdot\mathbf{t}+\lambda_h'\right\|_{0,e}^2 \;+\; h_e\,\|p_{{\rm D},h}-\lambda_h\|_{0,e}^2 \right\}.$$

Proof. It follows basically the same lines of the proof of Lemma 4.3.8. In fact, given $\mathbf{v}_{\rm D} \in \mathbf{H}_{\Gamma_{\rm D}}(\mathrm{div};\Omega_{\rm D})$ we first apply Lemma 4.3.3 to deduce the existence of $\mathbf{w} \in \mathbf{H}^1(\Omega_{\rm D})$ and $\beta \in H^1(\Omega_{\rm D})$ such that $\mathbf{v}_{\rm D} = \mathbf{w} + \mathrm{curl}\,\beta$ and

$$\|\mathbf{w}\|_{1,\Omega_{\rm D}} \;+\; \|\beta\|_{1,\Omega_{\rm D}} \;\leq\; C\,\|\mathbf{v}_{\rm D}\|_{\mathrm{div}\,;\Omega_{\rm D}}\,. \tag{4.29}$$

Then, since $R_2(\mathbf{v}_{{\rm D},h}) \;=\; 0 \quad \forall\,\mathbf{v}_{{\rm D},h} \in \mathbf{H}_h(\Omega_{\rm D})$, which corresponds to the first equation of the Galerkin scheme (4.11) with $(\boldsymbol{\tau}_{\rm S},\boldsymbol{\psi},\boldsymbol{\xi}) = (\mathbf{0},\mathbf{0},\mathbf{0})$, we obtain

$$R_2(\mathbf{v}_{\rm D}) \;=\; R_2(\mathbf{v}_{\rm D} \,-\, \mathbf{v}_{{\rm D},h}) \qquad \forall\,\mathbf{v}_{{\rm D},h} \in \mathbf{H}_h(\Omega_{\rm D})\,. \tag{4.30}$$

Next, we choose $\mathbf{v}_{{\rm D},h} \;=\; \Pi_h^{\rm D}\mathbf{w} \,+\, \mathrm{curl}\,\left(I_h^{\rm D}\beta\right)$, notice that

$$R_2(\mathbf{v}_{\rm D}) = R_2(\mathbf{w}-\Pi_h^{\rm D}\mathbf{w}) \,+\, R_2\left(\mathrm{curl}\,(\beta - I_h^{\rm D}\beta)\right)\,,$$

and apply Lemma 4.3.7. Noticing again that the number of triangles in $\Delta_{\rm D}(T)$ and $\Delta_{\rm D}(e)$ are bounded, and employing now the upper bound (4.29), we conclude (4.28). $\square$

We end this section by observing that the reliability estimate (4.14) (cf. Theorem 4.3.1) is a direct consequence of Lemmas 4.3.1, 4.3.2, 4.3.8, and 4.3.9.

4.3.2 Efficiency of the a posteriori error estimator

The main result of this section is stated as follows.

Theorem 4.3.2 *There exists $C_{\tt eff} > 0$, independent of h, such that*

$$C_{\tt eff}\,\Theta \;\leq\; \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} \,+\, \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \;+\; \text{h.o.t.}\,, \tag{4.31}$$

where h.o.t. *stands, eventually, for one or several terms of higher order.*

We remark in advance that the proof of (4.31) makes frequent use of the identities provided by Theorem 4.2.2. We begin with the estimates for the zero order terms appearing in the definition of $\Theta^2_{S,T}$ and $\Theta^2_{D,T}$.

Lemma 4.3.10 *There hold*

$$\|\mathbf{f}_S + \mathbf{div}\, \boldsymbol{\sigma}_{S,h}\|_{0,T} \le \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{\mathbf{div};T} \qquad \forall\, T \in \mathcal{T}_{S,h}$$

and

$$\|f_D - \operatorname{div} \mathbf{u}_{D,h}\|_{0,T} \le \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{\operatorname{div};T} \qquad \forall\, T \in \mathcal{T}_{D,h}\,.$$

Proof. It suffices to recall, as established by Theorem 4.2.2, that $\mathbf{f}_S = -\mathbf{div}\,\boldsymbol{\sigma}_S$ in Ω_S and $f_D = \operatorname{div} \mathbf{u}_D$ in Ω_D. □

In order to derive the upper bounds for the remaining terms defining the global a posteriori error estimator Θ (cf. (4.13), we proceed similarly as in [15], using results from [25], [28] and [40], and apply Helmholtz decomposition, inverse inequalities, and the localization technique based on element-bubble and edge-bubble functions. To this end, we now introduce further notations and preliminary results. Given $T \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$ and $e \in \mathcal{E}(T)$, we let ϕ_T and ϕ_e be the usual element-bubble and edge-bubble functions, respectively (see (1.5) and (1.6) in [81]). In particular, ϕ_T satisfies $\phi_T \in \mathbf{P}_3(T)$, $\operatorname{supp}\phi_T \subseteq T$, $\phi_T = 0$ on ∂T, and $0 \le \phi_T \le 1$ in T. Similarly, $\phi_e|_T \in \mathbf{P}_2(T)$, $\operatorname{supp}\phi_e \subseteq w_e := \cup\{T' \in \mathcal{T} \,:\, e \in \mathcal{E}(T')\}$, $\phi_e = 0$ on $\partial T\backslash e$, and $0 \le \phi_e \le 1$ in w_e. We also recall from [80] that, given $k \in \mathbb{N} \cup \{0\}$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p) \in \mathbf{P}_k(T)$ and $L(p)|_e = p \;\forall\, p \in \mathbf{P}_k(e)$. A corresponding vector version of L, that is the componentwise application of L, is denoted by $\mathbf{L}$. Additional properties of ϕ_T, ϕ_e, and L are collected in the following lemma.

Lemma 4.3.11 *Given $k \in \mathbb{N} \cup \{0\}$, there exist positive constants c_1, c_2 and c_3, depending only on k and the shape regularity of the triangulations (minimum angle condition), such that for each triangle T and $e \in \mathcal{E}(T)$, there hold*

$$\|q\|^2_{0,T} \le c_1\, \|\phi_T^{1/2} q\|^2_{0,T} \qquad \forall\, q \in \mathbf{P}_k(T), \tag{4.32}$$

$$\|q\|^2_{0,e} \le c_2\, \|\phi_e^{1/2} q\|^2_{0,e} \qquad \forall\, q \in \mathbf{P}_k(e), \tag{4.33}$$

and

$$\|\phi_e^{1/2} L(q)\|^2_{0,T} \le c_3\, h_e \|q\|^2_{0,e} \qquad \forall\, q \in \mathbf{P}_k(e)\,. \tag{4.34}$$

Proof. See Lemma 1.3 in [80]. □

The following inverse estimate for polynomials will also be used.

Lemma 4.3.12 *Let k, l, $m \in \mathbb{N} \cup \{0\}$ such that $l \leq m$. Then, there exists $c > 0$, depending only on k, l, m and the shape regularity of the triangulations, such that for each triangle T there holds*

$$|q|_{m,T} \leq c\, h_T^{l-m} |q|_{l,T}, \forall\, q \in \mathbf{P}_k(T)\,. \tag{4.35}$$

Proof. See Theorem 3.2.6 in [29]. □

In addition, we need to recall a discrete trace inequality, which establishes the existence of a positive constant c, depending only on the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ and $e \in \mathcal{E}(T)$, there holds

$$\|v\|_{0,e}^2 \leq c \left\{ h_e^{-1} \|v\|_{0,T}^2 + h_e\, |v|_{1,T}^2 \right\} \qquad \forall\, v \in H^1(T)\,. \tag{4.36}$$

For a proof of inequality (4.36) we refer to Theorem 3.10 in [1] (see also eq. (2.4) in [5]).

The following lemma summarizes known efficiency estimates for ten terms defining $\Theta_{\mathrm{S},T}^2$ and $\Theta_{\mathrm{D},T}^2$. Their proofs, which apply the preliminary results described above, are already available in the literature (see, e.g. [15], [18], [25], [40], [42], [47]). From now on we assume, without loss of generality, that $\mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D},h}$ is polynomial on each $T \in \mathcal{T}_h^{\mathrm{D}}$. Otherwise, additional higher order terms, given by the errors arising from suitable polynomial approximations, should appear in the corresponding bounds below, which explains the expression h.o.t. in (4.31).

Lemma 4.3.13 *There exist positive constants C_i, $i \in \{1, ..., 10\}$, independent of h, such that*

a) $h_T^2\, \|\mathrm{rot}\,(\mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 \leq C_1\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}$,

b) $h_T^2\, \|\mathbf{rot}\, \boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \leq C_2\, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}$,

c) $h_e\, \|[\mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D},h} \cdot \mathbf{t}]\|_{0,e}^2 \leq C_3\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,w_e}^2 \qquad \forall\, e \in \mathcal{E}_h(\Omega_{\mathrm{D}})$, *where the set w_e is given by* $w_e := \cup \left\{ T' \in \mathcal{T}_h^{\mathrm{D}} \,:\, e \in \mathcal{E}(T') \right\}$,

d) $h_e\, \|[\boldsymbol{\sigma}_{\mathrm{S},h}^d \mathbf{t}]\|_{0,e}^2 \leq C_4\, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,w_e}^2 \qquad \forall e \in \mathcal{E}_h(\Omega_{\mathrm{S}})$, *where the set w_e is given by* $w_e := \cup \left\{ T' \in \mathcal{T}_h^{\mathrm{S}} \,:\, e \in \mathcal{E}(T') \right\}$,

e) $h_e\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d \mathbf{t}\|_{0,e}^2 \leq C_5\, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \qquad \forall e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})$, *where T is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge,*

f) $h_T^2\, \|\mathbf{K}^{-1}\, \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \leq C_6 \left\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \right\} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}$,

g) $h_T^2\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \leq C_7 \left\{ \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2\, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \right\} \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}$,

h) $h_e\, \|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \leq C_8 \left\{ \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_e\, \|\lambda - \lambda_h\|_{0,e}^2 \right\} \quad \forall e \in \mathcal{E}_h(\Sigma)$, *where T is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge,*

i) $\displaystyle\sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\|\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} \,+\, \lambda_h'\right\|_{0,e}^2 \,\le\, C_9 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2 \,+\, \|\lambda - \lambda_h\|_{1/2,\Sigma}^2 \right\},$

where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge, and

j) $\displaystyle\sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e \left\|\nu^{-1}\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t} \,+\, \boldsymbol{\varphi}_h'\right\|_{0,e}^2 \,\le\, C_{10} \left\{ \sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{S}})} \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e}^2 \,+\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Sigma}^2 \right\},$

where, given $e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})$, T_e is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge.

Proof. For a) and b) we refer to [25, Lemma 6.1]. Alternatively, a) and b) follow from straightforward applications of the technical result provided in [18, Lemma 4.3] (see also [47, Lemma 4.9]). Similarly, for c), d), and e) we refer to [25, Lemma 6.2] or apply the technical result given by [18, Lemma 4.4] (see also [47, Lemma 4.10]). Then, for f) and g) we refer to [25, Lemma 6.3] (see also [47, Lemma 4.13] or [40, Lemma 5.5]). On the other hand, the estimate given by h) corresponds to [15, Lemma 4.12]. In particular, its proof makes use of the discrete trace inequality (4.36). Finally, the proofs of i) and j) follow from very slight modifications of the proof of [40, Lemma 5.7]. Alternatively, an *elasticity version* of i) and j), which is provided in [42, Lemma 20], can also be adapted to our case. □

The estimates i) and j) in the previous lemma provide the only non-local bounds of the present efficiency analysis. However, under additional regularity assumptions on λ and $\boldsymbol{\varphi}$, we can give the following local bounds instead.

Lemma 4.3.14 *Assume that $\lambda|_e \in H^1(e)$ for each $e \in \mathcal{E}_h(\Sigma)$, and that $\boldsymbol{\varphi}|_e \in \mathbf{H}^1(e)$ for each $e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})$. Then there exist $\tilde{C}_9$, $\tilde{C}_{10} > 0$, such that*

$$h_e \left\|\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} \,+\, \lambda_h'\right\|_{0,e}^2 \,\le\, \tilde{C}_9 \left\{ \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T_e}^2 \,+\, h_e \left\|\lambda' - \lambda_h'\right\|_{0,e}^2 \right\} \quad \forall\, e \in \mathcal{E}_h(\Sigma)\,,$$

and

$$h_e \left\|\nu^{-1}\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t} \,+\, \boldsymbol{\varphi}_h'\right\|_{0,e}^2 \,\le\, \tilde{C}_{10} \left\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T_e}^2 \,+\, h_e \left\|\boldsymbol{\varphi}' - \boldsymbol{\varphi}_h'\right\|_{0,e}^2 \right\} \quad \forall\, e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})\,.$$

Proof. Similarly as for i) and j) from Lemma 4.3.13, it follows by adapting the corresponding *elasticity version* from [42]. We omit details here and refer to [42, Lemma 21]. □

It remains to provide the efficiency estimates for three residual terms defined on the edges of the interface Σ. They have to do with the transmision conditions and with the trace equation $\mathbf{u}_{\mathrm{S}} + \boldsymbol{\varphi} = \mathbf{0}$ on Σ. More precisely, we have the following lemmas.

Lemma 4.3.15 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$, there holds*

$$h_e \, \|\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} + \boldsymbol{\varphi}_h \cdot \mathbf{n}\|_{0,e}^2 \leq C \left\{ \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_T^2 \|\mathrm{div}\,(\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 + h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge.

Proof. We proceed similarly as in [15, Lemma 4.7]. Given $e \in \mathcal{E}_h(\Sigma)$, we let T be the triangle of $\mathcal{T}_h^{\mathrm{D}}$ having e as an edge, and define $v_e := \mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} + \boldsymbol{\varphi}_h \cdot \mathbf{n}$ on e. Then, applying (4.33), recalling that $\phi_e = 0$ on $\partial T \backslash e$, extending $\phi_e \, L(v_e)$ by zero in $\Omega_{\mathrm{D}} \backslash T$ so that the resulting function belongs to $H^1(\Omega_{\mathrm{D}})$, and using that $\mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} + \boldsymbol{\varphi} \cdot \mathbf{n} = 0$ on Σ, we get

$$\begin{aligned} \|v_e\|_{0,e}^2 &\leq c_2 \, \|\phi_e^{1/2} \, v_e\|_{0,e}^2 = c_2 \int_e \phi_e \, v_e \, (\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} + \boldsymbol{\varphi}_h \cdot \mathbf{n}) = c_2 \, \langle \mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} + \boldsymbol{\varphi}_h \cdot \mathbf{n}, \phi_e \, L(v_e) \rangle_\Sigma \\ &= c_2 \, \langle \mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} - \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n}, \phi_e \, L(v_e) \rangle_\Sigma + c_2 \, \langle \boldsymbol{\varphi}_h \cdot \mathbf{n} - \boldsymbol{\varphi} \cdot \mathbf{n}, \phi_e \, L(v_e) \rangle_\Sigma \, . \end{aligned} \tag{4.37}$$

Next, integrating by parts in Ω_{D}, and noting that $\big(\boldsymbol{\varphi}_h \cdot \mathbf{n} - \boldsymbol{\varphi} \cdot \mathbf{n}\big) \in L^2(\Sigma)$, we find, respectively, that

$$\langle \mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} - \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n}, \phi_e \, L(v_e) \rangle_\Sigma = \int_T \nabla\big(\phi_e \, L(v_e)\big) \cdot (\mathbf{u}_{\mathrm{D},h} - \mathbf{u}_{\mathrm{D}}) + \int_T \phi_e \, L(v_e) \, \mathrm{div}\,(\mathbf{u}_{\mathrm{D},h} - \mathbf{u}_{\mathrm{D}}) \, ,$$

and

$$\langle \boldsymbol{\varphi}_h \cdot \mathbf{n} - \boldsymbol{\varphi} \cdot \mathbf{n}, \phi_e \, L(v_e) \rangle_\Sigma = \int_e \big(\boldsymbol{\varphi}_h \cdot \mathbf{n} - \boldsymbol{\varphi} \cdot \mathbf{n}\big) \, \phi_e \, v_e \, .$$

Thus, replacing the above expressions back into (4.37), applying the Cauchy-Schwarz inequality and the inverse estimate (4.35), and recalling that $0 \leq \phi_e \leq 1$, we obtain

$$\|v_e\|_{0,e}^2 \leq C \Big\{ h_T^{-1} \, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T} + \|\mathrm{div}\,(\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h})\|_{0,T} \Big\} \|\phi_e L(v_e)\|_{0,T} + c \, \|v_e\|_{0,e} \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}.$$

But, using again that $0 \leq \phi_e \leq 1$ and thanks to (4.34), we get

$$\|\phi_e \, L(v_e)\|_{0,T} \leq \|\phi_e^{1/2} L(v_e)\|_{0,T} \leq c_3^{1/2} \, h_e^{1/2} \, \|v_e\|_{0,e} \, , \tag{4.38}$$

whence the previous inequality yields

$$\|v_e\|_{0,e} \leq C \, h_e^{1/2} \, \left\{ h_T^{-1} \, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T} + \|\mathrm{div}\,(\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h})\|_{0,T} \right\} + c \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e} \, .$$

Finally, it is easy to see that this estimate and the fact that $h_e \leq h_T$ imply the required upper bound for $h_e \, \|v_e\|_{0,e}^2$, which finishes the proof. □

Lemma 4.3.16 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$, there holds*

$$\begin{aligned} & h_e \, \|\boldsymbol{\sigma}_{\mathrm{S},h} \, \mathbf{n} + \lambda_h \, \mathbf{n} - \nu \, \kappa^{-1} \, (\boldsymbol{\varphi}_h \cdot \mathbf{t}) \, \mathbf{t}\|_{0,e}^2 \\ & \quad \leq C \left\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2 \|\mathbf{div}\,(\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,T}^2 + h_e \, \|\lambda - \lambda_h\|_{0,e}^2 + h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \right\}, \end{aligned}$$

where T is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge.

Proof. We proceed as in the previous lemma (see also [15, Lemma 4.6]). Indeed, given $e \in \mathcal{E}_h(\Sigma)$, we let T be the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge, and define $\mathbf{v}_e := \boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}\,(\boldsymbol{\varphi}_h \cdot \mathbf{t})\,\mathbf{t}$ on e. Then, applying (4.33), recalling that $\phi_e = 0$ on $\partial T \backslash e$, extending $\phi_e\,\mathbf{L}(\mathbf{v}_e)$ by zero in $\Omega_{\mathrm{S}} \backslash T$ so that the resulting function belongs to $\mathbf{H}^1(\Omega_{\mathrm{S}})$, using that $\boldsymbol{\sigma}_{\mathrm{S}}\,\mathbf{n} + \lambda\,\mathbf{n} - \nu\,\kappa^{-1}\,(\boldsymbol{\varphi} \cdot \mathbf{t})\,\mathbf{t} = 0$ on Σ, and then integrating by parts in Ω_{S}, we arrive at

$$\begin{aligned} \|\mathbf{v}_e\|_{0,e}^2 \,&\le\, c_2\,\|\phi_e^{1/2}\,\mathbf{v}_e\|_{0,e}^2 \,=\, c_2 \int_e \phi_e\,\mathbf{v}_e \cdot \Big\{\boldsymbol{\sigma}_{\mathrm{S},h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa^{-1}\,(\boldsymbol{\varphi}_h \cdot \mathbf{t})\,\mathbf{t}\Big\} \\ &=\, c_2 \int_T \nabla(\phi_e\,\mathbf{L}(\mathbf{v}_e)) : (\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S}}) \,+\, c_2 \int_T \phi_e\,\mathbf{L}(\mathbf{v}_e) \cdot \mathbf{div}\,(\boldsymbol{\sigma}_{\mathrm{S},h} - \boldsymbol{\sigma}_{\mathrm{S}}) \\ &+\, c_2 \int_e \phi_e\,\mathbf{v}_e \cdot \Big\{(\lambda_h - \lambda)\,\mathbf{n} - \nu\,\kappa^{-1}\,(\boldsymbol{\varphi}_h \cdot \mathbf{t} - \boldsymbol{\varphi} \cdot \mathbf{t})\,\mathbf{t}\Big\}\,. \end{aligned}$$

Next, applying the Cauchy-Schwarz inequality and the inverse estimate (4.35), recalling that $0 \le \phi_e \le 1$, and employing the vector version of (4.38), we deduce that

$$\begin{aligned} \|\mathbf{v}_e\|_{0,e} \,&\le\, C\,h_e^{1/2}\,\Big\{h_T^{-1}\,\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0.T} \,+\, \|\mathbf{div}\,(\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h})\|_{0,T}\Big\} \\ &+\, C\,\Big\{\|\lambda - \lambda_h\|_{0,e} \,+\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}\Big\}\,, \end{aligned}$$

which easily yields the required estimate, thus finishing the proof. □

Lemma 4.3.17 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$, there holds*

$$h_e\,\|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \,\le\, C\,\Big\{\|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_T^2\,\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_e\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2\Big\}\,,$$

where T is the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge.

Proof. Let $e \in \mathcal{E}_h(\Sigma)$ and let T be the triangle of $\mathcal{T}_h^{\mathrm{S}}$ having e as an edge. We follow the proof of [15, Lemma 4.12] and obtain first an upper bound of $h_T^2\,|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T}^2$. Indeed, using that $\nabla \mathbf{u}_{\mathrm{S}} = \nu^{-1}\,\boldsymbol{\sigma}_{\mathrm{S}}^d$ in Ω_{S} (cf. Theorem 4.2.2) and that $\mathbf{u}_{\mathrm{S},h}$ is constant in T, adding and substracting $\boldsymbol{\sigma}_{\mathrm{S},h}^d$, and then applying the estimate g) from Lemma 4.3.13, we deduce that

$$\begin{aligned} h_T^2\,|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T}^2 \,&=\, \frac{h_T^2}{\nu^2}\,\|\boldsymbol{\sigma}_{\mathrm{S}}^d\|_{0,T}^2 \,\le\, C\,h_T^2\,\Big\{\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \,+\, \|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2\Big\} \\ &\le\, C\,\Big\{\|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_T^2\,\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2\Big\}\,. \end{aligned} \tag{4.39}$$

Next, since $\boldsymbol{\varphi} = -\,\mathbf{u}_{\mathrm{S}}$ on Σ (cf. Theorem 4.2.2), we find that

$$h_e\,\|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \,\le\, 2\,h_e\,\Big\{\|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,e}^2 \,+\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2\Big\}\,,$$

which, employing the discrete trace inequality (4.36) and the estimate (4.39), yields

$$\begin{aligned} h_e\,\|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \,&\le\, C\,\Big\{\|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_T^2\,|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}|_{1,T}^2 \,+\, h_e\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2\Big\} \\ &\le\, C\,\Big\{\|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_T^2\,\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \,+\, h_e\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2\Big\}\,, \end{aligned}$$

which completes the proof. □

We end this section by observing that the efficiency estimate (4.31) follows straightforwardly from Lemmas 4.3.10, 4.3.13, 4.3.15, 4.3.16, and 4.3.17. In particular, the terms $h_e \, \|\lambda - \lambda_h\|_{0,e}^2$ and $h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2$, which appear in Lemma 4.3.13 (item h)), 4.3.15, 4.3.16, and 4.3.17, are bounded as follows:

$$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \, \|\lambda - \lambda_h\|_{0,e}^2 \,\leq\, h \, \|\lambda - \lambda_h\|_{0,\Sigma}^2 \,\leq\, C \, h \, \|\lambda - \lambda_h\|_{1/2,\Sigma}^2 \,,$$

and

$$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \,\leq\, h \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,\Sigma}^2 \,\leq\, C \, h \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Sigma}^2 \,.$$

4.4 Numerical results

In [49, Section 5] we presented several numerical results illustrating the performance of the Galerkin scheme (4.11) with the subspaces $\mathbb{X}_h := \mathbb{H}_h(\Omega_S) \times \mathbf{H}_h(\Omega_D) \times \boldsymbol{\Lambda}_h(\Sigma) \times \Lambda_h(\Sigma)$ and $\mathbb{M}_h := \mathbf{L}_h(\Omega_S) \times L_{h,0}(\Omega_D)$ defined in Section 4.2.3. We now provide three examples confirming the reliability and efficiency of the respective a posteriori error estimator Θ derived in Section 4.3, and showing the behaviour of the associated adaptive algorithm.

In what follows, N stands for the number of degrees of freedom defining $\mathbb{X}_h$ and $\mathbb{M}_h$. The solution of (4.5) and (4.11) are denoted

$$(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \,:=\, ((\boldsymbol{\sigma}_S, \mathbf{u}_D, \boldsymbol{\varphi}, \lambda), (\mathbf{u}_S, p_D)) \,\in\, \mathbb{X} \times \mathbb{M}$$

and

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \,:=\, ((\boldsymbol{\sigma}_{S,h}, \mathbf{u}_{D,h}, \boldsymbol{\varphi}_h, \lambda_h), (\mathbf{u}_{S,h}, p_{D,h})) \,\in\, \mathbb{X}_h \times \mathbb{M}_h.$$

The separate and total errors are defined by:

$$\mathsf{e}(\boldsymbol{\sigma}_S) \,:=\, \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{\mathbf{div},\Omega_S} \,, \qquad \mathsf{e}(\mathbf{u}_S) \,:=\, \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{\mathrm{div};\Omega_S} \,,$$

$$\mathsf{e}(\mathbf{u}_D) \,:=\, \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{\mathrm{div};\Omega_D} \,, \qquad \mathsf{e}(p_D) \,:=\, \|p_D - p_{D,h}\|_{0,\Omega_D} \,,$$

$$\mathsf{e}(\boldsymbol{\varphi}) \,:=\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Sigma} \,, \qquad \mathsf{e}(\lambda) \,:=\, \|\lambda - \lambda_h\|_{1/2,\Sigma} \,,$$

and

$$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \,:=\, \Big\{ (\mathsf{e}(\boldsymbol{\sigma}_S))^2 + (\mathsf{e}(\mathbf{u}_S))^2 + (\mathsf{e}(\mathbf{u}_D))^2 + (\mathsf{e}(p_D))^2 + (\mathsf{e}(\boldsymbol{\varphi}))^2 + (\mathsf{e}(\lambda))^2 \Big\}^{1/2} \,.$$

The effectivity index with respect to Θ is given by

$$\mathrm{eff}(\Theta) \,:=\, \mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})/\Theta \,.$$

Also, we let $r(\boldsymbol{\sigma}_S)$, $r(\mathbf{u}_S)$, $r(\mathbf{u}_D)$, $r(p_D)$, $r(\boldsymbol{\varphi})$, $r(\lambda)$, and $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ be the individual and global experimental rates of convergence given by

$$r(\%) := \frac{\log(\mathsf{e}(\%)/\mathsf{e}'(\%))}{\log(h/h')} \qquad \text{for each} \quad \% \in \left\{\boldsymbol{\sigma}_S, \mathbf{u}_S, \mathbf{u}_D, p_D, \boldsymbol{\varphi}, \lambda\right\},$$

and

$$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) := \frac{\log(\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})/\mathsf{e}'(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}))}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with errors e and e'. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. In all of them we choose $\nu = 1$, $\mathbf{K} = \mathrm{I}$ and $\kappa = 1$. Example 1 is used to corroborate the reliability and efficiency of the a posteriori error estimator Θ. Examples 2 and 3 are utilized to illustrate the behavior of the associated adaptive algorithm, which applies the following procedure from [81]:

1) Start with a coarse mesh $\mathcal{T}_h := \mathcal{T}_h^D \cup \mathcal{T}_h^S$.

2) Solve the discrete problem (4.11) for the current mesh $\mathcal{T}_h$.

3) Compute $\Theta_T := \Theta_{\star,T}$ for each triangle $T \in \mathcal{T}_h^\star$, $\star \in \{D, S\}$.

4) Check the stopping criterion and decide whether to finish or go to next step.

5) Use *blue-green* refinement on those $T' \in \mathcal{T}_h$ whose indicator $\Theta_{T'}$ satisfies

$$\Theta_{T'} \geq \frac{1}{2} \max_{T \in \mathcal{T}_h} \{\Theta_T : \quad T \in \mathcal{T}_h\}.$$

6) Define resulting meshes as current meshes $\mathcal{T}_h^D$ and $\mathcal{T}_h^S$, and go to step 2.

In Example 1 we consider the regions $\Omega_D := (-0.5, 0.5)^2$ and $\Omega_S := (-1,1)^2 \setminus \bar{\Omega}_D$, which yields a porous medium completely surrounded by a fluid, and choose the data $\mathbf{f}_S$ and f_D so that the exact solution is given by the smooth functions

$$\mathbf{u}_S(\mathbf{x}) = \begin{pmatrix} -2\sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2) \\ \\ 2\sin(\pi x_1)\sin^2(\pi x_2)\cos(\pi x_1) \end{pmatrix} \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_S,$$

$$p_S(\mathbf{x}) = x_1^3 e^{x_2} \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_S,$$

and

$$p_{\rm D}(\mathbf{x}) \,=\, x_1^3 \sin(x_2) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\rm D} \,.$$

In Example 2 we consider $\Omega_{\rm D} := (-1,0)^2$ and let $\Omega_{\rm S}$ be the L-shaped domain given by $(-1,1)^2 \setminus \bar{\Omega}_{\rm D}$, which yields a porous medium partially surrounded by a fluid. Then we choose the data $\mathbf{f}_{\rm S}$ and $f_{\rm D}$ so that the exact solution is given by

$$\mathbf{u}_{\rm S}(\mathbf{x}) \,=\, \operatorname{curl}\Big(0.1\, \left(x_2^2 - 1\right)^2 \sin^2(\pi x_1)\Big) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\rm S} \,,$$

$$p_{\rm S}(\mathbf{x}) \,=\, \frac{1}{100\,(x_1^2 + x_2^2) + 0.1} \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\rm S} \,,$$

and

$$p_{\rm D}(\mathbf{x}) \,=\, \left(\frac{x_1 + 1}{10}\right)^2 \sin^3(2\pi\,(x_2 + 0.5)) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\rm D} \,.$$

Note that the fluid pressure $p_{\rm S}$ has high gradients around the origin.

