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ANALISIS DE ERROR A-POSTERIORI PARA FORMULACIONES MIXTAS DUALES DE PROBLEMAS DE VALORES DE CONTORNO LINEALES Y NO-LINEALES

Tesis para optar al grado de
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Mauricio Andrés Barrientos Barría

FACULTAD DE CIENCIAS FISICAS Y MATEMATICAS
DEPARTAMENTO DE INGENIERIA MATEMATICA
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Mauricio Andrés Barrientos Barría

Director de Tesis: Gabriel N. Gatica

Director de Programa: Norbert Heuer

COMISION EVALUADORA

Prof. Dr. Salim Meddahi, Universidad de Oviedo, España

Prof. Dr. Ernst P. Stephan, Universidad de Hannover, Alemania

COMISION EXAMINADORA

Firma: ______________________________
Prof. Dr. Ricardo Durán
Universidad de Buenos Aires, Argentina

Firma: ______________________________
Prof. Dr. Gabriel N. Gatica
Universidad de Concepción, Chile

Firma: ______________________________
Prof. Dr. Norbert Heuer
Universidad de Concepción, Chile

Fecha Examen de Grado: ______________________

Calificación: ______________________

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RESUMEN

En esta Tesis se aplican formulaciones variacionales mixtas duales para resolver problemas de valores de contorno lineales y nolineales. Más precisamente, nos interesan las formulaciones variacionales del tipo *dual-dual*, las cuales se llaman así por la estructura de punto de silla doble de las ecuaciones resultantes. El objetivo principal es realizar un estudio de error a-posteriori de estas formulaciones.

En efecto, se logran deducir estimaciones de error a-posteriori confiables de tipo explícito e implícito, para un problema de transmisión exterior lineal en teoría de potencial. También, se muestran experimentos numéricos que ilustran la eficiencia de estos estimadores. A continuación, se utiliza una estructura punto de silla doble en formulaciones variacionales de problemas de valor de frontera nolineal en hiperelasticidad plana y logramos deducir una estimacion de error a-posteriori confiable. Por último, seguimos el análisis descrito anteriormente y obtenemos una estimación de error a-posteriori confiable, para la formulación variacional dual-dual de un problema de transmisión lineal-nolineal en hiperelasticidad.

ABSTRACT

In this thesis we apply mixed-dual variational formulations to solve linear and nonlinear boundary value problems. More precisely, we are interested on *dual-dual* variational formulations, so called by the two-fold saddle point operator equations of the weak formulations. The main goal in this thesis is to realize an a-posteriori error analysis of these formulations.

In fact, we develop two different a-posteriori error analyses yielding explicit residual and implicit Bank-Weiser type reliable estimates, respectively, for a linear exterior transmission problems in the plane. Several numerical results illustrate the suitability of these estimators for the adaptive computation of the discrete solutions. Next, a two-fold saddle point nonlinear operator equation is used in a mixed variational formulation of hyperelasticity. Also, a reliable a-posteriori error estimate, based on the solution of local Dirichlet problems, and well suited for adaptive computations, is also given. Finally, we follow the above ideas to solve a linear-nonlinear transmission problem in plane hyperelasticity with mixed boundary conditions, which yields a twofold saddle point operator equation as the corresponding variational formulation. We derive a reliable a-posteriori error estimate.

Índice General

Capítulo 1

Introducción

En la actualidad, la aplicación de formulaciones variacionales mixtas ha llegado a ser un procedimiento estandar y necesario para la resolución de problemas de valores de frontera lineales y nolineales en la física y en la ingeniería. Por ejemplo, es bien sabido que en la mecánica de estructuras el uso de métodos de elementos finitos mixtos nos permiten calcular esfuerzos más exactos que los desplazamientos mismos. Además, para materiales casi incompresibles, la utilización de una aproximación sólo por elementos finitos, en los cuales los desplazamientos son las únicas incognitas, usualmente conduce al bloqueo de la solución. La bibliografía que versa sobre métodos mixtos para problemas lineales es extensa (ver [16], [44] y las propias referencias que estos dan, para más detalles). Sin embargo, formulaciones variacionales mixtas en problemas nolineales son poco conocidas en comparación con los lineales. Una de las ideas más comunes para el tratamiento de ecuaciones elípticas nolineales se basa en la inversión de la correspondiente ecuación constitutiva, gracias al teorema de la función implícita. A saber, en la conducción del calor el gradiente se expresa como una función de la temperatura y la variable de flujo. Este procedimiento ha sido estudiado, incluyendo las versiones h y p y extensiones a problemas parábolicos nolineales en varios trabajos recientes (ver, por ejemplo [48, 50, 52, 53, 58, 59]).

Ahora, cuando las ecuaciones constitutivas no son explícitamente invertibles, se sugiere un método como en [42, 43], en relación con el acoplamiento de elementos finitos mixtos y ecuaciones integrales de frontera para resolver problemas de transmisión. Este consiste en la introducción del gradiente (en teoría de potencial y conducción del calor) o el tensor de deformación (en elasticidad) como una incógnita adicional, el cual nos conduce a obtener ecuaciones de operadores con estructura punto de silla doble en las formulaciones débiles respectivas. Debido a su estructura, estas ecuaciones de operadores también han sido llamadas formulaciones variacionales *dual-dual* (ver [29, 31, 32, 33, 34, 40] para más detalles y aplicaciones). La idea de introducir este tipo de incógnitas también fue aplicada, independientemente, en [26, 27], en donde se les llamó método de elementos finitos mixtos extendido. Además, se debe hacer notar que el uso de una formulación mixta extendida ya había sido propuesta en algunos problemas de elasticidad (ver, por ejemplo, [28]). En todo caso, estructuras de punto de silla doble sólo han sido obtenidas y estudiadas en los trabajos anteriormente mencionados.

Por otro lado, para problemas nolineales usualmente no es posible obtener algún tipo de información *a priori* respecto de la solución que nos permitiría construir una malla conveniente para el esquema de elementos finitos. Por lo tanto, con la intención de obtener un buen comportamiento en la convergencia es que se requiere el aplicar un adecuado algoritmo de refinamiento, el cual usualmente se basa en estimaciones de error a-posteriori. La lista de referencias sobre análisis de error a-posteriori es bastante extensa para problemas lineales y no lineales (ver, por ejemplo, [4], [62] y las referencias que ellos contengan). Para formulaciones mixtas, nos referimos primero al trabajo de Verfürth [61], en el cual un estimador de error a-posteriori de tipo residual explícito se obtiene para el problema de Stokes. Entonces, Alonso en [5] usa espacios de Raviart-Thomas y Brezzi-Douglas-Marini para así obtener estimadores basados en evaluaciones residuales y en la solución de problemas locales, para ecuaciones diferenciales parciales elípticas de segundo orden. También, en relación con elementos de Raviart-Thomas, es bueno ver [15] y [20] en donde se obtienen estimadores de error residual confiables y eficientes. Además, una generalización del estimador de error residual jerárquico de Bank y Smith [9] para formulaciones mixtas se propuso en [1]. Aún más, el clásico estimador de Bank-Weiser de [10] y los resultados relacionados de [2] y [3], fueron recientemente ampliados en [17] para elasticidad con grandes deformaciones, incluyendo el caso incompresible. Este método, que involucra la solución de problemas de Neumann locales, fue también aplicado en [18] para obtener estimadores de error residual implícitos para el acoplamiento de elementos finitos y elementos de frontera.

La organización de esta Tesis es la siguiente. En el capítulo 2 se revisan todas las nociones básicas que son necesarias para una buena base matemática de los tópicos que se desarrollan más adelante. En el capítulo 3, el cual corresponde a lo presentado en [12], se combina el método de elementos finitos mixto-dual con una aplicación Dirichlet-to-Neumann (dada en términos de un operador integral de frontera) para resolver un problema de transmisión en el plano. Se elige como modelo una ecuación diferencial elíptica de segundo orden en forma de divergencia acoplada con la ecuación de Laplace en un dominio exterior. Se obtiene su formulación variacional mixta y su esquema de Galerkin asociado. Aquí, el objetivo principal es obtener estimaciones de error a-posteriori de este problema. En efecto, se deducen dos, una de tipo explícito y otra de tipo implícito, las que se basan en estimaciones confiables de tipo Bank-Weiser. También, se presentan varias pruebas numéricas para estas estimaciones.

A continuación, en el capítulo 4, el cual corresponde a lo presentado en [13], se extiende el método de elementos finitos mixtos para problemas lineales en elasticidad plana, tales como PEERS, a formulaciones variacionales en hiperelasticidad. Este tipo de aproximación se basa en la introducción del tensor de deformación como una incógnita adicional, lo cual produce una ecuación de operadores de tipo punto de silla doble para la formulación débil respectiva. Se obtiene la existencia y unicidad de los esquemas continuos y discretos, y se deduce la estimación de Cea usual para el error asociado. También, se deduce una estimación de error a-posteriori de tipo implícito, la cual es adecuada para implementaciones adaptivas.

Por último, en el capítulo 5, el cual corresponde a lo presentado en [11], se considera el acoplamiento de elementos finitos mixto-dual y elementos de frontera para resolver problemas de transmisión lineal-nolineal en hiperelasticidad plana con condiciones de frontera mixta. Aquí, usamos las ideas anteriores y nuevamente introducimos el tensor de deformación como una incógnita adicional para obtener en la formulación variacional una ecuación de operadores de tipo punto de silla doble. A continuación se obtiene una estimación de error a-posteriori implícita y confiable, en donde las soluciones de los problemas locales se obtienen en normas de Sobolev de orden negativo. Para ciertos subespacios específicos somos capaces de proporcionar dos estimaciones completamente locales del error a-posteriori, en donde los términos residuales se acotaron localmente por normas L^2. Además, una de las estimaciones de error no requiere la solución explícita de los problemas locales.

Capítulo 2

Algunos Conceptos Básicos

2.1 Nociones Básicas

Comencemos introduciendo la notación necesaria para desarrollar nuestro trabajo. Supondremos que nuestros problemas estarán definidos en un dominio Ω de $\mathbb{R}^n$ (en la práctica $n = 2$), con una frontera suficientemente suave $\partial\Omega = \Gamma$ (a saber, al menos Lipschitz continua). Tambien será necesario definir los espacios de Sobolev. Ellos están basados en

$$L^2(\Omega) = \left\{ v : \Omega \to \mathbb{R},\ \text{medible} : \int_\Omega |v|^2\, dx < +\infty \right\} . \tag{2.1}$$

Entonces, definimos en general, para un entero $m \geq 0$

$$H^m(\Omega) = \left\{ v :\ \partial^\alpha v \in L^2(\Omega) \qquad \forall\, |\alpha| \leq m \right\} , \tag{2.2}$$

donde $\alpha := (\alpha_1, \ldots, \alpha_n)$ y

$$\partial^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} , \quad |\alpha| = \alpha_1 + \ldots + \alpha_n ,$$

con las derivadas tomadas en el sentido de las distribuciones. Este espacio está provisto de la seminorma

$$|v|^2_{H^m(\Omega)} = \sum_{|\alpha|=m} |\partial^\alpha v|^2_{L^2(\Omega)} \tag{2.3}$$

y de la norma

$$||v||^2_{H^m(\Omega)} = \sum_{|\alpha|\leq m} |\partial^\alpha v|^2_{L^2(\Omega)} . \tag{2.4}$$

Además, denotaremos por $C_0^\infty(\Omega)$ el espacio de funciones infinitamente diferenciables que tienen soporte compacto en Ω y por $H_0^m(\Omega)$ la clausura de $C_0^\infty(\Omega)$ con respecto a la norma (2.4).

Si la frontera $\partial\Omega$ es suficientemente suave se pueden dar definiciones equivalentes de espacios sobre la frontera, los cuales serán de mucha utilidad en nuestro trabajo.

Definición 2.1.1 *Sea $0 < s < 1$. Se define $H^s(\Gamma)$ como la clausura de las funciones infinitamente diferenciables sobre Γ con respecto a la norma*

$$||\varphi||_{H^s(\Gamma)} := \left\{ ||\varphi||_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|\varphi(x) - \varphi(y)|^2}{||x-y||^{n-1+2s}} \, ds_x \, ds_y \right\}^{1/2} , \tag{2.5}$$

donde

$$||\varphi||_{L^2(\Gamma)} := \left\{ \int_\Gamma \varphi^2 \, ds \right\}^{1/2}$$

y n es la dimensión.

De acuerdo a esto, se define $H^{-s}(\Gamma)$ como el dual de $H^s(\Gamma)$ con respecto al producto interior en $L^2(\Gamma)$, es decir, la clausura de $L^2(\Gamma)$ con respecto a la norma

$$||\sigma||_{H^{-s}(\Gamma)} := \sup_{0 \neq \varphi \in H^s(\Gamma)} \frac{\left| \langle \sigma, \varphi \rangle_{L^2(\Gamma)} \right|}{||\varphi||_{H^s(\Gamma)}} . \tag{2.6}$$

Así, es posible dar el siguiente resultado de trazas.

Teorema 2.1.1 *Sea Ω un abierto acotado de $\mathbb{R}^n$ con frontera Γ de clase $C^{k,1}$, $k \in \mathbb{N} \cup \{0\}$ y $\frac{1}{2} < s \leq k+1 \in \mathbb{N}$. Entonces, la aplicación $v \longmapsto \gamma_0 v$ se extiende de manera única a un operador lineal y continuo de $H^s(\Omega) \to H^{s-1/2}(\Gamma)$.*

Demostración: Ver Teorema 6.2.40 en [46]. ■

En particular, existe un operador $\gamma_0 : H^1(\Omega) \longrightarrow L^2(\Gamma)$ lineal y continuo, tal que

$$\gamma_0 v = v \,|_\Gamma \quad \forall\, v \in C^\infty(\bar{\Omega}) .$$

En general, tenemos la siguiente relación:

$$H^1(\Gamma) \subset \gamma_0(H^1(\Omega)) \subset L^2(\Gamma) \equiv H^0(\Gamma) ,$$

donde cada inclusión es estricta.

2.2 Método de Elementos Finitos

Muchos problemas en elasticidad se representan matemáticamente mediante un problema de minimización. Una situación muy común es cuando la solución u, que corresponde al desplazamiento de un sistema mecánico, satisface

$$u \in V \quad \text{y} \quad J(u) = \inf_{v \in V} J(v) \tag{2.7}$$

donde V, el conjunto de desplazamientos admisibles, es un espacio de Hilbert.

Por medio del método de Ritz podemos aproximar la solución de (2.7), el cual consiste en buscar $u_m \in V_m$ tal que

$$J(u_m) = \inf_{v_m \in V_m} J(v_m) \ , \tag{2.8}$$

donde V_m es un subespacio de dimensión finita de V.

Por otro lado, el funcional J se puede representar en la forma

$$J(v) = \frac{1}{2} A(v, v) - L(v) \ ,$$

donde $L : V \to \mathbb{R}$ es una forma lineal y $A : V \times V \to \mathbb{R}$ es una forma bilineal que satisface las siguientes condiciones:

-) *Continuidad*: Existe una constante real $M > 0$, tal que

$$|A(u, v)| \leq M \, ||u||_V ||v||_V \quad \forall u, v \in V \ .$$

-) *Simetría*:

$$A(u, v) = A(v, u) \quad \forall u, v \in V \ .$$

De este modo, el problema (2.8) tiene la representación alternativa: *Hallar* $u_m \in V_m$ *tal que*

$$A(u_m, v_m) = L(v_m) \quad \forall v_m \in V_M \ . \tag{2.9}$$

Si se elige una base $\{\omega_1, \ldots, \omega_m\}$ de V_m, uno puede escribir

$$u_m = \sum_{i=1}^{m} \alpha_i \, \omega_i \tag{2.10}$$

y el problema (2.9) se reduce a encontrar la solución del sistema lineal

$$\sum_{i=1}^{m} a_{i\,j} \, \alpha_i = b_j \quad 1 \leq j \leq m \ , \tag{2.11}$$

donde

$$a_{i\,j} = A(\omega_i, \omega_j) \quad \text{y} \quad b_j = L(\omega_j) \ . \tag{2.12}$$

Esta formulación se puede extender al caso en que la forma bilineal A no es simétrica y el problema (2.9) ya no corresponde a un problema de minimización. Este caso pasa a llamarse *Método de Galerkin.*

Para su resolución, ahora necesitamos agregar la siguiente condición sobre A.

-) *Coercividad o V-elipticidad*: Existe una constante real $\beta > 0$, tal que

$$A(v,v) \geq \beta \, ||v||_V^2 \qquad \forall v \in V \, .$$

Así, tenemos el siguiente resultado de existencia y unicidad.

Teorema 2.2.1 *(Lax-Milgram) Sea V un espacio de Hilbert y $A : V \times V \to \mathbb{R}$ una forma bilineal, continua y coerciva. Entonces, para todo funcional lineal y acotado $L : V \to \mathbb{R}$, existe un único $u \in V$ tal que*

$$A(u,v) = L(v) \qquad \forall u \in V \, .$$

Además,

$$||u||_V \leq \frac{1}{\beta} \, ||L||_{V'} \, ,$$

donde β es la cosntante de coercividad.

Demostración: ver Teorema 1.7 en [44]. ■

Entonces, de acuerdo a lo que hemos visto, podemos decir que el método de elementos finitos es una técnica general para construir subespacios de dimensión finita de un espacio de Hilbert V con la intención de aplicar el *Método de Ritz-Galerkin* a un problema variacional planteado en V.

Esta técnica se basa en conceptos muy básicos. La idea fundamental es la partición del dominio Ω, de manera tal que el problema original se puede plantear en un conjunto de subdominios llamados elementos. Estos son usualmente triángulos, cuadrados, tetraedros, etc.

Así, un espacio de funciones V en Ω es aproximado por funciones simples, definidas sobre cada subdominio con adecuadas condiciones de interface. Estas funciones simples son por lo general polinomios o funciones que dependen de polinomios a través de algun cambio de variables.

Entonces, dado Ω un dominio poligonal con frontera $\partial\Omega$. Una partición (triangulación) admisible de elementos finitos $\{\mathcal{T}_h\}_{h>0}$ de Ω es una colección de elementos τ de diámetro h_τ, donde $h := \max_{\tau \in \mathcal{T}_h} h_\tau$, tal que:

1.- $\bar{\Omega} = \bigcup_{\tau \in \mathcal{T}_h} \tau$.

2.- *Cada τ es un polígono o poliedro de interior $\overset{\circ}{\tau}$ no vacío.*

3.- $\overset{\circ}{\tau}_i \cap \overset{\circ}{\tau}_j =$ *para todo τ_i, $\tau_j \in \mathcal{T}_h$, $i \neq j$.*

4.- *Si* $\ell := \tau_i \cap \tau_j \neq \emptyset$, *entonces* ℓ *es un lado común, cara común o vértice común de* τ_i *y* τ_j.

Sea ρ_τ el diámetro de la bola más grande contenida en τ. Entonces, se tiene la siguiente definición.

Definición 2.2.1 *Sea* $\{\mathcal{T}_h\}_{h>0}$ *una familia de triangulaciones admisibles de* $\bar{\Omega}$. *se dice que* $\{\mathcal{T}_h\}_{h>0}$ *es una familia regular si existe* $\sigma \geq 1$ *tal que para todo* $h > 0$:

$$\frac{h_\tau}{\rho_\tau} \leq \sigma \quad \forall \tau \in \mathcal{T}_h \,.$$

Es importante darse cuenta que los supuestos de regularidad nos permiten considerar las particiones del dominio Ω como mallas que pueden contener (localmente) elementos muy refinados y de muy disímiles tamaños. En particular, los supuestos de regularidad no se contraponen a los tipos de mallas que surgen de los refinamientos adaptivos, como veremos más adelante.

2.3 Formulación Variacional Mixta

Aquí nos dedicaremos a mostrar algunos aspectos que nos ayuden a entender mejor las nociones de las formulaciones mixtas y su tratamiento para obtener existencia y unicidad de solución. Para ello seguimos el análisis hecho en [44].

Sean X y M dos espacios de Hilbert con normas $||\cdot||_X$ y $||\cdot||_M$, respectivamente. Sean X' y M' sus respectivos espacios duales y denotemos por $||\cdot||'_X$ y $||\cdot||'_M$ a sus normas duales correspondientes. De aquí en adelante, $[\cdot,\cdot]$ denotará el producto de dualidad indicado por el subíndice correspondiente.

Ahora, definamos las formas bilineales continuas

$$a(\cdot,\cdot) : X \times X \to \mathbb{R}\,, \qquad b(\cdot,\cdot) : X \times M \to \mathbb{R}\,,$$

de normas

$$||a|| = \sup_{u,v\in X\,,u,v\neq 0} \frac{a(u,v)}{||u||_X\,||v||_X}\,, \qquad ||b|| = \sup_{v\in X,\tau\in M\,,v\neq 0\,,\tau\neq 0} \frac{b(v,\tau)}{||v||_X\,||\tau||_M}\,.$$

Entonces, se estudia el siguiente problema variacional: *Dado* $f \in X'$ *y* $g \in M'$, *hallar* $(u,\sigma) \in X \times M$ *tal que*

$$a(u,v) + b(v,\sigma) = [f,v]_{X'\times X} \tag{2.13}$$

$$b(u,\tau) = [g,\tau]_{M'\times M}$$

para todo $(v, \tau) \in X \times M$.

Las formas bilineales $a(\cdot,\cdot)$ y $b(\cdot,\cdot)$ tienen los operadores lineales asociados $A : X \to X'$ y $B : X \to M'$, respectivamente, definidos por:

$$[A\,u, v]_{X' \times X} = a(u, v) \qquad \forall\, u, v \in X \; , \tag{2.14}$$

$$[B\,v, \tau]_{M' \times M} = b(v, \tau) \qquad \forall\, v \in X \; , \quad \forall\, \tau \in M \; . \tag{2.15}$$

Además, el operador dual de B, $B' : M \to X'$, se define como:

$$[B'\,\tau, v]_{X' \times X} = [\tau, B\,v]_{M' \times M} = b(v, \tau) \qquad \forall\, v \in X \; , \quad \forall\, \tau \in M \; . \tag{2.16}$$

Con estos operadores, el problema (2.13) se escribe de forma equivalente: *Hallar* $(u, \sigma) \in X \times M$ *tal que*

$$\begin{bmatrix} A & B' \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \sigma \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \; , \tag{2.17}$$

o bien,

$$\begin{aligned} [A\,u, v]_{X' \times X} + [B'\,\sigma, v]_{X' \times X} &= [f, v]_{X' \times X} \\ [B\,u, \tau]_{M' \times M} &= [g, \tau]_{M' \times M} \end{aligned} \tag{2.18}$$

para todo $(v, \tau) \in X \times M$. Con la intención de dar las condiciones necesarias para que el problema (2.17) tenga unicidad de solución, es que para cada $g \in M'$ definimos el subespacio

$$V(g) = \{ v \in X : B\,v = g \} \tag{2.19}$$

y en particular,

$$V = V(0) = \{ v \in X : B\,v = 0 \}$$

o equivalentemente, $V = Ker(B)$. Debido a que B es continuo, se tiene que V es un subespacio cerrado de X. Además, sea $\Pi : X' \to V'$ la inyección canónica definida por $\Pi(f) = f|_V$ para todo $f \in X'$. Entonces, (2.17) tiene el siguiente problema asociado: *Hallar* $u \in V(g)$ *tal que* $\Pi\,A(u) = \Pi(f)$*, esto es,*

$$[A\,u, v]_{X' \times X} = [f, v]_{X' \times X} \tag{2.20}$$

para todo $v \in V$.

Luego, si $(u, \sigma) \in X \times M$ es solución del problema (2.17), entonces $u \in V(g)$ y es solución del problema (2.20) debido a que para todo $v \in V$

$$[B'\,\sigma, v]_{X' \times X} = [B\,v, \sigma]_{M' \times M} = 0 \; .$$

Con esto, podemos dar el resultado que buscabamos.

Teorema 2.3.1 *Para cada $(f,g) \in X' \times M'$ existe una única solución $(u,\sigma) \in X \times M$ del problema (2.17) sí y sólo si se cumplen las siguientes condiciones:*

(i) Existe una constante $\beta > 0$ tal que

$$\inf_{\tau \in M} \sup_{v \in X} \frac{[B\,v,\tau]_{M' \times M}}{||v||_X ||\tau||_M} \geq \beta\,. \tag{2.21}$$

(ii) Para cada $(f,g) \in X' \times M'$ existe una única solución $u \in V(g)$ solución de (2.20).

Demostración: ver Teorema 4.1, Capítulo I, en [44]. ■

Ahora, pasaremos a estudiar la unicidad de solución de formulaciones variacionales de tipo dual-dual, dado que conforman una parte importante de este trabajo.

Sean X_1 , M_1 y M espacios de Hilbert y se define $X = X_1 \times M_1$. Consideremos el operador nolineal $A_1 : X_1 \to X_1'$ y los operadores lineales y acotados $B : X \to M'$, $B : X_1 \to M_1'$ con operadores duales $B' : M \to X'$ y $B_1' : M_1 \to X_1'$, respectivamente.

Utilizando los operadores A_1, B_1 y B_1' se define el operador nolineal $A : X \to X'$ como

$$A(t,\sigma) = \begin{bmatrix} A_1 & B_1' \\ B_1 & 0 \end{bmatrix} \begin{bmatrix} t \\ \sigma \end{bmatrix} \in X' = X_1' \times M_1' \tag{2.22}$$

o de forma equivalente,

$$[A(t,\sigma),(s,\tau)]_{X' \times X} = [A_1(t),s]_{X_1' \times X_1} + [B_1'(\sigma),s]_{X_1' \times X_1} + [B_1(t),\tau]_{M_1' \times M_1} \tag{2.23}$$

para todo $(t,\sigma),(s,\tau) \in X$. Entonces, estamos interesados en el siguiente problema nolineal: *Dado $(F,G) \in X' \times M'$, hallar $((t,\sigma),u) \in X \times M$ tal que*

$$\begin{bmatrix} A & B' \\ B & 0 \end{bmatrix} \begin{bmatrix} (t,\sigma) \\ u \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}, \tag{2.24}$$

esto es,

$$\begin{aligned} [A(t,\sigma),(s,\tau)]_{X' \times X} + [B'(u),(s,\tau)]_{X' \times X} &= [F,(s,\tau)]_{X' \times X} \\ [B(t,\sigma),v]_{M' \times M} &= [G,v]_{M' \times M} \end{aligned} \tag{2.25}$$

para todo $((s,\tau),v) \in X \times M$.

Procediendo de manera análoga, definimos para cada $G \in M'$,

$$V(G) = \{(s,\tau) \in X : B(s,\tau) = G\}$$

y en el caso particular

$$V = V(0) = \{(s,\tau) \in X : B(s,\tau) = 0\}$$

o bien, $V = Ker(B)$. Entonces, debido a que B es acotado, tenemos que V es un subespacio cerrado de X. Ahora, damos la inyección canónica $\Pi : X' \to V'$ definida por $\Pi(F) = F|_V$ para todo $F \in X'$. Luego, (2.24) tiene el siguiente problema asociado: *Hallar* $(t,\sigma) \in V(G)$ *tal que* $\Pi\, A(t,\sigma) = \Pi(F)$ *, esto es*

$$[A(t,\sigma),(s,\tau)]_{X'\times X} = [F,(s,\tau)]_{X'\times X} \tag{2.26}$$

para todo $(s,\tau) \in V$. Así, es claro que si $((t,\sigma),u) \in X \times M$ es una solución de (2.24), entonces $(t,\sigma) \in V(G)$ y es solución de (2.26), debido a que para todo $(s,\tau) \in V$

$$[B'(u),(s,\tau)]_{X'\times X} = [B(s,\tau),u]_{M'\times M} = 0\ .$$

Con esto estamos en condiciones de dar el siguiente teorema.