Finally, in Example 3 we take $\Omega_{\rm D} := (-1,1) \times (-2,-1)$ and $\Omega_{\rm S} := (-1,1)^2 \setminus [0,1]^2$, which yields a porous medium below a fluid, and choose the data $\mathbf{f}_{\rm S}$ and $f_{\rm D}$ so that the exact solution is given by

$$\mathbf{u}_{\rm S}(r,\theta) \,=\, \operatorname{curl}\left(0.1\, r^{5/3}\,(r^2 \cos^2(\theta) - 1)^2\,(r \sin(\theta) - 1)^2 \sin^2\left(\frac{2\theta - \pi}{3}\right)\right) \quad \forall\,(r,\theta) \in \Omega_{\rm S} \,,$$

$$p_{\rm S}(\mathbf{x}) \,=\, 0.1\, x_1 \sin(x_2) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\rm S} \,,$$

and

$$p_{\rm D}(\mathbf{x}) \,=\, 0.1\,(x_2 + 2)^2 \sin^3(\pi x_1) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\rm D} \,.$$

Note that $\mathbf{u}_{\rm S}$ is defined in polar coordinates and that its derivatives are singular at the origin.

The numerical results shown below were obtained using a MATLAB code. In Table 4.1 we summarize the convergence history of the mixed finite element method (4.11), as applied to Example 1, for a sequence of quasi-uniform triangulations of the domain. We observe there, looking at the corresponding experimental rates of convergence, that the $O(h)$ predicted by Theorem 4.2.3 (here $\delta = 1$) is attained in all the unknowns. In addition, we notice that the effectivity index $\text{eff}(\Theta)$ remains always in a neighborhood of 0.91, which illustrates the reliability and efficiency of Θ in the case of a regular solution.

Next, in Tables 4.2 - 4.5 we provide the convergence history of the quasi-uniform and adaptive schemes, as applied to Examples 2 and 3. We observe that the errors of the adaptive procedures decrease faster than those obtained by the quasi-uniform ones, which is confirmed by the global experimental rates of convergence provided there. This fact is also illustrated in Figures 4.2

Table 4.1: EXAMPLE 1, quasi-uniform scheme

N	h	$\mathbf{e}(\boldsymbol{\sigma}_S)$	$r(\boldsymbol{\sigma}_S)$	$\mathbf{e}(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$\mathbf{e}(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$\mathbf{e}(p_D)$	$r(p_D)$
321	0.5000	35.4015	–	0.6875	–	0.1996	–	0.0117	–
1201	0.2500	20.0107	0.8647	0.4266	0.7234	0.1121	0.8743	0.0057	1.0798
4641	0.1250	10.0700	1.0160	0.1615	1.4370	0.0531	1.1046	0.0023	1.3213
18241	0.0625	5.0492	1.0087	0.0801	1.0238	0.0259	1.0490	0.0011	1.0967
72321	0.0312	2.5268	1.0052	0.0401	1.0064	0.0129	1.0178	0.0005	1.0234
288001	0.0156	1.2637	1.0029	0.0200	1.0031	0.0064	1.0062	0.0003	1.0062

N	h	$\mathbf{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathbf{e}(\lambda)$	$r(\lambda)$	$\mathtt{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$	Θ	$\mathrm{eff}(\Theta)$
321	0.5000	4.2653	–	0.0981	–	35.6649	–	39.0015	0.9144
1201	0.2500	4.3919	–	0.0973	0.0124	20.4920	0.8399	22.6847	0.9033
4641	0.1250	1.7410	1.3690	0.0537	0.8781	10.2209	1.0292	11.1965	0.9129
18241	0.0625	0.8088	1.1202	0.0259	1.0670	5.1144	1.0117	5.5954	0.9140
72321	0.0312	0.3949	1.0408	0.0126	1.0516	2.5579	1.0060	2.7969	0.9145
288001	0.0156	0.1962	1.0123	0.0062	1.0266	1.2791	1.0031	1.3982	0.9148

and 4.4 where we display the total errors $\mathtt{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ vs. the number of degrees of freedom N for both refinements. As shown by the values of $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$, the adaptive method is able to keep the quasi-optimal rate of convergence $\mathcal{O}(h)$ for the total error. Furthermore, the effectivity indexes remain bounded from above and below, which confirms the reliability and efficiency of Θ in these cases of non-smooth solutions. Intermediate meshes obtained with the adaptive refinements are displayed in Figures 4.3 and 4.5. Note that the method is able to recognize the region with high gradients in Example 2, and the singularity of the solution in Example 3.

Table 4.2: EXAMPLE 2, quasi-uniform scheme

N	h	$\mathbf{e}(\boldsymbol{\sigma}_{\rm S})$	$\mathbf{e}(\mathbf{u}_{\rm S})$	$\mathbf{e}(\mathbf{u}_{\rm D})$	$\mathbf{e}(p_{\rm D})$	$\mathbf{e}(\boldsymbol{\varphi})$	$\mathbf{e}(\lambda)$
608	0.3536	4.5187	0.1198	0.2649	0.0184	0.5760	0.1120
2332	0.1768	4.9963	0.0529	0.1520	0.0035	0.2653	0.0347
9140	0.0884	6.7481	0.0253	0.0778	0.0005	0.1485	0.0096
36196	0.0442	4.2857	0.0125	0.0392	0.0002	0.0771	0.0042
144068	0.0221	2.4834	0.0062	0.0196	0.0001	0.0348	0.0022

N	h	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$	Θ	$\text{eff}(\Theta)$
608	0.3536	4.5660	–	5.4033	0.8450
2332	0.1768	5.0060	–	5.2805	0.9480
9140	0.0884	6.7503	–	6.8230	0.9894
36196	0.0442	4.2866	0.6599	4.3158	0.9932
144068	0.0221	2.4837	0.7901	2.4958	0.9952

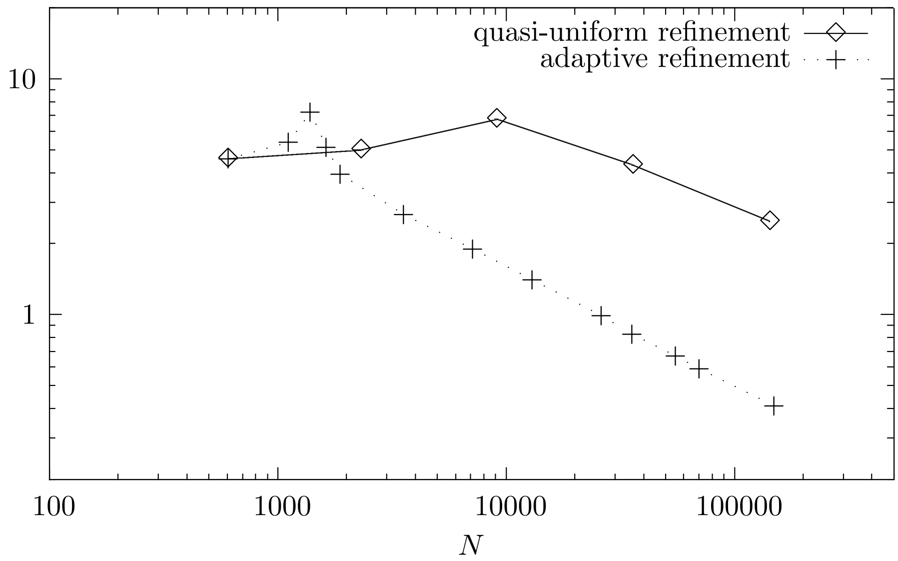

Figure 4.2: EXAMPLE 2, $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ vs. N for quasi-uniform/adaptive schemes

Table 4.3: EXAMPLE 2, adaptive scheme

N	$\mathbf{e}(\boldsymbol{\sigma}_S)$	$\mathbf{e}(\mathbf{u}_S)$	$\mathbf{e}(\mathbf{u}_D)$	$\mathbf{e}(p_D)$	$\mathbf{e}(\boldsymbol{\varphi})$	$\mathbf{e}(\lambda)$
608	4.5188	0.1199	0.2649	0.0184	0.5760	0.1121
1118	5.3792	0.0709	0.2262	0.0091	0.3185	0.0402
1391	7.2290	0.0661	0.2098	0.0082	0.2846	0.0215
1636	5.1151	0.0657	0.2094	0.0110	0.2591	0.0236
1884	3.9177	0.0657	0.2093	0.0108	0.2577	0.0229
3558	2.6519	0.0491	0.2020	0.0037	0.1626	0.0128
7164	1.8814	0.0320	0.1751	0.0067	0.1160	0.0171
13073	1.3945	0.0237	0.1591	0.0034	0.0742	0.0109
26227	0.9771	0.0165	0.1222	0.0030	0.0730	0.0103
35611	0.8163	0.0140	0.1089	0.0018	0.0384	0.0075
55318	0.6608	0.0114	0.0808	0.0005	0.0375	0.0039
70434	0.5825	0.0099	0.0747	0.0005	0.0357	0.0038
149402	0.4052	0.0070	0.0548	0.0003	0.0208	0.0023

N	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$	Θ	$\mathrm{eff}(\Theta)$
608	4.5660	–	5.4033	0.8450
1118	5.3940	–	5.7977	0.9304
1391	7.2379	–	7.4956	0.9656
1636	5.1264	4.2524	5.4334	0.9435
1884	3.9324	3.7572	4.3145	0.9114
3558	2.6650	1.2238	2.9662	0.8985
7164	1.8934	0.9768	2.0913	0.9054
13073	1.4057	0.9902	1.5394	0.9132
26227	0.9876	1.0142	1.0951	0.9018
35611	0.8246	1.1796	0.9191	0.8972
55318	0.6669	0.9637	0.7388	0.9026
70434	0.5885	1.0359	0.6505	0.9046
149402	0.4095	0.9644	0.4550	0.8999

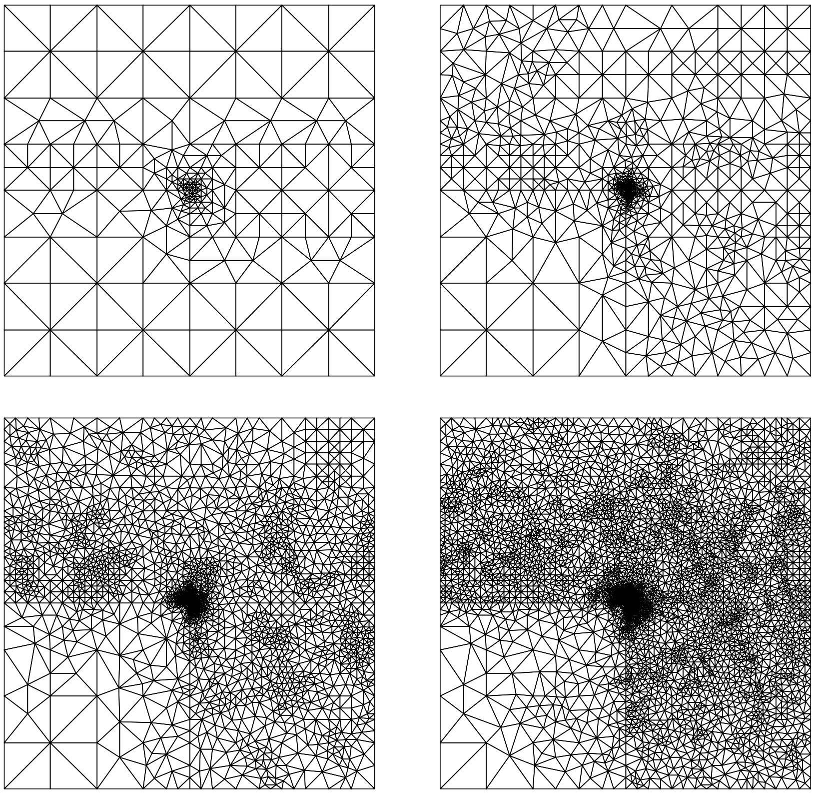

Figure 4.3: Example 2, adapted meshes with 1884, 7164, 26227, and 55318 degrees of freedom

Table 4.4: EXAMPLE 3, quasi-uniform scheme

N	h	$\mathbf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathbf{e}(\mathbf{u}_{\mathrm{S}})$	$\mathbf{e}(\mathbf{u}_{\mathrm{D}})$	$\mathbf{e}(p_{\mathrm{D}})$	$\mathbf{e}(\boldsymbol{\varphi})$	$\mathbf{e}(\lambda)$
344	0.5000	16.8563	0.4452	0.7130	0.0674	1.8109	0.1615
1324	0.2500	11.3317	0.3329	0.3846	0.0130	2.5160	0.0826
5204	0.1250	7.0011	0.0849	0.1980	0.0038	0.8665	0.0458
20644	0.0625	4.4530	0.0412	0.0992	0.0018	0.3859	0.0203
82244	0.0312	2.8037	0.0206	0.0496	0.0009	0.1877	0.0097

N	h	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$	Θ	$\mathrm{eff}(\Theta)$
344	0.5000	16.9751	–	18.8901	0.8986
1324	0.2500	11.6191	0.5626	13.1132	0.8861
5204	0.1250	7.0579	0.7284	7.8041	0.9044
20644	0.0625	4.4711	0.6626	5.0014	0.8940
82244	0.0312	2.8105	0.6717	3.1653	0.8879

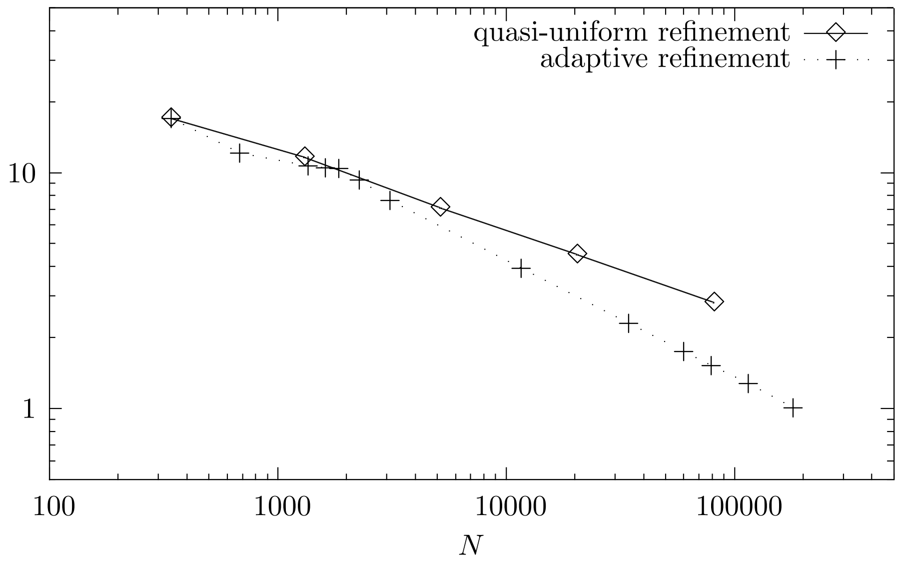

Figure 4.4: EXAMPLE 3, $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ vs. N for quasi-uniform/adaptive schemes

Table 4.5: EXAMPLE 3, adaptive scheme

N	$\mathbf{e}(\boldsymbol{\sigma}_{\rm S})$	$\mathbf{e}(\mathbf{u}_{\rm S})$	$\mathbf{e}(\mathbf{u}_{\rm D})$	$\mathbf{e}(p_{\rm D})$	$\mathbf{e}(\varphi)$	$\mathbf{e}(\lambda)$
344	16.8564	0.4453	0.7131	0.0675	1.8109	0.1616
684	11.8048	0.3406	0.5828	0.0177	2.5165	0.0863
1367	10.6242	0.1330	0.4426	0.0099	0.8682	0.0530
1625	10.4486	0.1314	0.4426	0.0099	0.8682	0.0530
1863	10.3440	0.1278	0.4426	0.0098	0.8678	0.0530
2291	9.2480	0.1173	0.4427	0.0097	0.8672	0.0526
3109	7.5456	0.1013	0.4425	0.0098	0.8670	0.0522
11719	3.9053	0.0530	0.3296	0.0072	0.3875	0.0271
34611	2.2713	0.0202	0.2614	0.0058	0.1901	0.0092
60159	1.7281	0.0153	0.1723	0.0034	0.1759	0.0083
79482	1.5031	0.0111	0.1644	0.0032	0.1154	0.0072
115241	1.2620	0.0167	0.1498	0.0019	0.1101	0.0055
182014	0.9954	0.0130	0.1226	0.0012	0.0900	0.0027

N	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$	Θ	$\mathrm{eff}(\Theta)$
344	16.9751	–	18.8901	0.8986
684	12.0893	0.9877	13.6112	0.8882
1367	10.6698	0.3608	11.3264	0.9420
1625	10.4949	0.1912	11.1221	0.9436
1863	10.3907	0.1460	10.8244	0.9599
2291	9.3000	1.0724	9.9113	0.9383
3109	7.6090	1.3146	8.2092	0.9269
11719	3.9388	1.0924	4.2413	0.9362
34611	2.2943	0.9981	2.4691	0.9292
60159	1.7456	0.9889	1.8902	0.9235
79482	1.5165	1.0102	1.5941	0.9513
115241	1.2757	0.9309	1.3418	0.9507
182014	1.0070	1.0350	1.0817	0.9309

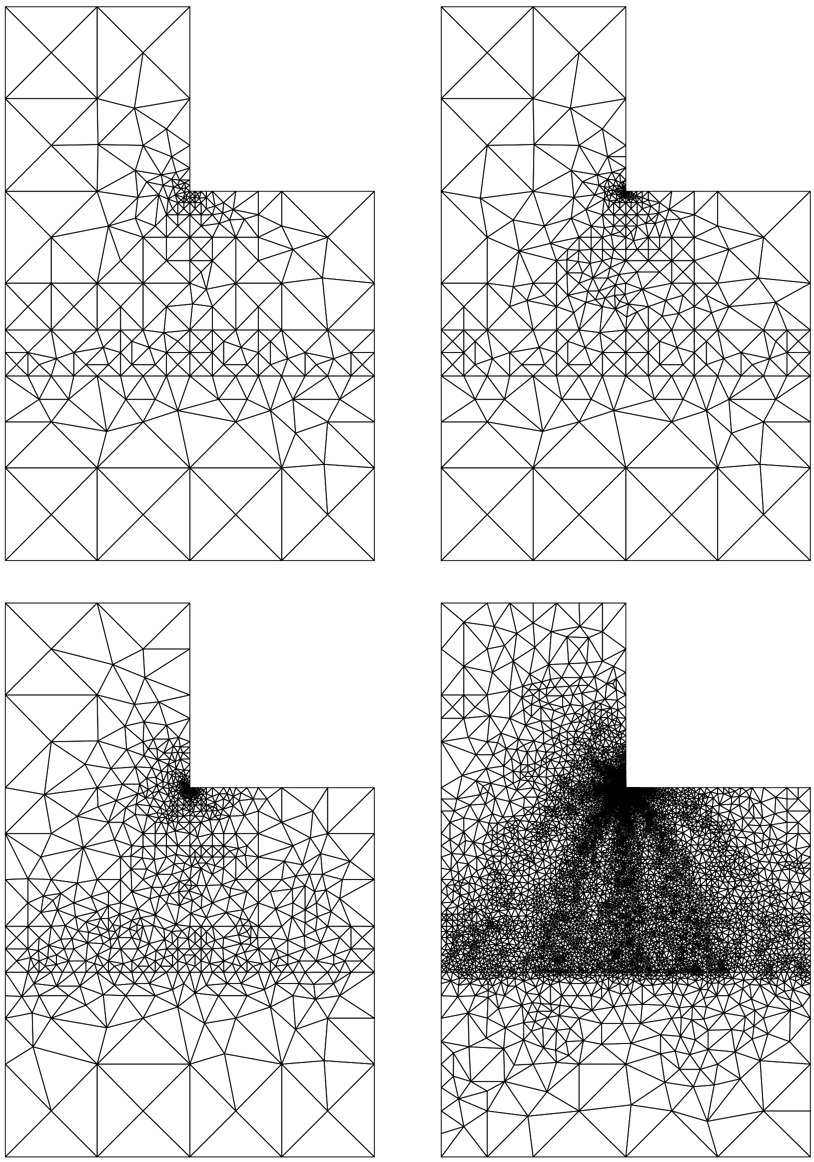

Figure 4.5: Example 3, adapted meshes with 1863, 3109, 11719, and 60159 degrees of freedom

Chapter 5

A twofold saddle point approach for the coupling of fluid flow with nonlinear porous media flow

5.1 Introduction

The development of appropriate numerical methods for the coupling of fluid flow (modeled by the Stokes equation) with porous media flow (modeled by the Darcy equation) has become a very active research area in recent years (see, e.g. [17], [34], [35], [37], [39], [45], [61], [63], [74], [78], [83] and the references therein). The above list includes porous media with cracks, the incorporation of the Brinkman equation in the model, and nonlinear problems. In particular, a mixed finite element method for a nonlinear Stokes-Darcy flow problem is introduced and analized in [37]. The fluid, being considered non-Newtonian in both domains, is modeled there by the generalized nonlinear Stokes equation in the free flow region and by the generalized nonlinear Darcy equation in the porous medium. In addition, the approach in [37] employs the primal method in the Stokes domain and the dual-mixed method in the Darcy region, which means that only the original velocity and pressure unknowns are considered in the fluid, whereas a further unknown (velocity) is added in the porous medium. The corresponding interface conditions are given, as usual lately, by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law. Further, since one of these conditions becomes essential, the trace of the Darcy pressure on the interface needs also to be incorporated as an additional Lagrange multiplier.

On the other hand, in the recent paper [49] we have developed the a priori error analysis of a new fully-mixed variational formulation for the 2D Stokes-Darcy coupled problem. This approach allows, on the one hand, the introduction of further unknowns of physical interest, and on the other hand, the utilization of the same family of finite element subspaces in both media, without requiring any stabilization term. More precisely, in [49] we consider dual-mixed formulations in both domains, which yields the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium, and the traces of the porous media pressure and the fluid velocity on the interface, as the resulting unknowns. The pressure and the velocity gradient in the fluid can then be computed as a very simple postprocess, in which no numerical differentiation is applied, and hence no further sources of error arise.

Now, it is well known that in order to guarantee a good convergence behaviour of most finite element solutions, specially under the eventual presence of singularities, one usually needs to apply an adaptive algorithm based on a posteriori error estimates. These are represented by global quantities $\boldsymbol{\theta}$ that are expressed in terms of local indicators θ_T defined on each element T of a given triangulation $\mathcal{T}$. The estimator $\boldsymbol{\theta}$ is said to be efficient (resp. reliable) if there exists $C_{\mathtt{eff}} > 0$ (resp. $C_{\mathtt{rel}} > 0$), independent of the meshsizes, such that

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} + \text{h.o.t.} \;\leq\; \|\text{error}\| \;\leq\; C_{\mathtt{rel}}\,\boldsymbol{\theta} + \text{h.o.t.}\,,$$

where h.o.t. is a generic expression denoting one or several terms of higher order. In spite of the many contributions available in the literature on the posteriori error analysis for variational formulations with saddle-point structure, the first results concerning the Stokes-Darcy coupled problem have been provided only in [15], where a reliable and efficient residual-based a posteriori error estimator for the variational formulation analyzed in [45] is derived. More recently, and following some of the techniques from [15] together with classical approaches, a reliable and efficient residual-based a posteriori error estimator for the fully-mixed variational method introduced in [49] was provided in [50].

The purpose of the present paper is to extend the results from [49] and [50] to the case of a nonlinear Stokes-Darcy coupled problem. More precisely, we develop the a priori and a posteriori error analyses of the fully mixed formulation from [49], as applied to the coupling of fluid flow with nonlinear porous media flow, where the nonlinearity in the latter region is given by the corresponding permeability. For this purpose, we consider a dual-mixed formulation in both domains, which yields the pseudostress and the velocity in the fluid, together with the velocity, the pressure and its gradient in the porous medium, as the main unknowns. Moreover, since the transmission conditions become essential, we impose them weakly and introduce the traces of the porous medium pressure and the fluid velocity as the corresponding Lagrange multipliers. As in [49], the remaining unknowns of physical interest can then be computed as a very simple postprocess that makes no use of any numerical differentiation procedure. Then, the corresponding variational formulation can be written as a two-fold saddle point operator equation, and hence the generalization of the Babuška-Brezzi theory developed in [41] is applied to prove the well-posedness of the continuous and discrete schemes. Furthermore, using some well known approaches (see, e.g. [8], [9], [10], [23], [25], [28], [43], [56], [64], [65], [72], [79], and the references therein), we derive a reliable and efficient residual-based a posteriori error estimator for our nonliner coupled problem. The proof of reliability makes use of a global inf-sup condition for a linearized version of the problem, Helmholtz decompositions in both media, and local approximation properties of the Clément interpolant and Raviart-Thomas operator. On the other hand, inverse inequalities, the localization technique based on element-bubble and edge-bubble functions, and known results from previous works, are the main tools for proving the efficiency of the estimator.

The rest of this work is organized as follows. In Section 5.2 we introduce the model problem, show that the resulting variational formulation can be written as a two-fold saddle-point operator equation, introduce an equivalent formulation, which is easier to analyze, and collect the main results of the generalized Babuška-Brezzi theory developed in [41] (see also [53]). This abstract framework is then applied in Section 5.3 to prove the unique solvability of the equivalent formulation, which in turn yields the well posedness of our continuous problem. Next, in Section 5.4 we define the Galerkin scheme and derive general hypotheses on the finite element subspaces ensuring that the discrete scheme becomes well posed. A specific choice of finite element subspaces satisfying these assumptions, namely Raviart-Thomas of lowest order and piecewise constants on both domains, and piecewise linears on the interface, is described in Section 5.5. In Section 5.6 we derive the a posteriori error estimator and prove its reliability and efficiency. Finally, the numerical results are presented in Section 5.7.

We end this section with some notations to be used below. In particular, in what follows we utilize the standard terminology for Sobolev spaces. In addition, if $\mathcal{O}$ is a domain, Γ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}, \quad \text{and} \quad \mathbf{H}^r(\Gamma) := [H^r(\Gamma)]^2.$$

However, for $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\Gamma}$ (for $H^r(\Gamma)$ and $\mathbf{H}^r(\Gamma)$). Also, the Hilbert space

$$\mathbf{H}(\mathrm{div}\,;\mathcal{O}) := \left\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\ \mathbf{w} \in L^2(\mathcal{O}) \right\},$$

is standard in the realm of mixed problems (see, e.g. [19]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$

and $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div}\,;\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div}\,;\mathcal{O}}$, respectively. On the other hand, the symbol for the $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ inner products

$$\langle \xi, \lambda \rangle_\Gamma := \int_\Gamma \xi\,\lambda \quad \forall\, \xi,\, \lambda \in L^2(\Gamma), \qquad \langle \boldsymbol{\xi}, \boldsymbol{\lambda} \rangle_\Gamma := \int_\Gamma \boldsymbol{\xi} \cdot \boldsymbol{\lambda} \quad \forall\, \boldsymbol{\xi},\, \boldsymbol{\lambda} \in \mathbf{L}^2(\Gamma)$$

will also be employed for their respective extensions as the duality products $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{1/2}(\Gamma)$. Hereafter, given a non-negative integer k and a subset S of $\mathbb{R}^2$, $\mathbf{P}_k(S)$ stands for the space of polynomials defined on S of degree $\leq k$. Finally, we employ $\mathbf{0}$ as a generic null vector, and use C, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

5.2 The continuous problem

5.2.1 Statement of the model problem

In order to describe the geometry, we let Ω_{S} and Ω_{D} be bounded and simply connected polygonal domains in $\mathbb{R}^2$ such that $\partial\Omega_{\mathrm{S}} \cap \partial\Omega_{\mathrm{D}} = \Sigma \neq \emptyset$ and $\Omega_{\mathrm{S}} \cap \Omega_{\mathrm{D}} = \emptyset$. Then, we let $\Gamma_{\mathrm{S}} := \partial\Omega_{\mathrm{S}} \backslash \bar{\Sigma}$, $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \backslash \bar{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ and Ω_{S} (and hence inward to Ω_{D} when seen on Σ). On Σ we also consider a unit tangent vector $\mathbf{t}$ (see Figure 5.1 below).

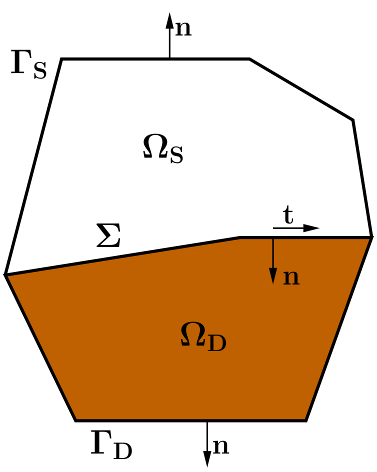

Figure 5.1: The domains for our 2D Stokes–Darcy model

The model consists of two separate groups of equations and a set of coupling terms. In Ω_{S}, the governing equations are those of the Stokes problem, which are written in the following velocity-pressure-pseudostress formulation:

$$
\begin{aligned}
\boldsymbol{\sigma}_{\mathrm{S}} = -p_{\mathrm{S}}\,\mathrm{I} + \nu\,\nabla\mathbf{u}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} + \mathbf{f}_{\mathrm{S}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \\
\operatorname{div}\mathbf{u}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}\,,
\end{aligned} \tag{5.1}
$$

where $\nu > 0$ is the viscosity of the fluid, $\mathbf{u}_{\mathrm{S}}$ is the fluid velocity, p_{S} is the pressure, $\boldsymbol{\sigma}_{\mathrm{S}}$ is the pseudostress tensor, $\mathbf{I}$ is the 2×2 identity matrix, $\mathbf{f}_{\mathrm{S}}$ are known source terms, and $\mathbf{div}$ is the usual divergence operator div acting row-wise on each tensor. Now, using that $\operatorname{tr}(\nabla\mathbf{u}_{\mathrm{S}}) = \operatorname{div}\mathbf{u}_{\mathrm{S}} = 0$ in Ω_{S}, we notice that the equations in (5.1) can be rewritten equivalently as

$$
\begin{aligned}
\nu^{-1}\,\boldsymbol{\sigma}_{\mathrm{S}}^{d} = \nabla\mathbf{u}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} + \mathbf{f}_{\mathrm{S}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \\
p_{\mathrm{S}} = -\tfrac{1}{2}\operatorname{tr}\boldsymbol{\sigma}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}\,,
\end{aligned} \tag{5.2}
$$

where tr stands for the usual trace of tensors, that is $\operatorname{tr}\boldsymbol{\tau} := \tau_{11} + \tau_{22}$, and

$$
\boldsymbol{\tau}^{d} := \boldsymbol{\tau} - \tfrac{1}{2}\,(\operatorname{tr}\boldsymbol{\tau})\,\mathrm{I}
$$

is the deviatoric part of the tensor $\boldsymbol{\tau}$. On the other hand, in Ω_{D} we consider the following nonlinear Darcy model:

$$
\begin{aligned}
\mathbf{u}_{\mathrm{D}} = -\boldsymbol{\kappa}\,(\cdot,|\nabla\,p_{\mathrm{D}}|)\,\nabla\,p_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \qquad \operatorname{div}\mathbf{u}_{\mathrm{D}} = f_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \\
\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\,,
\end{aligned} \tag{5.3}
$$

where $\mathbf{u}_{\mathrm{D}}$ and p_{D} denote the velocity and pressure, respectively, $\boldsymbol{\kappa} : \Omega_{\mathrm{D}}\times\mathbb{R}^{+} \to \mathbb{R}$ is a nonlinear operator representing the porous medium permeability, $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^2$, and f_{D} are known source terms satisfying $\displaystyle\int_{\Omega_{\mathrm{D}}} f_{\mathrm{D}} = 0$. Throughout the paper we assume that $\boldsymbol{\kappa} \in C^1(\Omega_{\mathrm{D}}\times\mathbb{R}^{+})$ and that there exist constants $k_0,\ k_1 > 0$ such that for all $(x,\rho) \in \Omega_{\mathrm{D}}\times\mathbb{R}^{+}$:

$$
\begin{aligned}
& k_0 \le \boldsymbol{\kappa}(x,\rho) \le k_1\,, \\
& k_0 \le \boldsymbol{\kappa}(x,\rho) + \rho\frac{\partial}{\partial\rho}\boldsymbol{\kappa}(x,\rho) \le k_1\,, \quad \text{and} \\
& |\nabla_x\boldsymbol{\kappa}(x,\rho)| \le k_1\,.
\end{aligned} \tag{5.4}
$$

In order to handle the nonlinearity in Ω_{D} we proceed as in [41] (see also [44] and [53]), and introduce the additional unknown $\mathbf{t}_{\mathrm{D}} := \nabla p_{\mathrm{D}}$ in Ω_{D}. In this way, the Darcy model is rewritten as follows:

$$
\begin{aligned}
\mathbf{t}_{\mathrm{D}} = \nabla p_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \qquad \mathbf{u}_{\mathrm{D}} + \boldsymbol{\kappa}\,(\cdot,|\mathbf{t}_{\mathrm{D}}|)\mathbf{t}_{\mathrm{D}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \\
\operatorname{div}\mathbf{u}_{\mathrm{D}} = f_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \qquad \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\,.
\end{aligned} \tag{5.5}
$$

Finally, the transmission conditions on Σ are given by

$$\begin{aligned} \mathbf{u}_{\mathrm{S}} \cdot \mathbf{n} \,&=\, \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} \quad \text{on} \quad \Sigma\,, \\ \boldsymbol{\sigma}_{\mathrm{S}}\, \mathbf{n} \,+\, \nu\, \kappa_f^{-1}\, (\mathbf{u}_{\mathrm{S}} \cdot \mathbf{t})\, \mathbf{t} \,&=\, -\, p_{\mathrm{D}}\, \mathbf{n} \quad \text{on} \quad \Sigma\,, \end{aligned} \tag{5.6}$$

where κ_f, the friction coefficient, is assumed to be constant.

5.2.2 The dual-mixed formulation

Let us first introduce further notations. In what follows, given $\star \in \{\mathrm{S}, \mathrm{D}\}$, we denote

$$(u, v)_\star \,:=\, \int_{\Omega_\star} u\, v, \qquad (\mathbf{u}, \mathbf{v})_\star \,:=\, \int_{\Omega_\star} \mathbf{u} \cdot \mathbf{v}, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_\star \,:=\, \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}\,,$$

where $\boldsymbol{\sigma} : \boldsymbol{\tau} \,=\, \mathrm{tr}(\boldsymbol{\sigma}^t \boldsymbol{\tau}) \,=\, \sum_{ij=1}^{2} \sigma_{ij} \tau_{ij}$.

The unknows in the dual-mixed formulation will be the unknows of (5.2) without the pressure p_{S} and the three unknows in (5.5). Hence, the corresponding spaces will be:

$$\boldsymbol{\sigma}_{\mathrm{S}} \,\in\, \mathbb{H}(\mathbf{div}; \Omega_{\mathrm{S}}), \quad \mathbf{u}_{\mathrm{S}} \,\in\, \mathbf{L}^2(\Omega_{\mathrm{S}}), \quad \mathbf{t}_{\mathrm{D}} \,\in\, \mathbf{L}^2(\Omega_{\mathrm{D}})\,, \quad \mathbf{u}_{\mathrm{D}} \,\in\, \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div}; \Omega_{\mathrm{D}}), \quad p_{\mathrm{D}} \,\in\, L^2(\Omega_{\mathrm{D}})\,,$$

where

$$\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div}; \Omega_{\mathrm{D}}) \,:=\, \{\mathbf{v} \,\in \mathbf{H}(\mathrm{div}\,; \Omega_{\mathrm{D}}) : \quad \mathbf{v} \cdot \mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\}\,.$$

In addition, we will need to define two unknowns on the coupling boundary

$$\boldsymbol{\varphi} \,:=\, -\mathbf{u}_{\mathrm{S}} \,\in\, \mathbf{H}^{1/2}(\Sigma)\,, \qquad \lambda \,:=\, p_{\mathrm{D}} \,\in\, H^{1/2}(\Sigma)\,, \tag{5.7}$$

where $\mathbf{H}^{1/2}(\Sigma) \,:=\, H_{00}^{1/2}(\Sigma) \times H_{00}^{1/2}(\Sigma)$ and

$$H_{00}^{1/2}(\Sigma) \,:=\, \left\{ v|_\Sigma : \quad v \,\in\, H^1(\Omega_{\mathrm{S}})\,, \quad v \,=\, 0 \quad \text{on} \quad \Gamma_{\mathrm{S}} \right\}.$$

Equivalently, if $E_{0,\mathrm{S}} : H^{1/2}(\Sigma) \to L^2(\partial\Omega_{\mathrm{S}})$ is the extension operator defined by

$$E_{0,\mathrm{S}}(\psi) \,:=\, \begin{cases} \psi & \text{on} \quad \Sigma \\ 0 & \text{on} \quad \Gamma_{\mathrm{S}} \end{cases} \qquad \forall\, \psi \,\in\, H^{1/2}(\Sigma)\,,$$

we have that

$$H_{00}^{1/2}(\Sigma) \,=\, \left\{ \psi \,\in\, H^{1/2}(\Sigma) : \quad E_{0,\mathrm{S}}(\psi) \,\in\, H^{1/2}(\partial\Omega_{\mathrm{S}}) \right\},$$

endowed with the norm $\|\psi\|_{1/2,00,\Sigma} \,:=\, \|E_{0,\mathrm{S}}(\psi)\|_{1/2,\partial\Omega_{\mathrm{S}}}$. The dual space of $\mathbf{H}^{1/2}(\Sigma)$ is denoted by $\mathbf{H}_{00}^{-1/2}(\Sigma)$. Note that, in principle, the spaces for $\mathbf{u}_{\mathrm{S}}$ and p_{D} do not allow enough regularity for the traces $\boldsymbol{\varphi}$ and λ to exist. However, solutions of (5.2) and (5.5) have these unknowns in $\mathbf{H}^1(\Omega_{\mathrm{S}})$ and $H^1(\Omega_{\mathrm{D}})$ respectively.

Next, for the derivation of the weak formulation of (5.2)-(5.5)-(5.6), we begin by testing the first equations of (5.2) and (5.5) with arbitrary $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ and $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$, respectively. Thus, integrating by parts, and using the identity $\boldsymbol{\sigma}_{\mathrm{S}}^d : \boldsymbol{\tau}_{\mathrm{S}} = \boldsymbol{\sigma}_{\mathrm{S}}^d : \boldsymbol{\tau}_{\mathrm{S}}^d$, we obtain

$$\nu^{-1}\,(\boldsymbol{\sigma}_{\mathrm{S}}^d,\boldsymbol{\tau}_{\mathrm{S}}^d)_{\mathrm{S}} \,+\, (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}})_{\mathrm{S}} \,+\, \langle \boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n},\boldsymbol{\varphi}\rangle_{\Sigma} \,=\, 0 \qquad \forall\,\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})\,, \tag{5.8}$$

and

$$(\mathbf{t}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \,+\, (\mathrm{div}\,\mathbf{v}_{\mathrm{D}},p_{\mathrm{D}})_{\mathrm{D}} \,+\, \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\lambda\rangle_{\Sigma} \,=\, 0 \qquad \forall\,\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\,. \tag{5.9}$$

In addition, the corresponding equilibrium equations become

$$(\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \,=\, -\,(\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \qquad \forall\,\mathbf{v}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})\,, \tag{5.10}$$

and

$$(\mathrm{div}\,\mathbf{u}_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}} \,=\, (f_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}} \qquad \forall\,q_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}})\,, \tag{5.11}$$

whereas the transmission conditions from (5.6), being essential due to the mixed nature of the coupled model, are imposed independently, which yields the introduction of the auxiliary unknowns (5.7) as the associated Lagrange multipliers. According to this, we get the equations

$$\langle \boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_{\Sigma} \,+\, \langle \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_{\Sigma} = 0 \qquad \forall\,\xi \in H^{1/2}(\Sigma) \tag{5.12}$$

and

$$\langle \boldsymbol{\sigma}_{\mathrm{S}}\mathbf{n},\boldsymbol{\psi}\rangle_{\Sigma} \,+\, \langle \boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_{\Sigma} \,-\, \nu\,\kappa_f^{-1}\,\langle \boldsymbol{\psi}\cdot\mathbf{t},\boldsymbol{\varphi}\cdot\mathbf{t}\rangle_{\Sigma} \,=\, 0 \qquad \forall\,\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Sigma)\,. \tag{5.13}$$

Finally, the equation relating $\mathbf{u}_{\mathrm{D}}$ to the new unknown $\mathbf{t}_{\mathrm{D}}$ is incorporated by:

$$(\boldsymbol{\kappa}(\cdot,|\mathbf{t}_{\mathrm{D}}|)\mathbf{t}_{\mathrm{D}},\mathbf{s}_{\mathrm{D}})_{\mathrm{D}} \,+\, (\mathbf{u}_{\mathrm{D}},\mathbf{s}_{\mathrm{D}})_{\mathrm{D}} \,=\, 0 \qquad \forall\,\mathbf{s}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})\,. \tag{5.14}$$

As a consequence of the above, we find that the resulting variational formulation reduces to a nonlinear system of seven unknowns and seven equations given by the set (5.8) – (5.14). However, it is easy to see that this system is not uniquely solvable since, given any solution $((\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{t}_{\mathrm{D}}),(\mathbf{u}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}},\boldsymbol{\varphi}),(p_{\mathrm{D}},\lambda))$ and $c \in \mathbb{R}$, $((\boldsymbol{\sigma}_{\mathrm{S}} - c\,\mathbf{I},\mathbf{t}_{\mathrm{D}}),(\mathbf{u}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}},\boldsymbol{\varphi}),(p_{\mathrm{D}}+c,\lambda+c))$ also becomes a solution. In order to avoid this non-uniqueness from now on we require that the Darcy pressure p_{D} belongs to $L_0^2(\Omega_{\mathrm{D}}) := \Big\{ v \in L^2(\Omega_{\mathrm{D}}) : \displaystyle\int_{\Omega_{\mathrm{D}}} v \,=\, 0 \Big\}$.

Now, it is quite clear that there are many different ways of ordering the variational system (5.8) – (5.14). Throughout the rest of the paper, and for convenience of the analysis, we adopt one leading to a twofold saddle point structure. To this end, we group unknowns and spaces as follows:

$$\begin{aligned}
&(\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{t}_{\mathrm{D}}) \,\in\, \mathbf{X} \,:=\, \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{L}^2(\Omega_{\mathrm{D}})\,,\\
&(\mathbf{u}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}},\boldsymbol{\varphi}) \,\in\, \mathbf{M} \,:=\, \mathbf{L}^2(\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbf{H}^{1/2}(\Sigma)\,,\\
&(p_{\mathrm{D}},\lambda) \,\in\, \mathbf{Q} \,:=\, L_0^2(\Omega_{\mathrm{D}}) \times H^{1/2}(\Sigma)\,,
\end{aligned}$$

and consider the following product norms

$$\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}} := \|\boldsymbol{\tau}_S\|_{\mathbf{div};\Omega_S} + \|\mathbf{s}_D\|_{0,\Omega_S} \qquad \forall\, \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \mathbf{s}_D) \in \mathbf{X},$$
$$\|\underline{\mathbf{v}}\|_{\mathbf{M}} := \|\mathbf{v}_S\|_{0,\Omega_S} + \|\mathbf{v}_D\|_{\mathrm{div};\Omega_D} + \|\boldsymbol{\psi}\|_{1/2,00,\Sigma} \qquad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_S, \mathbf{v}_D, \boldsymbol{\psi}) \in \mathbf{M},$$
$$\|\underline{\mathbf{q}}\|_{\mathbf{Q}} := \|q_D\|_{0,\Omega_D} + \|\xi\|_{1/2,\Sigma} \qquad \forall\, \underline{\mathbf{q}} := (q_D, \xi) \in \mathbf{Q}.$$

Next, we define the nonlinear operator $\mathbf{A} : \mathbf{X} \longrightarrow \mathbf{X}'$,

$$[\mathbf{A}(\boldsymbol{\sigma}_S, \mathbf{t}_D), (\boldsymbol{\tau}_S, \mathbf{s}_D)] := [\mathbf{A}_S(\boldsymbol{\sigma}_S), \boldsymbol{\tau}_S] + [\mathbf{A}_D(\mathbf{t}_D), \mathbf{s}_D] \tag{5.15}$$

where $\mathbf{A}_S : \mathbb{H}(\mathbf{div}; \Omega_S) \to \mathbb{H}(\mathbf{div}; \Omega_S)'$ and $\mathbf{A}_D : \mathbf{L}^2(\Omega_D) \to \mathbf{L}^2(\Omega_D)'$ are given, respectively, by

$$[\mathbf{A}_S(\boldsymbol{\sigma}_S), \boldsymbol{\tau}_S] := \nu^{-1}(\boldsymbol{\sigma}_S^d, \boldsymbol{\tau}_S^d)_S, \tag{5.16}$$

$$[\mathbf{A}_D(\mathbf{t}_D), \mathbf{s}_D] := (\boldsymbol{\kappa}(\cdot, |\mathbf{t}_D|)\mathbf{t}_D, \mathbf{s}_D)_D. \tag{5.17}$$

In addition, we define the bounded and linear operatos $\mathbf{B}_1 : \mathbf{X} \longrightarrow \mathbf{M}'$ and $\mathbf{B} : \mathbf{M} \longrightarrow \mathbf{Q}'$,

$$[\mathbf{B}_1(\boldsymbol{\tau}_S, \mathbf{s}_D), (\mathbf{v}_S, \mathbf{v}_D, \boldsymbol{\psi})] := (\mathbf{div}\, \boldsymbol{\tau}_S, \mathbf{v}_S)_S + (\mathbf{s}_D, \mathbf{v}_D)_D + \langle \boldsymbol{\tau}_S\, \mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma, \tag{5.18}$$

$$[\mathbf{B}(\mathbf{v}_S, \mathbf{v}_D, \boldsymbol{\psi}), (q_D, \xi)] := (\mathrm{div}\, \mathbf{v}_D, q_D)_D + \langle \mathbf{v}_D \cdot \mathbf{n}, \xi\rangle_\Sigma + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \xi\rangle_\Sigma, \tag{5.19}$$

the positive semi-definite and linear operator $\mathbf{S} : \mathbf{M} \longrightarrow \mathbf{M}'$,

$$[\mathbf{S}(\mathbf{u}_S, \mathbf{u}_D, \boldsymbol{\varphi}), (\mathbf{v}_S, \mathbf{v}_D, \boldsymbol{\psi})] := \nu \kappa_f^{-1} \langle \boldsymbol{\psi} \cdot \mathbf{t}, \boldsymbol{\varphi} \cdot \mathbf{t}\rangle_\Sigma, \tag{5.20}$$

and the functionals $\mathbf{F} \in \mathbf{X}'$, $\mathbf{G}_1 \in \mathbf{M}'$, and $\mathbf{G} \in \mathbf{Q}'$, given by

$$[\mathbf{F}, (\boldsymbol{\tau}_S, \mathbf{s}_D)] := 0, \quad [\mathbf{G}_1, (\mathbf{v}_S, \mathbf{v}_D, \boldsymbol{\psi})] := (\mathbf{f}_S, \mathbf{v}_S)_S, \quad \text{and} \quad [\mathbf{G}, (q_D, \xi)] := (f_D, q_D)_D. \tag{5.21}$$

Hereafter, $[\cdot, \cdot]$ denotes the duality pairing induced by the operators and functionals involved.

Hence, defining the global unknowns

$$\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_S, \mathbf{t}_D) \in \mathbf{X}, \quad \underline{\mathbf{u}} := (\mathbf{u}_S, \mathbf{u}_D, \boldsymbol{\varphi}) \in \mathbf{M}, \quad \text{and} \quad \underline{\mathbf{p}} := (p_D, \lambda) \in \mathbf{Q},$$

we realize that the variational system (5.8) – (5.14) can be stated as the twofold saddle point operator equation: Find $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ such that,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}_1' & \mathbf{0} \\ \mathbf{B}_1 & -\mathbf{S} & \mathbf{B}' \\ \mathbf{0} & \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{\sigma}} \\ \underline{\mathbf{u}} \\ \underline{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G}_1 \\ \mathbf{G} \end{bmatrix}. \tag{5.22}$$

The abstract theory for this kind of continuous formulation is already available (see, e.g. [41]), and its main results are collected in the following subsection.

5.2.3 Abstract theory for twofold saddle point operator equations

Let X, M and Q be Hilbert spaces with duals X', M' and Q', and consider a nonlinear operator $A : X \to X'$, and linear bounded operators $S : M \to M'$, $B_1 : X \to M'$, and $B : M \to Q'$, with corresponding adjoints $B_1' : M \to X'$ and $B' : Q \to M'$. Then we are interested in the following nonlinear variational problem: Given $(F, G_1, G) \in X' \times M' \times Q'$, find $(\sigma, u, p) \in X \times M \times Q$ such that

$$\begin{bmatrix} A & B_1' & O \\ B_1 & -S & B' \\ O & B & O \end{bmatrix} \begin{bmatrix} \sigma \\ u \\ p \end{bmatrix} = \begin{bmatrix} F_1 \\ G_1 \\ G \end{bmatrix} \tag{5.23}$$

We have the following theorem.

Theorem 5.2.1 *Let V be the kernel of B, that is*

$$V := \left\{ v \in M \,:\, [B(v), q] = 0 \qquad \forall\, q \in Q \right\}.$$

Assume that:

i) *A is strongly monotone and Lipschitz continuous , that is, there exist $\alpha,\, \gamma > 0$ such that*

$$[A(\tau) - A(\zeta), \tau - \zeta] \;\geq\; \alpha\, \|\tau - \zeta\|_X^2 \qquad \forall\, \tau,\, \zeta \in X\,,$$

and

$$\|A(\tau) - A(\zeta)\|_{X'} \;\leq\; \gamma\, \|\tau - \zeta\|_X \qquad \forall\, \tau,\, \zeta \in X\,;$$

ii) *S is positive semi-definite on V, that is*

$$[S(v), v] \;\geq\; 0 \qquad \forall\, v \in V\,;$$

iii) *B_1 satisfies the inf-sup condition on $X \times V$, that is, there exists $\beta_1 > 0$ such that*

$$\sup_{\substack{\tau \in X \\ \tau \neq \mathbf{0}}} \frac{[B_1(\tau), v]}{\|\tau\|_X} \;\geq\; \beta_1\, \|v\|_M \qquad \forall\, v \,\in\, V\,;$$

iv) *B satisfies the inf-sup condition on $M \times Q$, that is, there exists $\beta > 0$ such that*

$$\sup_{\substack{v \in M \\ v \neq \mathbf{0}}} \frac{[B(v), q]}{\|v\|_M} \;\geq\; \beta\, \|q\|_Q \qquad \forall\, q \,\in\, Q\,.$$

Then, for each $(F, G_1, G) \in X_1' \times M' \times Q'$*, there exists a unique* $(\sigma, u, p) \in X \times M \times Q$ *solution of* (5.23)*. In addition, there exists* $C > 0$*, depending only on* γ*,* α*,* β_1*,* β*,* $\|B_1\|$*, and* $\|S\|$*, such that*

$$\|\sigma\|_X + \|u\|_M + \|p\|_Q \leq C\left\{\|F_1\|_{X'} + \|G_1\|_{M'} + \|G\|_{Q'} + \|A(\mathbf{0})\|_{X'}\right\}. \tag{5.24}$$

Proof. See Theorem 2.1 in [41]. □

Now, let X_h, M_h and Q_h be finite-dimensional subspaces of X, M and Q, respectively. Then the Galerkin scheme associated with (5.23) reads as follows: Given $(F, G_1, G) \in X' \times M' \times Q'$, find $(\sigma_h, u_h, p_h) \in X_h \times M_h \times Q_h$ such that

$$\begin{aligned}
[A(\sigma_h), \tau_h] + [B_1(\tau_h), u_h] \qquad\qquad\qquad &= [F, \tau_h] \quad \forall\, \tau_h \in X_h\,,\\
[B_1(\sigma_h), v_h] - [S(u_h), v_h] + [B(v_h), p_h] &= [G_1, v_h] \quad \forall\, v_h \in M_h\,,\\
[B(u_h), q_h] &= [G, q_h] \quad \forall\, q_h \in Q_h\,.
\end{aligned} \tag{5.25}$$

The discrete analogue of Theorem 5.2.1 is established next.

Theorem 5.2.2 *Let* V_h *be the discrete kernel of* B*, that is*

$$V_h := \left\{v_h \in M_h : \quad [B(v_h), q_h] = 0 \quad \forall\, q_h \in Q_h\right\}.$$

Assume that

i) A *is strongly monotone and Lipschitz continuous (cf. hypothesis* i) *in Theorem* 5.2.1*);*

ii) S *is positive semi-definite on* V_h*, that is*

$$[S(v_h), v_h] \geq 0 \qquad \forall\, v_h \in V_h\,;$$

iii) B_1 *satisfies the inf-sup condition on* $X_h \times V_h$*, that is, there exists* $\beta_1^* > 0$ *such that*

$$\sup_{\substack{\tau_h \in X_h\\ \tau_h \neq \mathbf{0}}} \frac{[B_1(\tau_h), v_h]}{\|\tau_h\|_X} \geq \beta_1^*\, \|v_h\|_M \qquad \forall\, v_h \in V_h\,;$$

iv) B *satisfies the inf-sup condition on* $M_h \times Q_h$*, that is, there exists* $\beta > 0$ *such that*

$$\sup_{\substack{v_h \in M_h\\ v_h \neq \mathbf{0}}} \frac{[B(v_h), q_h]}{\|v_h\|_M} \geq \beta^*\, \|q_h\|_Q \qquad \forall\, q_h \in Q_h\,.$$

Then, there exists a unique $(\sigma_h, u_h, p_h) \in X_h \times M_h \times Q_h$ *solution of* (5.25)*. In addition, there exists* $C > 0$*, depending only on* γ*,* α*,* β_1^**,* β^**,* $\|B_1\|$*, and* $\|S\|$*, such that*

$$\|\sigma_h\|_X + \|u_h\|_M + \|p_h\|_Q \leq C\left\{\|F_h\|_{X_h'} + \|G_{1,h}\|_{M_h'} + \|G_h\|_{Q_h'} + \|A_h(\mathbf{0})\|_{X_h'}\right\},$$

where $F_h := F|_{X_h}$*,* $G_{1,h} := G_1|_{X_h}$*,* $G_h := G|_{Q_h}$*, and* $A_h(\mathbf{0}) := A(\mathbf{0})|_{X_h}$*.*

Proof. See Theorem 3.1 in [41]. □

Finally, concerning the error analysis, we have the following result.

Theorem 5.2.3 *Assume that the hypotheses of Theorem* 5.2.1 *and Theorem* 5.2.2 *hold and that the operator* $A : X \to X'$ *has a hemi-continuous first order Gâteaux derivative* $DA : X \to \mathcal{L}(X, X')$, *that is, for any* $\tau, \zeta \in X$, *the mapping* $\mathbb{R} \ni \mu \to DA(\zeta + \mu\,\tau)(\tau, \cdot) \in X'$ *is continuous. Let* $(\sigma, u, p) \in X \times M \times Q$ *and* $(\sigma_h, u_h, p_h) \in X_h \times M_h \times Q_h$ *be the unique solutions of* (5.23) *and* (5.25), *respectively. Then there exists* $C > 0$, *independent of* h, *such that*

$$\|(\sigma, u, p) - (\sigma_h, u_h, p_h)\| \le C \inf_{\substack{(\tau_h, v_h, q_h) \\ \in X_h \times M_h \times Q_h}} \|(\sigma, u, p) - (\tau_h, v_h, q_h)\| . \tag{5.26}$$

Proof. See Theorem 3.3 in [41]. □

5.2.4 An equivalent twofold saddle point formulation

In order to apply the abstract theory from Section 5.2.3 to our problem (5.22), we need first to introduce an equivalent formulation. To this end, we now reutilize the equilibrium equation of the Stokes problem in the form of the following Galerkin least squares-type term

$$(\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} = -(\mathbf{f}_{\mathrm{S}}, \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} \qquad \forall\, \boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div}\,; \Omega_{\mathrm{S}}) , \tag{5.27}$$

which is then added to the formulation (5.22) and placed within the operator $\mathbf{A}$, thus giving rise to a modified operator $\tilde{\mathbf{A}}$ (see (5.34), (5.35) below). In addition, we consider the decomposition

$$\mathbb{H}(\mathbf{div}\,; \Omega_{\mathrm{S}}) = \mathbb{H}_0(\mathbf{div}\,; \Omega_{\mathrm{S}}) \oplus \mathbf{P}_0(\Omega_{\mathrm{S}})\mathbf{I} , \tag{5.28}$$

where

$$\mathbb{H}_0(\mathbf{div}\,; \Omega_{\mathrm{S}}) := \left\{ \boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}\,; \Omega_{\mathrm{S}}) \,:\, \int_{\Omega_{\mathrm{S}}} \operatorname{tr} \boldsymbol{\sigma} = 0 \right\} ,$$

and set $\boldsymbol{\sigma}_{\mathrm{S}} = \tilde{\boldsymbol{\sigma}}_{\mathrm{S}} + c\mathbf{I}$, with the new unknowns $\tilde{\boldsymbol{\sigma}}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div}\,; \Omega_{\mathrm{S}})$ and $c \in \mathbb{R}$.