Teorema 2.3.2 *Para cada* $(F,G) \in X' \times M'$ *existe una única solución* $((t,\sigma),u) \in X \times M$ *de (2.24) sí y sólo si se cumplen las siguientes condiciones:*

(i) Existe una constante $\beta > 0$ *tal que*

$$\inf_{0\neq(s,\tau)\in X}\ \sup_{0\neq v\in M} \frac{[B(s,\tau),v]_{M'\times M}}{||(s,\tau)||_X ||v||_M} \geq \beta$$

(ii) Para cada $(F,G) \in X' \times M'$ *existe una única solución* $(t,\sigma) \in V(G)$ *de (2.26).*

Demostración: ver Teorema 2.1, Sección 2 en [29]. ■

También, es posible obtener resultados similares a los Teoremas 2.3.1 y 2.3.2 en la versión discreta de los problemas (2.17) y (2.24) respectivamente. Para estudiar esto en detalle, ver [16], [29] y [35].

2.4 El Método Integral de Frontera

La idea básica del Método Integral de Frontera (M.I.F.) consiste en la representación de la solución de un Problema de Valor de Contorno (P.V.C.) interior o exterior, cuya ecuación diferencial es lineal y homogénea, mediante potenciales de frontera. Luego se utilizan los datos de frontera para hallar las densidades de dichos potenciales.

Para entender mejor esto, consideremos Ω como un abierto acotado regular de $\mathbb{R}^2$ con frontera Γ seccionalmente de clase C^1. El M.I.F. se basa en el hecho que la solución u del problema

$$-\Delta\, u = 0 \qquad \text{en } \Omega\ , \tag{2.27}$$

$$u = f \qquad \text{en } \Gamma \,,$$

con $f \in C^0(\Gamma)$, puede ser representado mediante la fórmula de Green por

$$u(x) = \int_\Gamma E(x,y)\,\sigma(y)\,ds_y - \int_\Gamma \frac{\partial}{\partial\,\nu_y} E(x,y)\,\lambda(y)\,ds_y \tag{2.28}$$

pata todo $x \in \Omega$, donde

$$E(x,y) = -\frac{1}{2\,\pi}\,\log\,||x-y||$$

es la solución fundamental del operador de Laplace en $\mathbb{R}^2$, ν_y es el vector normal exterior a Ω para los y en Γ, y además, tenemos que

$$\lambda := u|_\Gamma \qquad \text{y} \qquad \sigma := \frac{\partial\, u}{\partial\,\nu}|_\Gamma \,.$$

De (2.28) se puede ver que la naturaleza de la solución u puede ser determinada a través de las integrales (sobre Γ) que la conforman. De hecho, haciendo tender x a Γ desde Ω, usando las condiciones de salto para las integrales en (2.28) y la condición de Dirichlet $u|_\Gamma = f$, se obtiene que σ está relacionada por la ecuación integral, llamada de primera clase,

$$\mathbf{V}\,\sigma(x) = \left(\frac{1}{2}\,\mathbf{I} + \mathbf{K}\right)\,f \quad \text{en } \Gamma \,, \tag{2.29}$$

donde $\mathbf{I}$ es el operador identidad y los operadores $\mathbf{V}$ y $\mathbf{K}$ son el potencial de capa simple y doble, respectivamente y están definidos por:

$$\mathbf{V}\,\sigma(x) = \int_\Gamma E(x,y)\,\sigma(y)\,ds_y \quad \forall x \in \Gamma$$

y

$$\mathbf{K}\,f(x) = \int_\Gamma \frac{\partial}{\partial\,\nu_y} E(x,y)\,f(y)\,ds_y \quad \forall x \in \Gamma \,.$$

Cabe señalar que a aprtir de (2.28), también es posible obtener una segunda ecuación integral (llamada de segunda clase) realizando el mismo procedimiento, pero sobre la derivada normal de (2.28), esto es,

$$\left(\frac{1}{2}\,\mathbf{I} - \mathbf{K}'\right)\,\sigma = \mathbf{W}\,f \quad \forall x \in \Gamma \tag{2.30}$$

donde $\mathbf{K}'$ y $\mathbf{W}$ son el operador adjunto al potencial de capa doble y el operador integral de frontera hipersingular, respectivamente, y están definidos por:

$$\mathbf{K}'\,\sigma(x) = \int_\Gamma \frac{\partial}{\partial\,\nu_x} E(x,y)\,\sigma(y)\,ds_y \quad \forall x \in \Gamma$$

y

$$\mathbf{W}\, f(x) = -\int_\Gamma \left\{ \frac{\partial}{\partial\, \nu_x} \frac{\partial}{\partial\, \nu_y} E(x,y) \right\} f(y)\, ds_y \quad \forall x \in \Gamma \; .$$

Para todos estos operadores puede demostrarse que (ver [24]):

$$\mathbf{V} : H^{-1/2+\gamma}(\Gamma) \longrightarrow H^{1/2+\gamma}(\Gamma)$$

$$\mathbf{K} : H^{1/2+\gamma}(\Gamma) \longrightarrow H^{1/2+\gamma}(\Gamma)$$

$$\mathbf{K}' : H^{-1/2+\gamma}(\Gamma) \longrightarrow H^{-1/2+\gamma}(\Gamma)$$

y

$$\mathbf{W} : H^{1/2+\gamma}(\Gamma) \longrightarrow H^{-1/2+\gamma}(\Gamma)$$

son operadores continuos para todo $\gamma \in (-\frac{1}{2}, \frac{1}{2})$ y que $\mathbf{V}$ y $\mathbf{W}$ son operadores simétricos. Aún más, el potencial de capa simple $\mathbf{V}$ es coercivo (elíptico) en $H_0^{-1/2}(\Gamma)$, vale decir, si definimos por $\langle \cdot, \cdot \rangle$ al producto de dualidad entre $H^{-1/2}(\Gamma)$ y $H^{1/2}(\Gamma)$ definido por el producto interior en $L^2(\Gamma)$, tenemos que existen constantes $C_1, C_2 > 0$ tales que

$$\langle \sigma, \mathbf{V}\, \sigma \rangle \geq C_1 \, ||\sigma||^2_{H^{-1/2}(\Gamma)} \quad \forall\, \sigma \in H_0^{-1/2}(\Gamma) \; ,$$

con

$$H_0^{-1/2}(\Gamma) := \left\{ \sigma \in H^{-1/2}(\Gamma) : \langle 1, \sigma \rangle = 0 \right\} \; ,$$

y

$$\langle \mathbf{W}\, \lambda, \lambda \rangle \geq C_2 \, ||\lambda||^2_{H^{1/2}(\Gamma)} \qquad \forall \lambda \in H^{1/2}(\Gamma)/\,\mathbb{R} \; .$$

También, es bueno señalar que $\mathbf{K}'$ es el adjunto de $\mathbf{K}$ con respecto a $\langle \cdot, \cdot \rangle$.

Por otro lado, notemos que la ecuación integral (2.29) es equivalente al siguiente problema variacional: *Hallar* $\delta \in H^{-1/2}(\Gamma)$ *tal que*

$$A(\delta, \tau) = F(\tau) \qquad \forall \tau \in H^{-1/2}(\Gamma) \; ,$$

donde

$$A(\delta, \tau) := \langle \tau, \mathbf{V}\, \delta \rangle \qquad \forall \delta, \tau \in H^{-1/2}(\Gamma)$$

y

$$F(\tau) := \langle \tau, (\frac{1}{2}\mathbf{I} + \mathbf{K})\, f \rangle \qquad \forall \tau \in H^{-1/2}(\Gamma) \; .$$

En general, tenemos el siguiente problema variacional: *Hallar* $\sigma \in V$ *tal que*

$$A(\sigma, \tau) = F(\tau) \qquad \forall\, \tau \in V \; , \tag{2.31}$$

donde V es un espacio de Hilbert, $A : V \times V \to \mathbb{R}$ es una forma bilineal continua y F es un funcional lineal acotado sobre V.

Ahora bien, por lo habitual, las formas bilineales que surgen de las formulaciones variacionales de ecuaciones integrales de frontera no son fuertemente elípticas, pero si satisfacen la desigualdad de Gårding. Esto es, existe $\alpha > 0$ tal que

$$A(\tau, \tau) \geq \alpha \, ||\tau||_V - B(\tau, \tau) \qquad \forall \, \tau \in V \, ,$$

donde B es una forma bilineal compacta sobre V. Por lo tanto, no es posible aplicar el Teorema de Lax-Milgram, pero en su defecto, si podemos utilizar la *alternativa de Fredholm*, lo cual significa que la existencia de solución en (2.31) implica unicidad y recíprocamente.

Resultados análogos para la versión discreta de (2.31) son también obtenibles. En efecto, sea $\{V_h\}_{h>0}$ una familia de subespacios de dimensión finita de V tal que, cuando $h \to 0$,

$$\inf_{\tau_h \in V_h} ||\tau - \tau_h||_V \longrightarrow 0 \qquad \forall \, \tau \in V \, . \tag{2.32}$$

Luego, se establece el *esquema de Galerkin*: *Hallar* $\sigma_h \in V_h$ *tal que*

$$A(\sigma_h, \tau_h) = F(\tau_h) \qquad \forall \, \tau_h \in V_h \, . \tag{2.33}$$

Entonces, si se dispone de (2.32), el operador A satisface la desigualdad de Gårding y (2.31) admite una única solución, se tiene que existen $\gamma, h_0 > 0$, tal que

$$\sup_{0 \neq \tau_h \in V_h} \frac{|A(\sigma_h, \tau_h)|}{||\tau_h||_V} \geq \gamma \, ||\sigma_h||_V \qquad \forall \, \sigma_h \in V_h \, , \quad \forall \, h \leq h_0 \, , \tag{2.34}$$

llamada *condición de Babuska-Brezzi asintótica*. Además, existe una única solución $\sigma_h \in V_h$ de (2.33) y se tiene la siguiente estimación de Cea,

$$||\sigma - \sigma_h||_V \leq C \inf_{\tau_h \in V_h} ||\sigma - \tau_h||_V \tag{2.35}$$

para todo $h \leq h_0$, donde $C > 0$ es una constante independiente de h y σ.

2.5 Análisis A-Posteriori

Desde el comienzo de las simulaciones computacionales de los sucesos físicos, la presencia de errores numéricos en los cálculos han sido uno de los principales motivos de estudio. El error numérico es intrínsico a tales simulaciones pues el proceso de discretización que transforma un modelo continuo en uno discreto y por ende (en teoría), posible de representar en un computador, no es capaz de incluir toda la información expresada por las ecuaciones diferenciales o integrales. Luego, el problema es obtener estimaciones *confiables* de las soluciones numéricas calculadas. Estimaciones de error a-priori dadas por los análisis de error estandar para métodos de elementos finitos, frecuentemente son insuficientes debido a que sólo dan información de acuerdo al comportamiento asintótico del error y requieren condiciones de regularidad de la solución que no se satisfacen en presencia de singularidades.

En efecto, otro hecho importante es el dominio donde uno esté trabajando, por ejemplo, una perdida frecuente de exactitud en la aproximación numérica es causada por singularidades que surgen, por ejemplo, por esquinas reentrantes.

Una solución inmediata sería refinar cerca de las regiones críticas, es decir, tener más elementos y más pequeños donde la solución es menos regular. Entonces, el problema medular es cómo identificar aquellas regiones y cómo obtener un buen equilibrio entre las regiones refinadas y las más gruesas, de modo que la precisión global de la solución sea óptima.

Todas estas situaciones muestran la necesidad de un estimador del error que se pueda obtener *a-posteriori*, es decir, a partir de la solución numérica ya calculada y del dato dado del problema en cuestión.

Para entender mejor esto, veamos el siguiente problema. Sea $\Omega \subset \mathbb{R}^2$ un dominio acotado con frontera Lipschitz $\partial\Omega = \bar{\Gamma}_N \bigcup \bar{\Gamma}_D$. Dado $f \in L^2(\Omega)$ y $g \in L^2(\Gamma_N)$, consideremos el problema de valor de frontera: *Hallar u tal que*

$$
\begin{aligned}
-\Delta u + u = f \qquad & \text{en } \Omega \\
u = 0 \qquad & \text{en } \Gamma_D \\
\frac{\partial u}{\partial \nu} = g \qquad & \text{en } \Gamma_N \,.
\end{aligned}
\tag{2.36}
$$

La formulación variacional de este problema es: *Hallar $u \in V$ tal que*

$$
A(u,v) = L(v) \qquad \forall v \in V \,, \tag{2.37}
$$

donde $V := \{v \in H^1(\Omega) : v = 0 \quad \text{en } \Gamma_D\}$,

$$
A(u,v) = \int_\Omega (\nabla u \cdot \nabla v + u\,v)\,dx \quad \text{y} \quad L(v) = \int_\Omega f\,v\,dx + \int_{\Gamma_N} g\,v\,ds \,.
$$

Sea V_h un subespacio de elementos finitos de V. La aproximación por elementos finitos de este problema es: *Hallar $u_h \in V_h$ tal que*

$$
A(u_h, v_h) = L(v_h) \qquad \forall v_h \in V_h \,. \tag{2.38}
$$

El error $e = u - u_h$ satisface

$$A(e,v) = A(u,v) - A(u_h,v) = L(v) - A(u_h,v) \quad \forall v \in V \ . \tag{2.39}$$

Además, se tiene la condición de ortogonalidad para el error en la proyección de Galerkin, esto es,

$$A(e,v_h) = 0 \qquad \forall v_h \in V \ . \tag{2.40}$$

Una ecuación residual equivalente a (2.38) se puede obtener integrando por partes sobre cada elemento, así tenemos que

$$\int_T (\nabla e \cdot \nabla v + e\, v)\; dx = \int_T r\, v\, dx + \int_{\partial T} v\; (\nabla u \cdot \nu_T - \nabla u_h \cdot \nu_T)\; ds \tag{2.41}$$

donde T es un elemento de la triangulación $\{\mathcal{T}_h\}_{h>0}$ de Ω, r es el *residuo* definido por

$$r = f + \Delta\, u_h - u_h$$

y ν_T es el vector normal exterior sobre la frontera ∂T. Bajo condiciones convenientes, se puede mostrar que la solución de (2.41) esta acotada por

$$||e||_T \leq C_1\, ||r||_{L^2(T)} + C_2\, ||\nabla e \cdot \nu_T||_{L^2(\partial T)} \tag{2.42}$$

donde C_1 y C_2 dependen del tamaño del elemento h_T y de otros parámetros de la malla. A este tipo de estimación se le denomina de tipo *explícito*.

Ahora bien, la presencia de las constantes C_1 y C_2 en la estimación del error a-posteriori explícito (2.42) nos lleva a considerar el tratar de resolver un problema de valor de frontera local aproximado para el error de la forma

$$\int_T (\nabla \varphi_{h,T}\, \nabla v + \varphi_{h,T}\, v)\; dx = \int_T r\, v\, dx + \int_{\partial T} v\; (g_T - \nabla u_h \cdot \nu_T)\; ds \ , \tag{2.43}$$

donde g_T es una aproximación al flujo en la frontera. La solución $\varphi_{h,T}$ se puede usar de la siguiente manera:

$$||\varphi_{h,T}||^2 = \int_T \left(|\nabla \varphi_{h,T}|^2 + \varphi_{h,T}^2 \right)\; dx \tag{2.44}$$

y así dar una medida del error contenida en la aproximación asociada al elemento T. De esta aproximación surgen las siguientes consideraciones:

-) El espacio de dimensión finita que contiene al error debe ser aproximado por un subespacio finito dimensional apropiado.

-) El flujo en la frontera $\nabla u \cdot \nu_T$ debe ser aproximado de manera efectiva.

A este tipo de estudio (resolución de problemas locales sobre cada elemento) se le denomina *método residual implícito del error*, y a la estimación del error $||\varphi_{h,T}||$ se le denomina de tipo implícito.

Por otro lado, si η_T es un estimador local del error en el elemento T, donde T es un elemento de una triangulación $\{\mathcal{T}_h\}_{h>0}$, entonces la estimación global del error η usualmente se considera como

$$\eta = \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\}^{1/2} , \tag{2.45}$$

al cual, usualmente se le exije, para considerarlo como un estimador del error completo, que existan constantes C_1, $C_2 > 0$ tales que

$$C_1 \, \eta \leq ||e|| \leq C_2 \, \eta , \tag{2.46}$$

donde $||e||$ es la magnitud del error global. entonces, η tiende a cero en la misma razón que el error verdadero. Cuando se logra obtener tanto la minoración como la mayoración, uno habla de eficiencia y confiabilidad del estimador.

Capítulo 3

An Exterior Linear Problem

The main purpose of the present chapter is to derive explicit and implicit reliable a-posteriori error estimates for linear exterior problems in the plane, whose variational formulations are obtained by the combination of dual-mixed FEM with DtN mappings. As a model, we consider the exterior transmission problem from potential theory studied in [51] (see also [22], [42] and [39]). In addition, we use the DtN mapping from [38] and [47], which is given in terms of the hypersingular boundary integral operator for the Laplacian. The rest of the chapter is organized as follows. In Section 2 we introduce the model problem, derive the associated mixed variational formulation, and prove the corresponding solvability and continuous dependence results. Actually, this is done through an equivalent formulation arising from a direct sum decomposition of one of the unknowns. In Section 3 we use Raviart-Thomas spaces to define the discrete scheme, show that it is stable and uniquely solvable, obtain the Cea error estimate, and state the associated rate of convergence. Then, a reliable a-posteriori error estimate of explicit residual type is derived in Section 4. Our analysis here follows very closely the techniques from [22] and [39]. In Section 5 we apply a Bank-Weiser type a-posteriori error analysis and provide a reliable estimate that depends on the solution of local problems. An explicit estimate, based on bounds of these local solutions and a suitable averaging technique, is also deduced in this section. Finally, several numerical experiments illustrating the efficiency of these estimators for the adaptive computation of the discrete solutions are given in Section 6.

In what follows, the symbols C, $\tilde{C}$, and $\bar{C}$ are used to denote generic positive constants with different values at different places.

3.1 The model problem

Let Ω_0 be a bounded and simply connected domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary Γ_0. Also, let Ω_1 be the annular domain bounded by Γ_0 and another Lipschitz-continuous closed curve Γ_1 whose interior region contains $\overline{\Omega}_0$. Then, given $f_1 \in L^2(\Omega_1)$, $g \in H^{1/2}(\Gamma_0)$ and a matrix valued function $\boldsymbol{\kappa}_1 \in C(\overline{\Omega}_1)$, we consider the exterior transmission problem: *Find* $u_1 \in H^1(\Omega_1)$ *and* $u_2 \in H^1_{loc}(\mathbb{R}^2 - \overline{\Omega}_0 \cup \overline{\Omega}_1)$ *such that*

$$\begin{gathered} u_1 = g \quad \text{on} \quad \Gamma_0\,, \qquad -\operatorname{div}(\boldsymbol{\kappa}_1 \nabla u_1) \,=\, f_1 \quad \text{in} \quad \Omega_1\,, \\ u_1 = u_2 \quad \text{and} \quad (\boldsymbol{\kappa}_1 \nabla u_1)\cdot \mathbf{n} \,=\, \frac{\partial u_2}{\partial \mathbf{n}} \quad \text{on} \quad \Gamma_1\,, \\ -\,\Delta u_2 = 0 \quad \text{in} \quad \mathbb{R}^2 - \overline{\Omega}_0 \cup \overline{\Omega}_1\,, \quad u_2(x) = O(1) \quad \text{as} \quad ||x|| \to +\infty\,, \end{gathered} \tag{3.1}$$

where $\mathbf{n} := (n_1, n_2)^{\mathbf{T}}$ denotes the unit outward normal to Γ_1.

We assume that $\boldsymbol{\kappa}_1$ induces a strongly elliptic differential operator, that is there exists $\alpha_1 > 0$ such that

$$\alpha_1\, ||\boldsymbol{\xi}||^2 \,\leq\, (\boldsymbol{\kappa}_1\, \boldsymbol{\xi})\cdot \boldsymbol{\xi} \qquad \forall\, \boldsymbol{\xi} \in \mathbb{R}^2\,. \tag{3.2}$$

We now introduce a sufficiently large circle Γ with center at the origin such that its interior region contains $\overline{\Omega}_0 \cup \overline{\Omega}_1$. Then we let Ω_2 be the annular region bounded by Γ_1 and Γ, put $\Omega := \Omega_1 \cup \Gamma_1 \cup \Omega_2$, and define the global unknown $u := \begin{cases} u_1 & \text{in} \quad \Omega_1 \\ u_2 & \text{in} \quad \Omega_2 \end{cases}$, the data $f := \begin{cases} f_1 & \text{in} \quad \Omega_1 \\ 0 & \text{in} \quad \Omega_2 \end{cases}$, and the flux variable $\boldsymbol{\sigma} := \boldsymbol{\kappa}\, \nabla u$ in Ω, where $\boldsymbol{\kappa} := \begin{cases} \boldsymbol{\kappa}_1 & \text{in} \quad \Omega_1 \\ \mathbf{I} & \text{in} \quad \Omega_2 \end{cases}$, and $\mathbf{I}$ denotes the identity matrix.

Next, we apply the boundary integral equation method in the region exterior to the circle Γ, and obtain the following Dirichlet-to-Neumann mapping (see [38], [47])

$$\boldsymbol{\sigma}\cdot\boldsymbol{\nu} \,=\, -\,2\,\mathbf{W}(\lambda) \quad \text{on} \quad \Gamma\,, \tag{3.3}$$

where $\boldsymbol{\nu}$ is the unit outward normal to $\partial\Omega := \Gamma_0 \cup \Gamma$, $\lambda := u|_\Gamma$ is a further unknown, and $\mathbf{W}$ is the hypersingular boundary integral operator.

We remark that if Γ is chosen as a polygonal boundary instead of a circle, then we would need all the boundary integral operators to express $\boldsymbol{\sigma}\cdot\boldsymbol{\nu}$ in terms of λ. The advantage of using a circle in this case lies in the simplicity of the resulting Dirichlet-to-Neumann mapping (3.3).

We recall here that $\mathbf{W}$ is the linear operator defined by

$$\mathbf{W}\mu(x) := -\frac{\partial}{\partial \boldsymbol{\nu}(x)} \int_\Gamma \frac{\partial}{\partial \boldsymbol{\nu}(y)}\, E(x,y)\, \mu(y)\, ds_y \quad \forall\, x \in \Gamma\,, \quad \forall\, \mu \in H^{1/2}(\Gamma)\,,$$

where $\boldsymbol{\nu}(z)$ stands for the unit outward normal at $z \in \Gamma$, and $E(x,y) := -\frac{1}{2\pi} \log ||x - y||$ is the fundamental solution of the two-dimensional Laplacian. It is well known that $\mathbf{W}$ maps continuously $H^{1/2+\delta}(\Gamma)$ into $H^{-1/2+\delta}(\Gamma)$ for all $\delta \in [-1/2, 1/2]$, and that there exists $\alpha_2 > 0$ such that

$$\langle \mathbf{W}(\mu), \mu \rangle_\Gamma \;\geq\; \alpha_2 \, ||\mu||^2_{H^{1/2}(\Gamma)} \quad \forall \mu \in H^{1/2}_0(\Gamma)\,, \tag{3.4}$$

where

$$H^{1/2}_0(\Gamma) := \{ \mu \in H^{1/2}(\Gamma) : \quad \langle 1, \mu \rangle_\Gamma = 0 \}\,.$$

In addition, $\mathbf{W}(1) = 0$ and $\mathbf{W}$ is symmetric in the sense that $\langle \mathbf{W}(\mu), \rho \rangle_\Gamma = \langle \mathbf{W}(\rho), \mu \rangle_\Gamma$ for all $\mu,\ \rho \in H^{1/2}(\Gamma)$.

Hereafter, $\langle \cdot, \cdot \rangle_\Gamma$ (resp. $\langle \cdot, \cdot \rangle_{\Gamma_0}$) denotes the duality pairing of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ (resp. $H^{-1/2}(\Gamma_0)$ and $H^{1/2}(\Gamma_0)$) with respect to the $L^2(\Gamma)$ (resp. $L^2(\Gamma_0)$) inner product.

In this way, the model problem (3.1) is reformulated as a boundary value problem in $\overline{\Omega}$ with the nonlocal boundary condition (3.3). Hence, by performing the usual integration by parts procedure in Ω, we find that the corresponding mixed variational formulation reads: *Find* $((\boldsymbol{\sigma}, \lambda), u) \in H \times Q$ *such that*

$$\begin{aligned} A((\boldsymbol{\sigma}, \lambda), (\boldsymbol{\tau}, \mu)) + B((\boldsymbol{\tau}, \mu), u) \;&=\; \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0}\,, \\ B((\boldsymbol{\sigma}, \lambda), v) \;&=\; -\int_\Omega f v \, dx\,, \end{aligned} \tag{3.5}$$

for all $((\boldsymbol{\tau}, \mu), v) \in H \times Q$, where $H := H(\mathrm{div}; \Omega) \times H^{1/2}(\Gamma)$, $Q := L^2(\Omega)$, and the bilinear forms $A : H \times H \to \mathbb{R}$ and $B : H \times Q \to \mathbb{R}$ are defined as follows:

$$A((\boldsymbol{\sigma}, \lambda), (\boldsymbol{\tau}, \mu)) := \int_\Omega (\boldsymbol{\kappa}^{-1} \boldsymbol{\sigma}) \cdot \boldsymbol{\tau} \, dx + 2 \langle \mathbf{W} \lambda, \mu \rangle_\Gamma - \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \lambda \rangle_\Gamma + \langle \boldsymbol{\sigma} \cdot \boldsymbol{\nu}, \mu \rangle_\Gamma\,, \tag{3.6}$$

$$B((\boldsymbol{\tau}, \mu), v) := \int_\Omega v \, \mathrm{div}\, \boldsymbol{\tau} \, dx\,, \tag{3.7}$$

for all $(\boldsymbol{\sigma}, \lambda)$, $(\boldsymbol{\tau}, \mu) \in H$, for all $v \in Q$.