In this way, the equations (5.8), (5.13) and (5.27) are rewritten, equivalently as

$$\nu^{-1}(\tilde{\boldsymbol{\sigma}}_{\mathrm{S}}^d, \boldsymbol{\tau}_{\mathrm{S}}^d)_{\mathrm{S}} + (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma = 0 \qquad \forall\, \boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div}\,; \Omega_{\mathrm{S}} , \tag{5.29}$$

$$d\, \langle \mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma = 0 \qquad \forall\, d \in \mathbb{R} , \tag{5.30}$$

$$\langle \tilde{\boldsymbol{\sigma}}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma - \nu\, \kappa_f^{-1} \langle \boldsymbol{\psi} \cdot \mathbf{t}, \boldsymbol{\varphi} \cdot \mathbf{t} \rangle_\Sigma + c\, \langle \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma = 0 \qquad \forall\, \boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Sigma) , \tag{5.31}$$

and

$$(\mathbf{div}\,\tilde{\boldsymbol{\sigma}}_{\mathrm{S}}, \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} = -(\mathbf{f}_{\mathrm{S}}, \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} \qquad \forall\, \boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div}\,; \Omega_{\mathrm{S}}) . \tag{5.32}$$

Then, we define the global unknowns

$$\underline{\tilde{\boldsymbol{\sigma}}} := (\tilde{\boldsymbol{\sigma}}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}) \in \tilde{\mathbf{X}} := \mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}}) \times \mathbf{L}^2(\Omega_{\mathrm{D}}), \quad \underline{\tilde{\mathbf{p}}} := (\underline{\mathbf{p}}, c) \in \tilde{\mathbf{Q}} := \mathbf{Q} \times \mathbb{R},$$

and group the equations (5.9)–(5.12), (5.14), (5.29)–(5.32), which yields the following variational formulation: Find $(\underline{\tilde{\boldsymbol{\sigma}}}, \underline{\mathbf{u}}, \underline{\tilde{\mathbf{p}}}) \in \tilde{\mathbf{X}} \times \mathbf{M} \times \tilde{\mathbf{Q}}$ such that

$$\begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{B}_1' & \mathbf{0} \\ \mathbf{B}_1 & -\mathbf{S} & \tilde{\mathbf{B}}' \\ \mathbf{0} & \tilde{\mathbf{B}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\tilde{\boldsymbol{\sigma}}} \\ \underline{\mathbf{u}} \\ \underline{\tilde{\mathbf{p}}} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{F}} \\ \mathbf{G}_1 \\ \tilde{\mathbf{G}} \end{bmatrix}. \tag{5.33}$$

Hereafter, the nonlinear operator $\tilde{\mathbf{A}} : \tilde{\mathbf{X}} \to \tilde{\mathbf{X}}'$ is given by

$$[\tilde{\mathbf{A}}(\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}), (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{s}_{\mathrm{D}})] := [\tilde{\mathbf{A}}_{\mathrm{S}}(\boldsymbol{\sigma}_{\mathrm{S}}), \boldsymbol{\tau}_{\mathrm{S}}] + [\mathbf{A}_{\mathrm{D}}(\mathbf{t}_{\mathrm{D}}), \mathbf{s}_{\mathrm{D}}], \tag{5.34}$$

with $\tilde{\mathbf{A}}_{\mathrm{S}} : \mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}}) \longrightarrow \mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}})'$ the linear and bounded operator defined by

$$[\tilde{\mathbf{A}}_{\mathrm{S}}(\boldsymbol{\sigma}_{\mathrm{S}}), \boldsymbol{\tau}_{\mathrm{S}}] := [\mathbf{A}_{\mathrm{S}}(\boldsymbol{\sigma}_{\mathrm{S}}), \boldsymbol{\tau}_{\mathrm{S}}] + (\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}},$$

which, according to the definition of $\mathbf{A}_{\mathrm{S}}$ (cf. (5.16)), yields

$$[\tilde{\mathbf{A}}_{\mathrm{S}}(\boldsymbol{\sigma}_{\mathrm{S}}), \boldsymbol{\tau}_{\mathrm{S}}] := \nu^{-1} (\boldsymbol{\sigma}_{\mathrm{S}}^d, \boldsymbol{\tau}_{\mathrm{S}}^d)_{\mathrm{S}} + (\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}}. \tag{5.35}$$

In addition, the linear and bounded operator $\tilde{\mathbf{B}} : \mathbf{M} \to \tilde{\mathbf{Q}}'$, and the functionals $\tilde{\mathbf{F}} \in \tilde{\mathbf{X}}'$ and $\tilde{\mathbf{G}} \in \tilde{\mathbf{Q}}'$, are given, respectively, by

$$\begin{aligned} [\tilde{\mathbf{B}}(\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}), (q_{\mathrm{D}}, \xi, d)] &:= [\mathbf{B}(\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}), (q_{\mathrm{D}}, \xi)] + d\,\langle \mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma \\ &= (\mathrm{div}\,\mathbf{v}_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}} + \langle \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, \xi\rangle_\Sigma + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \xi\rangle_\Sigma + d\,\langle \mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma, \end{aligned} \tag{5.36}$$

$$[\tilde{\mathbf{F}}, (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{s}_{\mathrm{D}})] = [\mathbf{F}, (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{s}_{\mathrm{D}})] - (\mathbf{f}_{\mathrm{S}}, \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} = -(\mathbf{f}_{\mathrm{S}}, \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}},$$

and

$$[\tilde{\mathbf{G}}, (q_{\mathrm{D}}, \xi, d)] = [\mathbf{G}, (q_{\mathrm{D}}, \xi)] = (f_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}}.$$

The following theorem establishes the equivalence between (5.22) and (5.33).

Theorem 5.2.4 *If* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) := ((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}), \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ *is a solution of* (5.22)*, where* $\boldsymbol{\sigma}_{\mathrm{S}} = \tilde{\boldsymbol{\sigma}}_{\mathrm{S}} + c\mathbf{I}$*, with* $\tilde{\boldsymbol{\sigma}}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}})$ *and* $c \in \mathbb{R}$*, then* $(\underline{\tilde{\boldsymbol{\sigma}}}, \underline{\mathbf{u}}, \underline{\tilde{\mathbf{p}}}) := ((\tilde{\boldsymbol{\sigma}}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}), \underline{\mathbf{u}}, (\underline{\mathbf{p}}, c)) \in \tilde{\mathbf{X}} \times \mathbf{M} \times \tilde{\mathbf{Q}}$ *is a solution of* (5.33)*. Conversely, if* $((\tilde{\boldsymbol{\sigma}}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}), \underline{\mathbf{u}}, (\underline{\mathbf{p}}, c)) \in \tilde{\mathbf{X}} \times \mathbf{M} \times \tilde{\mathbf{Q}}$ *is solution of* (5.33)*, then* $((\tilde{\boldsymbol{\sigma}}_{\mathrm{S}} + c\mathbf{I}, \mathbf{t}_{\mathrm{D}}), \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ *is a solution of* (5.22)*.*

Proof. It suffices to apply the decomposition (5.28) and observe that in either direction one deduces that $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} = \mathbf{div}\,\tilde{\boldsymbol{\sigma}}_{\mathrm{S}} = -\mathbf{f}_{\mathrm{S}}$ in Ω_{S}. We omit futher details. □

5.3 Analysis of the continuous problem

In this section we analyze the well posedness of (5.22) (equivalently (5.33)). To this end, we prove below in Section 5.3.2 that the formulation (5.33) satisfies the hypotheses of Theorem 5.2.1.

5.3.1 Preliminaries

Here we group some merely technical results and further notations that we will serve for the forthcoming analyis. The following lemma is already well known.

Lemma 5.3.1 *There exists $C > 0$, depending only on Ω_S, such that*

$$C\,\|\boldsymbol{\tau}_S\|^2_{0,\Omega_S} \,\le\, \|\boldsymbol{\tau}^d_S\|^2_{0,\Omega_S} + \|\mathbf{div}\,\boldsymbol{\tau}_S\|^2_{0,\Omega_S} \qquad \forall\,\boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}\,;\Omega_S)\,. \tag{5.37}$$

Proof. See [12, Lemma 3.1] or [19, Proposition 3.1, Chapter IV]. □

We also recall that, given $\mathbf{v}_D \in \mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)$, the boundary condition $\mathbf{v}_D\cdot\mathbf{n} = 0$ on Γ_D means $\langle \mathbf{v}_D\cdot\mathbf{n}, E_{0,D}(\mu)\rangle_{\partial\Omega_D} = 0 \quad \forall\,\mu \in H^{1/2}_{00}(\Gamma_D)$, where $\langle\cdot,\cdot\rangle_{\partial\Omega_D}$ stands for the duality pairing of $H^{-1/2}(\partial\Omega_D)$ and $H^{1/2}(\partial\Omega_D)$ with respect to the $L^2(\partial\Omega_D)$-inner product, $E_{0,D} : H^{1/2}(\Gamma_D) \to L^2(\partial\Omega_D)$ is the extension operator defined by

$$E_{0,D}(\mu) \,:=\, \begin{cases} \mu & \text{on} \quad \Gamma_D \\ 0 & \text{on} \quad \Sigma \end{cases} \qquad \forall\,\mu \in H^{1/2}(\Gamma_D)\,,$$

and

$$H^{1/2}_{00}(\Gamma_D) \,=\, \Big\{ \mu \in H^{1/2}(\Gamma_D) : \quad E_{0,D}(\mu) \in H^{1/2}(\partial\Omega_D) \Big\}\,,$$

endowed with the norm $\|\mu\|_{1/2,00,\Gamma_D} := \|E_{0,D}(\mu)\|_{1/2,\partial\Omega_D}$.

As a consequence, it is not difficult to show (see e.g. Section 2 in [39]) that the restriction of $\mathbf{v}_D\cdot\mathbf{n}$ to Σ can be identified with an element of $H^{-1/2}(\Sigma)$, namely

$$\langle \mathbf{v}_D\cdot\mathbf{n}, \xi\rangle_\Sigma \,:=\, \langle \mathbf{v}_D\cdot\mathbf{n}, E_D(\xi)\rangle_{\partial\Omega_D} \qquad \forall\,\xi \in H^{1/2}(\Sigma)\,, \tag{5.38}$$

where $E_D : H^{1/2}(\Sigma) \to H^{1/2}(\partial\Omega_D)$ is any bounded extension operator. In particular, given $\xi \in H^{1/2}(\Sigma)$, one could define $E_D(\xi) := z|_{\partial\Omega_D}$, where $z \in H^1(\Omega_D)$ is the unique solution of the boundary value problem: $\Delta z = 0 \;\text{ in }\; \Omega_D\,, \quad z = \xi \;\text{ on }\; \Sigma\,, \quad \nabla z\cdot\mathbf{n} = 0 \;\text{ on }\; \Gamma_D$. In addition, one can show that for all $\mu \in H^{1/2}(\partial\Omega_D)$, there exist unique elements $\mu_\Sigma \in H^{1/2}(\Sigma)$ and $\mu_{\Gamma_D} \in H^{1/2}_{00}(\Gamma_D)$ such that

$$\mu \,=\, E_D(\mu_\Sigma) + E_{0,D}(\mu_{\Gamma_D})\,, \tag{5.39}$$

and

$$C_1\,\big(\|\mu_\Sigma\|_{1/2,\Sigma} + \|\mu_{\Gamma_D}\|_{1/2,00,\Gamma_D}\big) \,\le\, \|\mu\|_{1/2,\partial\Omega_D} \,\le\, C_2\,\big(\|\mu_\Sigma\|_{1/2,\Sigma} + \|\mu_{\Gamma_D}\|_{1/2,00,\Gamma_D}\big)\,.$$

5.3.2 The main results

We begin by proving the continuous inf-sup condition for $\tilde{\mathbf{B}}$ (cf. (5.36)), which will follow from the next two lemmas that separate the required estimate into two parts.

Lemma 5.3.2 *There exist* $C_1,\, C_2 > 0$ *such that*

$$S_1(\xi, d) := \sup_{\substack{\boldsymbol{\psi}\in \mathbf{H}^{1/2}_{00}(\Sigma)\\ \boldsymbol{\psi}\neq \mathbf{0}}} \frac{d\,\langle \mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma + \langle \boldsymbol{\psi}\cdot\mathbf{n}, \xi\rangle_\Sigma}{\|\boldsymbol{\psi}\|_{1/2,00,\Sigma}} \;\geq\; C_1\,|d| - C_2\,\|\xi\|_{1/2,\Sigma}\,, \tag{5.40}$$

for all $(\xi, d) \in H^{1/2}(\Sigma)\times\mathbb{R}$.

Proof. Let $\boldsymbol{\psi}_0$ be a fixed element in $\mathbf{H}^{1/2}(\Sigma)$ such that $\langle \mathbf{n}, \boldsymbol{\psi}_0\rangle_\Sigma \neq 0$. Hence, given $(\xi, d) \in H^{1/2}(\Sigma)\times\mathbb{R}$, we find that

$$S_1(\xi, d) \;\geq\; \frac{\left| d\,\langle \mathbf{n}, \boldsymbol{\psi}_0\rangle_\Sigma + \langle \boldsymbol{\psi}_0\cdot\mathbf{n}, \xi\rangle_\Sigma \right|}{\|\boldsymbol{\psi}_0\|_{1/2,00,\Sigma}} \geq C_1\,|d| - C_2\,\|\xi\|_{1/2,\Sigma}\,, \tag{5.41}$$

where $C_1 := \frac{|\langle \mathbf{n}, \boldsymbol{\psi}_0\rangle_\Sigma|}{\|\boldsymbol{\psi}_0\|_{1/2,00,\Sigma}}$, and C_2 satisfies $|\,\langle \boldsymbol{\psi}_0\cdot\mathbf{n}, \xi\rangle_\Sigma\,| \leq C_2\,\|\boldsymbol{\psi}_0\|_{1/2,00,\Sigma}\,\|\xi\|_{1/2,\Sigma}$. □

Note that there is a very simple way of defining such an element $\boldsymbol{\psi}_0$. In fact, as explained in [49, Section 3.2], we pick one interior corner point of Σ and define a function v that is continuous, linear on each side of Σ, equal to one in the chosen vertex, and zero on all other ones. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors on the two sides of Σ that meet at the corner point, then $\boldsymbol{\psi}_0 := v\,(\mathbf{n}_1 + \mathbf{n}_2)$ satisfies that property. If the interface Σ were a line segment (without interior corners), we pick v as the continuous linear function on Σ, equal to one in any interior point and zero in the extreme points, and define $\boldsymbol{\psi}_0 := v\,\mathbf{n}$.

Lemma 5.3.3 *There exists* $C_3 > 0$ *such that*

$$S_2(q_\mathrm{D}, \xi) := \sup_{\substack{\mathbf{v}_\mathrm{D}\in \mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div};\Omega_\mathrm{D})\\ \mathbf{v}_\mathrm{D}\neq \mathbf{0}}} \frac{(\mathrm{div}\,\mathbf{v}_\mathrm{D}, q_\mathrm{D})_\mathrm{D} + \langle \mathbf{v}_\mathrm{D}\cdot\mathbf{n}, \xi\rangle_\Sigma}{\|\mathbf{v}_\mathrm{D}\|_{\mathrm{div}\,;\Omega_\mathrm{D}}} \;\geq\; C_3\left\{ \|q_\mathrm{D}\|_{0,\Omega_\mathrm{D}} + \|\xi\|_{1/2,\Sigma} \right\}, \tag{5.42}$$

for all $(q_\mathrm{D}, \xi) \in L_0(\Omega_\mathrm{D})\times H^{1/2}(\Sigma)$.

Proof. Let $(q_\mathrm{D}, \xi) \in L_0(\Omega_\mathrm{D})\times H^{1/2}(\Sigma)$. Then, we define $\mathbf{w}_\mathrm{D} := \nabla z$ in Ω_D, where $z\in H^1(\Omega_\mathrm{D})$ is the unique solution of the boundary value problem:

$$\Delta\, z = q_\mathrm{D} \quad\text{in}\quad \Omega_\mathrm{D}\,,\quad \nabla z\cdot\mathbf{n} = 0 \quad\text{on}\quad \partial\Omega_\mathrm{D}\,,\quad \int_{\Omega_\mathrm{D}} z = 0\,.$$

It is clear that div $\mathbf{w}_D = q_D$ in Ω_D, $\mathbf{w}_D \in \mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)$ (since actually $\mathbf{w}_D \cdot \mathbf{n} = 0$ on $\partial\Omega_D$), and $\|\mathbf{w}_D\|_{\mathrm{div};\Omega_D} \le C\,\|q_D\|_{0,\Omega_D}$. Hence, using from (5.38) that $\langle \mathbf{w}_D \cdot \mathbf{n}, \xi\rangle_\Sigma = \langle \mathbf{w}_D \cdot \mathbf{n}, E_D(\xi)\rangle_{\partial\Omega_D} = 0$, we deduce that

$$S_2(q_D,\xi) \ \ge\ \frac{(\mathrm{div}\,\mathbf{w}_D, q_D)_D}{\|\mathbf{w}_D\|_{\mathrm{div};\Omega_D}} \ \ge\ C_3\,\|q_D\|_{0,\Omega_D}\,. \tag{5.43}$$

On the other hand, given $\phi \in H^{-1/2}(\Sigma)$, we define $\eta \in H^{-1/2}(\partial\Omega_D)$ as

$$\langle \eta, \mu\rangle_{\partial\Omega_D} := \langle \phi, \mu_\Sigma\rangle_\Sigma \qquad \forall\,\mu \in H^{1/2}(\partial\Omega_D)\,, \tag{5.44}$$

where μ_Σ is given by the decomposition (5.39). It is not difficult to see that

$$\langle \eta, E_{0,D}(\rho)\rangle_{\partial\Omega_D} = 0 \qquad \forall\,\rho \in H^{1/2}_{00}(\Gamma_D)\,, \tag{5.45}$$

$$\langle \eta, E_D(\xi)\rangle_{\partial\Omega_D} = \langle \phi, \xi\rangle_\Sigma\,, \tag{5.46}$$

and

$$\|\eta\|_{-1/2,\partial\Omega_D} \le C\,\|\phi\|_{-1/2,\Sigma}\,. \tag{5.47}$$

Hence, we now define $\mathbf{w}_D := \nabla z$ in Ω_D, where $z \in H^1(\Omega_D)$ is the unique solution of the boundary value problem:

$$\Delta\, z = \frac{1}{|\Omega_D|}\,\langle \eta, 1\rangle_{\partial\Omega_D} \quad \text{in} \quad \Omega_D\,, \quad \nabla z \cdot \mathbf{n} = \eta \quad \text{on} \quad \partial\Omega_D\,, \quad \int_{\Omega_D} z = 0\,.$$

It follows that $\mathrm{div}\,\mathbf{w}_D = \frac{1}{|\Omega_D|}\langle \eta, 1\rangle_{\partial\Omega_D} \in \mathbf{P}_0(\Omega_D)$, $\mathbf{w}_D \cdot \mathbf{n} = \eta$ on $\partial\Omega_D$, and, using the estimate (5.47), $\|\mathbf{w}_D\|_{\mathrm{div};\Omega_D} \le C\,\|\eta\|_{-1/2,\partial\Omega_D} \le C\,\|\phi\|_{-1/2,\Sigma}$. In addition, according to (5.38) and (5.46), and (5.45), we find, respectively, that

$$\langle \mathbf{w}_D \cdot \mathbf{n}, \xi\rangle_\Sigma = \langle \mathbf{w}_D \cdot \mathbf{n}, E_D(\xi)\rangle_{\partial\Omega_D} = \langle \eta, E_D(\xi)\rangle_{\partial\Omega_D} = \langle \phi, \xi\rangle_\Sigma\,,$$

and

$$\langle \mathbf{w}_D \cdot \mathbf{n}, E_{0,D}(\rho)\rangle_{\partial\Omega_D} = \langle \eta, E_{0,D}(\rho)\rangle_{\partial\Omega_D} = 0 \qquad \forall \rho \in H^{1/2}_{00}(\Gamma_D)\,,$$

which implies that $\mathbf{w}_D \in \mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)$. In this way, since $q_D \in L^2_0(\Omega_D)$, we conclude that

$$S_2(q_D,\xi) \ge \frac{|\langle \mathbf{w}_D \cdot \mathbf{n}, \xi\rangle_\Sigma|}{\|\mathbf{w}_D\|_{\mathrm{div};\Omega_D}} \ge C\,\frac{|\langle \phi, \xi\rangle_\Sigma|}{\|\phi\|_{-1/2,\Sigma}} \qquad \forall\,\phi \in H^{-1/2}(\Sigma)\,,$$

and therefore

$$S_2(q_D,\xi) \ge C \sup_{\substack{\phi \in H^{-1/2}(\Sigma)\\ \phi \ne \mathbf{0}}} \frac{|\langle \phi, \xi\rangle_\Sigma|}{\|\phi\|_{-1/2,\Sigma}} = C\,\|\xi\|_{1/2,\Sigma}\,.$$

This estimate and (5.43) imply (5.42), which finishes the proof. □

The continuous inf-sup condition for $\tilde{\mathbf{B}}$ follows straightforwardly from the previous lemmas.

Lemma 5.3.4 *There exists $\beta > 0$ such that*

$$\sup_{\substack{\underline{\mathbf{v}}\in\mathbf{M}\\ \underline{\mathbf{v}}\neq\mathbf{0}}} \frac{[\tilde{\mathbf{B}}(\underline{\mathbf{v}}),(\underline{\mathbf{q}},d)]}{\|\underline{\mathbf{v}}\|_{\mathbf{M}}} \;\geq\; \beta\left\{\|\underline{\mathbf{q}}\|_{\mathbf{Q}} + |d|\right\} \qquad \forall\,(\underline{\mathbf{q}},d)\,\in\,\tilde{\mathbf{Q}} \,:=\, \mathbf{Q}\times\mathbb{R}\,. \tag{5.48}$$

Proof. It suffices to observe, recalling that $\mathbf{M} := \mathbf{L}^2(\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \times \mathbf{H}^{1/2}(\Sigma)$, that

$$\sup_{\substack{\underline{\mathbf{v}}\in\mathbf{M}\\ \underline{\mathbf{v}}\neq\mathbf{0}}} \frac{[\tilde{\mathbf{B}}(\underline{\mathbf{v}}),(\underline{\mathbf{q}},d)]}{\|\underline{\mathbf{v}}\|_{\mathbf{M}}} \;\geq\; \max\left\{S_1(\xi,d), S_2(q_{\mathrm{D}},\xi)\right\} \qquad \forall\,(\underline{\mathbf{q}},d) \,:=\, ((q_{\mathrm{D}},\xi),d)\,\in\,\tilde{\mathbf{Q}}\,,$$

and then perform a suitable linear combination of (5.40) and (5.42) (cf. Lemmas 5.3.2 and 5.3.3). □

We continue the analysis with the continuous inf-sup condition for $\mathbf{B}_1$ on $\tilde{\mathbf{X}}\times\mathbf{V}$, where $\mathbf{V}$, the kernel of $\tilde{\mathbf{B}}$, is given by

$$\mathbf{V} \,:=\, \left\{\underline{\mathbf{v}}\in\mathbf{M}: \quad [\tilde{\mathbf{B}}(\underline{\mathbf{v}}),(\underline{\mathbf{q}},d)] \,=\, 0, \quad \forall\,(\underline{\mathbf{q}},d)\,\in\tilde{\mathbf{Q}}\right\}.$$

More precisely, according to the definition of $\tilde{\mathbf{B}}$ (cf. (5.36)), we find that

$$\mathbf{V} \,:=\, \left\{(\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}},\boldsymbol{\psi})\in\mathbf{M}: \;\mathrm{div}\;\mathbf{v}_{\mathrm{D}} \,=\, 0 \text{ in } \Omega_{\mathrm{D}}\,,\; \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n} \,=\, -\,\boldsymbol{\psi}\cdot\mathbf{n} \text{ on } \Sigma\,,\; \langle\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma \,=\, 0\right\}.$$

Then, similarly as for $\tilde{\mathbf{B}}$, we recall from (5.18) the definition of $\mathbf{B}_1$, and separate the required estimate into the following two parts.

Lemma 5.3.5 *There holds*

$$S_3(\mathbf{v}_{\mathrm{D}}) \,:=\, \sup_{\substack{\mathbf{s}_{\mathrm{D}}\in\mathbf{L}^2(\Omega_{\mathrm{D}})\\ \mathbf{s}\neq\mathbf{0}}} \frac{(\mathbf{s}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}}}{\|\mathbf{s}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}} \;\geq\; \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}} \quad \forall\,(\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}},\boldsymbol{\psi})\,\in\,\mathbf{V}\,. \tag{5.49}$$

Proof. Given $(\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}},\boldsymbol{\psi})\,\in\,\mathbf{V}$, it suffices to bound $S_3(\mathbf{v}_{\mathrm{D}})$ by taking in particular $\mathbf{s}_{\mathrm{D}} = \mathbf{v}_{\mathrm{D}}$, and then use that $\|\mathbf{v}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \,=\, \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}}$. □

Lemma 5.3.6 *There exists $C_4 > 0$ such that*

$$S_4(\mathbf{v}_{\mathrm{S}},\boldsymbol{\psi}) \,:=\, \sup_{\substack{\boldsymbol{\tau}_{\mathrm{S}}\,\in\,\mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}})\\ \boldsymbol{\tau}_{\mathrm{S}}\neq\mathbf{0}}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \,+\, \langle\boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div}\,;\Omega_{\mathrm{S}}}} \;\geq\; C_4\left\{\|\mathbf{v}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\psi}\|_{1/2,00,\Sigma}\right\} \tag{5.50}$$

for all $(\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}},\boldsymbol{\psi})\,\in\,\mathbf{V}$.

Proof. Given $(\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}},\boldsymbol{\psi})\,\in\,\mathbf{V}$ and $\boldsymbol{\tau}_{\mathrm{S}} \,:=\, \boldsymbol{\tau}_{\mathrm{S},0} + c\,\mathbf{I}\,\in\,\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ with $\boldsymbol{\tau}_{\mathrm{S},0}\,\in\,\mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}})$ and $c\,\in\,\mathbf{P}_0(\Omega_{\mathrm{S}})$ (cf. (5.28)), we notice that $(\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \,=\, (\mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S},0},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}}$, $\langle\boldsymbol{\tau}_{\mathrm{S}}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma \,=\, \langle\boldsymbol{\tau}_{\mathrm{S},0}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma$, and $\|\boldsymbol{\tau}_{\mathrm{S}}\|^2_{\mathbf{div}\,;\Omega_{\mathrm{S}}} \,=\, \|\boldsymbol{\tau}_{\mathrm{S},0}\|^2_{\mathbf{div}\,;\Omega_{\mathrm{S}}} \,+\, 2\,c^2\,|\Omega_{\mathrm{S}}|$. Hence, the supremum S_4 remains the same if taken on $\mathbb{H}(\mathbf{div};\Omega_S)$ instead of $\mathbb{H}_0(\mathbf{div};\Omega_S)$. The rest proceeds exactly as in the proof of [14, Theorem 2.1] by defining suitable auxiliary problems. We omit further details. □

As a consequence of the previous lemmas, and recalling that $\tilde{\mathbf{X}} := \mathbb{H}_0(\mathbf{div};\Omega_S) \times \mathbf{L}^2(\Omega_D)$, we are able to establish the following result.

Lemma 5.3.7 *There exists $\beta_1 > 0$ such that*

$$\sup_{\substack{\underline{\tilde{\boldsymbol{\tau}}}\in\tilde{\mathbf{X}}\\ \underline{\tilde{\boldsymbol{\tau}}}\neq\mathbf{0}}} \frac{[\mathbf{B}_1(\underline{\tilde{\boldsymbol{\tau}}}),\underline{\mathbf{v}}]}{\|\underline{\tilde{\boldsymbol{\tau}}}\|_{\tilde{\mathbf{X}}}} \geq \beta_1 \|\underline{\mathbf{v}}\|_{\mathbf{M}} \qquad \forall\, \underline{\mathbf{v}} := (\mathbf{v}_S, \mathbf{v}_D, \boldsymbol{\psi}) \in \mathbf{V}. \tag{5.51}$$

Proof. It suffices to observe that

$$\sup_{\substack{\underline{\tilde{\boldsymbol{\tau}}}\in\tilde{\mathbf{X}}\\ \underline{\tilde{\boldsymbol{\tau}}}\neq\mathbf{0}}} \frac{[\mathbf{B}_1(\underline{\tilde{\boldsymbol{\tau}}}),\underline{\mathbf{v}}]}{\|\underline{\tilde{\boldsymbol{\tau}}}\|_{\tilde{\mathbf{X}}}} \geq \max\Big\{S_3(\mathbf{v}_D), S_4(\mathbf{v}_S,\boldsymbol{\psi})\Big\} \quad \forall\, (\mathbf{v}_S, \mathbf{v}_D, \boldsymbol{\psi}) \in \mathbf{V},$$

and then apply the estimates (5.3.5) and (5.3.6) (cf. Lemmas 5.3.5 and 5.3.6). □

We now come to the strong monotonicity and Lipschitz-continuity of $\tilde{\mathbf{A}} : \tilde{\mathbf{X}} \to \tilde{\mathbf{X}}'$.

Lemma 5.3.8 *There exist constants α, $\gamma > 0$ such that*

$$[\tilde{\mathbf{A}}(\underline{\tilde{\boldsymbol{\tau}}}) - \tilde{\mathbf{A}}(\underline{\tilde{\boldsymbol{\zeta}}}), \underline{\tilde{\boldsymbol{\tau}}} - \underline{\tilde{\boldsymbol{\zeta}}}] \geq \alpha \|\underline{\tilde{\boldsymbol{\tau}}} - \underline{\tilde{\boldsymbol{\zeta}}}\|^2_{\tilde{\mathbf{X}}}$$

and

$$\|\tilde{\mathbf{A}}(\underline{\tilde{\boldsymbol{\tau}}}) - \tilde{\mathbf{A}}(\underline{\tilde{\boldsymbol{\zeta}}})\|_{\tilde{\mathbf{X}}'} \leq \gamma \|\underline{\tilde{\boldsymbol{\tau}}} - \underline{\tilde{\boldsymbol{\zeta}}}\|_{\tilde{\mathbf{X}}},$$

for all $\underline{\tilde{\boldsymbol{\tau}}}$, $\underline{\tilde{\boldsymbol{\zeta}}} \in \tilde{\mathbf{X}}$.