At this point we recall that $H(\mathrm{div}; \Omega)$ is the space of functions $\boldsymbol{\tau} \in [L^2(\Omega)]^2$ such that $\mathrm{div}\, \boldsymbol{\tau} \in L^2(\Omega)$, which, provided with the inner product

$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_{H(\mathrm{div};\Omega)} \;:=\; \int_\Omega \mathrm{div}\, \boldsymbol{\sigma} \, \mathrm{div}\, \boldsymbol{\tau} \, dx \;+\; \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx\,,$$

becomes a Hilbert space. In addition, for all $\boldsymbol{\tau} \in H(\mathrm{div}; \Omega)$, $\boldsymbol{\tau} \cdot \boldsymbol{\nu}|_\Gamma \in H^{-1/2}(\Gamma)$, $\boldsymbol{\tau} \cdot \boldsymbol{\nu}|_{\Gamma_0} \in H^{-1/2}(\Gamma_0)$, and both $||\boldsymbol{\tau} \cdot \boldsymbol{\nu}||_{H^{-1/2}(\Gamma)}$ and $||\boldsymbol{\tau} \cdot \boldsymbol{\nu}||_{H^{-1/2}(\Gamma_0)}$ are bounded above by $||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}$.

On the other hand, each $\mu \in H^{1/2}(\Gamma)$ can be uniquely decomposed as $\mu := \tilde{\mu} + q$, with $\tilde{\mu} := \left(\mu - \frac{1}{|\Gamma|}\int_\Gamma \mu\, ds\right) \in H_0^{1/2}(\Gamma)$ and $q := \frac{1}{|\Gamma|}\int_\Gamma \mu\, ds \in \mathbb{R}$, which states that $H^{1/2}(\Gamma) = H_0^{1/2}(\Gamma) \oplus \mathbb{R}$. Further, it is easy to see that $||\mu||^2_{H^{1/2}(\Gamma)} = ||\tilde{\mu}||^2_{H^{1/2}(\Gamma)} + |\Gamma|\,|q|^2$, and hence $||\mu||_{H^{1/2}(\Gamma)}$ and $||(\tilde{\mu}, q)||_{H^{1/2}(\Gamma)\times\mathbb{R}} := ||\tilde{\mu}||_{H^{1/2}(\Gamma)} + |q|$ are equivalent.

Then we write $\lambda = \tilde{\lambda} + p$, with $\tilde{\lambda} \in H_0^{1/2}(\Gamma)$, $p \in \mathbb{R}$, and consider the alternative formulation: *Find* $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ *such that*

$$\begin{aligned} A((\boldsymbol{\sigma}, \tilde{\lambda}), (\boldsymbol{\tau}, \tilde{\mu})) + \tilde{B}((\boldsymbol{\tau}, \tilde{\mu}), (u, p)) &= \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0}\,, \\ \tilde{B}((\boldsymbol{\sigma}, \tilde{\lambda}), (v, q)) &= -\int_\Omega f v\, dx\,, \end{aligned} \tag{3.8}$$

for all $((\boldsymbol{\tau}, \tilde{\mu}), (v, q)) \in \tilde{H} \times \tilde{Q}$, where $\tilde{H} := H(\mathrm{div};\Omega) \times H_0^{1/2}(\Gamma)$, $\tilde{Q} := L^2(\Omega) \times \mathbb{R}$, and the bilinear form $\tilde{B} : \tilde{H} \times \tilde{Q} \to \mathbb{R}$ is defined as

$$\tilde{B}((\boldsymbol{\tau}, \tilde{\mu}), (v, q)) := \int_\Omega v \,\mathrm{div}\, \boldsymbol{\tau}\, dx \;-\; q\, \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma\,. \tag{3.9}$$

Then we have the following result.

Teorema 3.1.1 *Problems* (3.5) *and* (3.8) *are equivalent. More precisely:*

1. *If* $((\boldsymbol{\sigma}, \lambda), u) \in H \times Q$ *is a solution of* (3.5), *where* $\lambda := \tilde{\lambda} + p$, *with* $\tilde{\lambda} \in H_0^{1/2}(\Gamma)$ *and* $p \in \mathbb{R}$, *then* $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ *is a solution of* (3.8).

2. *If* $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ *is a solution of* (3.8), *then* $((\boldsymbol{\sigma}, \lambda), u) \in H \times Q$ *is a solution of* (3.5) *with* $\lambda := \tilde{\lambda} + p$.

Proof: Let $((\boldsymbol{\sigma}, \lambda), u) \in H \times Q$ be a solution of (3.5), where $\lambda := \tilde{\lambda} + p$, with $\tilde{\lambda} \in H_0^{1/2}(\Gamma)$ and $p \in \mathbb{R}$, and consider $((\boldsymbol{\tau}, \tilde{\mu}), (v, q)) \in \tilde{H} \times \tilde{Q}$. Since $\mathbf{W}(p) = 0$, it follows that

$$\begin{aligned} &A((\boldsymbol{\sigma}, \tilde{\lambda}), (\boldsymbol{\tau}, \tilde{\mu})) \;+\; \tilde{B}((\boldsymbol{\tau}, \tilde{\mu}), (u, p)) \\ &=\; A((\boldsymbol{\sigma}, \lambda), (\boldsymbol{\tau}, \tilde{\mu})) \;+\; B((\boldsymbol{\tau}, \tilde{\mu}), u) \;=\; \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0}\,. \end{aligned} \tag{3.10}$$

Now, taking $\mu = 1$ and $\boldsymbol{\tau} = 0$ in the first equation of (3.5), and using the symmetry of $\mathbf{W}$ and the fact that $\mathbf{W}(1) = 0$, we find that $\langle \boldsymbol{\sigma} \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = 0$, and hence

$$\tilde{B}((\boldsymbol{\sigma}, \tilde{\lambda}), (v, q)) = B((\boldsymbol{\sigma}, \tilde{\lambda}), v) = B((\boldsymbol{\sigma}, \lambda), v) = -\int_\Omega f v\, dx\,.$$

This equation and (3.10) prove that $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ is a solution of (3.8).

Conversely, let $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ be a solution of (3.8), and define $\lambda := \tilde{\lambda} + p$. Taking $v = 0$ and $q = 1$ in the second equation of (3.8), we deduce that $\langle \boldsymbol{\sigma} \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = 0$.

Then we consider $((\boldsymbol{\tau},\mu),v) \in H \times Q$, such that $\mu := \tilde{\mu} + q$, with $\tilde{\mu} \in H_0^{1/2}(\Gamma)$ and $q \in \mathbb{R}$, and observe that

$$\begin{aligned} & A((\boldsymbol{\sigma},\lambda),(\boldsymbol{\tau},\mu)) \;+\; B((\boldsymbol{\tau},\mu),u) \\ = \;& A((\boldsymbol{\sigma},\tilde{\lambda}),(\boldsymbol{\tau},\tilde{\mu})) \;+\; \tilde{B}((\boldsymbol{\tau},\tilde{\mu}),(u,p)) \;=\; \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}\,. \end{aligned} \tag{3.11}$$

Also, according to the second equation in (3.8), we find that

$$B((\boldsymbol{\sigma},\lambda),v) \;=\; \tilde{B}((\boldsymbol{\sigma},\tilde{\lambda}),(v,0)) \;=\; -\int_\Omega f\,v\,dx\,,$$

which, together with (3.11), shows that $((\boldsymbol{\sigma},\lambda),u) \in H \times Q$ is a solution of (3.5). ■

In virtue of Theorem 3.1.1, from now on we concentrate on the equivalent problem (3.8). The corresponding continuous and discrete analyses are based on the classical Babuška-Brezzi theory.

At this point we remark, which is easy to prove, that the bilinear forms A, B, and $\tilde{B}$ are all bounded.

We end this section with the following theorem providing the unique solvability and the continuous dependence result for the mixed variational formulation (3.8) (and hence also for (3.5)).

Teorema 3.1.2 *There exists a unique* $((\boldsymbol{\sigma},\tilde{\lambda}),(u,p)) \in \tilde{H} \times \tilde{Q}$ *solution of* (3.8). *Moreover, there exists* $C > 0$, *independent of the solution, such that*

$$||((\boldsymbol{\sigma},\tilde{\lambda}),(u,p))||_{\tilde{H}\times\tilde{Q}} \;\leq\; C\,\left\{\,||f||_{L^2(\Omega)} \;+\; ||g||_{H^{1/2}(\Gamma_0)}\,\right\}\,.$$

Proof: We first prove the continuous inf-sup condition for $\tilde{B}$. Thus, given $(v,q) \in \tilde{Q} := L^2(\Omega) \times \mathbb{R}$, we let $z \in H^1(\Omega)$ be the weak solution of the mixed boundary value problem:

$$-\Delta z = v \quad \text{in} \quad \Omega\,, \qquad z = 0 \quad \text{on} \quad \Gamma_0\,, \qquad \frac{\partial z}{\partial \boldsymbol{\nu}} = q \quad \text{on} \quad \Gamma\,,$$

for which one can easily show that $||z||_{H^1(\Omega)} \;\leq\; C\,\{||v||_{L^2(\Omega)} + |q|\}$. Then we set $\boldsymbol{\tau}_0 := -\nabla z$ and observe that $\operatorname{div}\boldsymbol{\tau}_0 = v$ in Ω, $\boldsymbol{\tau}_0\cdot\boldsymbol{\nu} = -q$ on Γ, and $||\boldsymbol{\tau}_0||_{H(\mathrm{div};\Omega)} \leq \tilde{C}\,\{||v||_{L^2(\Omega)} + |q|\}$. It follows that

$$\sup_{\substack{(\boldsymbol{\tau},\tilde{\mu})\in\tilde{H}\\(\boldsymbol{\tau},\tilde{\mu})\neq 0}} \frac{\tilde{B}((\boldsymbol{\tau},\tilde{\mu}),(v,q))}{||(\boldsymbol{\tau},\tilde{\mu})||_{\tilde{H}}} \;\geq\; \frac{\tilde{B}((\boldsymbol{\tau}_0,0),(v,q))}{||\boldsymbol{\tau}_0||_{H(\mathrm{div};\Omega)}} \;=\; \frac{||v||^2_{L^2(\Omega)} + |\Gamma|\,|q|^2}{||\boldsymbol{\tau}_0||_{H(\mathrm{div};\Omega)}} \;\geq\; \beta\,||(v,q)||_{\tilde{Q}}\,,$$

where β depends on $|\Gamma|$ and $\tilde{C}$.

We now let $\tilde{V}$ be the kernel of the operator induced by the bilinear form $\tilde{B}$, that is

$$\tilde{V} := \{ (\boldsymbol{\tau}, \tilde{\mu}) \in \tilde{H} : \quad B((\boldsymbol{\tau}, \tilde{\mu}), (v, q)) = 0 \quad \forall (v, q) \in \tilde{H} \},$$

which yields

$$\tilde{V} = \{ (\boldsymbol{\tau}, \tilde{\mu}) \in H(\mathrm{div}; \Omega) \times H_0^{1/2}(\Gamma) : \quad \mathrm{div}\, \boldsymbol{\tau} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = 0 \}.$$

It follows, using (3.6), (3.2), and (3.4), that A is strongly coercive on $\tilde{V}$, that is, for all $(\boldsymbol{\tau}, \tilde{\mu}) \in \tilde{V}$ it holds

$$A((\boldsymbol{\tau}, \tilde{\mu}), (\boldsymbol{\tau}, \tilde{\mu})) \,=\, \int_\Omega (\boldsymbol{\kappa}^{-1} \boldsymbol{\tau}) \cdot \boldsymbol{\tau}\, dx \,+\, 2\, \langle \mathbf{W}(\tilde{\mu}), \tilde{\mu} \rangle_\Gamma \,\geq\, \alpha\, ||(\boldsymbol{\tau}, \tilde{\mu})||^2_{H(\mathrm{div};\Omega) \times H^{1/2}(\Gamma)},$$

where α depends on α_1 and α_2.

Finally, a straightforward application of the abstract Theorem 1.1 in chapter II of [16] completes the proof. ■

3.2 The discrete scheme

Hereafter we assume, for simplicity, that Γ_0 and Γ_1 are polygonal boundaries. In order to discretize the circle Γ, we proceed similarly as in [40]. This means that given $n \in \mathbb{N}$, we let $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a uniform partition of $[0, 2\pi]$ with $t_{j+1} - t_j = \tilde{h} = \frac{2\pi}{n}$ for $j \in \{0, 1, ..., n-1\}$. In addition, we let $\mathbf{z} : [0, 2\pi] \to \Gamma$ be the parametrization of the circle Γ given by $\mathbf{z}(t) := r\,(\cos(t), \sin(t))^T$ for all $t \in [0, 2\pi]$. We denote by $\Omega_{\tilde{h}}$ the annular domain bounded by Γ_0 and the polygonal line $\Gamma_{\tilde{h}}$ whose vertices are $\{\mathbf{z}(t_1), \mathbf{z}(t_2), ..., \mathbf{z}(t_n)\}$.

Then we let $\mathcal{T}_{\tilde{h}}$ be a regular triangulation of $\Omega_{\tilde{h}}$ by triangles T of diameter h_T such that $h := \sup_{T \in \mathcal{T}_{\tilde{h}}} h_T$. We assume that for each $T \in \mathcal{T}_{\tilde{h}}$, either $T \subseteq \overline{\Omega}_1$ or $T \subseteq \overline{\Omega}_2$. Then, we replace each triangle $T \in \mathcal{T}_{\tilde{h}}$ with one side along $\Gamma_{\tilde{h}}$, by the corresponding curved triangle with one side along Γ. In this way, we obtain from $\mathcal{T}_{\tilde{h}}$ a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ made up of straight and curved triangles.

Next, we consider the canonical triangle with vertices $\hat{P}_1 = (0,0)^T$, $\hat{P}_2 = (1,0)^T$ and $\hat{P}_3 = (0,1)^T$ as a reference triangle $\hat{T}$, and introduce a family of bijective mappings $\{F_T\}_{T \in \mathcal{T}_h}$, such that $F_T(\hat{T}) = T$. In particular, if T is a straight triangle of $\mathcal{T}_h$, then F_T is the affine mapping defined by $F_T(\hat{x}) = B_T\, \hat{x} + b_T$, where B_T, a square matrix of order 2, and $b_T \in \mathbb{R}^2$ depend on the vertices of T.

On the other hand, if T is a curved triangle with vertices P_1, P_2 and P_3, such that $P_2 = \mathbf{z}(t_{j-1}) \in \Gamma$ and $P_3 = \mathbf{z}(t_j) \in \Gamma$, then $F_T(\hat{x}) = B_T\, \hat{x} + b_T + G_T(\hat{x})$ for all $\hat{x} := (\hat{x}_1, \hat{x}_2) \in \hat{T}$, where

$$G_T(\hat{x}) = \frac{\hat{x}_1}{1-\hat{x}_2}\left\{\mathbf{z}(t_{j-1}+\hat{x}_2(t_j-t_{j-1})) - [\mathbf{z}(t_{j-1}) + \hat{x}_2\,(\mathbf{z}(t_j)-\mathbf{z}(t_{j-1}))]\right\}. \quad (3.12)$$

We now let $\mathbf{J}(F_T)$ and $D(F_T)$ denote, respectively, the Jacobian and the Frêchet differential of the mapping F_T. Then we summarize their main properties in the following lemma.

Lema 3.2.1 *There exists $h_0 > 0$ such that for all $h \in (0, h_0)$ F_T is a diffeomorphism of class $\mathcal{C}^\infty$ that maps one-to-one $\hat{T}$ onto the curved triangle T in such a way that $F_T(\hat{P}_i) = P_i$ for all $i \in \{1,2,3\}$. In addition, $\mathbf{J}(F_T)$ does not vanish in a neighborhood of $\hat{T}$, and there exist positive constants C_i, $i \in \{1,...,5\}$, independent of T and h, such that for all $T \in \mathcal{T}_h$ there hold*

$$C_1\,h_T^2 \le |\mathbf{J}(F_T)| \le C_2\,h_T^2\,, \quad |\mathbf{J}(F_T)^k|_{W^{1,\infty}(\hat{T})} \le C_3\,h_T^{1+2k} \quad \forall\, k \in \{-1,1\}\,,$$

and

$$|(DF_T)|_{W^{k,\infty}(\hat{T})} \le C_4\,h_T^{k+1}\,, \quad |(DF_T)^{-1}|_{W^{k,\infty}(\hat{T})} \le C_5\,h_T^{k-1} \quad \forall\, k \in \{0,1\}\,.$$

Proof: See Theorem 22.4 in [64]. ■

Herafter, given $s \ge 0$, $\|\cdot\|_{W^{s,\infty}(\hat{T})}$ and $|\cdot|_{W^{s,\infty}(\hat{T})}$ (resp. $\|\cdot\|_{[W^{s,\infty}(\hat{T})]^{2\times2}}$ and $|\cdot|_{[W^{s,\infty}(\hat{T})]^{2\times2}}$) denote the norm and semi-norm of the usual Sobolev space $W^{s,\infty}(\hat{T})$ (resp. $[W^{s,\infty}(\hat{T})]^{2\times2}$). In addition, $|\cdot|_{[H^1(\hat{T})]^2}$ is the semi-norm of $[H^1(\hat{T})]^2$, and given a non-negative integer k and a subset S of $\mathbb{R}$ or $\mathbb{R}^2$, $\mathbf{P}_k(S)$ denotes the space of polynomials defined on S of degree $\le k$.

We now introduce the lowest order Raviart-Thomas spaces. For this purpose, we first let

$$\mathcal{R}T_0(\hat{T}) := \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix}\right\}, \quad (3.13)$$

and for each triangle $T \in \mathcal{T}_h$, we put

$$\mathcal{R}T_0(T) := \{\,\boldsymbol{\tau} : \quad \boldsymbol{\tau} = \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\boldsymbol{\tau}} \circ F_T^{-1}\,,\ \hat{\boldsymbol{\tau}} \in \mathcal{R}T_0(\hat{T})\,\}\,. \quad (3.14)$$

Then, we define the finite element subspaces for the unknowns $\boldsymbol{\sigma}$, λ, and u, as follows:

$$H_h^{\boldsymbol{\sigma}} := \{\,\boldsymbol{\tau}_h \in H(\text{div};\Omega) : \quad \boldsymbol{\tau}_h|_T \in \mathcal{R}T_0(T) \quad \forall\, T \in \mathcal{T}_h\,\}\,, \quad (3.15)$$

$$H_h^{\lambda} := \{\,\mu_h : \Gamma \to \mathbb{R}, \quad \mu_h = \hat{\mu}_h \circ \mathbf{z}^{-1},\ \hat{\mu}_h \in H_h^{\lambda}(0,2\pi)\,\}\,, \quad (3.16)$$

with

$$H_h^\lambda(0,2\pi) := \Big\{ \hat{\mu}_h : [0,2\pi] \to \mathbb{R}, \quad \hat{\mu}_h \text{ is continuous and periodic of period } 2\pi, \\ \hat{\mu}_h|_{[t_{j-1},t_j]} \in \mathbf{P}_1(t_{j-1},t_j) \quad \forall\, j \in \{1,...,n\} \Big\},$$

and

$$Q_h := \{ v_h \in L^2(\Omega) : \quad v_h|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h \}. \tag{3.17}$$

Thus, we set $H_h := H_h^{\boldsymbol{\sigma}} \times H_h^\lambda$ and state the Galerkin scheme associated with the continuous problem (3.5) as: *Find* $((\boldsymbol{\sigma}_h, \lambda_h), u_h) \in H_h \times Q_h$ *such that*

$$\begin{aligned} A((\boldsymbol{\sigma}_h,\lambda_h),(\boldsymbol{\tau}_h,\mu_h)) + B((\boldsymbol{\tau}_h,\mu_h),u_h) &= \langle \boldsymbol{\tau}_h\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}, \\ B((\boldsymbol{\sigma}_h,\lambda_h),v_h) &= -\int_\Omega f v_h\, dx, \end{aligned} \tag{3.18}$$

for all $((\boldsymbol{\tau}_h,\mu_h),v_h) \in H_h \times Q_h$.

Next, similarly as for the continuous problem, we introduce an alternative formulation, which is the discrete analogue of (3.8). To this end, we define

$$H_{h,0}^\lambda := H_h^\lambda \cap H_0^{1/2}(\Gamma), \quad \tilde{H}_h := H_h^{\boldsymbol{\sigma}} \times H_{h,0}^\lambda, \quad \tilde{Q}_h := Q_h \times \mathbb{R}, \tag{3.19}$$

and consider the Galerkin scheme: *Find* $((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(u_h,p_h)) \in \tilde{H}_h \times \tilde{Q}_h$ *such that*

$$\begin{aligned} A((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(\boldsymbol{\tau}_h,\tilde{\mu}_h)) + \tilde{B}((\boldsymbol{\tau}_h,\tilde{\mu}_h),(u_h,p_h)) &= \langle \boldsymbol{\tau}_h\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}, \\ \tilde{B}((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(v_h,q_h)) &= -\int_\Omega f v_h\, dx, \end{aligned} \tag{3.20}$$

for all $((\boldsymbol{\tau}_h,\tilde{\mu}_h),(v_h,q_h)) \in \tilde{H}_h \times \tilde{Q}_h$.

Then we have the following result.

Teorema 3.2.1 *Problems* (3.18) *and* (3.20) *are equivalent. More precisely:*

1. *If* $((\boldsymbol{\sigma}_h,\lambda_h),u_h) \in H_h \times Q_h$ *is a solution of* (3.18), *where* $\lambda_h := \tilde{\lambda}_h + p_h$, *with* $\tilde{\lambda}_h \in H_{h,0}^\lambda$ *and* $p_h \in \mathbb{R}$, *then* $((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(u_h,p_h)) \in \tilde{H}_h \times \tilde{Q}_h$ *is a solution of* (3.20).

2. *If* $((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(u_h,p_h)) \in \tilde{H}_h \times \tilde{Q}_h$ *is a solution of* (3.20), *then* $((\boldsymbol{\sigma}_h,\lambda_h),u_h) \in H_h \times Q_h$ *is a solution of* (3.18) *with* $\lambda_h := \tilde{\lambda}_h + p_h$.

Proof: It is similar to the proof of Theorem 3.1.1 since it is based on the decomposition $H_h^\lambda := H_{h,0}^\lambda \oplus \mathbb{R}$. We omit further details. ■

Our next goal is to show that the Galerkin scheme (3.20) is stable and uniquely solvable. To this end, we consider first the equilibrium interpolation operator $\mathcal{E}_h$:

$[H^1(\Omega)]^2 \to H_h^{\boldsymbol{\sigma}}$, which, according to the Piola transformation used in (3.14), is given by (see, e.g. [16], [60])

$$\mathcal{E}_h(\boldsymbol{\tau})|_T := \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\circ F_T^{-1} \qquad \forall\, T\in\mathcal{T}_h\,,$$

where $\hat{\boldsymbol{\tau}} := \mathbf{J}(F_T)\,(DF_T)^{-1}\boldsymbol{\tau}\circ F_T$ and $\hat{\mathcal{E}} : [H^1(\hat{T})]^2 \to \mathcal{R}T_0(\hat{T})$ is the local equilibrium interpolation operator on the reference triangle $\hat{T}$.

Lema 3.2.2 *There exists $C>0$, independent of h, such that*

$$||\boldsymbol{\tau}-\mathcal{E}_h(\boldsymbol{\tau})||_{[L^2(\Omega)]^2} \;\le\; C\,h\,||\boldsymbol{\tau}||_{[H^1(\Omega)]^2} \tag{3.21}$$

and

$$\|\mathrm{div}\,(\mathcal{E}_h(\boldsymbol{\tau}))\|_{L^2(\Omega)} \;\le\; C\,\|\mathrm{div}\,\boldsymbol{\tau}\|_{L^2(\Omega)} \tag{3.22}$$

for all $\boldsymbol{\tau}\in[H^1(\Omega)]^2$.

Proof: Using the change of variable $x = F_T(\hat{x})$, we find that

$$\|\boldsymbol{\tau}-\mathcal{E}_h(\boldsymbol{\tau})\|^2_{[L^2(T)]^2} \;=\; \int_T \|\boldsymbol{\tau}(x) - \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(F_T^{-1}(x))\|_2^2\,dx$$

$$= \int_{\hat{T}} |\mathbf{J}(F_T)|\,\left\|(\boldsymbol{\tau}\circ F_T)(\hat{x}) - \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(\hat{x})\right\|_2^2\,d\hat{x}$$

$$= \int_{\hat{T}} |\mathbf{J}(F_T)|\,\left\|\mathbf{J}(F_T)^{-1}\,(DF_T)\,\left[\hat{\boldsymbol{\tau}}(\hat{x}) - \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(\hat{x})\right]\right\|_2^2\,d\hat{x}$$

$$\le \int_{\hat{T}} |\mathbf{J}(F_T)|^{-1}\,\|(DF_T)\|_2^2\,\left\|\hat{\boldsymbol{\tau}}(\hat{x}) - \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(\hat{x})\right\|_2^2\,d\hat{x}\,, \tag{3.23}$$

where $\|\cdot\|_2$ is the usual euclidean norm for both vectors and matrices in $\mathbb{R}^2$ and $\mathbb{R}^{2\times 2}$, respectively.

Now, since $|\mathbf{J}(F_T)^{-1}| = O(h_T^{-2})$ and $\|(DF_T)\|_2 \le C_4\,h_T$ (see Lemma 3.2.1), and because of the approximation property of $\hat{\mathcal{E}}$, we deduce from (3.23) that

$$\|\boldsymbol{\tau}-\mathcal{E}_h(\boldsymbol{\tau})\|^2_{[L^2(T)]^2} \;\le\; \hat{C}\,\|\hat{\boldsymbol{\tau}}-\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\|^2_{[L^2(\hat{T})]^2} \;\le\; \hat{C}\,|\,\hat{\boldsymbol{\tau}}|^2_{[H^1(\hat{T})]^2}$$

$$= \hat{C}\,|\,\mathbf{J}(F_T)\,(DF_T)^{-1}\,(\boldsymbol{\tau}\circ F_T)\,|^2_{[H^1(\hat{T})]^2}$$

$$\begin{aligned}\le \hat{C}\Big\{ & |\mathbf{J}(F_T)|_{W^{1,\infty}(\hat{T})}\,\|(DF_T)^{-1})\|_{[W^{0,\infty}(\hat{T})]^{2\times 2}}\,\|\boldsymbol{\tau}\circ F_T\|_{[L^2(\hat{T})]^2} \\ & + \|\mathbf{J}(F_T)\|_{W^{0,\infty}(\hat{T})}\,|(DF_T)^{-1}|_{[W^{1,\infty}(\hat{T})]^{2\times 2}}\,\|\boldsymbol{\tau}\circ F_T\|_{[L^2(\hat{T})]^2} \\ & + \|\mathbf{J}(F_T)\|_{W^{0,\infty}(\hat{T})}\,\|(DF_T)^{-1}\|_{[W^{0,\infty}(\hat{T})]^{2\times 2}}\,|\boldsymbol{\tau}\circ F_T|_{[H^1(\hat{T})]^2}\Big\}^2\,,\end{aligned} \tag{3.24}$$

with a constant $\hat{C} > 0$, depending only on $\hat{T}$.

Next, applying the corresponding norm estimates for $\mathbf{J}(F_T)$ and $(DF_T)^{-1}$ (see again Lemma 3.2.1), changing back the variable $\hat{x}$ by $F_T^{-1}(x)$, and using chain rule in the term $|\boldsymbol{\tau} \circ F_T|_{[H^1(\hat{T})]^2}$, we conclude from (3.24) that

$$\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|^2_{[L^2(T)]^2} \leq \hat{C}\, h^2\, \|\boldsymbol{\tau}\|^2_{[H^1(T)]^2} \qquad \forall\, T \in \mathcal{T}_h\,. \tag{3.25}$$

On ther hand, we know from the conmuting diagram property on the reference triangle $\hat{T}$ that

$$\|\mathrm{div}\, \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\|_{L^2(\hat{T})} \leq \|\mathrm{div}\, \hat{\boldsymbol{\tau}}\|_{L^2(\hat{T})}\,.$$

Then we use the above inequality, identity (1.49) (cf. Lemma 1.5) in chapter III of [16], and Cauchy-Schwarz's inequality, to find that

$$\|\mathrm{div}\, \mathcal{E}_h(\boldsymbol{\tau})\|^2_{L^2(T)} := \int_T \mathrm{div}\, \mathcal{E}_h(\boldsymbol{\tau})\, \mathrm{div}\, \mathcal{E}_h(\boldsymbol{\tau})\, dx = \int_{\hat{T}} \widehat{\mathrm{div}\, \mathcal{E}_h(\boldsymbol{\tau})}\, \mathrm{div}\, \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\, d\hat{x}$$

$$\leq \|\widehat{\mathrm{div}\, \mathcal{E}_h(\boldsymbol{\tau})}\|_{L^2(\hat{T})}\, \|\mathrm{div}\, \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\|_{L^2(\hat{T})} \leq \|\widehat{\mathrm{div}\, \mathcal{E}_h(\boldsymbol{\tau})}\|_{L^2(\hat{T})}\, \|\mathrm{div}\, \hat{\boldsymbol{\tau}}\|_{L^2(\hat{T})}\,, \tag{3.26}$$

where $\widehat{\mathrm{div}\, \mathcal{E}_h(\boldsymbol{\tau})}$ stands for $\mathrm{div}\, \mathcal{E}_h(\boldsymbol{\tau}) \circ F_T$.

Then, applying the inequalities (1.40) (cf. Lemma 1.4) and (1.54) (cf. Lemma 1.6) in chapter III of [16], and the estimate for $\mathbf{J}(F_T)$ given in Lemma 3.2.1, we deduce that

$$\|\widehat{\mathrm{div}\, \mathcal{E}_h(\boldsymbol{\tau})}\|_{L^2(\hat{T})} \leq C\, h_T^{-1}\, \|\mathrm{div}\, \mathcal{E}_h(\boldsymbol{\tau})\|_{L^2(T)} \quad \text{and} \quad \|\mathrm{div}\, \hat{\boldsymbol{\tau}}\|_{L^2(\hat{T})} \leq C\, h_T\, \|\mathrm{div}\, \boldsymbol{\tau}\|_{L^2(T)}\,,$$

which replaced back into (3.26) yields

$$\|\mathrm{div}\, (\mathcal{E}_h(\boldsymbol{\tau}))\|_{L^2(T)} \leq C\, \|\mathrm{div}\, \boldsymbol{\tau}\|_{L^2(T)} \qquad \forall\, T \in \mathcal{T}_h\,. \tag{3.27}$$

Hence, summing up over all the triangles $T \in \mathcal{T}_h$ in (3.25) and (3.27), we conclude, respectively, (3.21) and (3.22). ■

We are now in a position to prove the discrete inf-sup condition for the bilinear form $\tilde{B}$.

Lema 3.2.3 *There exists $\beta^* > 0$, independent of h, such that for all $(v_h, q_h) \in \tilde{Q}_h$ it holds*

$$\sup_{\substack{(\boldsymbol{\tau}_h, \tilde{\mu}_h) \in \tilde{H}_h \\ (\boldsymbol{\tau}_h, \tilde{\mu}_h) \neq 0}} \frac{\tilde{B}((\boldsymbol{\tau}_h, \tilde{\mu}_h), (v_h, q_h))}{||(\boldsymbol{\tau}_h, \tilde{\mu}_h)||_H} \geq \beta^*\, ||(v_h, q_h)||_{\tilde{Q}}\,.$$

Proof: Given $(v_h, q_h) \in \tilde{Q}_h$, we note that

$$\sup_{\substack{(\boldsymbol{\tau}_h,\tilde{\mu}_h)\in \tilde{H}_h \\ (\boldsymbol{\tau}_h,\tilde{\mu}_h)\neq 0}} \frac{\tilde{B}((\boldsymbol{\tau}_h, \tilde{\mu}_h), (v_h, q_h))}{||(\boldsymbol{\tau}_h, \tilde{\mu}_h)||_H} \geq \sup_{\substack{\boldsymbol{\tau}_h\in H_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h\neq 0}} \frac{\tilde{B}((\boldsymbol{\tau}_h, 0), (v_h, q_h))}{||\boldsymbol{\tau}_h||_{H(\mathrm{div};\Omega)}}.$$

Then, we define $\tilde{v}_h := \begin{cases} v_h & \text{in} \quad \Omega \\ -\dfrac{1}{|\Omega_0|} \left(\displaystyle\int_\Omega v_h \, dx + q_h \, |\Gamma| \right) & \text{in} \quad \bar{\Omega}_0 \end{cases}$,

put $\tilde{\Omega} := \Omega \cup \bar{\Omega}_0$, and let $z \in H^1(\tilde{\Omega})$ be the weak solution of

$$-\Delta z = \tilde{v}_h \quad \text{in} \quad \tilde{\Omega}\,, \quad \frac{\partial z}{\partial \boldsymbol{\nu}} = q_h \quad \text{on} \quad \Gamma \quad \int_{\tilde{\Omega}} z \, dx = 0\,.$$

Since $\tilde{\Omega}$, being the interior region of the circle Γ, is clearly convex, the usual regularity result (see, e.g. [45]) implies that $z \in H^2(\tilde{\Omega})$ and

$$||z||_{H^2(\tilde{\Omega})} \leq C \{ \, ||v_h||_{L^2(\Omega)} + |q_h| \, \}\,.$$

Thus we define $\tilde{\boldsymbol{\tau}} := -\nabla z|_\Omega \in [H^1(\Omega)]^2$, and observe that $\mathrm{div}\, \tilde{\boldsymbol{\tau}} = v_h$ in Ω, $\tilde{\boldsymbol{\tau}} \cdot \boldsymbol{\nu} = -q_h$ on Γ, and

$$||\tilde{\boldsymbol{\tau}}||_{[H^1(\Omega)]^2} \; = \; ||\nabla z||_{[H^1(\Omega)]^2} \; \leq \; ||z||_{H^2(\tilde{\Omega})} \; \leq \; C \{ \, ||v_h||_{L^2(\Omega)} + |q_h| \, \}\,. \tag{3.28}$$

Further, it is easy to see that

$$||\tilde{\boldsymbol{\tau}}||_{H(\mathrm{div};\Omega)} \; \leq \; C \; \left\{ \, ||v_h||_{L^2(\Omega)} \; + \; |q_h| \, \right\}\,. \tag{3.29}$$

Then, using the approximation property (3.21) and the estimate (3.22) (cf. Lemma 3.2.2), we find that

$$||\mathcal{E}_h(\tilde{\boldsymbol{\tau}})||^2_{H(\mathrm{div};\Omega)} \; = \; ||\mathcal{E}_h(\tilde{\boldsymbol{\tau}})||^2_{[L^2(\Omega)]^2} \; + \; ||\mathrm{div}\,(\mathcal{E}_h(\tilde{\boldsymbol{\tau}}))||^2_{L^2(\Omega)}$$

$$\leq \; C \; \left\{ \, \|\tilde{\boldsymbol{\tau}} - \mathcal{E}_h(\tilde{\boldsymbol{\tau}})\|^2_{[L^2(\Omega)]^2} \; + \; \|\tilde{\boldsymbol{\tau}}\|^2_{[L^2(\Omega)]^2} \; + \; \|\mathrm{div}\, \tilde{\boldsymbol{\tau}}\|^2_{L^2(\Omega)} \, \right\}$$

$$\leq \; C \; \left\{ \, h^2 \, \|\tilde{\boldsymbol{\tau}}\|^2_{[H^1(\Omega)]^2} \; + \; \|\tilde{\boldsymbol{\tau}}\|^2_{H(\mathrm{div};\Omega)} \, \right\}\,,$$

which, using (3.28) and (3.29), implies

$$||\mathcal{E}_h(\tilde{\boldsymbol{\tau}})||_{H(\mathrm{div};\Omega)} \; \leq \; C \; \left\{ \, ||v_h||_{L^2(\Omega)} \; + \; |q_h| \, \right\}\,. \tag{3.30}$$

We now let $\mathcal{P}_h$ be the orthogonal projection from $L^2(\Omega)$ onto the finite element subspace Q_h. Then, using the identity (1.49) (cf. Lemma 1.5) in chapter III of

[16] and the conmuting diagram property on the reference triangle $\hat{T}$, similarly as we did in the proof of Lemma 3.2.2, we deduce that in this case there also holds $\mathcal{P}_h(\operatorname{div} \mathcal{E}_h(\tilde{\boldsymbol{\tau}})) = \mathcal{P}_h(\operatorname{div} \tilde{\boldsymbol{\tau}})$, which yields

$$\int_\Omega v_h \operatorname{div} \mathcal{E}_h(\tilde{\boldsymbol{\tau}})\, dx \;=\; \int_\Omega v_h \operatorname{div} \tilde{\boldsymbol{\tau}}\, dx \;=\; \|v_h\|^2_{L^2(\Omega)} \,.$$

Next, since $\int_e \mathcal{E}_h(\tilde{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}_e\, ds \;=\; \int_e \tilde{\boldsymbol{\tau}} \cdot \boldsymbol{\nu}_e\, ds$ for all the edges e of $\mathcal{T}_h$, with $\boldsymbol{\nu}_e$ being the unit outward normal to e, and since $\tilde{\boldsymbol{\tau}} \cdot \boldsymbol{\nu} = -q_h$ on Γ, we deduce that $\langle \mathcal{E}_h(\tilde{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = \, -\, q_h\, |\Gamma|$.

According to the above analysis we can write

$$\sup_{\substack{\boldsymbol{\tau}_h \in H_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h \neq 0}} \frac{\tilde{B}((\boldsymbol{\tau}_h, 0), (v_h, q_h))}{||\boldsymbol{\tau}_h||_{H(\operatorname{div};\Omega)}} \;\geq\; \frac{\tilde{B}((\mathcal{E}_h(\tilde{\boldsymbol{\tau}}), 0), (v_h, q_h))}{||\mathcal{E}_h(\tilde{\boldsymbol{\tau}})||_{H(\operatorname{div};\Omega)}}$$

$$= \frac{||v_h||^2_{L^2(\Omega)} \;+\; |\Gamma|\, q_h^2}{||\mathcal{E}_h(\tilde{\boldsymbol{\tau}})||_{H(\operatorname{div};\Omega)}} \;\geq\; \beta^* \, ||(v_h, q_h)||_{\tilde{Q}} \,,$$

where the last inequality follows from (3.30). This finishes the proof. ∎

We are now in a position to provide the stability and unique solvability of the Galerkin scheme (3.20), and the corresponding Cea estimate.

Teorema 3.2.2 *There exists a unique* $((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h)) \in \tilde{H}_h \times \tilde{Q}_h$ *solution of* (3.20). *In addition, there exists* $C > 0$, *independent of* h, *such that*

$$||((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h))||_{\tilde{H}\times\tilde{Q}} \;\leq\; C\; \left\{\, ||f||_{L^2(\Omega)} \;+\; ||g||_{H^{1/2}(\Gamma_0)} \,\right\} \,,$$

and

$$\begin{aligned} &||((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \;-\; ((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h))||_{\tilde{H}\times\tilde{Q}} \\ \leq\; C &\min_{((\boldsymbol{\tau}_h, \tilde{\mu}_h), v_h) \in \tilde{H}_h \times Q_h} ||((\boldsymbol{\sigma}, \tilde{\lambda}), u) \;-\; ((\boldsymbol{\tau}_h, \tilde{\mu}_h), v_h)||_{\tilde{H}\times Q} \,. \end{aligned}$$

Proof: Let $\tilde{V}_h$ be the discrete kernel of the operator induced by the bilinear form $\tilde{B}$. It is easy to show, according to the definition of $\tilde{B}$ (cf. (3.9)) and Lemma 5.7 in [40], that

$$\tilde{V}_h := \{\, (\boldsymbol{\tau}_h, \tilde{\mu}_h) \in \tilde{H}_h : \quad \langle \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{\tau}_h = 0 \quad \text{in} \quad \Omega \} \,,$$

and hence the bilinear form A is uniformly strongly coercive on $\tilde{V}_h$.

In this way, Lemma 3.2.3 and direct applications of the abstract Theorems 1.1 and 2.1 in chapter II of [16] complete the proof. ∎

We end this section with a result on the rate of convergence of the Galerkin scheme (3.20). For this purpose, we recall the following approximation properties of the subspaces $H_h^{\boldsymbol{\sigma}}$, $H_{h,0}^{\lambda}$, and Q_h, respectively (see, e.g. [8], [16], [49], [60]):

1. (AP$_h^{\boldsymbol{\sigma}}$): For all $\tau \in [H^1(\Omega)]^2$ with $\operatorname{div}\tau \in H^1(\Omega)$, it holds

$$||\tau - \mathcal{E}_h(\tau)||_{H(\operatorname{div};\Omega)} \leq C\,h\,\left\{ ||\tau||_{[H^1(\Omega)]^2} + ||\operatorname{div}\tau||_{H^1(\Omega)} \right\}.$$

2. (AP$_{h,0}^{\lambda}$): For all $\tilde{\mu} \in H^{3/2}(\Gamma) \cap H_0^{1/2}(\Gamma)$, there exists $\tilde{\mu}_h \in H_{h,0}^{\lambda}$ such that

$$||\tilde{\mu} - \tilde{\mu}_h||_{H^{1/2}(\Gamma)} \leq C\,h\,||\tilde{\mu}||_{H^{3/2}(\Gamma)}.$$

3. (AP$_h$): For all $v \in H^1(\Omega)$ it holds

$$||v - \mathcal{P}_h(v)||_{L^2(\Omega)} \leq C\,h\,||v||_{H^1(\Omega)},$$

   where $\mathcal{P}_h$ is the orthogonal projection from $L^2(\Omega)$ onto Q_h.

Then we can establish the following theorem.

Teorema 3.2.3 *Let* $((\boldsymbol{\sigma}, \tilde{\lambda}), (u,p))$ *and* $((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h))$ *be the unique solutions of the continuous and discrete mixed formulations* (3.8) *and* (3.20)*, respectively. Assume that* $\boldsymbol{\sigma} \in [H^s(\Omega)]^2$, $\operatorname{div}\boldsymbol{\sigma} \in H^s(\Omega)$, $\tilde{\lambda} \in H^{s+1/2}(\Gamma)$ *and* $u \in H^s(\Omega)$*, for some* $s \in (0,1]$*. Then there exists* $C > 0$*, independent of* h*, such that*

$$||((\boldsymbol{\sigma}, \tilde{\lambda}), (u,p)) - ((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h))||_{\tilde{H}\times\tilde{Q}}$$

$$\leq C\,h^s \left\{ ||\boldsymbol{\sigma}||_{[H^s(\Omega)]^2} + ||\operatorname{div}\boldsymbol{\sigma}||_{H^s(\Omega)} + ||\tilde{\lambda}||_{H^{s+1/2}(\Gamma)} + ||u||_{H^s(\Omega)} \right\}.$$

Proof: It follows from the Cea estimate in Theorem 3.2.2, the above approximation properties, and suitable interpolation theorems in the Sobolev spaces. ■

3.3 An explicit residual a-posteriori estimate

Let us first introduce some notations. We let $E(T)$ be the set of edges of $T \in \mathcal{T}_h$, and let E_h be the set of all edges of the triangulation $\mathcal{T}_h$. Then we write $E_h = E_h(\Omega) \cup E_h(\Gamma_0) \cup E_h(\Gamma)$, where $E_h(\Omega) := \{e \in E_h : e \subseteq \Omega\}$, $E_h(\Gamma) := \{e \in E_h : e \subseteq \Gamma\}$, and similarly for $E_h(\Gamma_0)$. In what follows, h_T and h_e stand for the diameters of the triangle $T \in \mathcal{T}_h$ and edge $e \in E_h$, respectively. Also, given a vector-valued function $\boldsymbol{\tau} := (\tau_1, \tau_2)^{\mathbf{T}}$ defined in Ω, an edge $e \in E(T) \cap E_h(\Omega)$, and the unit tangential vector $\mathbf{t}_T$ along e, we let $\boldsymbol{\tau}_T$ be the restriction of $\boldsymbol{\tau}$ to T, and let $J[\boldsymbol{\tau}\cdot\mathbf{t}_T]$ be the corresponding jump across e, that is $J[\boldsymbol{\tau}\cdot\mathbf{t}_T] := (\boldsymbol{\tau}_T - \boldsymbol{\tau}_{T'})|_e \cdot \mathbf{t}_T$, where T' is the other triangle of $\mathcal{T}_h$ having e as edge. Here, the tangential vector $\mathbf{t}_T$ is given by $(-\nu_2, \nu_1)^{\mathbf{T}}$ where $\boldsymbol{\nu}_T := (\nu_1, \nu_2)^{\mathbf{T}}$ is the unit outward normal to ∂T. Finally, we let $\operatorname{curl}(\boldsymbol{\tau})$ be the scalar $\frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}$.

Next, we define the finite element space

$$X_h := \{v_h \in C(\Omega): \quad v_h|_T = \hat{v}\circ F_T^{-1}, \quad \hat{v}\in \mathbf{P}_1(\hat{T}), \quad \forall T \in \mathcal{T}_h\},$$

and let $I_h : H^1(\Omega) \longrightarrow X_h$ be the usual Clément interpolation operator (see [14], [23]). The following lemma states the local approximation properties of I_h.

Lema 3.3.1 *There exist positive constants C_1 and C_2, independent of h, such that for all $\varphi \in H^1(\Omega)$ there holds*

$$||\varphi - I_h(\varphi)||_{L^2(T)} \le C_1 h_T ||\varphi||_{H^1(\Delta(T))} \qquad \forall T \in \mathcal{T}_h,$$

and

$$||\varphi - I_h(\varphi)||_{L^2(e)} \le C_2 h_e^{1/2} ||\varphi||_{H^1(\Delta(e))} \qquad \forall e \in E_h,$$

where $\Delta(T) := \cup\{T' \in \mathcal{T}_h: \quad T'\cap T \neq \emptyset\}$, and $\Delta(e) := \cup\{T' \in \mathcal{T}_h: \quad T'\cap e \neq \emptyset\}$.

Proof: See Theorem 4.1 in [14]. ■

The main goal of the present section is to prove the following theorem providing a reliable a-posteriori error estimate.

Teorema 3.3.1 *Let $((\boldsymbol{\sigma},\tilde{\lambda}),(u,p)) \in \tilde{H}\times\tilde{Q}$ and $((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(u_h,p_h)) \in \tilde{H}_h\times\tilde{Q}_h$ be the unique solutions of the continuous and discrete formulations* (3.8) *and* (3.20), *respectively. Assume that the Dirichlet data $g \in H^1(\Gamma_0)$ and that $\boldsymbol{\kappa}_1 \in C^1(\Omega_1)$. Then there exists $C > 0$, independent of h, such that*

$$\left|\left|((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \tilde{\lambda}-\tilde{\lambda}_h),(u-u_h,p-p_h))\right|\right|_{\tilde{H}\times\tilde{Q}} \le C \left\{\sum_{T\in\mathcal{T}_h} \eta_T^2\right\}^{1/2}, \tag{3.31}$$

where for any triangle $T \in \mathcal{T}_h$ we define

$$\eta_T^2 := ||f + \operatorname{div}\boldsymbol{\sigma}_h||^2_{L^2(T)} + h_T^2 \left|\left|\operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\right|\right|^2_{L^2(T)}$$

$$+ h_T^2 \left|\left|\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h\right|\right|^2_{[L^2(T)]^2} + \sum_{e\in E(T)\cap E_h(\Omega)} h_e \left|\left| J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T] \right|\right|^2_{L^2(e)}$$

$$+ \sum_{e\in E(T)\cap E_h(\Gamma_0)} h_e \left\{ \left|\left|(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h\cdot\mathbf{t}_T\right|\right|^2_{L^2(e)} + \left|\left|\frac{dg}{d\mathbf{t}_T}\right|\right|^2_{L^2(e)} + ||g-\hat{g}_h||^2_{L^2(e)} \right\}$$

$$+ \sum_{e \in E(T) \cap E_h(\Gamma)} h_e \left\{ \left|\left| (\boldsymbol{\kappa}^{-1} \boldsymbol{\sigma}_h) \cdot \mathbf{t}_T - \frac{d\tilde{\lambda}_h}{d\mathbf{t}_T} \right|\right|^2_{L^2(e)} + ||\xi_h||^2_{L^2(e)} + \left|\left| \tilde{\lambda}_h - \hat{\lambda}_h \right|\right|^2_{L^2(e)} \right\} , \tag{3.32}$$

with

$$\xi_h := \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} + 2\,\mathbf{W}(\tilde{\lambda}_h) \, , \tag{3.33}$$

$$\hat{g}_h|_e := \frac{1}{h_e} \int_e g \, ds \qquad \forall\, e \in E_h(\Gamma_0) \, , \tag{3.34}$$

and

$$\hat{\lambda}_h|_e := \frac{1}{h_e} \int_e \tilde{\lambda}_h \, ds \qquad \forall\, e \in E_h(\Gamma) \, . \tag{3.35}$$

In order to prove Theorem 3.3.1, we need some preliminary results. We begin with the following technical lemma.

Lema 3.3.2 *Let $\hat{\boldsymbol{\sigma}} := \boldsymbol{\sigma}_h + \boldsymbol{\sigma}^* \in H(\mathrm{div};\Omega)$, where $\boldsymbol{\sigma}^* := \nabla z$ and $z \in H^1(\Omega)$ is the weak solution of: $-\Delta z = f + \mathrm{div}\,\boldsymbol{\sigma}_h$ in Ω, $z = 0$ on Γ_0, $\frac{\partial z}{\partial \boldsymbol{\nu}} = 0$ on Γ. Then there exists $C > 0$, independent of h, such that*

$$\left|\left| (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}, \tilde{\lambda} - \tilde{\lambda}_h) \right|\right|^2_{\tilde{H}} \le C\, ||f + \mathrm{div}\,\boldsymbol{\sigma}_h||_{L^2(\Omega)} \left|\left| (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}, \tilde{\lambda} - \tilde{\lambda}_h) \right|\right|_{\tilde{H}}$$

$$+ \left| A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} - \boldsymbol{\tau}_h, \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h)) \right| + |\langle \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}, p_h \rangle_\Gamma| + |\langle\, (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} - \boldsymbol{\tau}_h) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0}| \, ,$$

for all $(\boldsymbol{\tau}_h, \tilde{\mu}_h) \in \tilde{H}_h$ with $\mathrm{div}\,\boldsymbol{\tau}_h = 0$.

Proof: It follows similarly as the proof of Lemma 4.2 in [39]. ∎

Now, we can give an a-posteriori error estimate for $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)$ and $(\tilde{\lambda} - \tilde{\lambda}_h)$ through the following theorem.

Teorema 3.3.2 *There exists $C > 0$, independent of h, such that*

$$\left|\left| (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h) \right|\right|_{\tilde{H}} \le C \left\{ \sum_{T \in \mathcal{T}_h} \eta^2_{1,T} \right\}^{1/2} , \tag{3.36}$$

where for any triangle $T \in \mathcal{T}_h$ we define

$$\eta^2_{1,T} := ||f + \mathrm{div}\,\boldsymbol{\sigma}_h||^2_{L^2(T)} + h_T^2 \left|\left| \mathrm{curl}(\boldsymbol{\kappa}^{-1} \boldsymbol{\sigma}_h) \right|\right|^2_{L^2(T)}$$

$$+ \sum_{e \in E(T) \cap E_h(\Omega)} h_e \left|\left|\, J[(\boldsymbol{\kappa}^{-1} \boldsymbol{\sigma}_h) \cdot \mathbf{t}_T] \,\right|\right|^2_{L^2(e)}$$

$$+ \sum_{e \in E(T)\cap E_h(\Gamma_0)} h_e \left\{ ||(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T||^2_{L^2(e)} + \left|\left|\frac{dg}{d\mathbf{t}_T}\right|\right|^2_{L^2(e)} \right\}$$

$$+ \sum_{e \in E(T)\cap E_h(\Gamma)} h_e \left\{ \left|\left|(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T - \frac{d\tilde{\lambda}_h}{d\mathbf{t}_T}\right|\right|^2_{L^2(e)} + ||\xi_h||^2_{L^2(e)} \right\}.$$

Proof: From Lemma 3.3.2 we know that $\boldsymbol{\sigma}^* \cdot \boldsymbol{\nu} = \frac{\partial z}{\partial \boldsymbol{\nu}} = 0$ on Γ and $\operatorname{div}(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) = 0$. In addition, the formulations (3.8) and (3.20) yield $\langle \boldsymbol{\sigma}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = \langle \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}, 1\rangle_\Gamma = 0$. Then, using Gauss's Theorem we deduce that $\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}})\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}})\cdot\boldsymbol{\nu}, 1\rangle_{\Gamma_0} = 0$.

Thus, since Ω is connected, there exists a stream function $\varphi \in H^1(\Omega)$, with $\int_\Omega \varphi\, dx = 0$, such that $\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} = \mathbf{curl}\,\varphi := \left(-\frac{\partial\varphi}{\partial x_2} \;\; \frac{\partial \varphi}{\partial x_1}\right)^{\mathbf{T}}$.

We now introduce the Clément interpolant $\varphi_h := I_h(\varphi) \in X_h$ and take from now on $\boldsymbol{\tau}_h := \mathbf{curl}\,\varphi_h$ in Lemma 3.3.2. In this way, $\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} - \boldsymbol{\tau}_h = \mathbf{curl}(\varphi - \varphi_h)$, and for all $\tilde{\mu}_h \in H^\lambda_{h,0}$ it holds

$$A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\mathbf{curl}(\varphi - \varphi_h), \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h))$$

$$= \int_\Omega (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{curl}(\varphi - \varphi_h)\, dx + \langle \frac{d}{d\mathbf{t}_T}(\varphi - \varphi_h), \tilde{\lambda}_h\rangle_\Gamma + \langle \xi_h, \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h\rangle_\Gamma\,, \quad (3.37)$$

where $\xi_h := \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} + 2\,\mathbf{W}\tilde{\lambda}_h$. Since $\boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}|_\Gamma \in L^2(\Gamma)$ and $\tilde{\lambda}_h \in H^1(\Gamma)$, it follows easily, using the mapping properties of $\mathbf{W}$, that $\xi_h \in L^2(\Gamma)$.