Proof. Let us have in mind the definition of $\tilde{\mathbf{A}}$ from (5.34). Then, thanks to the assumptions (5.4), one can show (see e.g. [53, Theorem 3.8] for details) that the nonlinear operator $\mathbf{A}_D$ (cf. (5.17)) is strongly monotone and Lipschitz continuous on $\mathbf{L}^2(\Omega_D)$. In addition, it is easy to see, using Lemma 5.3.1, that the bounded linear operator $\tilde{\mathbf{A}}_S$ (cf. (5.35)) is elliptic on $\mathbb{H}_0(\mathbf{div};\Omega_S)$. These results yield the required estimates for $\tilde{\mathbf{A}}$. □

We are now in a position to establish the well-posedness of (5.22).

Theorem 5.3.1 *For each $(\mathbf{F}, \mathbf{G}_1, \mathbf{G}) \in \mathbf{X}'\times\mathbf{M}'\times\mathbf{Q}'$ there exists a unique $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X}\times\mathbf{M}\times\mathbf{Q}$ solution of* (5.22)*. Moreover, there exists a constant $C > 0$, independent of the solution, such that*

$$\|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}} \leq C\left\{\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{G}_1\|_{\mathbf{M}'} + \|\mathbf{G}\|_{\mathbf{Q}'}\right\}. \tag{5.52}$$

Proof. It follows from Lemmas 5.3.4, 5.3.7 and 5.3.8, and a direct application of the abstract result given by Theorem 5.2.1, that problem (5.33) is well-posed and the analogue estimate (5.52) holds. Then, the equivalence result provided by Theorem 5.2.4 completes the proof. □

We end this section with the converse of the derivation of (5.22). More precisely, the following theorem establishes that the unique solution of (5.22) solves the original transmission problem described in Section 5.2.1. We remark that no extra regularity assumptions on the data, but only $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ and $f_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}})$, are required here.

Theorem 5.3.2 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X}\times\mathbf{M}\times\mathbf{Q}$ *be the unique solution of the variational formulation* (5.22) *with* $\mathbf{F}$*,* $\mathbf{G}_1$ *and* $\mathbf{G}$ *given by* (5.21)*. Then* $\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} = -\mathbf{f}_{\mathrm{S}}$ *in* Ω_{S}*,* $\nu^{-1}\boldsymbol{\sigma}_{\mathrm{S}}^{d} = \nabla\mathbf{u}_{\mathrm{S}}$ *in* Ω_{S}*,* $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$*,* $\operatorname{div}\mathbf{u}_{\mathrm{D}} = f_{\mathrm{D}}$ *in* Ω_{D}*,* $\mathbf{u}_{\mathrm{D}} = -\boldsymbol{\kappa}(\cdot,|\mathbf{t}_{\mathrm{D}}|)\,\mathbf{t}_{\mathrm{D}}$ *in* Ω_{D}*,* $\mathbf{t}_{\mathrm{D}} = \nabla p_{\mathrm{D}}$ *in* Ω_{D}*,* $p_{\mathrm{D}} \in H^1(\Omega_{\mathrm{D}})$*,* $\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} + \boldsymbol{\varphi}\cdot\mathbf{n} = 0$ *on* Σ*,* $\boldsymbol{\sigma}\,\mathbf{n} + \lambda\mathbf{n} - \nu\,\kappa_f^{-1}\,(\boldsymbol{\varphi}\cdot\mathbf{t})\,\mathbf{t} = 0$ *on* Σ*,* $\lambda = p_{\mathrm{D}}$ *on* Σ*,* $\boldsymbol{\varphi} = -\mathbf{u}_{\mathrm{S}}$ *on* Σ*,* $\mathbf{u}_{\mathrm{S}} = 0$ *on* Γ_{S}*, and* $\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0$ *on* Γ_{D}*.*

Proof. It basically follows by applying integration by parts backwardly in (5.22) and using suitable test functions. We omit further details. □

5.4 The mixed finite element scheme

In this section we analyze the well-posedness of the Galerkin scheme of (5.22). For this purpose, we also introduce the Galerkin scheme of the auxiliary problem (5.33), and establish suitable assumptions on the finite element subspaces ensuring that both discrete schemes are equivalent and that the latter is well-posed.

5.4.1 Preliminaries

We begin by selecting two collections of discrete spaces:

$$\begin{gathered}\mathbf{H}_h(\Omega_{\mathrm{S}}) \subseteq \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{S}})\,, \quad L_h(\Omega_{\mathrm{S}}) \subseteq L^2(\Omega_{\mathrm{S}})\,, \quad \Lambda_h^{\mathrm{S}}(\Sigma) \subseteq H_{00}^{1/2}(\Sigma)\,, \\ \mathbf{H}_h(\Omega_{\mathrm{D}}) \subseteq \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{D}})\,, \quad T_h(\Omega_{\mathrm{D}})\,, L_h(\Omega_{\mathrm{D}}) \subseteq L^2(\Omega_{\mathrm{D}})\,, \quad \Lambda_h^{\mathrm{D}}(\Sigma) \subseteq H^{1/2}(\Sigma)\,.\end{gathered} \tag{5.53}$$

According to this, for the Stokes domain we define the subspaces

$$\mathbf{L}_h(\Omega_{\mathrm{S}}) := L_h(\Omega_{\mathrm{S}}) \times L_h(\Omega_{\mathrm{S}}), \qquad \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) := \Lambda_h^{\mathrm{S}}(\Sigma) \times \Lambda_h^{\mathrm{S}}(\Sigma)\,,$$

$$\mathbb{H}_h(\Omega_{\mathrm{S}}) := \left\{\boldsymbol{\tau} : \Omega_{\mathrm{S}} \to \mathbb{R}^{2\times 2} : \quad \mathbf{a}^{\mathrm{t}}\boldsymbol{\tau} \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \quad \forall\,\mathbf{a} \in \mathbb{R}^2\right\},$$

$$\mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) := \mathbb{H}_h(\Omega_{\mathrm{S}}) \cap \mathbb{H}_0(\mathbf{div}\,;\Omega_{\mathrm{S}})\,,$$

and for the Darcy domain we set

$$\begin{aligned} \mathbf{T}_h(\Omega_D) &:= T_h(\Omega_D)\times T_h(\Omega_D)\,,\\ \mathbf{H}_{h,\Gamma_D}(\Omega_D) &:= \big\{\mathbf{v}\in\mathbf{H}_h(\Omega_D):\quad \mathbf{v}\cdot\mathbf{n} = 0 \quad\text{on}\quad \Gamma_D\big\}\,,\\ L_{h,0}(\Omega_D) &:= L_h(\Omega_D)\cap L^2_0(\Omega_D)\,. \end{aligned}\tag{5.54}$$

Then, the global unknowns and corresponding finite element subspaces are given by:

$$\begin{aligned} \underline{\boldsymbol{\sigma}}_h &:= (\boldsymbol{\sigma}_{S,h},\mathbf{t}_{D,h}) \in \mathbf{X}_h := \mathbb{H}_h(\Omega_S)\times\mathbf{T}_h(\Omega_D)\,,\\ \underline{\tilde{\boldsymbol{\sigma}}}_h &:= (\tilde{\boldsymbol{\sigma}}_{S,h},\mathbf{t}_{D,h}) \in \tilde{\mathbf{X}}_h := \mathbb{H}_{h,0}(\Omega_S)\times\mathbf{T}_h(\Omega_D)\,,\\ \underline{\mathbf{u}}_h &:= (\mathbf{u}_{S,h},\mathbf{u}_{D,h},\boldsymbol{\varphi}_h) \in \mathbf{M}_h := \mathbf{L}_h(\Omega_S)\times\mathbf{H}_{h,\Gamma_D}(\Omega_D)\times\boldsymbol{\Lambda}^S_h(\Sigma)\,,\\ \underline{\mathbf{p}}_h &:= (p_{D,h},\lambda_h) \in \mathbf{Q}_h := L_{h,0}(\Omega_D)\times\Lambda^D_h(\Sigma)\,,\\ \underline{\tilde{\mathbf{p}}}_h &:= (\underline{\mathbf{p}}_h,c_h) \in \tilde{\mathbf{Q}}_h := \mathbf{Q}_h\times\mathbb{R}\,. \end{aligned}$$

In this way, the Galerkin schemes for (5.22) and (5.33) read, respectively: Find $(\underline{\boldsymbol{\sigma}}_h,\underline{\mathbf{u}}_h,\underline{\mathbf{p}}_h) \in \mathbf{X}_h\times\mathbf{M}_h\times\mathbf{Q}_h$ such that

$$\begin{aligned} [\mathbf{A}(\underline{\boldsymbol{\sigma}}_h),\underline{\boldsymbol{\tau}}] + [\mathbf{B}_1(\underline{\boldsymbol{\tau}}),\underline{\mathbf{u}}_h] &= [\mathbf{F},\underline{\boldsymbol{\tau}}] \quad \forall\,\underline{\boldsymbol{\tau}}\in\mathbf{X}_h\,,\\ [\mathbf{B}_1(\underline{\boldsymbol{\sigma}}_h),\underline{\mathbf{v}}] - [\mathbf{S}(\underline{\mathbf{u}}_h),\underline{\mathbf{v}}] + [\mathbf{B}(\underline{\mathbf{v}}),\underline{\mathbf{p}}_h] &= [\mathbf{G}_1,\underline{\mathbf{v}}] \quad \forall\,\underline{\mathbf{v}}\in\mathbf{M}_h\,,\\ [\mathbf{B}(\underline{\mathbf{u}}_h),\underline{\mathbf{q}}] &= [\mathbf{G},\underline{\mathbf{q}}] \quad \forall\,\underline{\mathbf{q}}\in\mathbf{Q}_h\,, \end{aligned}\tag{5.55}$$

and: Find $(\underline{\tilde{\boldsymbol{\sigma}}}_h,\underline{\mathbf{u}}_h,\underline{\tilde{\mathbf{p}}}_h)\in\tilde{\mathbf{X}}_h\times\mathbf{M}_h\times\tilde{\mathbf{Q}}_h$ such that

$$\begin{aligned} [\tilde{\mathbf{A}}(\underline{\tilde{\boldsymbol{\sigma}}}_h),\underline{\boldsymbol{\tau}}] + [\mathbf{B}_1(\underline{\boldsymbol{\tau}}),\underline{\mathbf{u}}_h] &= [\widetilde{\mathbf{F}},\underline{\boldsymbol{\tau}}] \quad \forall\,\underline{\boldsymbol{\tau}}\in\tilde{\mathbf{X}}_h\,,\\ [\mathbf{B}_1(\underline{\tilde{\boldsymbol{\sigma}}}_h),\underline{\mathbf{v}}] - [\mathbf{S}(\underline{\mathbf{u}}_h),\underline{\mathbf{v}}] + [\tilde{\mathbf{B}}(\underline{\mathbf{v}}),\underline{\tilde{\mathbf{p}}}_h] &= [\mathbf{G}_1,\underline{\mathbf{v}}] \quad \forall\,\underline{\mathbf{v}}\in\mathbf{M}_h\,,\\ [\tilde{\mathbf{B}}(\underline{\mathbf{u}}_h),\underline{\tilde{\mathbf{q}}}] &= [\tilde{\mathbf{G}},\underline{\tilde{\mathbf{q}}}] \quad \forall\,\underline{\tilde{\mathbf{q}}}\in\tilde{\mathbf{Q}}_h\,. \end{aligned}\tag{5.56}$$

5.4.2 The main results

In what follows, we proceed analogously to [49, Section 4] and derive general hypotheses on the subspaces (5.53) that allow us to show that (5.55) and (5.56) are equivalent, and that (5.56) is well posed. Our approach consists of adapting to the present discrete setting the arguments employed in the corresponding continuous analyses (cf. Theorem 5.2.4 and Lemmas 5.3.2, 5.3.3, 5.3.5 and 5.3.6).

We observe first that, in order to have meaningful spaces $\mathbb{H}_{h,0}(\Omega_S)$ and $L_{h,0}(\Omega_D)$, we need to be able to eliminate multiples of the identity matrix from $\mathbb{H}_h(\Omega_S)$ and constant polynomials from $L_h(\Omega_D)$. This request is certainly satisfied if we assume the following:

(H.0) $[\mathbf{P}_0(\Omega_S)]^2 \subseteq \mathbf{H}_h(\Omega_S)$ and $\mathbf{P}_0(\Omega_D) \subseteq L_h(\Omega_D)$.

In particular, it follows that $\mathbf{I} \in \mathbb{H}_h(\Omega_S)$ for all h, and hence there holds the decomposition:

$$\mathbb{H}_h(\Omega_S) = \mathbb{H}_{h,0}(\Omega_S) \oplus \mathbf{P}_0(\Omega_S)\,\mathbf{I}. \tag{5.57}$$

Next, in order to prove the equivalence between (5.55) and (5.56), we assume that:

(H.1) $\operatorname{div}\mathbf{H}_h(\Omega_S) \subseteq L_h(\Omega_S)$.

As a consequence, we have the following theorem.

Theorem 5.4.1 *If* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) := ((\boldsymbol{\sigma}_{S,h}, \mathbf{t}_{D,h}), \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$ *is a solution of* (5.55)*, where* $\boldsymbol{\sigma}_{S,h} = \tilde{\boldsymbol{\sigma}}_{S,h} + c_h\mathbf{I}$*, with* $\tilde{\boldsymbol{\sigma}}_{S,h} \in \mathbb{H}_{h,0}(\Omega_S)$ *and* $c_h \in \mathbb{R}$*, then* $(\underline{\tilde{\boldsymbol{\sigma}}}_h, \underline{\mathbf{u}}_h, \underline{\tilde{\mathbf{p}}}_h) := ((\tilde{\boldsymbol{\sigma}}_{S,h}, \mathbf{t}_{D,h}), \underline{\mathbf{u}}_h, (\underline{\mathbf{p}}_h, c_h)) \in \tilde{\mathbf{X}}_h \times \mathbf{M}_h \times \tilde{\mathbf{Q}}_h$ *is a solution of* (5.56)*. Conversely, if* $(\underline{\tilde{\boldsymbol{\sigma}}}_h, \underline{\mathbf{u}}_h, \underline{\tilde{\mathbf{p}}}_h) \in \tilde{\mathbf{X}}_h \times \mathbf{M}_h \times \tilde{\mathbf{Q}}_h$ *is a solution of* (5.56)*, with* $\underline{\tilde{\boldsymbol{\sigma}}}_h = (\tilde{\boldsymbol{\sigma}}_{S,h}, \mathbf{t}_{D,h})$ *and* $\underline{\tilde{\mathbf{p}}}_h := (\underline{\mathbf{p}}_h, c_h)$*, then* $((\tilde{\boldsymbol{\sigma}}_{S,h} + c_h\mathbf{I}, \mathbf{t}_{D,h}), \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$ *is a solution of* (5.55).

Proof. Thanks to **(H.1)**, it suffices to apply the decomposition (5.57) and observe that in either direction one deduces that $\mathbf{div}\,\boldsymbol{\sigma}_{S,h} = \mathbf{div}\,\tilde{\boldsymbol{\sigma}}_{S,h} = -\mathbf{f}_S$. We omit futher details. □

As already announced, we now analyze the well-posedness of the Galerkin scheme (5.56), thanks to which we will conclude the well-posedness of the equivalent scheme (5.55). To this end, and in order to apply the abstract result given by Theorem 5.2.2, we need to introduce further hypotheses. We begin with the following:

(H.2) There exists $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}_{00}(\Sigma)$ such that

$$\boldsymbol{\psi}_0 \in \boldsymbol{\Lambda}^S_h(\Sigma) \quad \forall\, h \quad \text{and} \quad \langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, 1\rangle_\Sigma \neq 0\,. \tag{5.58}$$

It is easy to see that **(H.2)** yields the following inf-sup condition, which constitutes the discrete version of Lemma 5.3.2: There exist $\tilde{C}_1\,,\ \tilde{C}_2 > 0$, independent of h, such that

$$S_{1,h}(\xi_h, d_h) := \sup_{\substack{\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}^S_h(\Sigma) \\ \boldsymbol{\psi}_h \neq \mathbf{0}}} \frac{d_h\,\langle \mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma + \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, \xi_h\rangle_\Sigma}{\|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma}} \geq \tilde{C}_1\,|d_h| - \tilde{C}_2\,\|\xi_h\|_{1/2,\Sigma}\,, \tag{5.59}$$

for all $(\xi_h, d_h) \in \Lambda^D_h(\Sigma) \times \mathbb{R}$.

Next, we assume that the discrete version of Lemma 5.3.3 holds, that is:

(H.3) There exist $\tilde{C}_3 > 0$, independent of h, such that

$$S_{2,h}(q_h,\xi_h) := \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_{h,\Gamma_\mathrm{D}}(\Omega_\mathrm{D})\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{(\operatorname{div}\mathbf{v}_h, q_h)_\mathrm{D} + \langle \mathbf{v}_h\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_h\|_{\operatorname{div};\Omega_\mathrm{D}}} \geq \tilde{C}_3 \left\{ \|q_h\|_{0,\Omega_\mathrm{D}} + \|\xi_h\|_{1/2,\Sigma} \right\} \quad (5.60)$$

$\forall\,(q_h,\xi_h) \in L_{h,0}(\Omega_\mathrm{D}) \times \Lambda_h^\mathrm{D}(\Sigma)$.

On the other hand, we now look at the discrete kernel of $\tilde{\mathbf{B}}$, which is defined by

$$\mathbf{V}_h := \left\{ \underline{\mathbf{v}}_h \in \mathbf{M}_h : \quad [\tilde{\mathbf{B}}(\underline{\mathbf{v}}_h), (\underline{\mathbf{q}}_h, d_h)] = 0 \quad \forall\,(\underline{\mathbf{q}}_h, d_h) \in (\mathbf{Q}_h \times \mathbb{R}) \right\}.$$

Moreover, in order to deduce a more explicit definition of $\mathbf{V}_h$, we introduce the hypothesis:

(H.4) $\operatorname{div}\mathbf{H}_h(\Omega_\mathrm{D}) \subseteq L_h(\Omega_\mathrm{D})$ and $\mathbf{P}_0(\Sigma) \subseteq \Lambda_h^\mathrm{D}(\Sigma)$.

It follows, according to the definition of $\tilde{\mathbf{B}}$ (cf. (5.36)) and **(H.4)**, that $\underline{\mathbf{v}}_h := (\mathbf{v}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}, \boldsymbol{\psi}_h)$ belongs to $\mathbf{V}_h$ if and only if

$$\operatorname{div}\mathbf{v}_{\mathrm{D},h} \in \mathbf{P}_0(\Omega_\mathrm{D}), \; \langle \mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n}, \xi_h\rangle = -\,\langle \boldsymbol{\psi}_h\cdot\mathbf{n}, \xi_h\rangle_\Sigma \;\; \forall \xi_h \in \Lambda_h^\mathrm{D}(\Sigma), \text{ and } \langle \boldsymbol{\psi}_h\cdot\mathbf{n}, 1\rangle_\Sigma = 0.$$

In particular, taking $\xi_h := 1 \in \Lambda_h^\mathrm{D}(\Sigma)$ we find that $\langle \mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n}, 1\rangle_\Sigma = 0$, which implies that $\operatorname{div}\mathbf{v}_{\mathrm{D},h} = 0$ in Ω_D, and hence

$$\begin{aligned}\mathbf{V}_h := \Big\{ (\mathbf{v}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}, \boldsymbol{\psi}_h) \in \mathbf{M}_h := \mathbf{L}_h(\Omega_\mathrm{S}) \times \mathbf{H}_{h,\Gamma_\mathrm{D}}(\Omega_\mathrm{D}) \times \boldsymbol{\Lambda}_h^\mathrm{S}(\Sigma) :\; \operatorname{div}\mathbf{v}_{\mathrm{D},h} = 0 \text{ on } \Omega_\mathrm{D},\\ \langle \boldsymbol{\psi}_h\cdot\mathbf{n}, \xi_h\rangle_\Sigma = -\,\langle \mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n}, \xi_h\rangle \quad \forall\,\xi_h \in \Lambda_h^\mathrm{D}(\Sigma), \quad \langle \boldsymbol{\psi}_h\cdot\mathbf{n}, 1\rangle_\Sigma = 0 \Big\}.\end{aligned} \quad (5.61)$$

In virtue of the above, and aiming now to establish the discrete versions of Lemmas 5.3.5 and 5.3.6, we define

$$\mathbf{V}_h(\Omega_\mathrm{D}) := \left\{ \mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_h(\Omega_\mathrm{D}) : \quad \operatorname{div}\mathbf{v}_{\mathrm{D},h} = 0 \right\}, \quad (5.62)$$

and consider the following hypothesis:

(H.5) $\mathbf{V}_h(\Omega_\mathrm{D}) \subseteq \mathbf{T}_h(\Omega_\mathrm{D})$, and there exists $c_4 > 0$, independent of h, such that

$$S_{4,h}(v_h,\psi_h) := \sup_{\substack{\boldsymbol{\tau}_h\in\mathbf{H}_h(\Omega_\mathrm{S})\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{(\operatorname{div}\boldsymbol{\tau}_h, v_h)_\mathrm{S} + \langle \boldsymbol{\tau}_h\cdot\mathbf{n}, \psi_h\rangle_\Sigma}{\|\boldsymbol{\tau}_h\|_{\operatorname{div};\Omega_\mathrm{S}}} \geq c_4 \left\{ \|v_h\|_{0,\Omega_\mathrm{S}} + \|\psi_h\|_{1/2,00,\Sigma} \right\} \quad (5.63)$$

for all $(v_h,\psi_h) \in L_h(\Omega_\mathrm{S}) \times \Lambda_h^\mathrm{S}(\Sigma)$.

Hence, it is easy to see that the condition $\mathbf{V}_h(\Omega_D) \subseteq \mathbf{T}_h(\Omega_D)$ allows to extend the simple argument employed in the proof of Lemma 5.3.5 to the present discrete case, which yields

$$S_{3,h}(\mathbf{v}_{D,h}) := \sup_{\substack{\mathbf{s}_{D,h}\in \mathbf{L}_h(\Omega_D)\\ \mathbf{s}_{D,h}\neq \mathbf{0}}} \frac{(\mathbf{s}_{D,h}, \mathbf{v}_{D,h})}{\|\mathbf{s}_{D,h}\|_{0,\Omega_D}} \geq \|\mathbf{v}_{D,h}\|_{\mathrm{div}\,;\Omega_D} \quad \forall\,(\mathbf{v}_{S,h}, \mathbf{v}_{D,h}, \boldsymbol{\psi}_h) \in \mathbf{V}_h\,. \tag{5.64}$$

Furthermore, since $\mathbf{div}\,\mathbb{H}_h(\Omega_S) = \mathbf{div}\,\mathbb{H}_{h,0}(\Omega_S)$ (cf. 5.57), the inf-sup condition (5.63) implies the existence of $\tilde{C}_4 > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{S,h}\in \mathbb{H}_{h,0}(\Omega_S)\\ \boldsymbol{\tau}_{S,h}\neq \mathbf{0}}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_{S,h}, \mathbf{v}_{S,h})_S + \langle \boldsymbol{\tau}_{S,h}\,\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div}\,;\Omega_S}} \geq \tilde{C}_4 \left\{ \|\mathbf{v}_{S,h}\|_{0,\Omega_S} + \|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \right\} \tag{5.65}$$

for all $(\mathbf{v}_{S,h}, \mathbf{v}_{D,h}, \boldsymbol{\psi}_h) \in \mathbf{V}_h$.

We are now in a position to establish, under the hypotheses specified throughout this section, the well posedness of (5.55) and the associated Cea estimate, which follows straightforwardly from the corresponding results for the equivalent scheme (5.56).

Theorem 5.4.2 *Assume that* **(H.0)** – **(H.5)** *hold. Then the Galerkin scheme* (5.55) *has a unique solution* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$. *In addition, there exist* C, $\tilde{C} > 0$, *independent of* h, *such that*

$$\|(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}} \leq C \left\{ \|\mathbf{F}|_{\mathbf{X}_h}\|_{\mathbf{X}'_h} + \|\mathbf{G}_1|_{\mathbf{M}_h}\|_{\mathbf{M}'_h} + \|\mathbf{G}|_{\mathbf{Q}_h}\|_{\mathbf{Q}'_h} \right\}, \tag{5.66}$$

and

$$\begin{aligned} &\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbf{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbf{M}} + \|\underline{\mathbf{p}} - \underline{\mathbf{p}}_h\|_{\mathbf{Q}} \\ &\quad \leq \tilde{C} \left\{ \inf_{\underline{\boldsymbol{\tau}}_h\in\mathbf{X}_h} \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h\|_{\mathbf{X}} + \inf_{\underline{\mathbf{v}}_h\in\mathbf{M}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|_{\mathbf{M}} + \inf_{\underline{\mathbf{q}}_h\in\mathbf{Q}_h} \|\underline{\mathbf{p}} - \underline{\mathbf{q}}_h\|_{\mathbf{Q}} \right\}. \end{aligned} \tag{5.67}$$

Proof. We first observe, thanks to (5.59) (which follows from **(H.2)**) and **(H.3)**, and proceeding analogously to the proof of Lemma 5.3.4, that $\tilde{\mathbf{B}}$ satisfies the discrete inf-sup condition on $\mathbf{M}_h \times \tilde{\mathbf{Q}}_h$. Similarly, using (5.64) and (5.65) (which follows from **(H.4)** and **(H.5)**), and proceeding as in the proof of Lemma 5.3.7, one can easily show that $\mathbf{B}_1$ satisfies the discrete inf-sup condition on $\tilde{\mathbf{X}}_h \times \mathbf{V}_h$. In addition, we recall that the nonlinear operator $\tilde{\mathbf{A}}$ is strongly monotone and Lipschitz-continuous (cf. Lemma 5.3.8), and that $\mathbf{S}$ is positive semidefinite on $\mathbf{M}$ (cf. (5.20)). On the other hand, it is known from [11, Lemma 3] that the operator $\mathbf{A}_D$ (cf. (5.17)) has a continuous first order Gâteaux derivative $\mathcal{D}\mathbf{A}_D : \mathbf{L}^2(\Omega_D) \to \mathcal{L}(\mathbf{L}^2(\Omega_D), \mathbf{L}^2(\Omega_D)')$. Hence, due also to the boundedness of the linear operator $\tilde{\mathbf{A}}_S$ (cf. (5.35)), we conclude that $\tilde{\mathbf{A}}$ (cf. (5.34)) has a continuous first order Gâteaux derivative $\mathcal{D}\tilde{\mathbf{A}} : \tilde{\mathbf{X}} \to \mathcal{L}(\tilde{\mathbf{X}}, \tilde{\mathbf{X}}')$ as well. Consequently, straightforward applications of Theorems 5.2.2 and 5.2.3 imply the well-posedness of the auxiliary Galerkin scheme (5.56) and the associated Cea estimate. Finally, the equivalence results provided by Theorems 5.2.4 and 5.4.1 yield the unique solvability of the original Galerkin scheme (5.55) and the required estimates (5.66) and (5.67). □

5.5 A particular mixed finite element scheme

In this section we follow very closely the analysis and results from [49, Section 5] to define specific finite element subspaces verifying the hypotheses **(H.0)** – **(H.5)**. In this way, a particular mixed finite element scheme (5.55) satisfying the estimates (5.66), (5.67), and the corresponding rate of convergence, is derived.