Now, applying integration by parts, we obtain

$$\int_\Omega (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{curl}(\varphi - \varphi_h)\, dx$$

$$= \sum_{T\in\mathcal{T}_h} \Bigg\{ -\int_T \operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\,(\varphi - \varphi_h)\, dx + \frac{1}{2} \sum_{e\in E(T)\cap E_h(\Omega)} \langle J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T], \varphi - \varphi_h\rangle_{L^2(e)}$$

$$+ \sum_{e\in E(T)\cap E_h(\Gamma)} \langle (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T, \varphi - \varphi_h\rangle_{L^2(e)} + \sum_{e\in E(T)\cap E_h(\Gamma_0)} \langle (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T, \varphi - \varphi_h\rangle_{L^2(e)} \Bigg\}, \quad (3.38)$$

which, replaced back into (3.37), yields

$$A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\mathbf{curl}(\varphi - \varphi_h), \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h)) := \sum_{T\in\mathcal{T}_h} \Bigg\{ -\int_T \operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\,(\varphi - \varphi_h)\, dx$$

$$+\frac{1}{2} \sum_{e\in E(T)\cap E_h(\Omega)} \langle J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T], \varphi - \varphi_h\rangle_{L^2(e)} + \sum_{e\in E(T)\cap E_h(\Gamma_0)} \langle (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot \mathbf{t}_T, \varphi - \varphi_h\rangle_{L^2(e)}$$

$$+ \sum_{e \in E(T) \cap E_h(\Gamma)} \left(\langle (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T - \frac{d\tilde{\lambda}_h}{d\mathbf{t}_T}, \varphi - \varphi_h \rangle_{L^2(e)} + \langle \xi_h, \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h \rangle_{L^2(e)} \right) \Bigg\}, \quad (3.39)$$

for all $\tilde{\mu}_h \in H^{\lambda}_{h,0}$, where $\langle \cdot, \cdot \rangle_{L^2(e)}$ stands for the usual $L^2(e)-$inner product.

Next, we define $\mu_h := I_h(w)|_\Gamma$, where $w \in H^1(\Omega)$ is the solution of the boundary value problem: $\Delta w = 0$ in Ω, $w = \tilde{\lambda} - \tilde{\lambda}_h$ on Γ, and $\frac{\partial w}{\partial \boldsymbol{\nu}} = 0$ on Γ_0, and set $\tilde{\mu}_h := \left(\mu_h - \frac{1}{|\Gamma|} \int_\Gamma \mu_h \, ds \right) \in H^{\lambda}_{h,0}$. It is easy to see that

$$||w||_{H^1(\Omega)} \le C \left|\left| \tilde{\lambda} - \tilde{\lambda}_h \right|\right|_{H^{1/2}(\Gamma)}, \quad (3.40)$$

and from Lemma 3.3.1 it follows that

$$\left|\left| \tilde{\lambda} - \tilde{\lambda}_h - \mu_h \right|\right|_{L^2(e)} = ||w - I_h(w)||_{L^2(e)} \le C \, h_e^{1/2} \, ||w||_{H^1(\Delta(e))} .$$

Using the property $\langle \xi_h, 1 \rangle_\Gamma = 0$, the above inequality, and the fact that the number of triangles in $\Delta(e)$ is bounded (independently of h), we show that

$$\left| \sum_{T \in \mathcal{T}_h} \sum_{e \in E(T) \cap E_h(\Gamma)} \langle \xi_h, \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h \rangle_{L^2(e)} \right| = \left| \sum_{e \in E_h(\Gamma)} \langle \xi_h, \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h \rangle_{L^2(e)} \right|$$

$$= \left| \sum_{e \in E_h(\Gamma)} \langle \xi_h, \tilde{\lambda} - \tilde{\lambda}_h - \mu_h \rangle_{L^2(e)} \right| \le C \left\{ \sum_{e \in E_h(\Gamma)} h_e \, ||\xi_h||^2_{L^2(e)} \right\}^{1/2} ||w||_{H^1(\Omega)} . \quad (3.41)$$

In order to bound the remaining terms in (3.39) we apply Cauchy-Schwarz's inequality, Lemma 3.2.3, and the fact that the number of triangles in $\Delta(T)$ is also bounded. Thus, we find that

$$\left| \sum_{T \in \mathcal{T}_h} \int_T \operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \, (\varphi - \varphi_h) \, dx \right| \le C \sum_{T \in \mathcal{T}_h} h_T \left|\left| \operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \right|\right|_{L^2(T)} ||\varphi||_{1,\Delta(T)} , \quad (3.42)$$

$$\left| \sum_{T \in \mathcal{T}_h} \sum_{e \in E(T) \cap E_h(\Omega)} \langle J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T], \varphi - \varphi_h \rangle_{L^2(e)} \right|$$

$$\le C \sum_{T \in \mathcal{T}_h} \sum_{e \in E(T) \cap E_h(\Omega)} h_e^{1/2} \left|\left| J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T] \right|\right|_{L^2(e)} ||\varphi||_{1,\Delta(e)} , \quad (3.43)$$

$$\left| \sum_{T \in \mathcal{T}_h} \sum_{e \in E(T) \cap E_h(\Gamma_0)} \langle (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T, \varphi - \varphi_h \rangle_{L^2(e)} \right|$$

$$\leq C \sum_{T\in\mathcal{T}_h} \sum_{e\in E(T)\cap E_h(\Gamma_0)} h_e^{1/2} \left|\left|(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T\right|\right|_{L^2(e)} ||\varphi||_{H^1(\Delta(e))} \quad , \tag{3.44}$$

and

$$\left| \sum_{T\in\mathcal{T}_h} \sum_{e\in E(T)\cap E_h(\Gamma)} \langle(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T - \frac{d\tilde{\lambda}_h}{d\mathbf{t}_T}, \varphi-\varphi_h\rangle_{L^2(e)} \right|$$

$$\leq C \sum_{T\in\mathcal{T}_h} \sum_{e\in E(T)\cap E_h(\Gamma)} h_e^{1/2} \left|\left|(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T - \frac{d\tilde{\lambda}_h}{d\mathbf{t}_T}\right|\right|_{L^2(e)} ||\varphi||_{H^1(\Delta(e))} \,. \tag{3.45}$$

Also, we observe that $\langle \boldsymbol{\tau}_h\cdot\boldsymbol{\nu},1\rangle_\Gamma = \langle \mathbf{curl}(\varphi_h)\cdot\boldsymbol{\nu},1\rangle_\Gamma = \langle \frac{d}{d\mathbf{t}_T}\varphi_h,1\rangle_\Gamma = 0$, which shows that the third term on the right hand side of the inequality in Lemma 3.3.2 vanishes.

For the remaining term on Γ_0 we note that

$$|\langle\, (\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h)\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}| = |\langle \mathbf{curl}(\varphi-\varphi_h)\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}| = \left| \langle \varphi-\varphi_h, \frac{dg}{d\mathbf{t}_T}\rangle_{\Gamma_0} \right| ,$$

which, applying Lemma 3.3.1, leads to

$$|\langle\, (\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h)\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}| \leq C \left\{ \sum_{e\subseteq E_h(\Gamma_0)} h_e \left|\left|\frac{dg}{d\mathbf{t}_T}\right|\right|^2_{L^2(e)} \right\}^{1/2} ||\varphi||_{H^1(\Omega)} \,. \tag{3.46}$$

Therefore, using (3.39), (3.41), (3.42), (3.43), (3.44), (3.45) and (3.46), we deduce that

$$\left|A((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h,\tilde{\lambda}-\tilde{\lambda}_h-\tilde{\mu}_h))\right| + |\langle(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h)\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}|$$

$$\leq C\,\hat{\eta}_1 \left\{||\varphi||^2_{H^1(\Omega)} + ||w||^2_{H^1(\Omega)}\right\}^{1/2} , \tag{3.47}$$

where $\hat{\eta}_1 := \left\{\sum_{T\in\mathcal{T}_h}\left(\eta^2_{1,T} - ||f+\operatorname{div}\boldsymbol{\sigma}_h||^2_{L^2(T)}\right)\right\}^{1/2}$. Now, since $\int_\Omega \varphi\, dx = 0$, we obtain from (3.47) and (3.40),

$$\left|A((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h,\tilde{\lambda}-\tilde{\lambda}_h-\tilde{\mu}_h))\right| + |\langle(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h)\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}|$$

$$\leq C\,\hat{\eta}_1 \left\{|\varphi|^2_{H^1(\Omega)} + \left|\left|\tilde{\lambda}-\tilde{\lambda}_h\right|\right|^2_{H^{1/2}(\Gamma)}\right\}^{1/2} = C\,\hat{\eta}_1 \left|\left|(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}},\tilde{\lambda}-\tilde{\lambda}_h)\right|\right|^2_{\tilde{H}} .$$

Hence, in virtue of Lemma 3.3.2 and the continuous dependence result given by the estimate $||\boldsymbol{\sigma}^*||_{H(\mathrm{div};\Omega)} \leq \bar{C}\,||f+\operatorname{div}\boldsymbol{\sigma}_h||_{L^2(\Omega)}$, we conclude that

$$\left|\left|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\lambda}-\tilde{\lambda}_h)\right|\right|_{\tilde{H}} \leq C\left\{||f+\operatorname{div}\boldsymbol{\sigma}_h||^2_{L^2(\Omega)} + \hat{\eta}_1^2\right\}^{1/2} = C\left\{\sum_{T\in\mathcal{T}_h}\eta^2_{1,T}\right\}^{1/2} , \tag{3.48}$$

which ends the proof. ■

In order to complete our a-posteriori error estimate, we need to provide the estimate for $(u-u_h)$ and $(p-p_h)$. For this purpose, the following lemma is necessary.

Lema 3.3.3 *For any* $\boldsymbol{\tau} \in H(\text{div}, \Omega)$ *there exists* $\mathbf{r}_{\boldsymbol{\tau}} \in [H^1(\Omega)]^2$ *such that* $\text{div}(\mathbf{r}_{\boldsymbol{\tau}}) = \text{div}\,\boldsymbol{\tau}$ *in* Ω, $\langle \mathbf{r}_{\boldsymbol{\tau}} \cdot \boldsymbol{\nu}, 1\rangle_\Gamma = \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, 1\rangle_\Gamma$, *and*

$$||\mathbf{r}_{\boldsymbol{\tau}}||_{[H^1(\Omega)]^2} \leq \bar{C}\, ||\boldsymbol{\tau}||_{H(\text{div};\Omega)}\,,$$

with a constant $\bar{C} > 0$, *independent of* $\boldsymbol{\tau}$.

Proof: We proceed similarly as the proof of Lemma 4.4 in [39]. Let $\mathcal{O}$ be the convex domain whose boundary is the circle Γ, that is $\mathcal{O} := \bar{\Omega}_0 \cup \Omega$. Then, given $\boldsymbol{\tau} \in H(\text{div};\Omega)$ we consider the function $\tilde{f} \in L^2(\mathcal{O})$ defined by

$$\tilde{f} := \begin{cases} \text{div}\,\boldsymbol{\tau} \quad \text{in} \quad \Omega \\[2ex] -\dfrac{1}{|\Omega_0|} \left\{ \displaystyle\int_\Omega \text{div}\,\boldsymbol{\tau}\, dx \;-\; \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, 1\rangle_\Gamma \right\} \quad \text{in} \quad \Omega_0\,. \end{cases}$$

Since $\int_{\mathcal{O}} \tilde{f}\, dx - \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, 1\rangle_\Gamma = 0$, we deduce that the weak solution $z \in H^1(\mathcal{O})$ of: $\Delta\, z = \tilde{f}$ in $\mathcal{O}$, $\frac{\partial z}{\partial \boldsymbol{\nu}} = \frac{1}{|\Gamma|}\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, 1\rangle_\Gamma$ on Γ, and $\int_{\mathcal{O}} z\, dx = 0$, is uniquely determined. In addition, a classical regularity result and the trace theorem in $H(\text{div};\Omega)$ imply that $z \in H^2(\mathcal{O})$ and

$$||z||_{H^2(\mathcal{O})} \;\leq\; C \left\{ ||\tilde{f}||_{L^2(\mathcal{O})} \;+\; |\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, 1\rangle_\Gamma| \right\} \;\leq\; \bar{C}\, ||\boldsymbol{\tau}||_{H(\text{div};\Omega)}\,. \tag{3.49}$$

Thus, we put $\mathbf{r}_{\boldsymbol{\tau}} := \nabla z|_\Omega$ and observe that $\mathbf{r}_{\boldsymbol{\tau}} \in [H^1(\Omega)]^2$, $\text{div}\,(\mathbf{r}_{\boldsymbol{\tau}}) = \tilde{f} = \text{div}\,\boldsymbol{\tau}$ in Ω, and $\langle \mathbf{r}_{\boldsymbol{\tau}} \cdot \boldsymbol{\nu}, 1\rangle_\Gamma = \langle \frac{\partial z}{\partial \boldsymbol{\nu}}, 1\rangle_\Gamma = \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, 1\rangle_\Gamma$. Finally, (3.49) yields

$$||\mathbf{r}_{\boldsymbol{\tau}}||_{[H^1(\Omega)]^2} \;\leq\; ||z||_{H^2(\Omega)} \;\leq\; ||z||_{H^2(\mathcal{O})} \;\leq\; \bar{C}\, ||\boldsymbol{\tau}||_{H(\text{div};\Omega)}\,,$$

which completes the proof of the lemma. ■

The a-posteriori error estimate for $(u-u_h, p-p_h) \in \tilde{Q}$ is established now.

Teorema 3.3.3 *There exists* $C > 0$, *independent of* h, *such that*

$$||(u-u_h, p-p_h)||_{\tilde{Q}} \leq C \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\}^{1/2}\,.$$

Proof: The continuous inf-sup condition for $\tilde{B}$ (cf. proof of Theorem 3.1.2) yields the inequality

$$||(u - u_h, p - p_h)||_{\tilde{Q}} \leq \tilde{C} \sup_{\substack{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \frac{\tilde{B}((\boldsymbol{\tau}, 0), (u, p)) - \tilde{B}((\boldsymbol{\tau}, 0), (u_h, p_h))}{||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}} . \tag{3.50}$$

Now, given $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$ we consider the function $\mathbf{r}_{\boldsymbol{\tau}}$ provided by Lemma 3.3.3 and note that

$$\tilde{B}((\boldsymbol{\tau}, 0), (u, p)) := \int_\Omega u \,\mathrm{div}\, \boldsymbol{\tau} \, dx - p \,\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma$$

$$= \int_\Omega u \,\mathrm{div}\,(\mathbf{r}_{\boldsymbol{\tau}}) \, dx - p \,\langle \mathbf{r}_{\boldsymbol{\tau}} \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = \tilde{B}((\mathbf{r}_{\boldsymbol{\tau}}, 0), (u, p)) \,,$$

which, according to the first equation of (3.8), gives

$$\tilde{B}((\boldsymbol{\tau}, 0), (u, p)) = - A((\boldsymbol{\sigma}, \tilde{\lambda}), (\mathbf{r}_{\boldsymbol{\tau}}, 0)) + \langle \mathbf{r}_{\boldsymbol{\tau}} \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0} . \tag{3.51}$$

Similarly, using now the properties of the operator $\mathcal{E}_h$, we easily deduce that

$$\tilde{B}((\boldsymbol{\tau}, 0)(u_h, p_h)) := \int_\Omega u_h \,\mathrm{div}\, \boldsymbol{\tau} \, dx - p_h \,\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma$$

$$= \int_\Omega u_h \,\mathrm{div}\,(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})) \, dx - p_h \,\langle \mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = \tilde{B}((\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}), 0), (u_h, p_h)) \,,$$

which, in virtue of the first equation of (3.20), yields

$$\tilde{B}((\boldsymbol{\tau}, 0), (u_h, p_h)) = - A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}), 0)) + \langle \mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0} . \tag{3.52}$$

Then, by replacing (3.51) and (3.52) back into (3.50), we obtain

$$||(u - u_h, p - p_h)||_{\tilde{Q}} \leq$$

$$C \sup_{\substack{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \left\{ \frac{\begin{array}{c} A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}, 0)) \; - \; A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (\mathbf{r}_{\boldsymbol{\tau}}, 0)) \\ - \; \langle (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0} \end{array}}{|||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}} \right\} . \tag{3.53}$$

We now bound the terms on the right hand side of (3.53). First, the boundedness of A, Theorem 3.3.2, and Lemma 3.3.3 imply that

$$|\, A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (\mathbf{r}_{\boldsymbol{\tau}}, 0)) \,| \;\leq\; C \left\{ \sum_{T \in \mathcal{T}_h} \eta_{1,T}^2 \right\}^{1/2} ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)} . \tag{3.54}$$

Next, since $\mathcal{E}_h$ satisfies $\int_e \mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}\, ds = \int_e \mathbf{r}_{\boldsymbol{\tau}} \cdot \boldsymbol{\nu}\, ds$ for all $e \in E_h$, we deduce that

$$\langle (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0} \;=\; \langle (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, g - s_h \rangle_{\Gamma_0} \quad \forall\, s_h \in S_h \,,$$

where S_h is the space of piecewise constant functions on the partition of Γ_0 induced by the triangulation $\mathcal{T}_h$, and hence

$$|\, \langle (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0} \,| \;\leq\; \sum_{e \in E_h(\Gamma_0)} ||g - s_h||_{L^2(e)}\; ||(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}||_{L^2(e)} \tag{3.55}$$

for all $s_h \in S_h$. But, with the same interpolation results used in the proof of Theorem 4.5 in [39], we can prove that

$$||(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}||_{L^2(e)} \;\leq\; \tilde{C}\, h_e^{1/2}\, ||\mathbf{r}_{\boldsymbol{\tau}}||_{[H^1(T_e)]^2} \,, \tag{3.56}$$

where T_e is the triangle to which e belongs, and $\tilde{C}$, a constant independent of h, may depend on the minimum angle of $\mathcal{T}_h$.

In this way, (3.55), (3.56), Cauchy-Schwarz's inequality, and Lemma 3.3.3 lead to

$$\begin{aligned} |\, \langle (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0} \,| \;&\leq\; C \inf_{s_h \in S_h} \left\{ \sum_{e \in E_h(\Gamma_0)} h_e\, ||g - s_h||^2_{L^2(e)} \right\}^{1/2} ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)} \\ &=\; C \left\{ \sum_{e \in E_h(\Gamma_0)} h_e\, ||g - \hat{g}_h||^2_{L^2(e)} \right\}^{1/2} ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)} \,, \end{aligned} \tag{3.57}$$

where $\hat{g}_h|_e := \frac{1}{h_e} \int_e g\, ds$ for all $e \in E_h(\Gamma_0)$.

In order to bound the first term on the right hand side of (3.53), we recall from (3.25) that

$$||\mathcal{E}_h(\boldsymbol{\zeta}) - \boldsymbol{\zeta}||_{[L^2(T)]^2} \;\leq\; C\, h_T\, ||\boldsymbol{\zeta}||_{[H^1(T)]^2} \quad \forall\, \boldsymbol{\zeta} \in [H^1(\Omega)]^2 \,, \quad \forall\, T \in \mathcal{T}_h \,. \tag{3.58}$$

Thus, applying Cauchy-Schwarz's inequality, (3.58) with $\boldsymbol{\zeta} = \mathbf{r}_{\boldsymbol{\tau}}$, Lemma 3.3.3, and following a similar analysis to the one yielding (3.57), we can show that

$$\Big|\, A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}, 0)) \,\Big|$$

$$\leq \left| \int_\Omega (\boldsymbol{\kappa}^{-1} \boldsymbol{\sigma}_h) \cdot (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}})\, dx \right| + \left| \langle (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, \tilde{\lambda}_h \rangle_\Gamma \right|$$

$$\leq C \left\{ \sum_{T \in \mathcal{T}_h} h_T^2 \, ||\boldsymbol{\kappa}^{-1} \boldsymbol{\sigma}_h||^2_{[L^2(T)]^2} + \sum_{e \in E_h(\Gamma)} h_e \, ||\tilde{\lambda}_h - \hat{\lambda}_h||^2_{L^2(e)} \right\}^{1/2} ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)} \,, \tag{3.59}$$

where $\hat{\lambda}_h|_e := \frac{1}{h_e} \int_e \tilde{\lambda}_h \, ds$ for all $e \in E_h(\Gamma)$.

Therefore, by inserting (3.54), (3.57), and (3.59) back into (3.53), we conclude the required estimate. ■

Finally, the proof of Theorem 3.3.1, which is the main contribution of this section, follows straightforward from Theorems 3.3.2 and 3.3.3.

3.4 An implicit a-posteriori estimate

In this section we apply a Bank-Weiser type procedure (similarly as in [36] and [41]) to our model problem. For the classical Bank-Weiser's approach we refer to [10]. As a result of our analysis we obtain a second reliable a-posteriori error estimate of implicit type, which depends on the solution of local problems. In addition, we bound these local solutions, introduce a suitable averaging technique, and transform the original estimate into an explicit one.

We first need a symmetric, bounded, and strongly coercive bilinear form $\mathbf{A}$ on the space $\tilde{H} := H(\mathrm{div};\Omega) \times H_0^{1/2}(\Gamma)$. In particular, from now on we consider

$$\mathbf{A}((\boldsymbol{\zeta}, \rho), (\boldsymbol{\tau}, \mu)) := \langle \boldsymbol{\zeta}, \boldsymbol{\tau} \rangle_{H(\mathrm{div};\Omega)} + \langle \mathbf{W}(\rho), \mu \rangle_\Gamma \quad \forall \, (\boldsymbol{\zeta}, \rho), \, (\boldsymbol{\tau}, \mu) \in \tilde{H} \,. \tag{3.60}$$

Then, given the solutions $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ and $((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h)) \in \tilde{H}_h \times \tilde{Q}_h$ of the continuous and Galerkin schemes (3.8) and (3.20), respectively, we define the $\tilde{H}$-Ritz projection of the error with respect to $\mathbf{A}$, as the unique $(\bar{\boldsymbol{\sigma}}, \bar{\lambda}) \in \tilde{H}$ such that

$$\mathbf{A}((\bar{\boldsymbol{\sigma}}, \bar{\lambda}), (\boldsymbol{\tau}, \mu)) = A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (\boldsymbol{\tau}, \mu)) + \tilde{B}((\boldsymbol{\tau}, \mu), (u - u_h, p - p_h)) \tag{3.61}$$

for all $(\boldsymbol{\tau}, \mu) \in \tilde{H}$. The existence of such a $(\bar{\boldsymbol{\sigma}}, \bar{\lambda})$ is guaranteed by the fact that the right hand side of (3.61) (as a mapping acting on $(\boldsymbol{\tau}, \mu)$) constitutes a linear and bounded functional on $\tilde{H}$.

Now, given $T \in \mathcal{T}_h$ and $e \in E(T)$, we denote by $\langle \cdot, \cdot \rangle_{H(\mathrm{div};T)}$ the inner product of $H(\mathrm{div};T)$ and let $\langle \cdot, \cdot \rangle_{\partial T}$ be the duality pairing between $H^{-1/2}(\partial T)$ and $H^{1/2}(\partial T)$ with respect to the $L^2(\partial T)$-inner product. In addition, we let $H_{00}^{1/2}(e)$ be the space of functions in $H^{1/2}(e)$ whose extensions by zero to the rest of ∂T are in $H^{1/2}(\partial T)$, and denote by $\langle \cdot, \cdot \rangle_e$ the duality pairing between $H_{00}^{-1/2}(e)$ and $H_{00}^{1/2}(e)$ with respect to the $L^2(e)$-inner product. As before, $\boldsymbol{\nu}_T$ stands for the unit outward normal to ∂T.

The following theorem provides an important upper bound for the Ritz projection $(\bar{\boldsymbol{\sigma}}, \bar{\lambda}) \in \tilde{H}$.

Teorema 3.4.1 *Assume there exists $s > 2$ such that $g \in H^{1/2}(\Gamma_0) \cap W^{1-1/s,s}(\Gamma_0)$ and let $\tilde{\varphi}_h$ be a function in $H^1(\Omega) \cap W^{1,s}(\Omega)$ such that $\tilde{\varphi}_h(\bar{x}) = g(\bar{x})$ for each vertex $\bar{x}$ of $\mathcal{T}_h$ lying on Γ_0, and $\tilde{\varphi}_h(\bar{x}) = \tilde{\lambda}_h(\bar{x}) + p_h$ for each vertex $\bar{x}$ of $\mathcal{T}_h$ lying on Γ. Further, for each $T \in \mathcal{T}_h$ let $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div}; T)$ be the unique solution of the local problem*

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{H(\mathrm{div};T)} \;=\; G_{h,T}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in H(\mathrm{div}; T)\,, \tag{3.62}$$

where $G_{h,T} \in H(\mathrm{div}; T)'$ is defined by

$$G_{h,T}(\boldsymbol{\tau}) := -\int_T (\boldsymbol{\kappa}^{-1} \boldsymbol{\sigma}_h) \cdot \boldsymbol{\tau}\, dx \;-\; \int_T u_h \,\mathrm{div}\, \boldsymbol{\tau}\, dx \;+\; \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \tilde{\varphi}_h \rangle_{\partial T}$$
$$+ \sum_{e \in E(T) \cap E_h(\Gamma)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h \rangle_e \;+ \sum_{e \in E(T) \cap E_h(\Gamma_0)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, g - \tilde{\varphi}_h \rangle_e\,. \tag{3.63}$$

Then there holds

$$\mathbf{A}((\bar{\boldsymbol{\sigma}}, \bar{\lambda}), (\bar{\boldsymbol{\sigma}}, \bar{\lambda})) \;\le\; \sum_{T \in \mathcal{T}_h} ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} \;+\; ||\mathbf{W}^{-1}||\; ||\, 2\, \mathbf{W}(\tilde{\lambda}_h) \;+\; \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}\, ||^2_{H^{-1/2}(\Gamma)}\,. \tag{3.64}$$

Proof: We first observe from (3.8) that

$$A((\boldsymbol{\sigma}, \tilde{\lambda}), (\boldsymbol{\tau}, \mu)) \;+\; \tilde{B}((\boldsymbol{\tau}, \mu), (u, p)) \;=\; \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0}\,,$$

and hence

$$\mathbf{A}((\bar{\boldsymbol{\sigma}}, \bar{\lambda}), (\boldsymbol{\tau}, \mu)) = \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0} - A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\boldsymbol{\tau}, \mu)) - \tilde{B}((\boldsymbol{\tau}, \mu), (u_h, p_h)) \tag{3.65}$$

for all $(\boldsymbol{\tau}, \mu) \in \tilde{H}$. Thus, since $\mathbf{A}$ is symmetric and strongly coercive on $\tilde{H}$, we have that

$$-\frac{1}{2}\, \mathbf{A}((\bar{\boldsymbol{\sigma}}, \bar{\lambda}), (\bar{\boldsymbol{\sigma}}, \bar{\lambda})) \;=\; \min_{(\boldsymbol{\tau}, \mu) \in \tilde{H}} \left\{ \frac{1}{2}\, \mathbf{A}((\boldsymbol{\tau}, \mu), (\boldsymbol{\tau}, \mu)) - \mathbf{A}((\bar{\boldsymbol{\sigma}}, \bar{\lambda}), (\boldsymbol{\tau}, \mu)) \right\}, \tag{3.66}$$

which, according to (3.65), becomes

$$-\frac{1}{2}\, \mathbf{A}((\bar{\boldsymbol{\sigma}}, \bar{\lambda}), (\bar{\boldsymbol{\sigma}}, \bar{\lambda})) \;=\; \min_{(\boldsymbol{\tau}, \mu) \in \tilde{H}} \mathcal{J}(\boldsymbol{\tau}, \mu)\,, \tag{3.67}$$

with

$$\mathcal{J}(\boldsymbol{\tau}, \mu) := \frac{1}{2}\, \mathbf{A}((\boldsymbol{\tau}, \mu), (\boldsymbol{\tau}, \mu)) + A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\boldsymbol{\tau}, \mu)) + \tilde{B}((\boldsymbol{\tau}, \mu), (u_h, p_h)) - \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0}\,. \tag{3.68}$$

On the other hand, the hypotheses on g and $\tilde{\varphi}_h$ imply, according to the Sobolev imbedding theorems, that $(g - \tilde{\varphi}_h)|_e \in H_{00}^{1/2}(e)$ for each $e \in E_h(\Gamma_0)$ and $(\tilde{\lambda}_h + p_h - \tilde{\varphi}_h)|_e \in H_{00}^{1/2}(e)$ for each $e \in E_h(\Gamma)$, whence

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g - \tilde{\varphi}_h \rangle_{\Gamma_0} = \sum_{e \in E_h(\Gamma_0)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g - \tilde{\varphi}_h \rangle_e \tag{3.69}$$

and

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h \rangle_{\Gamma} = \sum_{e \in E_h(\Gamma)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h \rangle_e . \tag{3.70}$$

Further, we also get $- \sum_{T \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \tilde{\varphi}_h \rangle_{\partial T} \; + \; \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \tilde{\varphi}_h \rangle_{\Gamma} \; + \; \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \tilde{\varphi}_h \rangle_{\Gamma_0} \; = \; 0$, which is then added to the quadratic functional $\mathcal{J}$.

In this way, recalling the definitions of $\mathbf{A}$, A, and $\tilde{B}$, and using (3.69) and (3.70), we obtain

$$\mathcal{J}(\boldsymbol{\tau}, \mu) \; = \; \sum_{T \in \mathcal{T}_h} \mathcal{J}_{1,T}(\boldsymbol{\tau}_T) \; + \; \mathcal{J}_2(\mu) \, , \tag{3.71}$$

where $\boldsymbol{\tau}_T$ is the restriction $\boldsymbol{\tau}|_T$,

$$\mathcal{J}_{1,T}(\boldsymbol{\tau}_T) := \frac{1}{2} \, ||\boldsymbol{\tau}_T||^2_{H(\mathrm{div};T)} \; - \; G_{h,T}(\boldsymbol{\tau}_T) \, , \tag{3.72}$$

and

$$\mathcal{J}_2(\mu) := \frac{1}{2} \, \langle \mathbf{W}(\mu), \mu \rangle_{\Gamma} \; + \; \langle 2 \, \mathbf{W}(\tilde{\lambda}_h) \; + \; \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}, \mu \rangle_{\Gamma} \, . \tag{3.73}$$

We observe here that

$$\min_{\boldsymbol{\tau}_T \in H(\mathrm{div};T)} \mathcal{J}_{1,T}(\boldsymbol{\tau}_T) \; = \; - \, \frac{1}{2} \, ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} \, , \tag{3.74}$$

where $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div}; T)$ is the unique solution of the local problem (3.62).

Hence, replacing (3.71) up to (3.73) back into (3.67), noting that $H(\mathrm{div}; \Omega)$ is contained in the *broken* space

$$H(\mathrm{div}; \Omega)^{br} := \{ \boldsymbol{\tau} \in [L^2(\Omega)]^2 : \quad \boldsymbol{\tau}_T \in H(\mathrm{div}; T) \quad \forall T \in \mathcal{T}_h \} \, ,$$

and using (3.74), we deduce that

$$\begin{aligned} - \, \frac{1}{2} \, \mathbf{A}((\bar{\boldsymbol{\sigma}}, \bar{\lambda}), (\bar{\boldsymbol{\sigma}}, \bar{\lambda})) \; &= \; \min_{\boldsymbol{\tau} \in H(\mathrm{div};\Omega)} \left\{ \sum_{T \in \mathcal{T}_h} \mathcal{J}_{1,T}(\boldsymbol{\tau}_T) \right\} + \min_{\mu \in H_0^{1/2}(\Gamma)} \mathcal{J}_2(\mu) \\ &\geq \sum_{T \in \mathcal{T}_h} \min_{\boldsymbol{\tau}_T \in H(\mathrm{div};T)} \mathcal{J}_{1,T}(\boldsymbol{\tau}_T) \; + \; \min_{\mu \in H_0^{1/2}(\Gamma)} \mathcal{J}_2(\mu) \end{aligned}$$

$$= -\frac{1}{2} \sum_{T\in\mathcal{T}_h} ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} - \frac{1}{2} \langle \mathbf{W}(\rho), \rho\rangle_\Gamma , \tag{3.75}$$

where $\rho \in H_0^{1/2}(\Gamma)$ is the unique solution to the equation

$$\langle \mathbf{W}(\rho), \mu\rangle_\Gamma = - \langle 2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}, \mu\rangle_\Gamma \qquad \forall\, \mu \in H_0^{1/2}(\Gamma) . \tag{3.76}$$

It follows from (3.76) that

$$-\frac{1}{2} \langle \mathbf{W}(\rho), \rho\rangle_\Gamma \geq -\frac{1}{2} ||\mathbf{W}^{-1}||\; ||2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}||^2_{H^{-1/2}(\Gamma)} ,$$

whence (3.75) yields

$$-\frac{1}{2} \mathbf{A}((\bar{\boldsymbol{\sigma}}, \bar{\lambda}), (\bar{\boldsymbol{\sigma}}, \bar{\lambda})) \geq -\frac{1}{2} \sum_{T\in\mathcal{T}_h} ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} - \frac{1}{2} ||\mathbf{W}^{-1}||\; ||2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}||^2_{H^{-1/2}(\Gamma)} ,$$

which completes the proof. ∎

It is important to remark that the above theorem does not require any further condition on $\tilde{\varphi}_h$, and hence, in principle, this function can be chosen in many different ways. However, we will prove below that the proposed a-posteriori error estimate becomes efficient up to a term depending on $(u - \tilde{\varphi}_h)$. This property is called *quasi-efficiency*. Therefore, one should try to choose $\tilde{\varphi}_h$ as close as possible, at least empirically, to the exact solution u.

We now give the main reliable a-posteriori error estimate for the Galerkin scheme (3.20), which makes use of the $\tilde{H}$-Ritz projection $(\bar{\boldsymbol{\sigma}}, \bar{\lambda})$ and the associated upper bound provided by Theorem 3.4.1.

Teorema 3.4.2 *Let $\tilde{\varphi}_h$ be as indicated in Theorem 3.4.1, and for each $T \in \mathcal{T}_h$ let $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div};T)$ be the unique solution of the local problem (3.62). Then there exists $C > 0$, independent of h, such that*

$$\left|\left|((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (u - u_h, p - p_h))\right|\right|_{\tilde{H}\times\tilde{Q}} \leq C \left\{ \sum_{T\in\mathcal{T}_h} \theta_T^2 + R_\Gamma^2 \right\}^{1/2} ,$$

where

$$\theta_T^2 := ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} + ||\, f + \mathrm{div}\, \boldsymbol{\sigma}_h \,||^2_{L^2(T)}$$

and

$$R_\Gamma^2 := ||\, 2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} \,||^2_{H^{-1/2}(\Gamma)} .$$

Proof: The continuous dependence result given by Theorem 3.1.2 is equivalent to stating that the variational formulation (3.8) satisfies a global inf-sup condition, which means that there exists $\tilde{C} > 0$ such that

$$||((\boldsymbol{\zeta},\rho),(w,r))||_{\tilde{H}\times\tilde{Q}}$$
$$\leq \tilde{C} \sup_{\substack{((\boldsymbol{\tau},\mu),(v,q))\in\tilde{H}\times\tilde{Q}\\ ||((\boldsymbol{\tau},\mu),(v,q))||\leq 1}} \Big\{ A((\boldsymbol{\zeta},\rho),(\boldsymbol{\tau},\mu)) + \tilde{B}((\boldsymbol{\tau},\mu),(w,r)) + \tilde{B}((\boldsymbol{\zeta},\rho),(v,q)) \Big\}$$

for all $((\boldsymbol{\zeta},\rho),(w,r)) \in \tilde{H}\times\tilde{Q}$.

In particular, taking $((\boldsymbol{\zeta},\rho),(w,r)) := ((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\lambda}-\tilde{\lambda}_h),(u-u_h,p-p_h))$ in the above inequality, and using the definition of the Ritz projection $(\bar{\boldsymbol{\sigma}},\bar{\lambda}) \in \tilde{H}$ (cf. (3.61)), and the statements of the continuous and Galerkin schemes (3.8) and (3.20), we obtain that

$$||((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\lambda}-\tilde{\lambda}_h),(u-u_h,p-p_h))||_{\tilde{H}\times\tilde{Q}}$$
$$\leq \tilde{C} \sup_{\substack{((\boldsymbol{\tau},\mu),(v,q))\in\tilde{H}\times\tilde{Q}\\ ||((\boldsymbol{\tau},\mu),(v,q))||\leq 1}} \left\{ \mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\boldsymbol{\tau},\mu)) - \int_\Omega (f + \operatorname{div}\boldsymbol{\sigma}_h)\, v\, dx \right\}.$$

Hence, using the properties of $\mathbf{A}$, and applying Cauchy-Schwarz's inequality, we deduce that there exists $\bar{C} > 0$ such that

$$||((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\lambda}-\tilde{\lambda}_h),(u-u_h,p-p_h))||_{\tilde{H}\times\tilde{Q}}$$
$$\leq \bar{C} \left\{ \mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\bar{\boldsymbol{\sigma}},\bar{\lambda})) + \sum_{T\in\mathcal{T}_h} ||f + \operatorname{div}\boldsymbol{\sigma}_h||^2_{L^2(T)} \right\}^{1/2},$$

which, together with the upper bound (3.64), finishes the proof. ∎

The following lemma provides a-priori estimates for the solution of the local problem (3.62). They will be used to show the *quasi-efficiency* of the estimate provided by Theorem 3.4.2, and also to deduce an explicit reliable a-posteriori error estimate based on a suitable averaging technique.

Lema 3.4.1 *Let $\tilde{\varphi}_h$ be as indicated in Theorem 3.4.1, and for each $T \in \mathcal{T}_h$ let $\hat{\boldsymbol{\sigma}}_T \in H(\operatorname{div};T)$ be the unique solution of the local problem (3.62). Then there exists $C > 0$, independent of h and T, such that*

$$||\hat{\boldsymbol{\sigma}}_T||_{H(\operatorname{div};T)} \leq C \Big\{ ||(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) - \nabla\tilde{\varphi}_h||^2_{[L^2(T)]^2} + ||u_h - \tilde{\varphi}_h||^2_{L^2(T)}$$
$$+ \sum_{e\in E(T)\cap E_h(\Gamma)} ||\tilde{\lambda}_h + p_h - \tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)} + \sum_{e\in E(T)\cap E_h(\Gamma_0)} ||g - \tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)} \Big\}^{1/2}. \quad (3.77)$$

In addition, for any $z \in H^1(\Omega) \cap W^{1,s}(\Omega)$*, with* $s > 2$*, such that* $z = g$ *on* Γ_0*, we get*

$$||\hat{\boldsymbol{\sigma}}_T||_{H(\mathrm{div};T)} \le C \Big\{ ||\,(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \,-\, \nabla z\,||^2_{[L^2(T)]^2}$$

$$+ ||\,u_h \,-\, z\,||^2_{L^2(T)} \,+\, ||\mathbf{J}_{h,T}(z)||^2_{H^{1/2}(\partial T)}\Big\}^{1/2}, \tag{3.78}$$

where $\mathbf{J}_{h,T}(z) := \begin{cases} 0 \quad on \quad \partial T \cap \Gamma_0 \\ z - (\tilde{\lambda}_h + p_h) \quad on \quad \partial T \cap \Gamma \\ z - \tilde{\varphi}_h \quad otherwise \end{cases}.$

Proof: We first recall from (3.62) that $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div};T)$ and $\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau}\rangle_{H(\mathrm{div};T)} = G_{h,T}(\boldsymbol{\tau})$ for all $\boldsymbol{\tau} \in H(\mathrm{div};T)$, where

$$G_{h,T}(\boldsymbol{\tau}) := -\int_T (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\boldsymbol{\tau}\,dx \,-\, \int_T u_h \,\mathrm{div}\,\boldsymbol{\tau}\,dx \,+\, \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \tilde{\varphi}_h\rangle_{\partial T}$$

$$+ \sum_{e\in E(T)\cap E_h(\Gamma)} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h\rangle_e \,+\, \sum_{e\in E(T)\cap E_h(\Gamma_0)} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, g - \tilde{\varphi}_h\rangle_e\,. \tag{3.79}$$

Since $\tilde{\varphi}_h \in H^1(\Omega)$, we apply Gauss's formula to obtain

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \tilde{\varphi}_h\rangle_{\partial T} \,=\, \int_T \nabla\tilde{\varphi}_h \cdot \boldsymbol{\tau}\,dx \,+\, \int_T \tilde{\varphi}_h \,\mathrm{div}\,\boldsymbol{\tau}\,dx\,.$$

Then, replacing this expression back into (3.79), applying Cauchy-Schwarz's inequality, and using the fact that $||\hat{\boldsymbol{\sigma}}_T||_{H(\mathrm{div};T)} = ||G_{h,T}||_{H(\mathrm{div};T)'}$, we arrive to (3.77).

On the other hand, given $z \in H^1(\Omega) \cap W^{1,s}(\Omega)$, with $s > 2$, such that $z = g$ on Γ_0, we obtain

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \tilde{\varphi}_h\rangle_{\partial T} + \sum_{e\in E(T)\cap E_h(\Gamma)} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h\rangle_e + \sum_{e\in E(T)\cap E_h(\Gamma_0)} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, g - \tilde{\varphi}_h\rangle_e$$

$$= \quad \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, z\rangle_{\partial T} \,-\, \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, z - \tilde{\varphi}_h\rangle_{\partial T}$$

$$+ \sum_{e\in E(T)\cap E_h(\Gamma)} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h\rangle_e \,-\, \sum_{e\in E(T)\cap E_h(\Gamma_0)} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \tilde{\varphi}_h - z\rangle_e$$

$$= \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, z\rangle_{\partial T} \,-\, \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \mathbf{J}_{h,T}(z)\rangle_{\partial T}\,,$$

which, replaced back into (3.79), yields (3.78) and ends the proof. ∎

We show next that the reliable a-posteriori error estimate from Theorem 3.4.2 is *quasi-efficient*, that is, it is efficient up to a term depending on the traces of $(u - \tilde{\varphi}_h)$ on the edges of $\mathcal{T}_h$. Indeed, we have the following lemma.

Lema 3.4.2 *Let $\tilde{\varphi}_h$ be as indicated in Theorem 3.4.1, and assume that $u \in W^{1,s}(\Omega)$, with $s > 2$. Then there exists $C > 0$, independent of h, such that for all $T \in \mathcal{T}_h$*

$$\theta_T^2 \;\le\; C \left\{ ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{H(\mathrm{div};T)} \;+\; ||u - u_h||^2_{L^2(T)} \;+\; \|\mathbf{J}_{h,T}(u)\|^2_{H^{1/2}(\partial T)} \right\}, \tag{3.80}$$

where $\mathbf{J}_{h,T}(u) := \begin{cases} 0 \quad on \quad \partial T \cap \Gamma_0 \\ \lambda - \lambda_h \quad on \quad \partial T \cap \Gamma \\ u - \tilde{\varphi}_h \quad otherwise \end{cases}.$

Further, there exists $\tilde{C} > 0$, independent of h, such that

$$\sum_{T \in \mathcal{T}_h} \theta_T^2 \;+\; R_\Gamma^2 \;\le$$

$$\tilde{C} \left\{ ||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (u - u_h, p - p_h))||^2_{\tilde{H} \times \tilde{Q}} \;+\; \sum_{T \in \mathcal{T}_h} \|\mathbf{J}_{h,T}(u)\|^2_{H^{1/2}(\partial T)} \right\}. \tag{3.81}$$

Proof: From the second equation of (3.8) we get $\mathrm{div}\,\boldsymbol{\sigma} = -f$ in Ω and $\langle \boldsymbol{\sigma} \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = 0$. In addition, from the first equation of (3.8) we deduce that $\boldsymbol{\kappa}^{-1}\,\boldsymbol{\sigma} = \nabla u$ in Ω, $u = \tilde{\lambda} + p$ on Γ, $u = g$ on Γ_0, and $2\,\mathbf{W}(\tilde{\lambda}) + \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = 0$ on Γ.

Then, applying Lemma 3.4.1 (cf. (3.78)) with $z = u$, we obtain that

$$||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} \;\le$$

$$C \left\{ ||(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) - (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma})||^2_{[L^2(T)]^2} \;+\; ||u_h - u||^2_{L^2(T)} \;+\; ||\mathbf{J}_{h,T}(u)||^2_{H^{1/2}(\partial T)} \right\}. \tag{3.82}$$

Hence, (3.80) follows from (3.82) and the fact that

$$\theta_T^2 := ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} \;+\; ||f + \mathrm{div}\,\boldsymbol{\sigma}_h||_{L^2(T)} \;=\; ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} \;+\; ||\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||_{L^2(T)}\,.$$

On the other hand, using that $(2\,\mathbf{W}(\tilde{\lambda}) + \boldsymbol{\sigma} \cdot \boldsymbol{\nu}) = 0$ on Γ, and applying the boundedness of $\mathbf{W}$ and the trace theorem in $H(\mathrm{div};\Omega)$, we obtain that

$$\begin{aligned} R_\Gamma^2 \;&:=\; ||2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}||^2_{H^{-1/2}(\Gamma)} \\ &\le\; C \left\{ ||\tilde{\lambda}_h \,-\, \tilde{\lambda}||^2_{H^{1/2}(\Gamma)} \;+\; ||\boldsymbol{\sigma}_h \,-\, \boldsymbol{\sigma}||^2_{H(\mathrm{div};\Omega)} \right\}. \end{aligned} \tag{3.83}$$

Finally, summing up in (3.80) over all $T \in \mathcal{T}_h$, and adding (3.83), we conclude (3.81) and finish the proof. ■

The *quasi-efficiency* provided by Lemma 3.4.2 is in agreement with the properties of the classical Bank-Weiser approach. In fact, it is well known that this a-posteriori error analysis only yields reliability, and that it is possible to obtain an explicit lower bound of the error through the utilization of a different estimator, usually of residual type.

Our next purpose is to bound the global quantity R_Γ by computable local indicators on the edges $e \in E_h(\Gamma)$. Indeed, we have the following lemma.

Lema 3.4.3 *There exists $C > 0$, independent of h, such that*

$$R_\Gamma^2 \,\le\, C\, \log[1 + C_h(\Gamma)] \sum_{e \in E_h(\Gamma)} h_e\, ||2\, \mathbf{W}(\tilde{\lambda}_h) \,+\, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}||_{L^2(e)}^2 \,, \tag{3.84}$$

where

$$C_h(\Gamma) := \max \left\{ \frac{h_e}{h_{e'}} : \quad e \text{ and } e' \text{ are neighbour edges of } \Gamma \right\}.$$

Proof: We first observe from the definitions of the finite element subspaces $H_h^{\boldsymbol{\sigma}}$ and $H_{h,0}^\lambda$ (cf. (3.15) and (3.19)) that $(\boldsymbol{\sigma}_h \cdot \boldsymbol{\nu})|_\Gamma \in L^2(\Gamma)$ and $\tilde{\lambda}_h \in H^1(\Gamma)$, and hence, a mapping property of $\mathbf{W}$ implies that $(\, 2\mathbf{W}(\tilde{\lambda}_h) \,+\, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}) \in L^2(\Gamma)$.

Now, taking $\boldsymbol{\tau}_h \,=\, 0$ in the first equation of (3.20), and $(v_h, q_h) \,=\, (0, 1)$ in the second one, we deduce, respectively, that $\langle 2\mathbf{W}(\tilde{\lambda}_h) \,+\, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}, \tilde{\mu}_h \rangle_\Gamma \,=\, 0$ for all $\tilde{\mu}_h \in H_{h,0}^\lambda$, and $\langle \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = 0$.

Therefore, using the decomposition $H_h^\lambda \,=\, H_{h,0}^\lambda \oplus \mathbb{R}$, the symmetry of $\mathbf{W}$, and the fact that $\mathbf{W}(1) = 0$, we conclude that $(2\mathbf{W}(\tilde{\lambda}_h) \,+\, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu})$ is $L^2(\Gamma)$-orthogonal to H_h^λ. Thus, a straightforward application of Theorem 2 in [21] yields the estimate (3.84) and ends the proof. ■

As a consequence of Theorem 3.4.2 and Lemma 3.4.3, we obtain the following result.

Teorema 3.4.3 *Let $\tilde{\varphi}_h$ be as indicated in Theorem 3.4.1, and for each $T \in \mathcal{T}_h$ let $\hat{\boldsymbol{\sigma}}_T \in H(\text{div}; T)$ be the unique solution of the local problem (3.62). Then there exists $C > 0$, independent of h, such that*

$$\left|\left|((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (u - u_h, p - p_h))\right|\right|_{\tilde{H} \times \tilde{Q}} \,\le\, C \left\{ \sum_{T \in \mathcal{T}_h} \tilde{\theta}_T^2 \right\}^{1/2} ,$$

where

$$\tilde{\theta}_T^2 := ||\hat{\boldsymbol{\sigma}}_T||_{H(\text{div};T)}^2 \,+\, ||\, f + \text{div}\, \boldsymbol{\sigma}_h \,||_{L^2(T)}^2$$
$$+ \log[1 + C_h(\Gamma)] \sum_{e \in E(T) \cap E_h(\Gamma)} h_e\, ||2\, \mathbf{W}(\tilde{\lambda}_h) \,+\, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}||_{L^2(e)}^2 \,.$$

It is important to remark here that the local problem defining $\hat{\boldsymbol{\sigma}}_T$ lives in the infinite dimensional space $H(\mathrm{div};T)$, and therefore, it can only be solved approximately by considering suitable finite dimensional subspaces. To this respect, as indicated in [2], we suggest to apply the p or the $h-p$ version.

Alternatively, we propose to utilize the upper bound (3.77) from Lemma 3.4.1 to derive a fully explicit reliable a-posteriori error estimate that does not require neither the exact nor any approximate solution of the local problem (3.62). More precisely, our main explicit reliable a-posteriori error estimate for the Galerkin scheme (3.20) is stated as follows.

Teorema 3.4.4 *Let $\tilde{\varphi}_h$ be as indicated in Theorem 3.4.1. Then there exists $C>0$, independent of h, such that*

$$\Big|\Big|((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\lambda}-\tilde{\lambda}_h),(u-u_h,p-p_h))\Big|\Big|_{\tilde{H}\times\tilde{Q}} \leq C\left\{\sum_{T\in\mathcal{T}_h}\hat{\theta}_T^2\right\}^{1/2}, \tag{3.85}$$

where

$$\begin{aligned}\hat{\theta}_T^2 := &\ ||\,(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) - \nabla\tilde{\varphi}_h\,||^2_{[L^2(T)]^2} + ||\,u_h - \tilde{\varphi}_h\,||^2_{L^2(T)} + ||\,f+\mathrm{div}\,\boldsymbol{\sigma}_h\,||^2_{L^2(T)}\\ &+ \sum_{e\in E(T)\cap E_h(\Gamma)}||\tilde{\lambda}_h+p_h-\tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)} + \sum_{e\in E(T)\cap E_h(\Gamma_0)}||g-\tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)}\\ &+ \log[1+C_h(\Gamma)]\sum_{e\in E(T)\cap E_h(\Gamma)}h_e\,||2\,\mathbf{W}(\tilde{\lambda}_h)+\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}||^2_{L^2(e)}\,.\end{aligned} \tag{3.86}$$

Proof: It follows directly from Theorem 3.4.3 and Lemma 3.4.1. ∎

We end this section by setting an appropriate choice for $\tilde{\varphi}_h$. As suggested by the quasi-efficiency result provided by Lemma 3.4.2, this function needs to be as close as possible to the exact solution u. Hence, we follow an averaging technique and define $\tilde{\varphi}_h:\bar{\Omega}\to\mathbb{R}$ as the unique continuous function satisfying the following conditions.

1. $(\tilde{\varphi}_h|_T\circ F_T)\in\mathbf{P}_1(\hat{T})$ for all $T\in\mathcal{T}_h$, where F_T is the diffeomorphism mapping the reference triangle $\hat{T}$ onto T (cf. Section 3.2).

2. For each vertex $\bar{x}$ of $\mathcal{T}_h$ lying on Γ_0: $\tilde{\varphi}_h(\bar{x})=g(\bar{x})$.

3. For each vertex $\bar{x}$ of $\mathcal{T}_h$ lying on Γ: $\tilde{\varphi}_h(\bar{x})=\tilde{\lambda}_h(\bar{x})+p_h$.

4. For each vertex $\bar{x}$ of $\mathcal{T}_h$ not lying on $\Gamma_0\cup\Gamma$: $\tilde{\varphi}_h(\bar{x})$ is the weighted average of the constant values of u_h on all the triangles $T\in\mathcal{T}_h$ to which $\bar{x}$ belongs. Here, the weighting is according to the relative area of each triangle.

Finally, we observe that for implementation purposes, the $H^{1/2}$-norms appearing in the definition of the local indicators $\hat{\theta}_T$ can be bounded using an interpolation theorem. More precisely, given an edge $e \in E_h(\Gamma) \cup E_h(\Gamma_0)$, and a function $\rho \in H_0^1(e)$, we have

$$||\rho||^2_{H^{1/2}_{00}(e)} \;\leq\; ||\rho||_{L^2(e)}\; ||\rho||_{H^1_0(e)}\,.$$

3.5 Numerical results

We now provide several numerical results illustrating the performance of the discrete scheme (3.18), and supporting the quality and efficiency of the a-posteriori error estimates given by (3.31)-(3.32) and (3.85)-(3.86). We emphasize, according to Theorem 3.2.1, that it suffices to solve (3.18) instead of the equivalent Galerkin scheme (3.20).

For the geometry of the problem, we let Γ_0 ($\partial\Omega_0$) and Γ_1 be the boundaries of the squares with center at $(0,0)$ and side lengths given by 1 and 4, respectively. In other words, Γ_0 is the polygonal curve determined by the vertices $(1/2,1/2)$, $(-1/2,1/2)$, $(-1/2,-1/2)$, and $(1/2,-1/2)$, and Γ_1 is the one determined by $(2,2)$, $(-2,2)$, $(-2,-2)$, and $(2,-2)$. In all our computations we consider $\boldsymbol{\kappa}_1$ equals the identity matrix $\mathbf{I}$, and choose the data f and g so that the exact solution of (3.1) is

$$u(x,y) \;:=\; \frac{x}{(x-0.45)^2+y^2}\,\chi\left(\sqrt{x^2+y^2}\right) \qquad \forall\,(x,y)\in\mathbb{R}^2-\bar{\Omega}_0\,,$$

where $\chi \in C^2([\frac{1}{2},+\infty))$ is the cut-off function defined by

$$\chi(r) \;:=\; \begin{cases} r^3-3r^2+3r\,, & \text{if } \frac{1}{2}\leq r\leq 1 \\ \\ 1\,, & \text{if } 1\leq r. \end{cases}$$

Hence, we take Γ as the circle with center at $(0,0)$ and radius 4, and recall that the computational domain Ω is the annular region bounded by Γ_0 and Γ.