5.5.1 The finite element subspaces

Let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D}, which are formed by shape-regular triangles of diameter h_T, and assume that they match in Σ so that $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. In addition, $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ are supposed to be quasiuniform in a neighborhood of Σ. Then, for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we consider the local Raviart–Thomas space of the lowest order

$$\mathrm{RT}_0(T) := \mathrm{span}\Big\{(1,0),\, (0,1),\, (x_1,x_2)\Big\}.$$

We also define one Raviart–Thomas space on each subdomain and their usual companion spaces of piecewise constant functions: for $\star \in \{\mathrm{S},\mathrm{D}\}$

$$\begin{aligned} \mathbf{H}_h(\Omega_\star) &:= \Big\{\mathbf{v}_h \in \mathbf{H}(\mathrm{div}\,;\Omega_\star) : \quad \mathbf{v}_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star\Big\}, \\ L_h(\Omega_\star) &:= \Big\{q_h : \Omega_\star \to \mathbb{R} : \quad q_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star\Big\}. \end{aligned} \tag{5.68}$$

It is clear that **(H.0)**, **(H.1)**, and the condition $\mathrm{div}\,\mathbf{H}_h(\Omega_{\mathrm{D}}) \subseteq L_h(\Omega_{\mathrm{D}})$ in **(H.4)** are satisfied. In addition, it is easy to see that in this case $\mathbf{V}_h(\Omega_{\mathrm{D}})$ (cf. (5.62)) is contained in $L_h(\Omega_{\mathrm{D}}) \times L_h(\Omega_{\mathrm{D}})$, and hence, in order to have the condition $\mathbf{V}_h(\Omega_{\mathrm{D}}) \subseteq \mathbf{T}_h(\Omega_{\mathrm{D}})$ in **(H.5)**, it suffices to choose $T_h(\Omega_{\mathrm{D}}) = L_h(\Omega_{\mathrm{D}})$, that is

$$T_h(\Omega_{\mathrm{D}}) := \Big\{q_h : \Omega_{\mathrm{D}} \to \mathbb{R} : \quad q_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}\Big\}. \tag{5.69}$$

Furthermore, it is well known (see, e.g. [19, Chapter IV] or [69, Chapter 7]) that the pairs of subspaces $(\mathbf{H}_h(\Omega_{\mathrm{S}}), L_h(\Omega_{\mathrm{S}}))$ and $(\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}), L_{h,0}(\Omega_{\mathrm{D}}))$ (cf. (5.54) and (5.68)) satisfy the usual discrete inf-sup conditions, that is there exist $\tilde{\beta}_{\mathrm{S}}$, $\tilde{\beta}_{\mathrm{D}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{(\mathrm{div}\,\boldsymbol{\tau}_h, v_h)_{\mathrm{S}}}{\|\boldsymbol{\tau}_h\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}}} \geq \tilde{\beta}_{\mathrm{S}}\, \|v_h\|_{0,\Omega_{\mathrm{S}}} \qquad \forall\, v_h \in L_h(\Omega_{\mathrm{S}}), \tag{5.70}$$

and

$$\sup_{\substack{\mathbf{v}_h\in\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{(\operatorname{div}\mathbf{v}_h, q_h)_{\mathrm{D}}}{\|\mathbf{v}_h\|_{\operatorname{div};\Omega_{\mathrm{D}}}} \,\geq\, \tilde{\beta}_{\mathrm{D}}\,\|q_h\|_{0,\Omega_{\mathrm{D}}} \qquad \forall\, q_h\in L_{h,0}(\Omega_{\mathrm{D}})\,. \tag{5.71}$$

In addition, the set of discrete normal traces on Σ of $\mathbf{H}_h(\Omega_{\mathrm{S}})$ and $\mathbf{H}_h(\Omega_{\mathrm{D}})$ is given by

$$\Phi_h(\Sigma) \,:=\, \Big\{\phi_h:\Sigma\,\rightarrow\,\mathbb{R}: \quad \phi_h|_e\,\in\,\mathbb{P}_0(e) \qquad \forall\text{ edge } e\,\in\,\Sigma_h\Big\}\,, \tag{5.72}$$

where, hereafter, Σ_h denotes the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$). Note that the local quasiuniformity around Σ and the shape regularity property of the triangulations imply that Σ_h is also quasiuniform, which yields a classical inverse inequality for $\Phi_h(\Sigma)$ (see [50, eq. (5.3)]).

Next, in order to introduce the particular subspaces $\Lambda_h^{\mathrm{S}}(\Sigma)$ and $\Lambda_h^{\mathrm{D}}(\Sigma)$, we first assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. Note that because Σ_h is inherited from the interior triangulations, it is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded) and, therefore, so is Σ_{2h}. Now, if the number of edges of Σ_h is odd, we simply reduce it to the even case by joining any pair of two adjacent elements, and then construct Σ_{2h} from this reduced partition. In this way, denoting by x_0 and x_N the extreme points of Σ, we define

$$\Lambda_h^{\mathrm{S}}(\Sigma) \,:=\, \Big\{\psi_h\,\in\,\mathcal{C}(\Sigma): \quad \psi_h|_e\,\in\,\mathbf{P}_1(e) \quad \forall\, e\in\Sigma_{2h}\,, \quad \psi_h(x_0)\,=\,\psi_h(x_N)\,=\,0\Big\}\,, \tag{5.73}$$

$$\Lambda_h^{\mathrm{D}}(\Sigma) \,=\, \Big\{\xi_h\,\in\,\mathcal{C}(\Sigma): \quad \xi_h|_e\,\in\mathbf{P}_1(e) \quad \forall\, e\,\in\,\Sigma_{2h}\Big\}\,. \tag{5.74}$$

It is clear from (5.74) that $\mathbf{P}_0(\Sigma)\,\subseteq\,\Lambda_h^{\mathrm{D}}(\Sigma)$, which completes the requirements of **(H.4)**. In addition, if we assume that the elements of Σ_{2h} are segments, that is no element of Σ_{2h} crosses a corner point, then we can construct $\boldsymbol{\psi}_0$ satisfying **(H.2)**, exactly as explained at the end of the proof of Lemma 5.3.2.

Furthermore, at this point we recall from [49, Lemma 5.2] that there exist $\widehat{\beta}_{\mathrm{S}}$, $\widehat{\beta}_{\mathrm{D}}\,>\,0$, independent of h, such that the pairs of subspaces $(\Phi_h(\Sigma), \Lambda_h^{\mathrm{S}}(\Sigma))$ and $(\Phi_h(\Sigma), \Lambda_h^{\mathrm{D}}(\Sigma))$ satisfy, respectively, the following discrete inf-sup conditions:

$$\sup_{\substack{\phi_h\,\in\,\Phi_h(\Sigma)\\ \phi_h\neq 0}} \frac{\langle\phi_h,\psi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \,\geq\, \widehat{\beta}_{\mathrm{S}}\,\|\psi_h\|_{1/2,00,\Sigma} \qquad \forall\,\psi_h\,\in\,\Lambda_h^{\mathrm{S}}(\Sigma)\,, \tag{5.75}$$

and

$$\sup_{\substack{\phi_h\,\in\,\Phi_h(\Sigma)\\ \phi_h\neq 0}} \frac{\langle\phi_h,\xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \,\geq\, \widehat{\beta}_{\mathrm{D}}\,\|\xi_h\|_{1/2,\Sigma} \qquad \forall\,\xi_h\,\in\,\Lambda_h^{\mathrm{D}}(\Sigma)\,. \tag{5.76}$$

5.5.2 The discrete inf-sup conditions

In what follows we complete the verification of the hypotheses required by Theorem 5.4.2. More precisely, according to our previous analysis, it only remains to show the discrete inf-sup conditions (5.60) and (5.63), which yield **(H.3)** and **(H.5)**, respectively. This is the purpose of the following two lemmas.

Lemma 5.5.1 *Let us recall from* (5.54) *that* $\mathbf{H}_{h,\Gamma_D}(\Omega_D) := \left\{ \mathbf{v} \in \mathbf{H}_h(\Omega_D) : \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_D \right\}$ *and* $L_{h,0}(\Omega_D) := L_h(\Omega_D) \cap L_0(\Omega_D)$*, with* $\mathbf{H}_h(\Omega_D)$ *and* $L_h(\Omega_D)$ *given by* (5.68)*, and let* $\Lambda_h^D(\Sigma)$ *be defined by* (5.74)*. Then, there exists* $\tilde{C}_3 > 0$*, independent of* h*, such that*

$$S_{2,h}(q_h,\xi_h) := \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_{h,\Gamma_D}(\Omega_D)\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{(\operatorname{div}\mathbf{v}_h, q_h)_D + \langle \mathbf{v}_h\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_h\|_{\operatorname{div};\Omega_D}} \ge \tilde{C}_3 \left\{ \|q_h\|_{0,\Omega_D} + \|\xi_h\|_{1/2,\Sigma} \right\}$$

$\forall\, (q_h,\xi_h) \in L_{h,0}(\Omega_D)\times\Lambda_h^D(\Sigma)$.

Proof. Let $(q_h,\xi_h) \in L_{h,0}(\Omega_D)\times\Lambda_h^D(\Sigma)$. It is easy to see, using the estimate (5.71) and the boundedness of the normal trace of $\mathbf{H}(\operatorname{div};\Omega_D)$, that

$$S_{2,h}(q_h,\xi_h) \ge \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_{h,\Gamma_D}(\Omega_D)\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{(\operatorname{div}\mathbf{v}_h, q_h)_D}{\|\mathbf{v}_h\|_{\operatorname{div};\Omega_D}} - \|\xi_h\|_{1/2,\Sigma} \ge \tilde{\beta}_D \|q_h\|_{0,\Omega_D} - \|\xi_h\|_{1/2,\Sigma} \tag{5.77}$$

On the other hand, given $\phi_h \in \Phi_h(\Sigma)$, we proceed similarly to the proof of Lemma 5.3.3 and define $\eta_h \in H^{-1/2}(\partial\Omega_D)$ as

$$\langle \eta_h, \mu\rangle_{\partial\Omega_D} = \langle \phi_h, \mu_\Sigma\rangle_\Sigma \qquad \forall\,\mu \in H^{1/2}(\partial\Omega_D)\,, \tag{5.78}$$

which satisfies

$$\langle \eta_h, E_{0,D}(\rho)\rangle_{\partial\Omega_D} = 0 \qquad \forall\,\rho \in H_{00}^{1/2}(\Gamma_D)\,, \tag{5.79}$$

$$\langle \eta_h, E_D(\xi_h)\rangle_{\partial\Omega_D} = \langle \phi_h, \xi_h\rangle_\Sigma\,, \tag{5.80}$$

and

$$\|\eta_h\|_{-1/2,\partial\Omega_D} \le C\,\|\phi_h\|_{-1/2,\Sigma}\,. \tag{5.81}$$

Then, according to the result provided by [49, Lemma 5.1] for the Darcy domain Ω_D, we deduce the existence of $\bar{\mathbf{v}}_h \in \mathbf{H}_h(\Omega_D)$ such that

$$\operatorname{div}\bar{\mathbf{v}}_h \in \mathbf{P}_0(\Omega_D) \quad \text{in} \quad \Omega_D\,, \quad \bar{\mathbf{v}}_h\cdot\mathbf{n} = \eta_h \quad \text{on} \quad \partial\Omega_D\,, \tag{5.82}$$

and

$$\|\bar{\mathbf{v}}_h\|_{\operatorname{div};\Omega_D} \le C\,\|\eta_h\|_{-1/2,\partial\Omega_D}\,. \tag{5.83}$$

In this way, thanks to (5.38) and (5.80), and (5.79), we find, respectively, that

$$\langle \bar{\mathbf{v}}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma \,=\, \langle \bar{\mathbf{v}}_h \cdot \mathbf{n}, E_{\mathrm{D}}(\xi_h) \rangle_{\partial\Omega_{\mathrm{D}}} \,=\, \langle \eta_h, E_{\mathrm{D}}(\xi_h) \rangle_{\partial\Omega_{\mathrm{D}}} \,=\, \langle \phi_h, \xi_h \rangle_\Sigma \,,$$

and

$$\langle \bar{\mathbf{v}}_h \cdot \mathbf{n}, E_{0,\mathrm{D}}(\rho) \rangle_{\partial\Omega_{\mathrm{D}}} \,=\, \langle \eta_h, E_{0,\mathrm{D}}(\rho) \rangle_{\partial\Omega_{\mathrm{D}}} \,=\, 0 \qquad \forall \rho \in H^{1/2}_{00}(\Gamma_{\mathrm{D}}) \,,$$

which implies that $\bar{\mathbf{v}}_h \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div}; \Omega_{\mathrm{D}})$. Moreover, it is clear from (5.81) and (5.83) that

$$\|\bar{\mathbf{v}}_h\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}} \,\le\, C\, \|\phi_h\|_{-1/2,\Sigma} \,. \tag{5.84}$$

Hence, bounding from below with $\mathbf{v}_h \,=\, \bar{\mathbf{v}}_h$, and recalling that $q_h \in L^2_0(\Omega_{\mathrm{D}})$, we deduce that

$$S_{2,h}(q_h, \xi_h) \,\ge\, \frac{|\,(\mathrm{div}\,\bar{\mathbf{v}}_h, q_h)_{\mathrm{D}} \,+\, \langle \bar{\mathbf{v}}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma\,|}{\|\bar{\mathbf{v}}_h\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}}} \,=\, \frac{|\,\langle \bar{\mathbf{v}}_h \cdot \mathbf{n}, \xi_h \rangle_\Sigma\,|}{\|\bar{\mathbf{v}}_h\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}}} \,\ge\, \bar{C}\, \frac{|\,\langle \phi_h, \xi_h \rangle_\Sigma\,|}{\|\phi_h\|_{-1/2,\Sigma}} \,,$$

which, noting that ϕ_h is arbitrary in $\Phi_h(\Sigma)$, yields

$$S_{2,h}(q_h, \xi_h) \,\ge\, C \sup_{\substack{\phi_h \,\in\, \Phi_h(\Sigma) \\ \phi_h \neq 0}} \frac{\langle \phi_h, \xi_h \rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \,.$$

This inequality and (5.76) imply that $S_{2,h}(q_h, \xi_h) \,\ge\, C\, \|\xi_h\|_{1/2,\Sigma}$, which, combined with (5.77), completes the proof. □

Lemma 5.5.2 *Let* $\mathbf{H}_h(\Omega_{\mathrm{S}})$ *and* $L_h(\Omega_{\mathrm{S}})$ *be given by* (5.68)*, and let* $\Lambda^{\mathrm{S}}_h(\Sigma)$ *be defined by* (5.73)*. Then there exists* $c_4 > 0$*, independent of* h*, such that*

$$S_{4,h}(v_h, \boldsymbol{\psi}_h) \,:=\, \sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{(\mathrm{div}\,\boldsymbol{\tau}_h, v_h)_{\mathrm{S}} \,+\, \langle \boldsymbol{\tau}_h \cdot \mathbf{n}, \psi_h \rangle_\Sigma}{\|\boldsymbol{\tau}_h\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}}} \,\ge\, c_4 \Big\{ \|v_h\|_{0,\Omega_{\mathrm{S}}} \,+\, \|\psi_h\|_{1/2,00,\Sigma} \Big\}$$

for all $(v_h, \psi_h) \,\in\, L_h(\Omega_{\mathrm{S}}) \times \Lambda^{\mathrm{S}}_h(\Sigma)$.

Proof. Let $(v_h, \psi_h) \,\in\, L_h(\Omega_{\mathrm{S}}) \times \Lambda^{\mathrm{S}}_h(\Sigma)$. We first observe, using (5.70) and the boundedness of the normal trace of $\mathbf{H}(\mathrm{div}\,; \Omega_{\mathrm{S}})$, that

$$S_{4,h}(v_h, \boldsymbol{\psi}_h) \,\ge\, \sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{(\mathrm{div}\,\boldsymbol{\tau}_h, v_h)_{\mathrm{S}}}{\|\boldsymbol{\tau}_h\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}}} \,-\, \|\psi_h\|_{1/2,00,\Sigma} \,\ge\, \tilde{\beta}_{\mathrm{S}}\, \|v_h\|_{0,\Omega_{\mathrm{S}}} \,-\, \|\psi_h\|_{1/2,00,\Sigma} \,. \tag{5.85}$$

Next, given $\phi_h \in \Phi_h(\Sigma)$, we apply a slight modification of [49, Lemma 5.1] for the Stokes domain Ω_{S}, and deduce the existence of $\bar{\boldsymbol{\tau}}_h \in \mathbf{H}_h(\Omega_{\mathrm{S}})$ such that

$$\mathrm{div}\,\bar{\boldsymbol{\tau}}_h \,=\, 0 \quad \text{in} \quad \Omega_{\mathrm{S}} \,, \quad \bar{\boldsymbol{\tau}}_h \cdot \mathbf{n} \,=\, \phi_h \quad \text{on} \quad \Sigma \,, \tag{5.86}$$

and

$$\|\bar{\boldsymbol{\tau}}_h\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}} \,\le\, C\, \|\phi_h\|_{-1/2,\Sigma} \,. \tag{5.87}$$

Therefore, bounding from below with $\boldsymbol{\tau}_h = \bar{\boldsymbol{\tau}}_h$, we deduce in this case that

$$S_{4,h}(v_h,\psi_h) \,\ge\, \frac{|\,(\operatorname{div}\bar{\boldsymbol{\tau}}_h, v_h)_{\mathrm{S}} \,+\, \langle \bar{\boldsymbol{\tau}}_h\cdot\mathbf{n},\psi_h\rangle_\Sigma\,|}{\|\bar{\boldsymbol{\tau}}_h\|_{\operatorname{div};\Omega_{\mathrm{S}}}} \,=\, \frac{|\,\langle \bar{\boldsymbol{\tau}}_h\cdot\mathbf{n},\psi_h\rangle_\Sigma\,|}{\|\bar{\boldsymbol{\tau}}_h\|_{\operatorname{div};\Omega_{\mathrm{S}}}} \,\ge\, \bar{C}\,\frac{|\,\langle\phi_h,\psi_h\rangle_\Sigma\,|}{\|\phi_h\|_{-1/2,\Sigma}}\,,$$

which, noting that ϕ_h is arbitrary in $\Phi_h(\Sigma)$, yields

$$S_{4,h}(v_h,\psi_h) \,\ge\, C \sup_{\substack{\phi_h\in\Phi_h(\Sigma)\\ \phi_h\neq 0}} \frac{\langle\phi_h,\psi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}}\,.$$

This inequality and (5.75) imply that $S_{4,h}(v_h,\psi_h) \,\ge\, C\,\|\psi_h\|_{1/2,00,\Sigma}$, which, combined with (5.85), completes the proof. □

5.5.3 The main results

In this section we prove the unique solvability of (5.55) for the subspaces introduced in Section 5.5.1, and establish the associated rate of convergence.

Theorem 5.5.1 *Assume that $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ are quasiuniform in a neighborhood of Σ. Let $\mathbf{H}_h(\Omega_{\mathrm{S}})$, $\mathbf{H}_h(\Omega_{\mathrm{D}})$, $L_h(\Omega_{\mathrm{S}})$, $L_h(\Omega_{\mathrm{D}})$, $T_h(\Omega_{\mathrm{D}})$, $\Lambda_h^{\mathrm{S}}(\Sigma)$, and $\Lambda_h^{\mathrm{D}}(\Sigma)$ be the finite element subspaces defined in* (5.68), (5.69), (5.73), *and* (5.74), *respectively, and let*

$$\begin{aligned}
\mathbb{H}_h(\Omega_{\mathrm{S}}) &:= \{\,\boldsymbol{\tau}:\Omega_{\mathrm{S}}\to\mathbb{R}^{2\times 2}:\quad \mathbf{a}^{\mathrm{t}}\boldsymbol{\tau}\,\in\,\mathbf{H}_h(\Omega_{\mathrm{S}})\quad \forall\,\mathbf{a}\in\mathbb{R}^2\,\}\,,\\
\mathbf{T}_h(\Omega_{\mathrm{D}}) &:= T_h(\Omega_{\mathrm{D}})\times T_h(\Omega_{\mathrm{D}})\,,\\
\mathbf{L}_h(\Omega_{\mathrm{S}}) &:= L_h(\Omega_{\mathrm{S}})\times L_h(\Omega_{\mathrm{S}})\,,\\
\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) &:= \big\{\mathbf{v}\in\mathbf{H}_h(\Omega_{\mathrm{D}}):\quad \mathbf{v}\cdot\mathbf{n} \,=\, 0 \quad on \quad \Gamma_{\mathrm{D}}\big\}\,,\\
L_{h,0}(\Omega_{\mathrm{D}}) &:= L_h(\Omega_{\mathrm{D}})\,\cap\,L_0^2(\Omega_{\mathrm{D}})\,,\\
\boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) &:= \Lambda_h^{\mathrm{S}}(\Sigma)\times\Lambda_h^{\mathrm{S}}(\Sigma)\,.
\end{aligned}$$

Then the Galerkin scheme (5.55) *with the discrete spaces $\mathbf{X}_h := \mathbb{H}_h(\Omega_{\mathrm{S}})\times\mathbf{T}_h(\Omega_{\mathrm{D}})$, $\mathbf{M}_h := \mathbf{L}_h(\Omega_{\mathrm{S}})\times\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})\times\boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma)$, and $\mathbf{Q}_h := L_{h,0}(\Omega_{\mathrm{D}})\times\Lambda_h^{\mathrm{D}}(\Sigma)$ has a unique solution $(\underline{\boldsymbol{\sigma}}_h,\underline{\mathbf{u}}_h,\underline{\mathbf{p}}_h)\in\mathbf{X}_h\times\mathbf{M}_h\times\mathbf{Q}_h$, which satisfies the estimates* (5.66) *and* (5.67).

Proof. Since the hypotheses **(H.0)** – **(H.5)** are satisfied by the specific finite element subspaces $\mathbf{X}_h$, $\mathbf{M}_h$, and $\mathbf{Q}_h$, the conclusion follows from a straightforward application of Theorem 5.4.2. □

Our next goal is to provide the rate of convergence of the Galerkin scheme (5.55). To this end, we now recall the approximation properties of the subspaces involved (see, e.g. [13], [19], [58]). Note that each one of them is named after the unknown to which it is applied later on.

$(\mathbf{AP}_h^{\boldsymbol{\sigma}_\mathrm{S}})$ For each $\delta \in (0,1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^\delta(\Omega_\mathrm{S})$ with $\mathbf{div}\ \boldsymbol{\tau} \in \mathbf{H}^\delta(\Omega_\mathrm{S})$, there exists $\boldsymbol{\tau}_h \in \mathbb{H}_h(\Omega_\mathrm{S})$ such that

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}\,;\Omega_\mathrm{S}} \,\le\, C\, h^\delta \left\{ \|\boldsymbol{\tau}\|_{\delta,\Omega_\mathrm{S}} \,+\, \|\mathbf{div}\ \boldsymbol{\tau}\|_{\delta,\Omega_\mathrm{S}} \right\}.$$

$(\mathbf{AP}_h^{\mathbf{t}_\mathrm{D}})$ For each $\delta \in [0,1]$, and for each $\mathbf{s} \in \mathbf{H}^\delta(\Omega_\mathrm{D})$, there exists $\mathbf{s}_h \in \mathbf{T}_h(\Omega_\mathrm{D})$ such that

$$\|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega_\mathrm{D}} \,\le\, C\, h^\delta\, \|\mathbf{s}\|_{\delta,\Omega_\mathrm{D}}\,.$$

$(\mathbf{AP}_h^{\mathbf{u}_\mathrm{S}})$ For each $\delta \in [0,1]$, and for each $\mathbf{v} \in \mathbf{H}^\delta(\Omega_\mathrm{S})$, there exists $\mathbf{v}_h \in \mathbf{L}_h(\Omega_\mathrm{S})$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{0,\Omega_\mathrm{S}} \,\le\, C\, h^\delta\, \|\mathbf{v}\|_{\delta,\Omega_\mathrm{S}}\,.$$

$(\mathbf{AP}_h^{\mathbf{u}_\mathrm{D}})$ For each $\delta \in (0,1]$, and for each $\mathbf{v} \in \mathbf{H}^\delta(\Omega_\mathrm{D}) \cap \mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div}\,;\Omega_\mathrm{D})$ with $\mathrm{div}\ \mathbf{v} \in H^\delta(\Omega_\mathrm{D})$, there exists $\mathbf{v}_h \in \mathbf{H}_{h,\Gamma_\mathrm{D}}(\Omega_\mathrm{D})$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathrm{div}\,;\Omega_\mathrm{D}} \,\le\, C\, h^\delta \left\{ \|\mathbf{v}\|_{\delta,\Omega_\mathrm{D}} \,+\, \|\mathrm{div}\ \mathbf{v}\|_{\delta,\Omega_\mathrm{D}} \right\}.$$

$(\mathbf{AP}_h^{p_\mathrm{D}})$ For each $\delta \in [0,1]$, and for each $q \in H^\delta(\Omega_\mathrm{D}) \cap L^2_0(\Omega_\mathrm{D})$, there exists $q_h \in L_{h,0}(\Omega_\mathrm{D})$ such that

$$\|q - q_h\|_{0,\Omega_\mathrm{D}} \,\le\, C\, h^\delta\, \|q\|_{\delta,\Omega_\mathrm{D}}\,.$$

$(\mathbf{AP}_h^{\boldsymbol{\varphi}})$ For each $\delta \in [0,1]$ and for each $\boldsymbol{\psi} \in \mathbf{H}^{1/2+\delta}(\Sigma) \cap \mathbf{H}^{1/2}_{00}(\Sigma)$, there exists $\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}^\mathrm{S}_h(\Sigma)$ such that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \,\le\, C\, h^\delta\, \|\boldsymbol{\psi}\|_{1/2+\delta,\Sigma}\,.$$

$(\mathbf{AP}_h^{\lambda})$ For each $\delta \in [0,1]$ and for each $\xi \in H^{1/2+\delta}(\Sigma)$, there exists $\xi_h \in \Lambda^\mathrm{D}_h(\Sigma)$ such that

$$\|\xi - \xi_h\|_{1/2,\Sigma} \,\le\, C\, h^\delta\, \|\xi\|_{1/2+\delta,\Sigma}\,.$$

The following theorem provides the theoretical rate of convergence of the Galerkin scheme (5.55) under suitable regularity assumptions on the exact solution.

Theorem 5.5.2 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ *and* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$ *be the unique solutions of the continuous and discrete formulations* (5.22) *and* (5.55)*, respectively. Assume that there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_\mathrm{S} \in \mathbb{H}^\delta(\Omega_\mathrm{S})$*,* $\mathbf{div}\ \boldsymbol{\sigma}_\mathrm{S} \in \mathbf{H}^\delta(\Omega_\mathrm{S})$*,* $\mathbf{t}_\mathrm{D} \in \mathbf{H}^\delta(\Omega_\mathrm{D})$*,* $\mathbf{u}_\mathrm{D} \in \mathbf{H}^\delta(\Omega_\mathrm{D})$*, and* $\mathrm{div}\ \mathbf{u}_\mathrm{D} \in H^\delta(\Omega_\mathrm{D})$*. Then,* $\mathbf{u}_\mathrm{S} \in \mathbf{H}^{1+\delta}(\Omega_\mathrm{S})$*,* $p_\mathrm{D} \in H^{1+\delta}(\Omega_\mathrm{D})$*,* $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$*,* $\lambda \in H^{1/2+\delta}(\Sigma)$*, and there exists* $C > 0$*, independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned} \|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \,-\, (\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}} \,&\le\, C\, h^\delta \Big\{ \|\boldsymbol{\sigma}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}} \,+\, \|\mathbf{div}\ \boldsymbol{\sigma}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}} \\ +\quad \|\mathbf{t}_\mathrm{D}\|_{\delta,\Omega_\mathrm{D}} \,+\, \|\mathbf{u}_\mathrm{S}\|_{1+\delta,\Omega_\mathrm{S}} \,+\, \|\mathbf{u}_\mathrm{D}\|_{\delta,\Omega_\mathrm{D}} \,&+\, \|\mathrm{div}\ \mathbf{u}_\mathrm{D}\|_{\delta,\Omega_\mathrm{D}} \,+\, \|p_\mathrm{D}\|_{1+\delta,\Omega_\mathrm{D}} \Big\}. \end{aligned} \tag{5.88}$$

Proof. We first recall from Theorem 5.3.2 that $\nabla \mathbf{u}_{\mathrm{S}} = \nu^{-1} \boldsymbol{\sigma}_{\mathrm{S}}^{d}$ and $\nabla p_{\mathrm{D}} = \mathbf{t}_{\mathrm{D}}$, which implies that $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1+\delta}(\Omega_{\mathrm{S}})$ and $p_{\mathrm{D}} \in H^{1+\delta}(\Omega_{\mathrm{D}})$, whence $\boldsymbol{\varphi} = -\mathbf{u}_{\mathrm{S}}|_{\Sigma} \in \mathbf{H}^{1/2+\delta}(\Sigma)$ and $\lambda = p_{\mathrm{D}}|_{\Sigma} \in H^{1/2+\delta}(\Sigma)$. The rest of the proof follows from the corresponding Cea estimate, the above approximation properties, and the fact that, thanks to the trace theorem in Ω_{S} and Ω_{D}, respectively, there holds

$$\|\boldsymbol{\varphi}\|_{1/2+\delta,\Sigma} \leq c\,\|\mathbf{u}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}} \quad \text{and} \quad \|\lambda\|_{1/2+\delta,\Sigma} \leq c\,\|p_{\mathrm{D}}\|_{1+\delta,\Omega_{\mathrm{D}}}\,.$$