We observe that u has a singularity at $(0.45,0)$, $u \in C^2(\Omega)$, and $u \notin C^3(\Omega)$. In fact, because of the definition of χ, the third order derivatives of u are not continuous on the unit circle.

We let N be the number of degrees of freedom defining the subspaces H_h and Q_h, that is $N :=$ number of edges of $\mathcal{T}_h$ + number of nodes on Γ + number of triangles of $\mathcal{T}_h$. Also, we use the following notations for the individual and global errors

$$\mathbf{e}(u) \;:=\; ||u-u_h||_{L^2(\Omega)}\,,\quad \mathbf{e}(\lambda) \;:=\; ||\lambda-\lambda_h||_{L^2(\Gamma)}\,,\quad \mathbf{e}(\boldsymbol{\sigma}) \;:=\; ||\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||_{H(\mathrm{div};\Omega)}\,,$$

and

$$\mathbf{e} := \left\{ [\mathbf{e}(u)]^2 + [\mathbf{e}(\lambda)]^2 + [\mathbf{e}(\boldsymbol{\sigma})]^2 \right\}^{1/2},$$

where $((\boldsymbol{\sigma}, \lambda), u) \in H \times Q$ and $((\boldsymbol{\sigma}_h, \lambda_h), u_h) \in H_h \times Q_h$ are the solutions of (3.5) and (3.18), respectively. In addition, we consider the error estimates given by

$$\eta := \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\}^{1/2} \quad \text{and} \quad \hat{\theta} := \left\{ \sum_{T \in \mathcal{T}_h} \hat{\theta}_T^2 \right\}^{1/2},$$

where η_T and $\hat{\theta}_T$ are defined by (3.32) and (3.86).

The adaptive algorithm used in our computations follows a standard approach from [62] (see also [54]). More precisely, given a parameter $\boldsymbol{\gamma} \in (0, 1)$, we apply the following scheme:

1. Start with a coarse mesh $\mathcal{T}_h$.

2. Solve the discrete problem (3.18) for the actual mesh $\mathcal{T}_h$.

3. Compute η_T $(\hat{\theta}_T)$ for each triangle $T \in \mathcal{T}_h$.

4. Evaluate stopping criterion and decide to finish or go to next step.

5. Use *blue-green* procedure to refine each T whose indicator η_T $(\hat{\theta}_T)$ is among the $100\boldsymbol{\gamma}\,\%$ of the largest indicators. Define resulting mesh as actual mesh $\mathcal{T}_h$, update h and go to step 2.

In Tables 6.1 throughout 6.5 we display the errors for each unknown, the error estimates η and $\hat{\theta}$, and the effectivity indices $\mathbf{e}/\eta$ and $\mathbf{e}/\hat{\theta}$, for uniform and adaptive refinements. In addition, Figures 6.1 and 6.2 show the global error $\mathbf{e}$ versus the degrees of freedom N. We consider here two choices of the refinement parameter $\boldsymbol{\gamma}$, namely 0.1 and 0.25. We remark that the errors on each triangle $T \in \mathcal{T}_h$ are computed using a 16 points Gaussian quadrature rule.

As expected, the errors $\mathbf{e}$ for the adaptive refinements decrease considerably faster than for the uniform one. Also, it is observed in all cases that $\mathbf{e}$ is mainly dominated by the individual error $\mathbf{e}(\boldsymbol{\sigma})$. Further, the indices $\mathbf{e}/\eta$ and $\mathbf{e}/\hat{\theta}$ are always bounded above, which provides experimental evidences for the estimates (3.31) and (3.85). We note, at least for this example, that the adaptive algorithm based on $\hat{\theta}$ is more efficient than the one based on η. Nevertheless, as shown in Figures 6.1 and 6.2 the adaptive refinement using η converges a bit faster than the one using $\hat{\theta}$. Now, it is also clear from Figures 6.1 and 6.2 that the adaptive meshes generated with $\boldsymbol{\gamma} = 0.1$ yield a much faster decreasing of $\mathbf{e}$ than with $\boldsymbol{\gamma} = 0.25$. However, after about $N = 15000$ degrees of freedom, this process saturates and no further significant improvement is obtained. On the other hand, the decreasing obtained with $\boldsymbol{\gamma} = 0.25$ shows a closer behaviour to the expected quasi-optimal linear rate of convergence. These facts can also be verified from Tables 6.2 up to 6.5 by computing the experimental rates of convergence, that is the quantities $-\frac{2\log(\mathbf{e}/\mathbf{e}')}{\log(N/N')}$, where $\mathbf{e}$ and $\mathbf{e}'$ are the global errors associated with two consecutive adaptive meshes with N and N' degrees of freedom, respectively.

Next, in Figures 6.3 and 6.4 we display initial and intermediate meshes obtained with the refinement strategies. We observe that the adaptive algorithms, based on both η and $\hat{\theta}$, are able to recognize a neighborhood of $(0.5, 0)$, which is close to the singular point $(0.45, 0)$. Also, they clearly identify the unit circle, on which, as mentioned before, the exact solution u looses smoothness.

Finally, we emphasize that the numerical results presented in this section provide enough support for the adaptive methods being much more efficient than a uniform discretization when solving linear exterior problems.

Table 6.1: Individual errors, error estimates η and $\hat{\theta}$, and effectivity indices for the uniform refinement.

N	$\mathbf{e}(u)$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\sigma})$	η	$\mathbf{e}/\eta$	$\hat{\theta}$	$\mathbf{e}/\hat{\theta}$
1222	0.9602	0.3206	21.0856	39.2721	0.5375	29.4685	0.7164
4764	0.6674	0.1793	18.0624	36.2539	0.4986	25.1351	0.7191
18808	0.5291	0.1765	12.4059	34.1627	0.3635	17.7766	0.6986
74736	0.4847	0.1796	6.8630	25.9430	0.2653	10.2980	0.6683

Table 6.2: Individual errors, error estimate, and effectivity index for the adaptive refinement based on η, with $\boldsymbol{\gamma} = 0.1$.

N	$\mathbf{e}(u)$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\sigma})$	η	$\mathbf{e}/\eta$
1222	0.9602	0.3206	21.0856	39.2721	0.5375
1705	0.6835	0.1922	18.0697	36.2829	0.4984
2339	0.5532	0.1837	12.4267	34.2311	0.3634
3226	0.5135	0.1842	6.9190	26.0892	0.2660
4344	0.5046	0.1842	3.7602	17.0105	0.2233
5913	0.5027	0.1843	2.3152	11.3346	0.2097
8340	0.4947	0.1813	1.7358	8.2160	0.2208
12553	0.4907	0.1809	1.5224	6.3317	0.2542
19094	0.4846	0.1798	1.3737	4.9846	0.2945
28893	0.4826	0.1801	1.2419	3.9749	0.3382

Table 6.3: Individual errors, error estimate, and effectivity index for the adaptive refinement based on $\hat{\theta}$, with $\boldsymbol{\gamma} = 0.1$.

N	$\mathbf{e}(u)$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\sigma})$	$\hat{\theta}$	$\mathbf{e}/\hat{\theta}$
1222	0.9602	0.3206	21.0856	29.4685	0.7164
1802	0.6806	0.1889	18.0673	25.1435	0.7191
2587	0.5460	0.1796	12.4188	17.8093	0.6981
3676	0.5054	0.1811	6.8987	10.3820	0.6665
5215	0.4944	0.1807	3.7113	6.0897	0.6155
11078	0.4885	0.1803	1.5939	3.2497	0.5160
17558	0.4834	0.1795	1.3610	2.9870	0.4873
27947	0.4819	0.1794	1.2214	2.8579	0.4637
45060	0.4798	0.1795	1.1351	2.7639	0.4506
73261	0.4777	0.1796	1.1031	2.6731	0.4547

Table 6.4: Individual errors, error estimate, and effectivity index for the adaptive refinement based on η, with $\boldsymbol{\gamma} = 0.25$.

N	$\mathbf{e}(u)$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\sigma})$	η	$\mathbf{e}/\eta$
1222	0.9602	0.3206	21.0856	39.2721	0.5375
2305	0.6725	0.1823	18.0628	36.2611	0.4985
4654	0.5356	0.1772	12.4080	34.1781	0.3634
9264	0.4899	0.1786	6.8699	25.9641	0.2654
18693	0.4769	0.1789	3.6320	16.6864	0.2198
37606	0.4736	0.1795	2.0264	10.5919	0.1972

Table 6.5: Individual errors, error estimate, and effectivity index for the adaptive refinement based on $\hat{\theta}$, with $\boldsymbol{\gamma} = 0.25$.

N	$\mathbf{e}(u)$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\sigma})$	$\hat{\theta}$	$\mathbf{e}/\hat{\theta}$
1222	0.9602	0.3206	21.0856	29.4685	0.7164
2345	0.6725	0.1825	18.0628	25.1380	0.7191
4941	0.5355	0.1772	12.4075	17.7864	0.6983
10986	0.4887	0.1785	6.8666	10.3175	0.6674
24320	0.4758	0.1790	3.6255	5.9069	0.6198
54177	0.4729	0.1798	2.0168	3.6833	0.5645

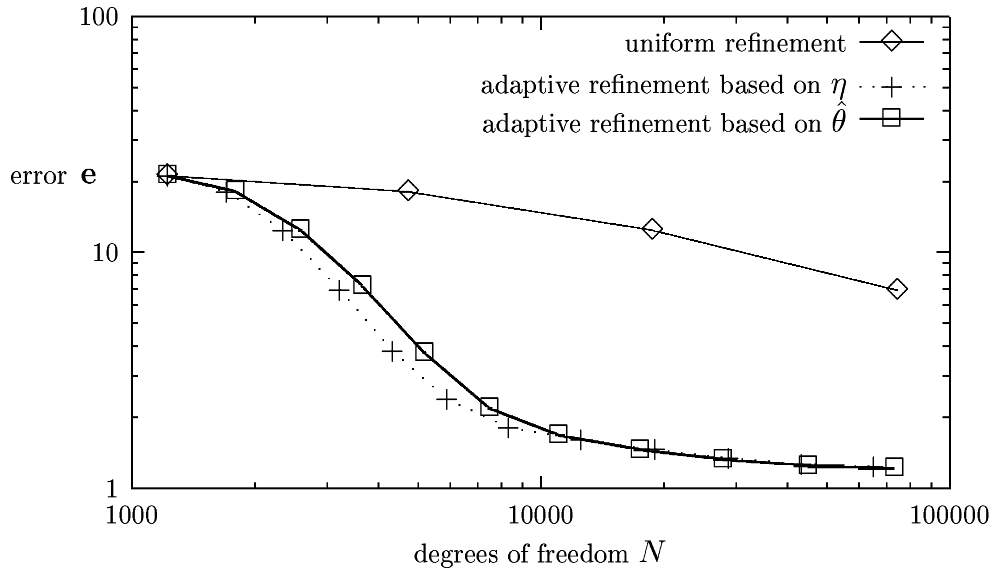

Figure 6.1: Error $\mathbf{e}$ for uniform and adaptive refinements (with $\boldsymbol{\gamma} = 0.1$).

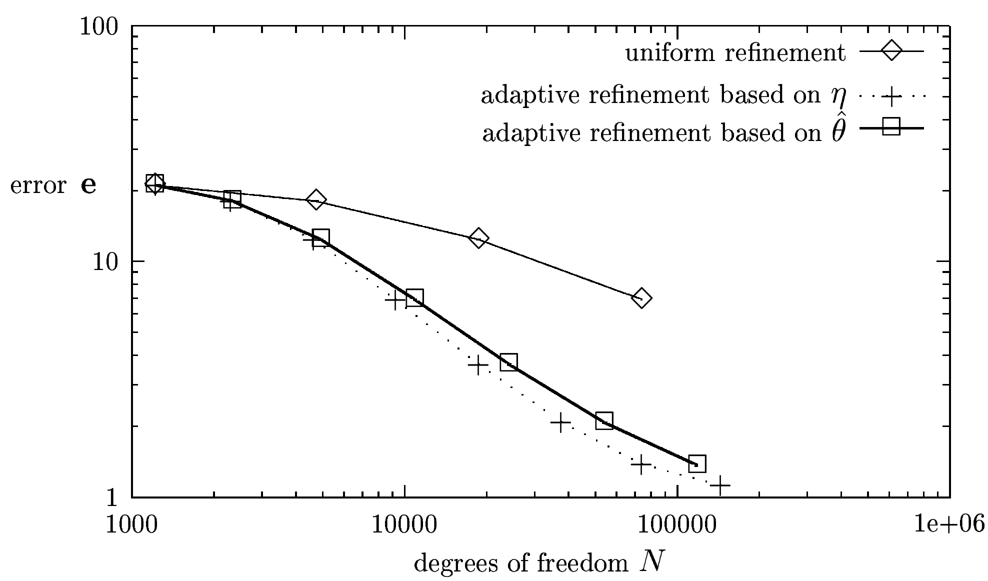

Figure 6.2: Error $\mathbf{e}$ for uniform and adaptive refinements (with $\boldsymbol{\gamma} = 0.25$).

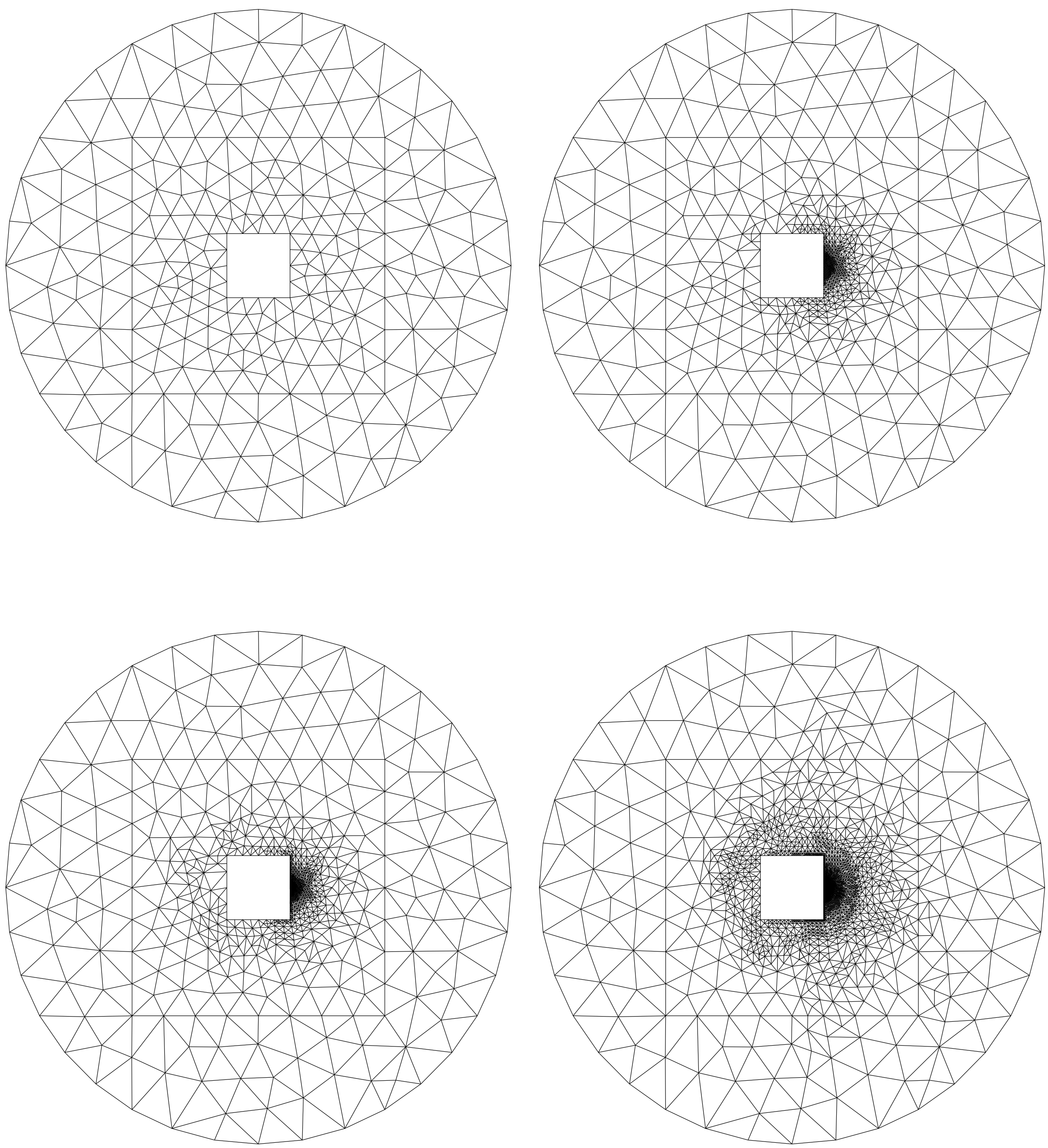

Figure 6.3: Initial and intermediate meshes with 1222, 4344, 8340, and 28893 degrees of freedom, respectively, for the adaptive refinement based on η, with $\boldsymbol{\gamma} = 0.1$.

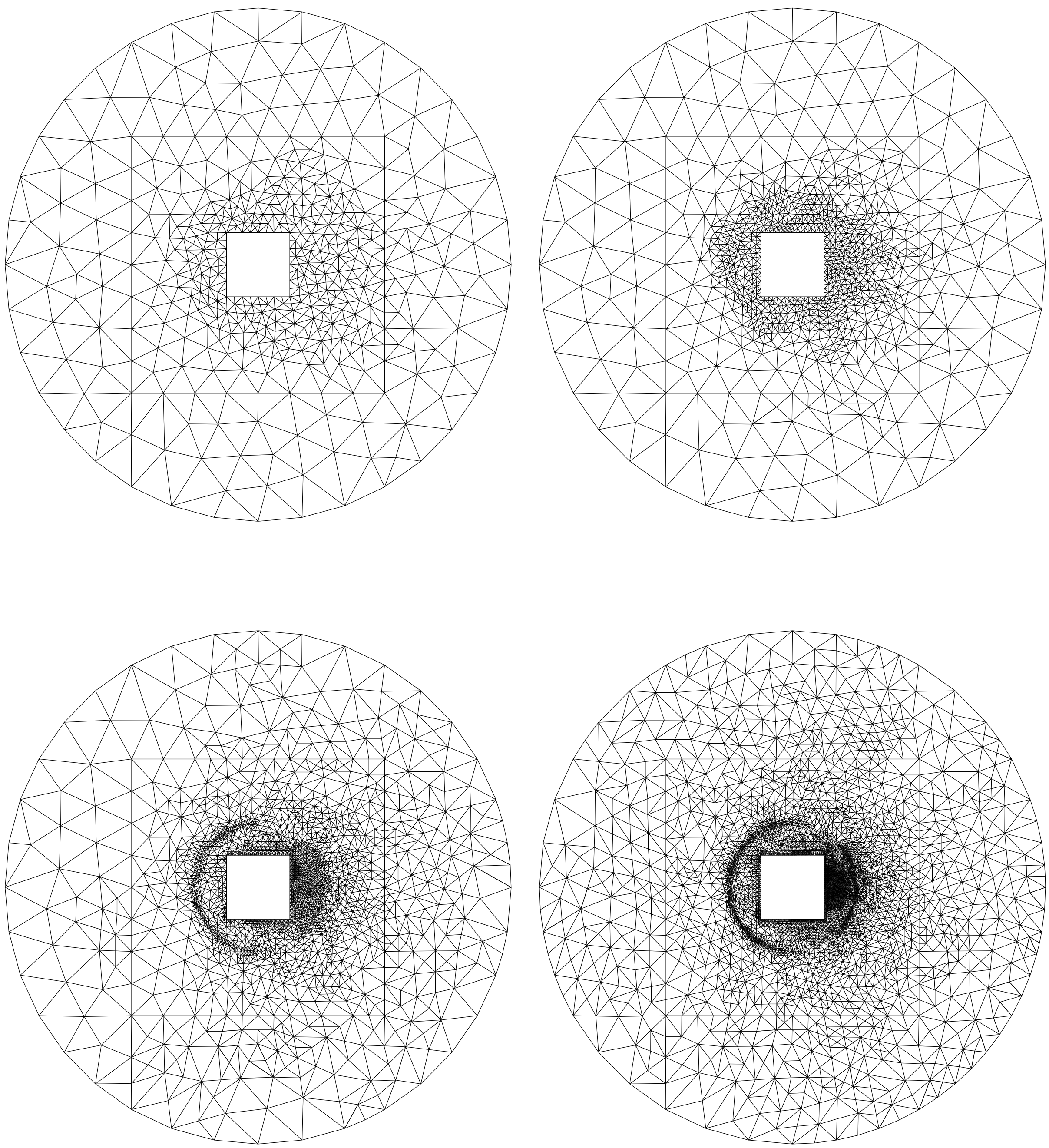

Figure 6.4: Intermediate meshes with 2345, 4941, 10986, and 24320 degrees of freedom, respectively, for the adaptive refinement based on $\hat{\theta}$, with $\boldsymbol{\gamma} = 0.25$.

Capítulo 4

A Nonlinear Problem in Elasticity

The objective of this chapter is to utilize the two-fold saddle point approach from [29, 31, 32, 33, 34, 40] to extend the applicability of stable mixed finite elements from linear elasticity, to a nonlinear boundary value problem in plane hyperelasticity. In order to illustrate our results, we consider the PEERS-space introduced in [7] as a particular example. In addition, we follow [17] and [18], together with the ideas from [19] and [30], to derive a reliable a-posteriori error estimate for the resulting Galerkin scheme. We remark that the extensions to our dual-dual case of the a-posteriori error estimates of explicit residual type or of those given in terms of hierarchical bases, have not been obtained yet. Hence, up to the authors's knowledge, the present chapter is the first one providing an a-posteriori error analysis for the two-fold saddle point variational formulations. The remaining of the paper is organized as follows. In Sect. 2 we describe the nonlinear boundary value problem from plane hyperelasticity. The two-fold saddle point formulation is derived in Sect. 3 and the corresponding solvability result is provided in Sect. 4. Then, the finite element solution is analyzed in Sect. 5. Finally, Sect. 6 is devoted to the a-posteriori error estimate.

4.1 The boundary value problem

Let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary Γ. Further, let Γ_D and Γ_N be two disjoint subsets of Γ such that $|\Gamma_D| \neq 0$ and $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. Our goal is to determine the displacements $\mathbf{u}$ and stresses $\boldsymbol{\sigma}$ of a hyperelastic material occupying the region Ω.

In what follows, given any Hilbert space U, U^2 and $U^{2\times 2}$ denote, respectively, the space of vectors and square matrices of order 2 with entries in U. In particular, $\mathbb{R}^{2\times 2}$ is the space of square matrices of order 2 with real entries, $\mathbf{I} := (\delta_{ij})$ is the identity matrix of $\mathbb{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we use the notations

$$\text{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii} \quad , \quad \boldsymbol{\zeta} : \boldsymbol{\tau} := \sum_{i,j=1}^{2} \zeta_{ij}\, \tau_{ij}\,.$$

As a description of the hyperelasticity we assume the validity of the Hencky-Mises stress-strain relation as discussed in [56] (see, also, [57] and [63]). In other words, if $\boldsymbol{\sigma}(\mathbf{u}) := (\sigma_{ij}(\mathbf{u})) \in \mathbb{R}^{2\times 2}$ denotes the Cauchy stress tensor and $\mathbf{e}(\mathbf{u}) := (e_{ij}(\mathbf{u})) \in \mathbb{R}^{2\times 2}$ is the strain tensor of small deformations, with $e_{ij}(\mathbf{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, then the constitutive equation in Ω is given by

$$\sigma_{ij}(\mathbf{u}) \;=\; \frac{\partial}{\partial e_{ij}}\, \Psi(\mathbf{e}(\mathbf{u})) \quad \forall\, i,\, j \in \{1,2\}\,, \tag{4.1.1}$$

where $\Psi : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is the stored energy function

$$\Psi(\boldsymbol{\tau}) = \frac{1}{2}\,\kappa\,(\text{tr}\,(\boldsymbol{\tau}))^2 \,+\, \bar{\mu}\,\Phi(\mathbf{dev}(\boldsymbol{\tau})) \qquad \forall\,\boldsymbol{\tau} \in \mathbb{R}^{2\times 2}\,. \tag{4.1.2}$$

Here, κ and $\bar{\mu}$ are positive constants ($\bar{\mu}$ is called the ground state shear modulus), $\Phi : [0,+\infty) \to \mathbb{R}$ is a function of class C^2 and $\mathbf{dev} : \mathbb{R}^{2\times 2} \to \mathbb{R}^{+}$ is defined by $\mathbf{dev}(\boldsymbol{\tau}) := \big(\boldsymbol{\tau} - \frac{1}{2}\text{tr}\,(\boldsymbol{\tau})\mathbf{I}\big) : \big(\boldsymbol{\tau} - \frac{1}{2}\text{tr}\,(\boldsymbol{\tau})\mathbf{I}\big)$. In addition, we assume $\Phi'(0) = 1$ and that there exist constants C_1, C_2 and C_3 such that

$$\begin{gathered} 0 < C_1 \le \bar{\mu}\,\Phi'(\rho) < \kappa \quad , \quad |\rho\,\Phi''(\rho)| \le C_2 \quad \text{and} \\ \Phi'(\rho) + 2\,\rho\,\Phi''(\rho) \ge C_3 > 0 \end{gathered} \tag{4.1.3}$$

for all $\rho \in \mathbb{R}^{+} := [0,+\infty)$.