□

5.6 The a-posteriori error analysis

In this section we derive a reliable and efficient residual-based a-posteriori error estimate for our mixed finite element scheme (5.55) with the discrete spaces introduced in Section 5.5. Most of our analysis makes extensive use of the estimates derived in [50] and [15] for the corresponding linear case. We begin with some notations. For each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we let $\mathcal{E}(T)$ be the set of edges of T, and we denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, subdivided as follows:

$$\mathcal{E}_h = \mathcal{E}_h(\Gamma_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{D}}) \cup \mathcal{E}_h(\Sigma)\,,$$

where $\mathcal{E}_h(\Gamma_{\mathrm{S}}) := \{ e \in \mathcal{E}_h : e \subseteq \Gamma_{\mathrm{S}} \}$, $\mathcal{E}_h(\Omega_\star) := \{ e \in \mathcal{E}_h : e \subseteq \Omega_\star \}$ for each $\star \in \{\mathrm{S},\mathrm{D}\}$, and $\mathcal{E}_h(\Sigma) := \{ e \in \mathcal{E}_h : e \subseteq \Sigma \}$. Note that $\mathcal{E}_h(\Sigma)$ is the set of edges defining the partition Σ_h. Analogously, we let $\mathcal{E}_{2h}(\Sigma)$ be the set of *double* edges defining the partition Σ_{2h}. In what follows, h_e stands for the diameter of a given edge $e \in \mathcal{E}_h \cup \mathcal{E}_{2h}(\Sigma)$. Now, let $\star \in \{\mathrm{D},\mathrm{S}\}$ and let $q \in [L^2(\Omega_\star)]^m$, with $m \in \{1,2\}$, such that $q|_T \in [C(T)]^m$ for each $T \in \mathcal{T}_h^\star$. Then, given $e \in \mathcal{E}_h(\Omega_\star)$, we denote by $[q]$ the jump of q across e, that is $[q] := (q|_{T'})|_e - (q|_{T''})|_e$, where T' and T'' are the triangles of $\mathcal{T}_h^\star$ having e as an edge. Also, we fix a unit normal vector $\mathbf{n}_e := (n_1, n_2)^{\mathtt{t}}$ to the edge e (its particular orientation is not relevant) and let $\mathbf{t}_e := (-n_2, n_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Hence, given $\mathbf{v} \in \mathbf{L}^2(\Omega_\star)$ and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$ such that $\mathbf{v}|_T \in [C(T)]^2$ and $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2}$, respectively, for each $T \in \mathcal{T}_h^\star$, we let $[\mathbf{v}\cdot\mathbf{t}_e]$ and $[\boldsymbol{\tau}\,\mathbf{t}_e]$ be the tangential jumps of $\mathbf{v}$ and $\boldsymbol{\tau}$, across e, that is $[\mathbf{v}\cdot\mathbf{t}_e] := \{(\mathbf{v}|_{T'})|_e - (\mathbf{v}|_{T''})|_e\}\cdot\mathbf{t}_e$ and $[\boldsymbol{\tau}\,\mathbf{t}_e] := \{(\boldsymbol{\tau}|_{T'})|_e - (\boldsymbol{\tau}|_{T''})|_e\}\,\mathbf{t}_e$, respectively. From now on, when no confusion arises, we will simply write $\mathbf{t}$ and $\mathbf{n}$ instead of $\mathbf{t}_e$ and $\mathbf{n}_e$, respectively. Finally, for sufficiently smooth scalar, vector and tensors fields q, $\mathbf{v} := (v_1, v_2)^{\mathtt{t}}$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, we let

$$\mathbf{curl}\,\mathbf{v} := \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix}, \qquad \mathrm{curl}\, q := \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1} \right)^{\mathtt{t}},$$

$$\operatorname{rot} \mathbf{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \text{and} \quad \mathbf{rot}\, \boldsymbol{\tau} := \left(\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \right)^{\mathtt{t}}.$$

In what follows, $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) := ((\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{t}_{\mathrm{D},h}), (\mathbf{u}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h), (p_{\mathrm{D},h}, \lambda_h)) \in \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h$ and $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$ denote the solutions of (5.55) and (5.22), respectively. Then we introduce the global a posteriori error estimator:

$$\Theta := \left\{ \sum_{T \in \mathcal{T}_h^S} \Theta_{\mathrm{S},T}^2 + \sum_{T \in \mathcal{T}_h^D} \Theta_{\mathrm{D},T}^2 \right\}^{1/2}, \tag{5.89}$$

where, for each $T \in \mathcal{T}_h^{\mathrm{S}}$

$$\begin{aligned} \Theta_{\mathrm{S},T}^2 &:= \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\, \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2 \|\mathbf{rot}\, \boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 + h_T^2 \|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \\ &+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_{\mathrm{S}})} h_e \|[\boldsymbol{\sigma}_{\mathrm{S},h}^d \mathbf{t}]\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e \|\boldsymbol{\sigma}_{\mathrm{S},h}^d \mathbf{t}\|_{0,e}^2 + \sum_{e \in \mathcal{E}_h(T) \cap \mathcal{E}_h(\Sigma)} h_e \|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \\ &+ \sum_{e \in \mathcal{E}_h(T) \cap \mathcal{E}_h(\Sigma)} \left\{ h_e \left\| \boldsymbol{\sigma}_{\mathrm{S},h} \mathbf{n} + \lambda_h \mathbf{n} - \nu \kappa_f^{-1} (\boldsymbol{\varphi}_h \cdot \mathbf{t}) \mathbf{t} \right\|_{0,e}^2 + h_e \left\| \nu^{-1} \boldsymbol{\sigma}_{\mathrm{S},h}^d \mathbf{t} + \boldsymbol{\varphi}_h' \right\|_{0,e}^2 \right\}, \end{aligned}$$

and for each $T \in \mathcal{T}_h^{\mathrm{D}}$

$$\begin{aligned} \Theta_{\mathrm{D},T}^2 &:= \|f_{\mathrm{D}} - \operatorname{div} \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_T^2 \|\mathbf{t}_{\mathrm{D},h}\|_{0,T}^2 + \|\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D},h}|)\, \mathbf{t}_{\mathrm{D},h} + \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \\ &+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_{\mathrm{D}}))} h_e \|[\mathbf{t}_{\mathrm{D},h} \cdot \mathbf{t}]\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma_{\mathrm{D}})} h_e \|\mathbf{t}_{\mathrm{D},h} \cdot \mathbf{t}\|_{0,e}^2 \\ &+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} \left\{ h_e \left\| \mathbf{t}_{\mathrm{D},h} \cdot \mathbf{t} - \lambda_h' \right\|_{0,e}^2 + h_e \|\mathbf{u}_{\mathrm{D},h} \cdot \mathbf{n} + \boldsymbol{\varphi}_h \cdot \mathbf{n}\|_{0,e}^2 + h_e \|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2 \right\}. \end{aligned}$$

Here, $\boldsymbol{\varphi}_h'$ andλ_h' have to be understood as tangential derivatives, that is in the direction imposed by the tangential vector field $\mathbf{t}$ on Σ. In addition, it is important to remark, as announced at the beginning of this section, that some components of the a posteriori error estimator (5.89) coincide with those obtained in [50] and [15]. In particular, $\Theta_{\mathrm{S},T}$ is exactly the same estimator for the Stokes domain provided in [50].

The main result of this section is stated as follows.

Theorem 5.6.1 *There exist positive constants $C_{\mathtt{rel}}$ and $C_{\mathtt{eff}}$, independent of h, such that*

$$C_{\mathtt{eff}}\, \Theta \le \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbf{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbf{M}} + \|\underline{\mathbf{p}} - \underline{\mathbf{p}}_h\|_{\mathbf{Q}} \le C_{\mathtt{rel}}\, \Theta. \tag{5.90}$$

The efficiency of Θ (lower bound in (5.90)) is proved below in Section 5.6.2, whereas the corresponding reliability estimate (upper bound in (5.90)) is proved next in Section 5.6.1.

5.6.1 Reliability of the a posteriori error estimator

We begin by noticing, thanks to the assumptions (5.4), that the Gâteaux derivative of $\mathbf{A}_{\rm D}$ at any $\mathbf{r}_{\rm D} \in \mathbf{L}^2(\Omega_{\rm D})$, say $\mathcal{D}\mathbf{A}_{\rm D}(\mathbf{r}_{\rm D})$, is a uniformly bounded and uniformly elliptic bilinear form on $\mathbf{L}^2(\Omega_{\rm D}) \times \mathbf{L}^2(\Omega_{\rm D})$ (see, e.g. [53, Theorem 3.8] for details). Hence, as a consequence of the continuous dependence result provided by the linear version of Theorem 5.2.1 (cf. (5.24) with A linear), we conclude that the linear operator obtained by adding the three equations of the left hand side of (5.22), after replacing $\mathbf{A}_{\rm D}$ by $\mathcal{D}\mathbf{A}_{\rm D}(\mathbf{r}_{\rm D})$, satisfies a global inf-sup condition. Furthermore, we observe that the continuity of $\mathcal{D}\mathbf{A}_{\rm D}$ guarantees that there exists a particular $\tilde{\mathbf{r}}_{\rm D} \in \mathbf{L}^2(\Omega_{\rm D})$, which is a convex combination of $\mathbf{t}_{\rm D}$ and $\mathbf{t}_{{\rm D},h}$, such that

$$[\mathcal{D}\mathbf{A}_{\rm D}(\tilde{\mathbf{r}}_{\rm D})(\mathbf{t}_{\rm D} - \mathbf{t}_{{\rm D},h}), \mathbf{s}_{\rm D}] \,=\, [\mathbf{A}_{\rm D}(\mathbf{t}_{\rm D}) - \mathbf{A}_{\rm D}(\mathbf{t}_{{\rm D},h}), \mathbf{s}_{\rm D}] \qquad \forall\, \mathbf{s}_{\rm D} \in \mathbf{L}^2(\Omega_{\rm D})\,. \tag{5.91}$$

Hence, applying the above described inf-sup estimate (with $\mathbf{r}_{\rm D} = \tilde{\mathbf{r}}_{\rm D}$) to our Galerkin error $(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h, \underline{\mathbf{p}} - \underline{\mathbf{p}}_h) \in \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$, we find that

$$\|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h, \underline{\mathbf{p}} - \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}} \,\le\, C \sup_{\substack{(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}},\underline{\mathbf{q}})\in\mathbf{X}\times\mathbf{M}\times\mathbf{Q}\\ (\underline{\boldsymbol{\tau}},\underline{\mathbf{v}},\underline{\mathbf{q}})\neq\mathbf{0}}} \frac{R(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}, \underline{\mathbf{q}})}{\|(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}, \underline{\mathbf{q}})\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}}}\,, \tag{5.92}$$

where, according to (5.22), (5.91), and the definitions of $\mathbf{B}_1$, $\mathbf{B}$ and $\mathbf{S}$, the residual functional $R : \mathbf{X} \times \mathbf{M} \times \mathbf{Q} \to \mathbb{R}$ is given by

$$R(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}, \underline{\mathbf{q}}) \,:=\, R_1(\boldsymbol{\tau}_{\rm S}) \,+\, R_2(\mathbf{s}_{\rm D}) \,+\, R_3(\mathbf{v}_{\rm S}) \,+\, R_4(\mathbf{v}_{\rm D}) \,+\, R_5(\boldsymbol{\psi}) \,+\, R_6(q_{\rm D}) \,+\, R_7(\xi)\,,$$

for each $\underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\rm S}, \mathbf{s}_{\rm D}) \in \mathbf{X}$, $\underline{\mathbf{v}} := (\mathbf{v}_{\rm S}, \mathbf{v}_{\rm D}, \boldsymbol{\psi}) \in \mathbf{M}$, and $\underline{\mathbf{q}} := (q_{\rm D}, \xi) \in \mathbf{Q}$, with

$$\begin{aligned}
R_1(\boldsymbol{\tau}_{\rm S}) \;&:=\; -\nu^{-1} \int_{\Omega_{\rm S}} \boldsymbol{\sigma}^d_{{\rm S},h} : \boldsymbol{\tau}^d_{\rm S} \,-\, \int_{\Omega_{\rm S}} \mathbf{u}_{{\rm S},h} \cdot \mathbf{div}\, \boldsymbol{\tau}_{\rm S} \,-\, \langle \boldsymbol{\tau}_{\rm S}\, \mathbf{n}, \boldsymbol{\varphi}_h \rangle_\Sigma\,,\\
R_2(\mathbf{s}_{\rm D}) \;&:=\; -\int_{\Omega_{\rm D}} (\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{{\rm D},h}|)\, \mathbf{t}_{{\rm D},h} + \mathbf{u}_{{\rm D},h}) \cdot \mathbf{s}_{\rm D}\,,\\
R_3(\mathbf{v}_{\rm S}) \;&:=\; -\int_{\Omega_{\rm S}} \mathbf{v}_{\rm S} \cdot (\mathbf{f}_{\rm S} + \mathbf{div}\, \boldsymbol{\sigma}_{{\rm S},h})\,,\\
R_4(\mathbf{v}_{\rm D}) \;&:=\; -\int_{\Omega_{\rm D}} \mathbf{t}_{{\rm D},h} \cdot \mathbf{v}_{\rm D} \,-\, \int_{\Omega_{\rm D}} p_{{\rm D},h}\, \mathrm{div}\, \mathbf{v}_{\rm D} \,-\, \langle \mathbf{v}_{\rm D} \cdot \mathbf{n}, \lambda_h \rangle_\Sigma\,,\\
R_5(\boldsymbol{\psi}) \;&:=\; -\langle \boldsymbol{\sigma}_{{\rm S},h}\, \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma \,-\, \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda_h \rangle_\Sigma \,+\, \nu\, \kappa_f^{-1} \langle \boldsymbol{\psi} \cdot \mathbf{t}, \boldsymbol{\varphi}_h \cdot \mathbf{t} \rangle_\Sigma\,,\\
R_6(q_{\rm D}) \;&:=\; \int_{\Omega_{\rm D}} q_{\rm D}\, (f_{\rm D} - \mathrm{div}\, \mathbf{u}_{{\rm D},h})\,,\\
R_7(\xi) \;&:=\; \langle \mathbf{u}_{{\rm D},h} \cdot \mathbf{n}, \xi \rangle_\Sigma \,+\, \langle \boldsymbol{\varphi}_h \cdot \mathbf{n}, \xi \rangle_\Sigma\,.
\end{aligned}$$

Hence, the supremum in (5.92) can be bounded in terms of R_i, $i \in \{1, ..., 7\}$, which yields

$$\begin{aligned}\|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h, \underline{\mathbf{p}} - \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{M}\times\mathbf{Q}} \,\le\, C \Big\{ \|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'} + \|R_2\|_{\mathbf{L}^2(\Omega_D)'} \\ + \|R_3\|_{\mathbf{L}^2(\Omega_S)'} + \|R_4\|_{\mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)'} + \|R_5\|_{\mathbf{H}^{1/2}_{00}(\Sigma)'} + \|R_6\|_{L^2_0(\Omega_D)'} + \|R_7\|_{H^{1/2}(\Sigma)'} \Big\}.\end{aligned} \tag{5.93}$$

Throughout the rest of this section we provide suitable upper bounds for each one of the terms on the right hand side of (5.93). The following lemma, whose proof follows from straightforward applications of the Cauchy-Schwarz inequality, is stated first (see also [50, Lemma 3.1] for the estimates (5.95) and (5.96) below).

Lemma 5.6.1 *There hold*

$$\|R_2\|_{\mathbf{L}^2(\Omega_D)'} \,=\, \|\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{D,h}|)\,\mathbf{t}_{D,h} + \mathbf{u}_{D,h}\|_{0,\Omega_D} \,=\, \left\{ \sum_{T\in\mathcal{T}_h^D} \|\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{D,h}|)\,\mathbf{t}_{D,h} + \mathbf{u}_{D,h}\|^2_{0,T} \right\}^{1/2}, \tag{5.94}$$

$$\|R_3\|_{\mathbf{L}^2(\Omega_S)'} \,=\, \|\mathbf{f}_S \,+\, \mathbf{div}\,\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S} \,=\, \left\{ \sum_{T\in\mathcal{T}_h^S} \|\mathbf{f}_S \,+\, \mathbf{div}\,\boldsymbol{\sigma}_{S,h}\|^2_{0,T} \right\}^{1/2}, \tag{5.95}$$

$$\|R_6\|_{L^2_0(\Omega_D)'} \,\le\, \|f_D \,-\, \mathrm{div}\,\mathbf{u}_{D,h}\|_{0,\Omega_D} \,=\, \left\{ \sum_{T\in\mathcal{T}_h^D} \|f_D \,-\, \mathrm{div}\,\mathbf{u}_{D,h}\|^2_{0,T} \right\}^{1/2}. \tag{5.96}$$

Next, we give the estimates for the suprema on the spaces defined in the interface Σ.

Lemma 5.6.2 *There exist C_5, $C_7 > 0$, independent of h, such that*

$$\|R_5\|_{\mathbf{H}^{1/2}_{00}(\Sigma)'} \,\le\, C_5 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \, \left\| \boldsymbol{\sigma}_{S,h}\,\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\,\kappa_f^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t} \right\|^2_{0,e} \right\}^{1/2}, \tag{5.97}$$

and

$$\|R_7\|_{H^{1/2}(\Sigma)'} \,\le\, C_7 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \, \|\mathbf{u}_{D,h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|^2_{0,e} \right\}^{1/2}. \tag{5.98}$$

Proof. See [50, Lemma 3.2] for details. □

It remains to provide the upper bounds for $\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'}$ and $\|R_4\|_{\mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)'}$. For this purpose, we also proceed as in [50] and apply Helmholtz decompositions of $\mathbb{H}(\mathbf{div};\Omega_S)$ and $\mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)$ (see, e.g. [50, Lemma 3.3]), the usual integration by parts on each element, and the approximation properties of the Clément and Raviart-Thomas interpolation operators in both domains. More precisely, applying the same analysis suggested by [50, Lemmas 3.6 and 3.7], we observe that the estimate for $\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'}$ is exactly the one provided by [50, Lemma 3.8], whereas the estimate for $\|R_4\|_{\mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)'}$ arises from a slight modification of the proof of [50, Lemma 3.9]. These results are established as follows.

Lemma 5.6.3 *There exists $C_1 > 0$, independent of h, such that*

$$\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_S)'} \;\le\; C_1 \left\{ \sum_{T\in\mathcal{T}_h^S} \widehat{\Theta}_{S,T}^2 \right\}^{1/2}, \tag{5.99}$$

where, for each $T \in \mathcal{T}_h^S$

$$\widehat{\Theta}_{S,T}^2 \;:=\; h_T^2\,\|\mathbf{rot}\,\boldsymbol{\sigma}_{S,h}^d\|_{0,T}^2 \;+\; h_T^2\,\|\boldsymbol{\sigma}_{S,h}^d\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_S)} h_e\,\|[\boldsymbol{\sigma}_{S,h}^d\mathbf{t}]\|_{0,e}^2$$

$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_S)} h_e\,\|\boldsymbol{\sigma}_{S,h}^d\mathbf{t}\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\left\|\nu^{-1}\boldsymbol{\sigma}_{S,h}^d\mathbf{t} + \boldsymbol{\varphi}_h'\right\|_{0,e}^2 + h_e\,\|\mathbf{u}_{S,h} + \boldsymbol{\varphi}_h\|_{0,e}^2 \right\}$$

Proof. See [50, Lemma 3.8]. □

Lemma 5.6.4 *There exists $C_4 > 0$, independent of h such that*

$$\|R_4(\mathbf{v}_D)\|_{\mathbf{H}_{\Gamma_D}(\mathrm{div};\Omega_D)'} \;\le\; C_4 \left\{ \sum_{T\in\mathcal{T}_h^D} \widehat{\Theta}_{D,T}^2 \right\}^{1/2}, \tag{5.100}$$

where, for each $T \in \mathcal{T}_h^D$

$$\hat{\Theta}_{D,T}^2 \;:=\; h_T^2\,\|\mathbf{t}_{D,h}\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_D)} h_e\,\|[\mathbf{t}_{D,h}\cdot\mathbf{t}]\|_{0,e}^2$$

$$+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_D)} h_e\,\|\mathbf{t}_{D,h}\cdot\mathbf{t}\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \left\{ h_e\,\left\|\mathbf{t}_{D,h}\cdot\mathbf{t} \,-\, \lambda_h'\right\|_{0,e}^2 + h_e\,\|p_{D,h} - \lambda_h\|_{0,e}^2 \right\}.$$

Proof. It suffices to apply [50, Lemma 3.9] with $\mathbf{t}_{D,h}$ instead of $\mathbf{K}^{-1}\,\mathbf{u}_{D,h}$, noting that $\mathrm{rot}\,(\mathbf{t}_{D,h})$ vanishes since $\mathbf{t}_{D,h}$ is piecewise constant, and then recalling that in the present geometry the boundary of Ω_D includes also the additional part given by Γ_D. □

We end this section by observing that the reliability estimate (upper bound in (5.90)) is a direct consequence of Lemmas 5.6.1, 5.6.2, 5.6.3, and 5.6.4.

5.6.2 Efficiency of the a posteriori error estimator

We now aim to prove the eficiency of Θ, that is the lower bound in (5.90). We begin with the estimates for the zero order terms appearing in the definition of $\Theta_{S,T}^2$ and $\Theta_{D,T}^2$.

Lemma 5.6.5 *There hold*

$$\|\mathbf{f}_S + \mathbf{div}\;\boldsymbol{\sigma}_{S,h}\|_{0,T} \;\le\; \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{\mathbf{div};T} \qquad \forall\, T\in\mathcal{T}_h^S,$$

$$\|f_D - \mathrm{div}\;\mathbf{u}_{D,h}\|_{0,T} \;\le\; \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{\mathrm{div};T} \qquad \forall\, T\in\mathcal{T}_h^D,$$

and there exists $c > 0$*, depending on* κ_1 *(cf.* (5.4)*), such that*

$$\|\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D},h}|)\mathbf{t}_{\mathrm{D},h} + \mathbf{u}_{\mathrm{D},h}\|_{0,T} \le c\Big\{ \|\mathbf{t}_{\mathrm{D}} - \mathbf{t}_{\mathrm{D},h}\|_{0,T} + \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div}\,;T} \Big\} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}\,.$$

Proof. For the first two estimates it suffices to recall, as established by Theorem 5.3.2, that $\mathbf{f}_{\mathrm{S}} = -\,\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}}$ in Ω_{S} and $f_{\mathrm{D}} = \mathrm{div}\,\mathbf{u}_{\mathrm{D}}$ in Ω_{D}. Next, adding and subtracting $\mathbf{u}_{\mathrm{D}}$, and using also from Theorem 5.3.2 that $\mathbf{u}_{\mathrm{D}} = -\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D}}|)\mathbf{t}_{\mathrm{D}}$, we find that

$$\|\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D},h}|)\mathbf{t}_{\mathrm{D},h} + \mathbf{u}_{\mathrm{D},h}\|_{0,T} \le \|\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D},h}|)\mathbf{t}_{\mathrm{D},h} - \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D}}|)\mathbf{t}_{\mathrm{D}}\|_{0,T} + \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div}\,;T}\,.$$

Then, proceeding similarly as in the proof of [11, Lemma 3] and using the assumptions on $\boldsymbol{\kappa}$ (cf. (5.4)), we deduce that

$$\|\boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D},h}|)\mathbf{t}_{\mathrm{D},h} - \boldsymbol{\kappa}(\cdot, |\mathbf{t}_{\mathrm{D}}|)\mathbf{t}_{\mathrm{D}}\|_{0,T} \le 3\,k_1\,\|\mathbf{t}_{\mathrm{D}} - \mathbf{t}_{\mathrm{D},h}\|_{0,T}\,,$$

which, replaced back into the previous estimate, completes the proof. □

The derivation of the upper bounds for the remaining terms defining the global a posteriori error estimator proceeds similarly as in [50] (see also [15]), using known results from [25], [28], and [40], and applying Helmholtz decompositions, inverse inequalities, and the localization technique based on element-bubble and edge-bubble functions. We omit further details and just provide the following lemma that summarizes known efficiency estimates for thirteen terms defining $\Theta_{\mathrm{S},T}^2$ and $\Theta_{\mathrm{D},T}^2$. The corresponding proofs, as detailed below, can be found in [15], [18], [25], [40], [42], [47], and [50]).

Lemma 5.6.6 *There exist positive constants* c_i , $i \in \{1, ..., 13\}$*, independent of* h*, such that*

a) $h_T^2\,\|\mathbf{rot}\,\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 \le c_1\,\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}$,

b) $h_e\,\|[\mathbf{t}_{\mathrm{D},h}\cdot\mathbf{t}]\|_{0,e}^2 \le c_2\,\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,w_e}^2 \qquad \forall\, e \in \mathcal{E}_h(\Omega_{\mathrm{D}})$*, where the set* w_e *is given by* $w_e := \cup\Big\{T' \in \mathcal{T}_h^{\mathrm{D}} : \quad e \in \mathcal{E}(T')\Big\}$,

c) $h_e\,\|[\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}]\|_{0,e}^2 \le c_3\,\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,w_e}^2 \qquad \forall e \in \mathcal{E}_h(\Omega_{\mathrm{S}})$*, where the set* w_e *is given by* $w_e := \cup\Big\{T' \in \mathcal{T}_h^{\mathrm{S}} : \quad e \in \mathcal{E}(T')\Big\}$,

d) $h_e\,\|\mathbf{t}_{\mathrm{D},h}\cdot\mathbf{t}\|_{0,e}^2 \le c_4\,\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 \qquad \forall e \in \mathcal{E}_h(\Gamma_{\mathrm{D}})$*, where* T *is the triangle of* $\mathcal{T}_h^{\mathrm{D}}$ *having* e *as an edge,*

e) $h_e\,\|\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}\|_{0,e}^2 \le c_5\,\|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 \qquad \forall e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})$*, where* T *is the triangle of* $\mathcal{T}_h^{\mathrm{S}}$ *having* e *as an edge,*

f) $h_T^2\,\|\mathbf{t}_{\mathrm{D},h}\|_{0,T}^2 \le c_6\,\Big\{\|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\,\|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{0,T}^2\Big\} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}$,

g) $h_T^2\,\|\boldsymbol{\sigma}^d_{S,h}\|^2_{0,T} \,\le\, c_7 \Big\{ \|\mathbf{u}_S - \mathbf{u}_{S,h}\|^2_{0,T} \,+\, h_T^2\,\|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|^2_{0,T} \Big\} \quad \forall\, T \,\in\, \mathcal{T}_h^S ,$

h) $h_e\,\|p_{D,h} - \lambda_h\|^2_{0,e} \,\le\, c_8 \Big\{ \|p_D - p_{D,h}\|^2_{0,T} + h_T^2\,\|\mathbf{u}_D - \mathbf{u}_{D,h}\|^2_{0,T} + h_e\,\|\lambda - \lambda_h\|^2_{0,e} \Big\} \quad \forall e \,\in\, \mathcal{E}_h(\Sigma),$
where T is the triangle of $\mathcal{T}_h^D$ having e as an edge,

i) $\displaystyle\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\big\|\mathbf{t}_{D,h}\cdot\mathbf{t} \,-\, \lambda'_h\big\|^2_{0,e} \,\le\, c_9 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \|\mathbf{u}_D - \mathbf{u}_{D,h}\|^2_{0,T_e} \,+\, \|\lambda - \lambda_h\|^2_{1/2,\Sigma} \right\},$
where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^D$ having e as an edge.

j) $\displaystyle\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\left\|\nu^{-1}\boldsymbol{\sigma}^d_{S,h}\mathbf{t} \,+\, \boldsymbol{\varphi}'_h\right\|^2_{0,e} \,\le\, c_{10} \left\{ \sum_{e\in\mathcal{E}_h(\Gamma_S)} \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|^2_{0,T_e} \,+\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{1/2,00,\Sigma} \right\},$
where, given $e \,\in\, \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^S$ having e as an edge.

k) $h_e\,\|\mathbf{u}_{D,h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|^2_{0,e} \,\le\, c_{11} \Big\{ \|\mathbf{u}_D - \mathbf{u}_{D,h}\|^2_{0,T} + h_T^2\,\|\mathrm{div}\,(\mathbf{u}_D - \mathbf{u}_{D,h})\|^2_{0,T} + h_e\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{0,e} \Big\},$
for all $e \,\in\, \mathcal{E}_h(\Sigma)$, where T is the triangle of $\mathcal{T}_h^D$ having e as an edge,

l) $h_e\,\|\boldsymbol{\sigma}_{S,h}\,\mathbf{n} \,+\, \lambda_h\,\mathbf{n} \,-\, \nu\,\kappa_f^{-1}\,(\boldsymbol{\varphi}_h\cdot\mathbf{t})\,\mathbf{t}\|^2_{0,e}$
$\le\, c_{12} \Big\{ \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|^2_{0,T} \,+\, h_T^2\,\|\mathbf{div}\,(\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h})\|^2_{0,T} \,+\, h_e\,\|\lambda - \lambda_h\|^2_{0,e} \,+\, h_e\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{0,e} \Big\},$
for all $e \,\in\, \mathcal{E}_h(\Sigma)$, where T is the triangle of $\mathcal{T}_h^S$ having e as an edge, and

m) $h_e\,\|\mathbf{u}_{S,h} \,+\, \boldsymbol{\varphi}_h\|^2_{0,e} \,\le\, c_{13} \Big\{ \|\mathbf{u}_S - \mathbf{u}_{S,h}\|^2_{0,T} \,+\, h_T^2\,\|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|^2_{0,T} \,+\, h_e\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{0,e} \Big\},$
for all $e \,\in\, \mathcal{E}_h(\Sigma)$, where T is the triangle of $\mathcal{T}_h^S$ having e as an edge.

Proof. For a) we refer to [25, Lemma 6.1]. Alternatively, a) follows from straightforward applications of the technical result provided in [18, Lemma 4.3] (see also [47, Lemma 4.9]). Similarly, for b), c), d), and e) we refer to [25, Lemma 6.2] or apply the technical result given by [18, Lemma 4.4] (see also [47, Lemma 4.10]). Then, for f) and g) we refer to [25, Lemma 6.3] (see also [47, Lemma 4.13] or [40, Lemma 5.5]). On the other hand, the estimate given by h) corresponds to [15, Lemma 4.12]. The proofs of i) and j) follow from very slight modifications of the proof of [40, Lemma 5.7]. Alternatively, an *elasticity version* of i) and j), which is provided in [42, Lemma 20], can also be adapted to our case. Finally, for k), l) and m) we refer to [50, Lemmas 3.15, 3.16 and 3.17]. □

The estimates i) and j) in the previous lemma provide the only non-local bounds of the present efficiency analysis. However, under additional regularity assumptions on λ and $\boldsymbol{\varphi}$, we can give the following local bounds instead.

Lemma 5.6.7 *Assume that $\lambda|_e \,\in\, H^1(e)$ for each $e \,\in\, \mathcal{E}_h(\Sigma)$, and that $\boldsymbol{\varphi}|_e \,\in\, \mathbf{H}^1(e)$ for each $e \,\in\, \mathcal{E}_h(\Gamma_S)$. Then there exist $\tilde{c}_9$, $\tilde{c}_{10} \,>\, 0$, such that*

$$h_e\,\big\|\mathbf{t}_{D,h}\cdot\mathbf{t} \,+\, \lambda'_h\big\|^2_{0,e} \,\le\, \tilde{c}_9\,\Big\{ \|\mathbf{u}_D - \mathbf{u}_{D,h}\|^2_{0,T_e} \,+\, h_e\,\big\|\lambda' - \lambda'_h\big\|^2_{0,e} \Big\} \quad \forall\, e \in \mathcal{E}_h(\Sigma)\,,$$

and

$$h_e \left\| \nu^{-1}\boldsymbol{\sigma}^d_{\mathrm{S},h}\mathbf{t} \,+\, \boldsymbol{\varphi}'_h \right\|^2_{0,e} \,\leq\, \tilde{c}_{10} \left\{ \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T_e} \,+\, h_e\, \|\boldsymbol{\varphi}' - \boldsymbol{\varphi}'_h\|^2_{0,e} \right\} \quad \forall\, e \in \mathcal{E}_h(\Gamma_{\mathrm{S}})\,.$$

Proof. Similarly as for i) and j) from Lemma 5.6.6, it follows by adapting the corresponding *elasticity version* from [42]. We omit details here and refer to [42, Lemma 21]. □

We end this section by observing that the required efficiency estimate follows straightforwardly from Lemmas 5.6.5, 5.6.6, and 5.6.7. In particular, the terms $h_e\,\|\lambda - \lambda_h\|^2_{0,e}$ and $h_e\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{0,e}$, which appear in Lemma 5.6.6 (items h), k), l), and m)), are bounded as follows:

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\lambda - \lambda_h\|^2_{0,e} \,\leq\, h\,\|\lambda - \lambda_h\|^2_{0,\Sigma} \,\leq\, C\,h\,\|\lambda - \lambda_h\|^2_{1/2,\Sigma}\,,$$

and

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{0,e} \,\leq\, h\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{0,\Sigma} \,\leq\, C\,h\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|^2_{1/2,00,\Sigma}\,.$$

5.7 Numerical results

In this section we provide three examples illustrating the performance of the Galerkin scheme (5.55) with the subspaces $\mathbf{X}_h := \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \mathbf{T}_h(\Omega_{\mathrm{D}})$, $\mathbf{M}_h := \mathbf{L}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \times \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma)$ and $\mathbf{Q}_h := L_{h,0}(\Omega_{\mathrm{D}}) \times \Lambda^{\mathrm{D}}_h(\Sigma)$ defined in Section 5.5, confirming the reliability and efficiency of the a posteriori error estimator Θ, and showing the behaviour of the associated adaptive algorithm.

In what follows, N stands for the number of degrees of freedom defining $\mathbb{X}_h$ and $\mathbb{M}_h$. The solution of (5.22) and (5.55) are denoted

$$(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \,:=\, ((\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}), (\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}, \boldsymbol{\varphi}), (p_{\mathrm{D}}, \lambda)) \,\in\, \mathbf{X} \times \mathbf{M} \times \mathbf{Q}$$

and

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{p}}_h) \,:=\, ((\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{t}_{\mathrm{D},h}), (\mathbf{u}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h), (p_{\mathrm{D},h}, \lambda_h)) \,\in\, \mathbf{X}_h \times \mathbf{M}_h \times \mathbf{Q}_h\,.$$

The individual and global errors are defined by:

$$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}) \,:=\, \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div}\,;\Omega_{\mathrm{S}}}\,, \qquad \mathsf{e}(\mathbf{u}_{\mathrm{S}}) \,:=\, \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{\mathrm{div}\,;\Omega_{\mathrm{S}}}\,,$$

$$\mathsf{e}(\mathbf{t}_{\mathrm{D}}) \,:=\, \|\mathbf{t}_{\mathrm{D}} - \mathbf{t}_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,, \qquad \mathsf{e}(\mathbf{u}_{\mathrm{D}}) \,:=\, \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div}\,;\Omega_{\mathrm{D}}}\,, \qquad \mathsf{e}(p_{\mathrm{D}}) \,:=\, \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}}\,,$$

$$\mathsf{e}(\boldsymbol{\varphi}) \,:=\, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00,\Sigma}\,, \qquad \mathsf{e}(\lambda) \,:=\, \|\lambda - \lambda_h\|_{1/2,\Sigma}\,,$$

and

$$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) \,:=\, \Big\{(\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}}))^2 + (\mathsf{e}(\mathbf{u}_{\mathrm{S}}))^2 + (\mathsf{e}(\mathbf{t}_{\mathrm{D}}))^2 + (\mathsf{e}(\mathbf{u}_{\mathrm{D}}))^2 + (\mathsf{e}(p_{\mathrm{D}}))^2 + (\mathsf{e}(\boldsymbol{\varphi}))^2 + (\mathsf{e}(\lambda))^2\Big\}^{1/2}\,,$$

whereas the effectivity index with respect to Θ is given by

$$\mathrm{eff}(\Theta) := \mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})/\Theta\,.$$

Also, we let $r(\boldsymbol{\sigma}_{\mathrm{S}})$, $r(\mathbf{u}_{\mathrm{S}})$, $r(\mathbf{t}_{\mathrm{D}})$, $r(\mathbf{u}_{\mathrm{D}})$, $r(p_{\mathrm{D}})$, $r(\boldsymbol{\varphi})$, $r(\lambda)$, and $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$ be the individual and global experimental rates of convergence given by

$$r(\%) := \frac{\log(\mathsf{e}(\%)/\mathsf{e}'(\%))}{\log(h/h')} \qquad \text{for each} \quad \% \in \big\{\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \mathbf{t}_{\mathrm{D}}, \mathbf{u}_{\mathrm{D}}, p_{\mathrm{D}}, \boldsymbol{\varphi}, \lambda\big\}\,,$$

and

$$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}) := \frac{\log(\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})/\mathsf{e}'(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}}))}{\log(h/h')}\,,$$

where h and h' denote two consecutive meshsizes with errors e and e'. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. In all of them we choose $\nu = 1$, $\kappa_f = 1$, and $\boldsymbol{\kappa}(\cdot, s) = 2 + 1/(1+s)$. It is easy to check that $\boldsymbol{\kappa}$ satisfies the assumptions (5.4) with $k_0 = 1$ and $k_1 = 3$. Example 1 is used to illustrate the performance of the Galerkin scheme (5.55) and to corroborate the reliability and efficiency of the a posteriori error estimator Θ. Then, Examples 2 and 3 are utilized to illustrate the behavior of the associated adaptive algorithm, which applies the following procedure from [81]:

1) Start with a coarse mesh $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{D}} \cup \mathcal{T}_h^{\mathrm{S}}$.

2) Solve the discrete problem (5.55) for the current mesh $\mathcal{T}_h$.

3) Compute $\Theta_T := \Theta_{\star,T}$ for each triangle $T \in \mathcal{T}_h^{\star}$, $\star \in \{\mathrm{D}, \mathrm{S}\}$.

4) Check the stopping criterion and decide whether to finish or go to next step.

5) Use *blue-green* refinement on those $T' \in \mathcal{T}_h$ whose indicator $\Theta_{T'}$ satisfies

$$\Theta_{T'} \geq \frac{1}{2}\max_{T\in\mathcal{T}_h}\{\Theta_T : \quad T \in \mathcal{T}_h\}\,.$$

6) Define resulting meshes as current meshes $\mathcal{T}_h^{\mathrm{D}}$ and $\mathcal{T}_h^{\mathrm{S}}$, and go to step 2.

In Example 1 we consider the regions $\Omega_{\mathrm{S}} := (-1,1)\times(0,1)$ and $\Omega_{\mathrm{D}} := (-1,1)\times(-1,0)$, and choose the data $\mathbf{f}_{\mathrm{S}}$ and f_{D} so that the exact solution is given by the smooth functions

$$\mathbf{u}_{\mathrm{S}}(\mathbf{x}) = \mathrm{curl}\big(x_2^2 \sin(\pi x_1)\big) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{S}}\,,$$

$$p_S(\mathbf{x}) \,=\, x_1^3 + x_2^3 \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_S\,,$$

and

$$p_D(\mathbf{x}) \,=\, x_1\,\left(x_1^2 - 1\right)^2\,(x_2+1)^2 \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_D\,.$$

In Example 2 we consider $\Omega_D := (-1,1) \times (-2,-1)$ and let Ω_S be the L-shaped domain given by $(-1,1)^2 \setminus [0,1]^2$. Then we choose $\mathbf{f}_S$ and f_D so that the exact solution is given by

$$\mathbf{u}_S(\mathbf{x}) \,=\, \operatorname{curl}\left(3\,(x_1^2+x_2^2)^{4/3}\,(x_2+1)^2\right) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_S\,,$$

$$p_S(\mathbf{x}) \,=\, (x_2+1)^2\,e^{x_1} \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_S\,,$$

and

$$p_D(\mathbf{x}) \,=\, \frac{1}{5}\,(x_1^3 - 3x_1)\,\cos(\pi\,x_2) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_D\,.$$

Note that $\nabla \mathbf{u}_S$ and $\boldsymbol{\sigma}_S$ have a singularity at the origin.

In Example 3 we consider the same geometry of Example 1 and choose the data $\mathbf{f}_S$ and f_D so that the exact solution is given by the smooth functions

$$\mathbf{u}_S(\mathbf{x}) \,=\, \operatorname{curl}\left(0.2\,x_2^3\,e^{x_1+x_2}\right) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_S\,,$$

$$p_S(\mathbf{x}) \,=\, x_2^2\,e^{x_1} \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_S\,,$$

and

$$p_D(\mathbf{x}) \,=\, \frac{x_1\,\left(x_1^2-1\right)^2}{\left(x_1^2 + (x_2+1)^2 + 0.05\right)} \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_D\,,$$

In this case, p_D and hence $\mathbf{t}_D = \nabla p_D$ and $\mathbf{u}_D = -\boldsymbol{\kappa}\,(\cdot, |\nabla p_D|)\,\nabla p_D$ show a numerical singularity in a neighborhood of the point $(0,-1)$.

The numerical results shown below were obtained using a MATLAB code. In Table 5.1 we summarize the convergence history of the mixed finite element method (5.55), as applied to Example 1, for a sequence of quasi-uniform triangulations of the domain. We observe there, looking at the corresponding experimental rates of convergence, that the $O(h)$ predicted by Theorem 5.5.2 (here $\delta = 1$) is attained in all the unknowns. In addition, we notice that the effectivity index $\mathrm{eff}(\Theta)$ remains always in a neighborhood of 0.87, which illustrates the reliability and efficiency of Θ in the case of a regular solution.

Next, in Tables 5.2 - 5.5 we provide the convergence history of the quasi-uniform and adaptive schemes, as applied to Examples 2 and 3. We observe that the errors of the adaptive procedures

Table 5.1: EXAMPLE 1, quasi-uniform scheme

N	h	$\mathsf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(\mathbf{t}_{\mathrm{D}})$	$r(\mathbf{t}_{\mathrm{D}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$
168	0.707	7.359	–	0.865	–	0.489	–	1.694	–
640	0.354	4.312	0.799	0.457	0.953	0.220	1.196	1.375	0.312
2496	0.177	2.195	0.992	0.230	1.008	0.106	1.078	0.829	0.743
9856	0.088	1.103	1.002	0.115	1.007	0.052	1.036	0.476	0.810
39168	0.044	0.552	1.003	0.058	1.004	0.026	1.011	0.260	0.878
156160	0.022	0.276	1.002	0.029	1.002	0.013	1.004	0.137	0.929

N	$\mathsf{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathsf{e}(\lambda)$	$r(\lambda)$	$\mathsf{e}(\varphi)$	$r(\varphi)$	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$\mathrm{eff}(\Theta)$
168	0.126	–	0.683	–	0.037	–	7.648	–	0.862
640	0.045	1.524	0.497	0.475	0.139	–	4.583	0.765	0.879
2496	0.018	1.371	0.244	1.041	0.042	1.734	2.373	0.967	0.898
9856	0.008	1.179	0.120	1.030	0.014	1.549	1.213	0.976	0.845
39168	0.004	1.061	0.060	1.011	0.005	1.513	0.616	0.982	0.875
156160	0.002	1.018	0.030	1.004	0.001	1.505	0.311	0.988	0.871

decrease faster than those obtained by the quasi-uniform ones, which is confirmed by the global experimental rates of convergence provided there. This fact is also illustrated in Figures 5.2 and 5.4 where we display the total errors $\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$ vs. the number of degrees of freedom N for both refinements. As shown by the values of $r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$, the adaptive method is able to keep the quasi-optimal rate of convergence $\mathcal{O}(h)$ for the total error. Furthermore, the effectivity indexes remain bounded from above and below, which confirms the reliability and efficiency of Θ in these cases of non-smooth solutions. Intermediate meshes obtained with the adaptive refinements are displayed in Figures 5.3 and 5.5. Note that the method is able to recognize the singularity of the solution in Example 2 and the region with high gradients in Example 3.

Table 5.2: EXAMPLE 2, quasi-uniform scheme

N	h	$\mathbf{e}(\boldsymbol{\sigma}_S)$	$\mathbf{e}(\mathbf{u}_S)$	$\mathbf{e}(\mathbf{t}_D)$	$\mathbf{e}(\mathbf{u}_D)$	$\mathbf{e}(p_D)$
404	0.5000	29.3565	5.8914	0.3784	2.2553	0.0806
1576	0.2500	19.7820	3.0327	0.1895	1.2565	0.0409
6224	0.1250	13.2561	1.5276	0.0948	0.6588	0.0204
24736	0.0625	8.4281	0.7652	0.0474	0.3369	0.0102
98624	0.0312	5.5354	0.3828	0.0237	0.1703	0.0051

N	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\varphi})$	$\mathtt{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	eff(Θ)
404	0.3325	0.2636	30.0322	–	0.5258
1576	0.1713	0.1226	20.0546	0.5933	0.5631
6224	0.0887	0.0477	13.3608	0.5914	0.5627
24736	0.0450	0.0172	8.4697	0.6607	0.5986
98624	0.0226	0.0060	5.5513	0.6109	0.5598

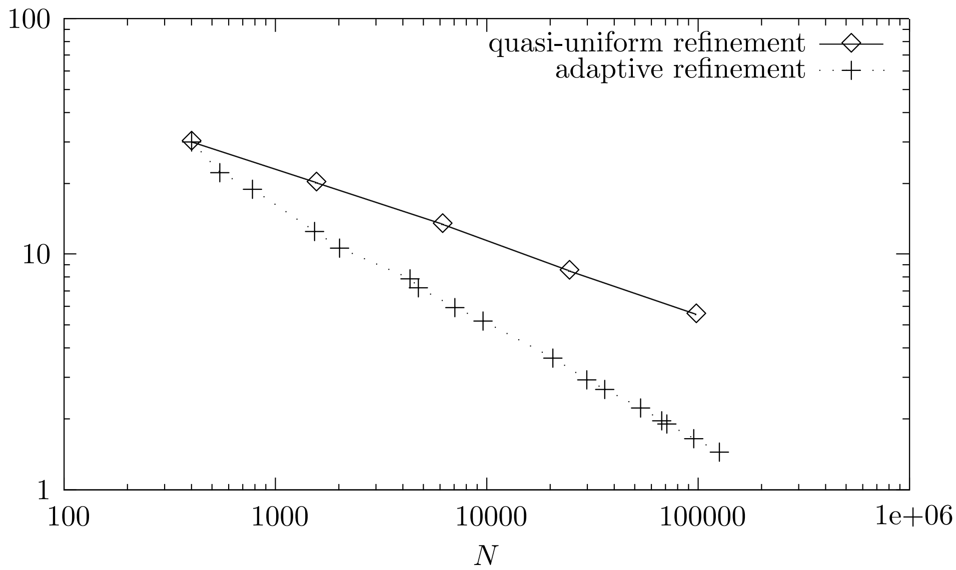

Figure 5.2: EXAMPLE 2, $\mathtt{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$ vs. N for the quasi-uniform and adaptive schemes

Table 5.3: EXAMPLE 2, adaptive scheme

N	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	Θ	eff(Θ)
404	30.0322	–	57.1171	0.5258
548	22.2145	1.9781	34.9558	0.6355
784	18.8764	0.9093	27.8872	0.6769
1544	12.4998	1.2164	18.5554	0.6736
2026	10.6033	1.2113	15.9807	0.6635
4373	7.8376	0.7856	11.0736	0.7078
4781	7.2224	1.8328	10.3884	0.6952
7105	5.9397	0.9872	8.4901	0.6996
9673	5.2169	0.8411	7.3908	0.7059
20712	3.6174	0.9618	5.0386	0.7179
29906	2.9286	1.1501	4.1342	0.7084
36304	2.6731	0.9416	3.7189	0.7188
53634	2.2272	0.9353	3.0884	0.7212
67436	1.9670	1.0850	2.7358	0.7190
71449	1.9011	1.1802	2.6419	0.7196
96176	1.6508	0.9499	2.2885	0.7213
126900	1.4424	0.9737	2.0029	0.7201

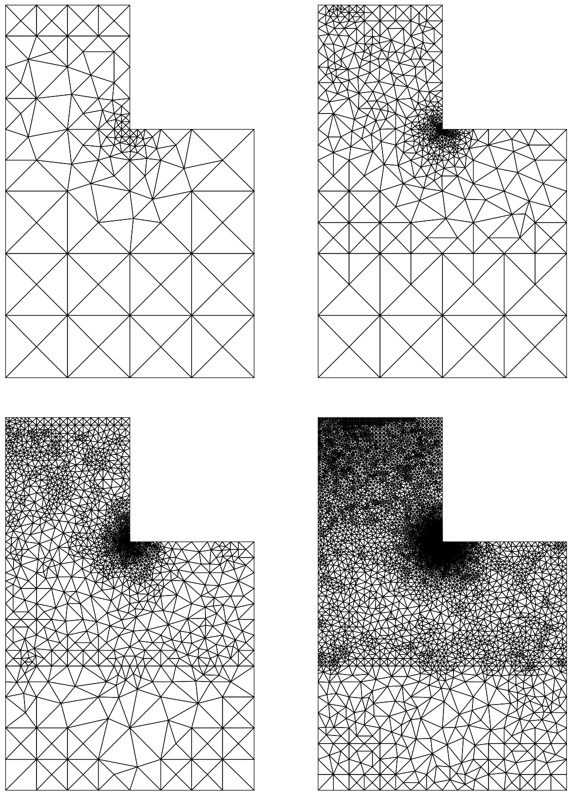

Figure 5.3: EXAMPLE 2, adapted meshes with 1544, 4781, 20712, and 67436 degrees of freedom

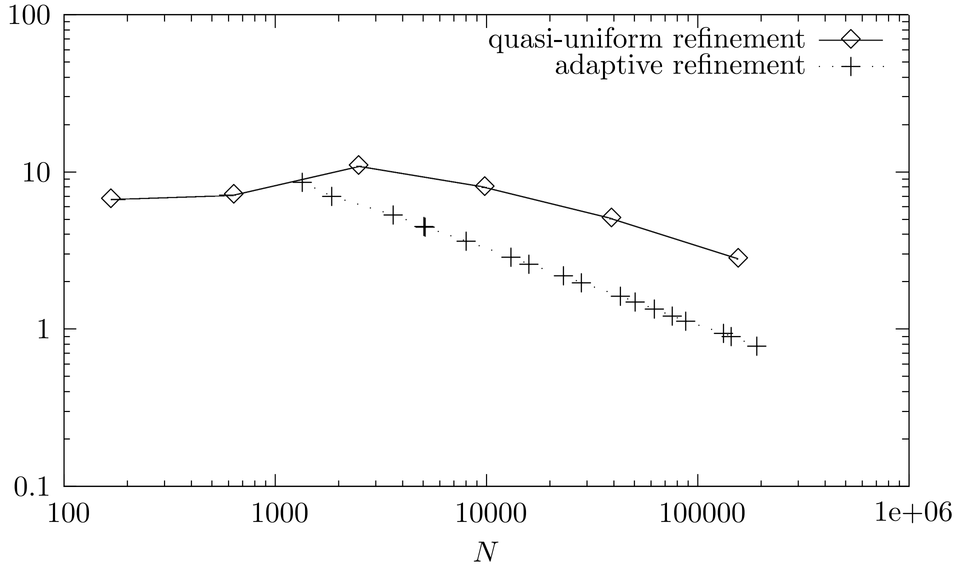

Figure 5.4: EXAMPLE 3, $\mathtt{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$ vs. N for the quasi-uniform and adaptive schemes

Table 5.4: EXAMPLE 3, quasi-uniform scheme

N	h	$\mathbf{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathbf{e}(\mathbf{u}_{\mathrm{S}})$	$\mathbf{e}(\mathbf{t}_{\mathrm{D}})$	$\mathbf{e}(\mathbf{u}_{\mathrm{D}})$	$\mathbf{e}(p_{\mathrm{D}})$
168	0.7071	1.6203	0.1781	2.7631	5.7862	0.4991
640	0.3536	0.8166	0.0920	2.5015	6.6021	0.2920
2496	0.1768	0.4069	0.0446	1.3345	10.7401	0.1288
9856	0.0884	0.2035	0.0222	0.6524	7.9304	0.0642
39168	0.0442	0.1017	0.0111	0.3264	4.9954	0.0322
156160	0.0221	0.0509	0.0055	0.1632	2.7788	0.0161

N	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\varphi})$	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$\mathrm{eff}(\Theta)$
168	0.4683	0.0570	6.6515	–	0.9877
640	0.6154	0.0736	7.1407	–	1.0202
2496	0.2349	0.0159	10.8337	–	1.0145
9856	0.1023	0.0044	7.9607	0.4487	1.0077
39168	0.0478	0.0014	5.0074	0.6720	1.0050
156160	0.0253	0.0005	2.7842	0.8488	1.0041

Table 5.5: Example 3, adaptive scheme

N	$\mathsf{e}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	$r(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \underline{\mathbf{p}})$	Θ	eff(Θ)
1346	8.5936	–	8.5943	0.9999
1866	6.9966	1.2588	7.0143	0.9975
3633	5.3139	0.8258	5.3029	1.0021
5069	4.4949	1.0051	4.4942	1.0001
5146	4.4662	0.8474	4.4546	1.0026
8042	3.6365	0.9207	3.6203	1.0045
13148	2.8766	0.9538	2.8588	1.0062
15921	2.5961	1.0722	2.5742	1.0085
23197	2.1824	0.9225	2.1712	1.0051
28262	1.9700	1.0365	1.9556	1.0074
43218	1.6240	0.9096	1.6176	1.0039
50762	1.4914	1.0589	1.4833	1.0055
62798	1.3415	0.9958	1.3341	1.0055
76352	1.2116	1.0424	1.2053	1.0052
88422	1.1253	1.0064	1.1186	1.0060
133093	0.9381	0.8898	0.9318	1.0068
144737	0.8932	1.1703	0.8877	1.0062
191228	0.7814	0.9597	0.7767	1.0062

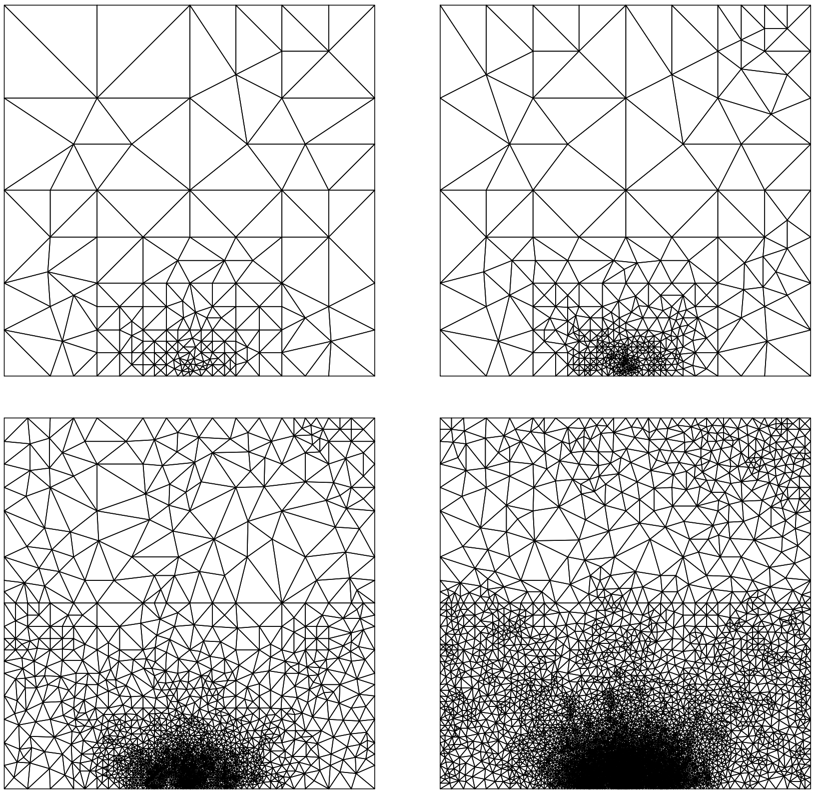

Figure 5.5: Example 3, adapted meshes with 1346, 3633, 15921, and 62798 degrees of freedom

Chapter 6

Conclusiones y trabajo futuro

6.1 Conclusiones

El objetivo principal de la tesis presentada ha sido el desarrollo de métodos de elementos finitos mixtos conformes para el problema acoplado de Stokes-Darcy. Lo anterior se logra, en primer lugar, mejorando resultados previos existentes en la literatura, y en segundo lugar, proponiendo nuevos métodos que permiten aproximar las distintas variables físicas del problema. Además, con el fin de verificar el buen funcionamiento de cada uno de los métodos propuestos, se han desarrollado códigos computacionales y se han presentado ejemplos numéricos que corroboran los resultados teóricos obtenidos.

Las conclusiones principales de esta tesis, en orden de desarrollo, son:

1. Se mejoran los resultados obtenidos en [45] y se demuestra que es posible utilizar cualquier par de elementos finitos estables para los problemas de Stokes y Darcy en el esquema de Galerkin de la formulación primal-mixta propuesta en [63]. En particular, para el dominio de Stokes se pueden utilizar elementos de Taylor-Hood, Bernardi-Raugel y el elemento MINI, mientras que en el dominio de Darcy se pueden utilizar elementos de Raviart-Thomas de cualquier orden.

2. Se introduce una nueva formulación variacional, dual-mixta en ambos dominios, para el problema acoplado de Stokes-Darcy, la cual permite la utilización de la misma familia de elementos finitos en ambos dominios. La estructura dual-mixta se obtiene mediante la introducción del pseudo-esfuerzo y la velocidad en el fluido, junto con la velocidad y la presión en el medio poroso, como incógnitas principales del modelo.

3. Se desarrolla un análisis de error a posteriori para la formulación variacional, dual-mixta en ambos dominios, descrita en **2.**, y se obtiene un estimador de error a posteriori residual, confiable y eficiente, para el problema acoplado. Los elementos finitos considerandos son: elementos de Raviart-Thomas para el pseudostress en el fluido y la velocidad de filtración en el medio poroso, elementos constantes a trozos para la velocidad del fluido y la presión en el medio poroso, y elementos continuos lineales a trozos para los multiplicadores de Lagrange definidos en la interfase.

4. Se desarrolla un análisis a priori y a posteriori para la formulación variacional de un acoplamiento no lineal de Stokes-Darcy. El modelo considerado describe la interacción de un fluido viscoso cuyo comportamiento es descrito por la ecuación de Stokes, con un medio poroso modelado por un sistema de Darcy no lineal. Las incógnitas principales consideradas en el modelo son: el pseudo-esfuerzo y la velocidad en el fluido; la velocidad, la presión y el gradiente de presión en el medio poroso; la presión del fluido en el medio poroso y la velocidad del fluido libre en la interfase. Con ello se obtiene una estructura dual-mixta en el fluido y dual-dual-mixta en el medio poroso. A nivel discreto el esquema propuesto permite la utilización de la misma familia de elementos finitos en ambos dominios. Finalmente, se desarrolla un estimador de error a posteriori residual, confiable y eficiente.

6.2 Trabajo futuro

1. Se desarrollará el análisis a priori y a posteriori de una versión aumentada del método de elementos finitos mixtos para el problema acoplado de Stokes-Darcy introducido en el Capítulo 3. Esto apunta a la posibilidad de relajar las hipótesis sobre los espacios de elementos finitos a utilizar.

2. Se comenzará un análisis a priori y a posteriori de métodos de elementos finitos mixtos para el acoplamiento de fluidos con medios porosos, considerando no linealidades en el dominio Stokes, y/o en ambos dominios. Se desarrollará un análisis teórico, utilizando herramientas disponibles en la literatura y se elaborarán codigos computacionales que corroboren los resultados teóricos obtenidos.

3. Se extenderán los resultados obtenidos en este trabajo de tesis al problema evolutivo de acoplamiento de fluidos con medios porosos, cuyo modelo se determina por un sistema acoplado de las ecuaciones evolutivas de Stokes y Darcy.

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