In this way, by computing the right hand side of (4.1.1) according to (4.1.2), the constitutive equation in Ω becomes

$$\boldsymbol{\sigma}(\mathbf{u}) := \tilde{\lambda}(\mathbf{dev}(\mathbf{e}(\mathbf{u})))\,\text{tr}\,(\mathbf{e}(\mathbf{u}))\,\mathbf{I} + \tilde{\mu}(\mathbf{dev}(\mathbf{e}(\mathbf{u})))\,\mathbf{e}(\mathbf{u})\,, \tag{4.1.4}$$

where $\tilde{\lambda}$, $\tilde{\mu} : \mathbb{R}^{+} \to \mathbb{R}$ are the nonlinear Lamé functions defined by

$$\tilde{\mu}(\rho) := 2\,\bar{\mu}\,\Phi'(\rho) \quad \text{and} \quad \tilde{\lambda}(\rho) := \kappa - \frac{1}{2}\,\tilde{\mu}(\rho) \qquad \forall\,\rho \in \mathbb{R}^{+}\,.$$

Further, by using the assumptions from (4.1.3), one can easily show that there exist positive constants μ_0, μ_1 and μ_2 such that for all $\rho \in \mathbb{R}^{+}$

$$0 < \mu_0 \le \tilde{\mu}(\rho) < 2\,\kappa \quad \text{and} \quad 0 < \mu_1 \le \tilde{\mu}(\rho) + 2\rho\,\tilde{\mu}'(\rho) \le \mu_2\,. \tag{4.1.5}$$

Then, given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma_D)]^2$, our nonlinear boundary value problem reads as follows: *Find a tensor field* $\boldsymbol{\sigma}$ *and a vector field* $\mathbf{u}$ *such that*

$$\begin{aligned} \boldsymbol{\sigma} := \tilde{\lambda}(\mathbf{dev}(\mathbf{e}(\mathbf{u})))\,\mathrm{tr}\,(\mathbf{e}(\mathbf{u}))\,\mathbf{I} + \tilde{\mu}(\mathbf{dev}(\mathbf{e}(\mathbf{u})))\,\mathbf{e}(\mathbf{u}) \quad \text{in} \quad \Omega\,, \\ -\,\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma_D\,, \quad \boldsymbol{\sigma}\,\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma_N\,, \end{aligned} \tag{4.1.6}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal to Γ and for any $\boldsymbol{\tau} := (\tau_{ij}) \in \mathbb{R}^{2\times 2}$,

$$\mathbf{div}\,\boldsymbol{\tau} := \begin{bmatrix} \mathrm{div}\,(\tau_{11}\;\tau_{12}) \\ \mathrm{div}\,(\tau_{21}\;\tau_{22}) \end{bmatrix}.$$

Our subsequent analysis also applies to a non-homogeneous Neumann boundary condition. For simplicity, here we have chosen $\boldsymbol{\sigma}\,\boldsymbol{\nu} = 0$ on Γ_N.

4.2 The two-fold saddle point formulation

We now derive a variational formulation for the nonlinear boundary value problem (4.1.6) and show that it can be written as a two-fold saddle point operator equation, also called dual-dual mixed formulation.

First, we define the tensor spaces

$$H(\mathbf{div};\Omega) := \{\, \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}\,\boldsymbol{\tau} \in [L^2(\Omega)]^2 \,\}\,,$$

$$H_0(\mathbf{div},\Omega) := \{\, \boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}\,\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma_N \,\}\,,$$

with inner product

$$\langle \boldsymbol{\zeta}, \boldsymbol{\tau} \rangle_{H(\mathbf{div};\Omega)} := \langle \boldsymbol{\zeta}, \boldsymbol{\tau} \rangle_{[L^2(\Omega)]^{2\times 2}} + \langle \mathbf{div}\,\boldsymbol{\zeta}, \mathbf{div}\,\boldsymbol{\tau} \rangle_{[L^2(\Omega)]^2}\,,$$

where

$$\langle \boldsymbol{\zeta}, \boldsymbol{\tau} \rangle_{[L^2(\Omega)]^{2\times 2}} := \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau}\,dx \quad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$$

and

$$\langle \mathbf{z}, \mathbf{v} \rangle_{[L^2(\Omega)]^2} := \int_\Omega \mathbf{z}\cdot\mathbf{v}\,dx \quad \forall\, \mathbf{z},\, \mathbf{v} \in [L^2(\Omega)]^2\,.$$

The corresponding induced norms are $||\cdot||_{H(\mathbf{div};\Omega)}$, $||\cdot||_{[L^2(\Omega)]^{2\times 2}}$ and $||\cdot||_{[L^2(\Omega)]^2}$.

Now, as in [43] and [34], we introduce the auxiliary unknown

$$\mathbf{t} := \mathbf{e}(\mathbf{u}) \quad \text{in} \quad \Omega\,, \tag{4.2.1}$$

whence (4.1.4) becomes

$$\boldsymbol{\sigma} = \hat{\lambda}(\mathbf{t})\,\mathrm{tr}\,(\mathbf{t})\,\mathbf{I} + \hat{\mu}(\mathbf{t})\,\mathbf{t} \quad \text{in} \quad \Omega\,, \tag{4.2.2}$$

where, for any $\mathbf{r} \in [L^2(\Omega)]^{2\times 2}$ we put

$$\hat{\lambda}(\mathbf{r}) := \tilde{\lambda}(\mathbf{dev}(\mathbf{r})) \quad \text{and} \quad \hat{\mu}(\mathbf{r}) := \tilde{\mu}(\mathbf{dev}(\mathbf{r})) \,. \tag{4.2.3}$$

In virtue of the definition of $\mathbf{e}(\mathbf{u})$ we can rewrite (4.2.1) as

$$\mathbf{t} \,=\, \nabla \mathbf{u} \,-\, \boldsymbol{\gamma} \,, \tag{4.2.4}$$

where

$$\boldsymbol{\gamma} \,:=\, \frac{1}{2}\,(\,\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathbf{T}}\,)$$

represents rotations and lives in the space

$$\mathcal{R} \,:=\, \{\, \boldsymbol{\eta} \in [L^2(\Omega)]^{2\times 2} : \quad \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathbf{T}} = 0 \,\} \,. \tag{4.2.5}$$

Hereafter, the super index $^{\mathbf{T}}$ stands for the transpose of any vector or matrix.

Then, we multiply (4.2.4) by a test function $\boldsymbol{\tau} \in H_0(\mathbf{div};\Omega)$, integrate by parts on Ω and make use of $\mathbf{u} = \mathbf{g}$ on Γ_D, to obtain

$$\int_\Omega \mathbf{t} : \boldsymbol{\tau}\, dx + \int_\Omega \mathbf{u} \cdot \mathbf{div}\, \boldsymbol{\tau}\, dx + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau}\, dx \,=\, \int_{\Gamma_D} \mathbf{g} \cdot \boldsymbol{\tau}\boldsymbol{\nu}\, ds \tag{4.2.6}$$

for all $\boldsymbol{\tau} \in H_0(\mathbf{div};\Omega)$.

Also, the constitutive equation (4.2.2) yields

$$\int_\Omega \Big[\, \hat{\lambda}(\mathbf{t})\, \mathrm{tr}\,(\mathbf{t})\, \mathrm{tr}\,(\mathbf{s}) + \hat{\mu}(\mathbf{t})\, \mathbf{t} : \mathbf{s} \,\Big]\, dx - \int_\Omega \boldsymbol{\sigma} : \mathbf{s}\, dx = 0 \tag{4.2.7}$$

for all $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$, and the equilibrium equation becomes

$$-\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma}\, dx \,=\, \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx \quad \forall\, \mathbf{v} \in [L^2(\Omega)]^2 \,. \tag{4.2.8}$$

Finally, the symmetry of $\boldsymbol{\sigma}$ is weakly required by

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta}\, dx \,=\, 0 \quad \forall\, \boldsymbol{\eta} \in \mathcal{R} \,. \tag{4.2.9}$$

In this way, collecting (4.2.6), (4.2.7), (4.2.8) and (4.2.9), we arrive at the following variational formulation of the boundary value problem (4.1.6): *Find* $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \in [L^2(\Omega)]^{2\times 2} \times H_0(\mathbf{div};\Omega) \times [L^2(\Omega)]^2 \times \mathcal{R}$ *such that*

$$\begin{aligned}
\int_\Omega \Big[\hat{\lambda}(\mathbf{t})\, \mathrm{tr}\,(\mathbf{t})\, \mathrm{tr}\,(\mathbf{s}) + \hat{\mu}(\mathbf{t})\, \mathbf{t} : \mathbf{s}\Big]\, dx - \int_\Omega \boldsymbol{\sigma} : \mathbf{s}\, dx \quad &= 0 \\
-\int_\Omega \mathbf{t} : \boldsymbol{\tau}\, dx - \int_\Omega \mathbf{u} \cdot \mathbf{div}\, \boldsymbol{\tau}\, dx - \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau}\, dx \quad &= -\int_{\Gamma_D} \mathbf{g} \cdot \boldsymbol{\tau}\boldsymbol{\nu}\, ds \\
-\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma}\, dx - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta}\, dx \quad &= \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx
\end{aligned}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in [L^2(\Omega)]^{2\times 2} \times H_0(\mathbf{div};\Omega) \times [L^2(\Omega)]^2 \times \mathcal{R}$.

As in [34, 43], and in order to simplify the analysis, we now rewrite the above formulation as an appropriate operator equation. To this purpose, we introduce the spaces

$$X_1 := [L^2(\Omega)]^{2\times 2}\,, \quad M_1 := H_0(\mathbf{div};\Omega)\,,$$
$$X := X_1 \times M_1\,, \quad M := [L^2(\Omega)]^2 \times \mathcal{R}\,,$$

and the operators

$$\mathbf{A}_1 : X_1 \to X_1'\,, \quad \mathbf{B}_1 : X_1 \to M_1'\,,$$
$$\mathbf{A} : X \to X'\,, \quad \mathbf{B} : M_1 \to M'\,,$$

as follows:

$$[\mathbf{A}_1(\mathbf{r}), \mathbf{s}] := \int_\Omega \left[\hat{\lambda}(\mathbf{r})\,\mathrm{tr}\,(\mathbf{r})\,\mathrm{tr}\,(\mathbf{s}) + \hat{\mu}(\mathbf{r})\,\mathbf{r} : \mathbf{s}\right] dx\,,$$

$$[\mathbf{B}_1(\mathbf{r}), \boldsymbol{\tau}] := -\int_\Omega \mathbf{r} : \boldsymbol{\tau}\, dx\,,$$

$$[\mathbf{A}(\mathbf{r}, \boldsymbol{\zeta}), (\mathbf{s}, \boldsymbol{\tau})] := [\mathbf{A}_1(\mathbf{r}), \mathbf{s}] + [\mathbf{B}_1(\mathbf{r}), \boldsymbol{\tau}] + [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\zeta}]\,,$$

$$[\mathbf{B}(\boldsymbol{\zeta}), (\mathbf{v}, \boldsymbol{\eta})] := -\int_\Omega \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\zeta}\, dx - \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\eta}\, dx\,,$$

for all $\mathbf{r}$, $\mathbf{s} \in X_1$, $\boldsymbol{\zeta}$, $\boldsymbol{\tau} \in M_1$ and $(\mathbf{v}, \boldsymbol{\eta}) \in M$. Hereafter, the brackets $[\,\cdot,\cdot\,]$ denote the duality pairings induced by the corresponding operators.

We also set $\mathbf{O}$ as a generic null operator/functional and define $\mathbf{F} := (\mathbf{F}_1, \mathbf{F}_2) \in X' := X_1' \times M_1'$, $\mathbf{G} \in M'$ by

$$\mathbf{F}_1 = \mathbf{O}\,, \quad [\mathbf{F}_2, \boldsymbol{\tau}] := -\int_{\Gamma_D} \mathbf{g} \cdot \boldsymbol{\tau\nu}\, ds\,, \quad [\mathbf{G}, (\mathbf{v}, \boldsymbol{\eta})] := \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx\,.$$

In terms of these operators and functionals, the variational formulation of (4.1.6) can be equivalently stated as the following matrix operator equation:
Find $(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in X_1 \times M_1 \times M$ *such that*

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1^* & \mathbf{O} \\ \mathbf{B}_1 & \mathbf{O} & \mathbf{B}^* \\ \mathbf{O} & \mathbf{B} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \boldsymbol{\sigma} \\ (\mathbf{u}, \boldsymbol{\gamma}) \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{G} \end{bmatrix}. \tag{4.2.10}$$

We remark that $\mathbf{A}_1$ is a nonlinear operator and that $\mathbf{B}_1$ and $\mathbf{B}$ are linear and bounded. Here, $\mathbf{B}_1^* : M_1 \to X_1'$ and $\mathbf{B}^* : M \to M_1'$ are the adjoints of $\mathbf{B}_1$ and $\mathbf{B}$, respectively. In addition, since

$$\mathbf{A} := \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1^* \\ \mathbf{B}_1 & \mathbf{O} \end{bmatrix}$$

is a saddle point operator, we observe that (4.2.10) presents a two-fold saddle point structure, which justifies the name given to our formulation. At this point we recall that in previous works we also used the name *dual-dual* mixed formulation for the same kind of matrix operator equations (see, e.g. [40, 31, 32, 33, 34]).

In the next two sections we apply some abstract results from [29] (see also [40]) for studying the formulation (4.2.10) and the associated Galerkin scheme with specific finite element subspaces

4.3 Solvability of the two-fold saddle point formulation

In order to conclude the unique solvability of (4.2.10), and according to Theorem 2.4 in [29] (see also Theorem 4.1 in [40]), we need to show that $\mathbf{A}_1$ is strongly monotone and Lipschitz-continuous, and that $\mathbf{B}$ and $\mathbf{B}_1$ satisfy appropriate inf-sup conditions.

Lema 4.3.1 *The nonlinear operator $\mathbf{A}_1$ is strongly monotone and Lipschitz-continuous, i.e., there exist constants α_1, $\alpha_2 > 0$ such that*

$$[\mathbf{A}_1(\mathbf{r}) - \mathbf{A}_1(\mathbf{s}), \mathbf{r} - \mathbf{s}] \;\geq\; \alpha_1 \,||\mathbf{r} - \mathbf{s}||_{X_1}^2$$

and

$$||\mathbf{A}_1(\mathbf{r}) - \mathbf{A}_1(\mathbf{s})||_{X_1'} \;\leq\; \alpha_2 \,||\mathbf{r} - \mathbf{s}||_{X_1}$$

for all $\mathbf{r}$, $\mathbf{s} \in X_1 := [L^2(\Omega)]^{2\times 2}$.

Proof: Let $a_{ij} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be the nonlinear mapping defined by

$$a_{ij}(\mathbf{r}) := \hat{\lambda}(\mathbf{r})\,\mathrm{tr}\,(\mathbf{r})\,\delta_{ij} \;+\; \hat{\mu}(\mathbf{r})\,r_{ij} \tag{4.3.1}$$

for all $\mathbf{r} := (r_{ij}) \in \mathbb{R}^{2\times 2}$.

Because of the properties of κ and $\tilde{\mu}$ (cf. (4.1.5)) one can show (cf. [37] or Lemmata 4.1 and 4.2 in [6]) that $a_{ij}(\cdot)$ is continuous and has first order partial derivatives in $\mathbb{R}^{2\times 2}$, and there exist C_1, $C_2 > 0$ such that

$$|a_{ij}(\mathbf{r})| \;\leq\; C_1 \,||\mathbf{r}||_{\mathbb{R}^{2\times 2}}$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}}\, a_{ij}(\mathbf{r})\, s_{kl}\, s_{ij} \;\geq\; C_2 \sum_{i,j=1}^{2} s_{ij}^2 \tag{4.3.2}$$

for all $\mathbf{r} := (r_{ij})$, $\mathbf{s} := (s_{ij}) \in \mathbb{R}^{2\times 2}$.

Now, for all $\mathbf{r}(\cdot) := (r_{ij}(\cdot))$, $\mathbf{s}(\cdot) := (s_{ij}(\cdot)) \in X_1$ we can write

$$[\mathbf{A}_1(\mathbf{r}), \mathbf{s}] \;=\; \sum_{i,j=1}^{2} \int_\Omega a_{ij}(\mathbf{r})\, s_{ij}\, dx\,. \tag{4.3.3}$$

In addition, we note that

$$\begin{aligned} &a_{ij}(\mathbf{r}(x)) - a_{ij}(\mathbf{s}(x)) \\ &= \int_0^1 \left\{ \sum_{k,l=1}^{2} \frac{\partial}{\partial r_{kl}} a_{ij}(\hat{\mathbf{r}}(x,\rho))[r_{kl}(x) - s_{kl}(x)] \right\} d\rho\,, \end{aligned} \tag{4.3.4}$$

with $\hat{\mathbf{r}}(x,\rho) := \mathbf{s}(x) + \rho[\mathbf{r}(x) - \mathbf{s}(x)]$ $\forall\ \rho \in [0,1]$, $\forall\ x \in \Omega$.

It follows, using (4.3.3), (4.3.4) and (4.3.2), that

$$[\mathbf{A}_1(\mathbf{r}) - \mathbf{A}_1(\mathbf{s}), \mathbf{r} - \mathbf{s}] = \sum_{i,j=1}^{2} \int_\Omega [a_{ij}(\mathbf{r}(x)) - a_{ij}(\mathbf{s}(x))]\,[r_{ij}(x) - s_{ij}(x)]\, dx$$

$$= \int_\Omega \int_0^1 \sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}} a_{ij}(\hat{\mathbf{r}}(x,\rho))[r_{kl}(x) - s_{kl}(x)]\,[r_{ij}(x) - s_{ij}(x)]\, d\rho\, dx$$

$$\geq\; C_2 \int_\Omega \sum_{i,j=1}^{2} [r_{ij}(x) - s_{ij}(x)]^2\, dx \;=\; C_2\, ||\mathbf{r} - \mathbf{s}||_{X_1}^2\,,$$

which proves the strong monotonicity of $\mathbf{A}_1$.

On the other hand, we recall from [37] (see also Lemma 4.3 in [6]) that the functions $\frac{\partial}{\partial r_{kl}} a_{ij}(\cdot)$ are continuous in $\mathbb{R}^{2\times 2}$, and there exists $\bar{C} > 0$ such that

$$\left| \frac{\partial}{\partial r_{kl}} a_{ij}(\mathbf{r}) \right| \;\leq\; \bar{C} \quad \forall\, \mathbf{r} \in \mathbb{R}^{2\times 2}\,. \tag{4.3.5}$$

Then, applying again the identity (4.3.4), and proceeding similarly as above, one can prove, using (4.3.5) now, that $\mathbf{A}_1$ is a Lipschitz-continuous operator. For further details we refer to Theorem 5.2 in [6] or Lemma 3.2 in [30]. ∎

In order to continue our analysis, we need to recall the following previous lemma from [43].

Lema 4.3.2 *Let* $H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) : \; v = 0 \quad on \quad \Gamma_D\}$ *and let* $\tilde{H}_0(\mathbf{div};\Omega)$ *be the space*

$$\{\boldsymbol{\tau} \in H_0(\mathbf{div};\Omega) : \; \frac{1}{2}(\boldsymbol{\tau} + \boldsymbol{\tau}^{\mathbf{T}}) = \mathbf{e}(\mathbf{z}) \textit{ for some } \mathbf{z} \in [H^1_{\Gamma_D}(\Omega)]^2\}\,.$$

Then, for every $\boldsymbol{\eta} \in \mathcal{R}$ *there exists a unique* $\boldsymbol{\tau}(\boldsymbol{\eta}) \in \tilde{H}_0(\mathbf{div};\Omega)$ *such that* $\mathbf{div}\,\boldsymbol{\tau}(\boldsymbol{\eta}) = 0$ *in* Ω *and* $\frac{1}{2}(\boldsymbol{\tau}(\boldsymbol{\eta}) - \boldsymbol{\tau}(\boldsymbol{\eta})^{\mathbf{T}}) = \boldsymbol{\eta}$*. Moreover, there exists* $\tilde{C} > 0$*, independent of* $\boldsymbol{\eta}$*, such that*

$$||\boldsymbol{\tau}(\boldsymbol{\eta})||_{H(\mathbf{div};\Omega)} \;\leq\; \tilde{C}\,||\boldsymbol{\eta}||_{[L^2(\Omega)]^{2\times 2}} \quad \forall\,\boldsymbol{\eta} \in \mathcal{R}\,. \tag{4.3.6}$$

Proof: See Lemma 4.4 in [43]. ■

We are ready to prove the inf-sup condition for $\mathbf{B}$.

Lema 4.3.3 *There exists* $\beta > 0$*, depending only on* Ω*, such that*

$$\sup_{\substack{\boldsymbol{\tau}\in M_1\\ \boldsymbol{\tau}\neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\eta})]}{||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)}} \;\geq\; \beta\,||(\mathbf{v},\boldsymbol{\eta})||_M$$

for all $(\mathbf{v},\boldsymbol{\eta}) \in M := [L^2(\Omega)]^2 \times \mathcal{R}$*.*

Proof: We follow the proof of Lemma 4.5 in [43]. Thus, given $(\mathbf{v},\boldsymbol{\eta}) \in M$, we let $\boldsymbol{\tau}(\boldsymbol{\eta}) \in \tilde{H}_0(\mathbf{div};\Omega) \subseteq M_1 := H_0(\mathbf{div};\Omega)$ be the element provided by Lemma 4.3.2.

Then, we can write

$$\sup_{\substack{\boldsymbol{\tau}\in M_1\\ \boldsymbol{\tau}\neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\eta})]}{||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)}} \;\geq\; \frac{[\mathbf{B}(-\boldsymbol{\tau}(\boldsymbol{\eta})),(\mathbf{v},\boldsymbol{\eta})]}{||\boldsymbol{\tau}(\boldsymbol{\eta})||_{H(\mathbf{div};\Omega)}}$$

$$= \frac{1}{||\boldsymbol{\tau}(\boldsymbol{\eta})||_{H(\mathbf{div};\Omega)}}\left\{\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau}(\boldsymbol{\eta})\,dx + \int_\Omega \boldsymbol{\tau}(\boldsymbol{\eta}) : \boldsymbol{\eta}\,dx\right\}.$$

Since $\mathbf{div}\,\boldsymbol{\tau}(\boldsymbol{\eta}) = 0$ in Ω and $\int_\Omega \boldsymbol{\tau}(\boldsymbol{\eta}) : \boldsymbol{\eta}\,dx = ||\boldsymbol{\eta}||^2_{[L^2(\Omega)]^{2\times 2}}$, we deduce, using (4.3.6), that

$$\sup_{\substack{\boldsymbol{\tau}\in M_1\\ \boldsymbol{\tau}\neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\eta})]}{||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)}} \;\geq\; \frac{1}{\tilde{C}}\,||\boldsymbol{\eta}||_{[L^2(\Omega)]^{2\times 2}}\,. \tag{4.3.7}$$

Now, let $\tilde{\boldsymbol{\tau}} := \mathbf{e}(\mathbf{z})$ where $\mathbf{z} \in [H^1_{\Gamma_D}(\Omega)]^2$ is the unique solution of the weak formulation

$$\int_\Omega \mathbf{e}(\mathbf{z}) : \mathbf{e}(\mathbf{w})\,dx \;=\; -\int_\Omega \mathbf{v}\cdot\mathbf{w}\,dx \quad \forall\,\mathbf{w} \in [H^1_{\Gamma_D}(\Omega)]^2\,, \tag{4.3.8}$$

which formally solves the boundary value problem

$$\mathbf{div}\,\mathbf{e}(\mathbf{z}) = \mathbf{v} \quad \text{in} \quad \Omega\,, \quad \mathbf{e}(\mathbf{z})\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma_N\,, \quad \mathbf{z} = 0 \quad \text{on} \quad \Gamma_D\,.$$

It follows that $\mathbf{div}\,\tilde{\boldsymbol{\tau}} = \mathbf{v}$ in Ω and $\tilde{\boldsymbol{\tau}}\boldsymbol{\nu} = 0$ on Γ_N, whence $\tilde{\boldsymbol{\tau}} \in M_1$. Moreover, applying identity (4.3.8) with $\mathbf{w} = \mathbf{z}$, and using the equivalence in $[H^1_{\Gamma_D}(\Omega)]^2$ of $||\cdot||_{[H^1(\Omega)]^2}$ and $||\mathbf{e}(\cdot)||_{[L^2(\Omega)]^{2\times 2}}$ (because of Korn's inequality and the fact that $|\Gamma_D| \neq 0$), we get the estimate

$$\begin{aligned} ||\tilde{\boldsymbol{\tau}}||^2_{H(\mathbf{div};\Omega)} &= ||\mathbf{e}(\mathbf{z})||^2_{[L^2(\Omega)]^{2\times 2}} + ||\mathbf{v}||^2_{[L^2(\Omega)]^2} \\ &\leq \bar{C}\, ||\mathbf{v}||^2_{[L^2(\Omega)]^2}\,. \end{aligned} \tag{4.3.9}$$

In this way, since $\int_\Omega \tilde{\boldsymbol{\tau}} : \boldsymbol{\eta}\, dx = 0$, due to the symmetry of $\tilde{\boldsymbol{\tau}}$, we find that

$$\begin{aligned} \sup_{\substack{\boldsymbol{\tau}\in M_1\\ \boldsymbol{\tau}\neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\eta})]}{||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)}} &\geq \frac{[\mathbf{B}(-\tilde{\boldsymbol{\tau}}),(\mathbf{v},\boldsymbol{\eta})]}{||\tilde{\boldsymbol{\tau}}||_{H(\mathbf{div};\Omega)}} \\ &= \frac{||\mathbf{v}||^2_{[L^2(\Omega)]^2}}{||\tilde{\boldsymbol{\tau}}||_{H(\mathbf{div};\Omega)}} \geq \frac{1}{\sqrt{\bar{C}}}\, ||\mathbf{v}||_{[L^2(\Omega)]^2}\,, \end{aligned} \tag{4.3.10}$$

where the last inequality follows from (4.3.9).

Finally, (4.3.7) and (4.3.10) complete the proof of the lemma. ∎

The inf-sup condition for $\mathbf{B}_1$, on the kernel of $\mathbf{B}$, is proved next.

Lema 4.3.4 *Let* $\tilde{M}_1 := \{\boldsymbol{\tau} \in M_1 :\ [\mathbf{B}(\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\eta})] = 0\ \forall\,(\mathbf{v},\boldsymbol{\eta}) \in M\}$. *Then, there exists* $\beta_1 > 0$ *such that for all* $\boldsymbol{\tau} \in \tilde{M}_1$

$$\sup_{\substack{\mathbf{s}\in X_1\\ \mathbf{s}\neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{||\mathbf{s}||_{X_1}} \geq \beta_1\, ||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)}\,.$$

Proof: We first observe, according to the definition of $\mathbf{B}$, that

$$\tilde{M}_1 := \{\, \boldsymbol{\tau} \in M_1 : \quad \mathbf{div}\,\boldsymbol{\tau} = 0 \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathbf{T}} \quad \text{in} \quad \Omega \,\}\,. \tag{4.3.11}$$

Then, given $\boldsymbol{\tau} \in \tilde{M}_1$ we have

$$\begin{aligned} \sup_{\substack{\mathbf{s}\in X_1\\ \mathbf{s}\neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{||\mathbf{s}||_{X_1}} &= \sup_{\substack{\mathbf{s}\in X_1\\ \mathbf{s}\neq 0}} \frac{-\int_\Omega \mathbf{s} : \boldsymbol{\tau}\, dx}{||\mathbf{s}||_{[L^2(\Omega)]^{2\times 2}}} \\ &= ||\boldsymbol{\tau}||_{[L^2(\Omega)]^{2\times 2}} = ||\boldsymbol{\tau}||_{H(\mathbf{div};\Omega)} \end{aligned}$$

since $\tilde{M}_1 \subseteq X_1$. ∎

We are now in position to provide our main result concerning the solvability of the continuous problem (4.2.10).

Teorema 4.3.1 *There exists a unique* $(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in X_1 \times M_1 \times M$ *solution of the two-fold saddle point operator equation* (4.2.10).

Proof: In virtue of Lemmata 4.3.1, 4.3.3 and 4.3.4, the proof follows from a straightforward application of the abstract Theorem 2.4 in [29] (see also Theorem 4.1 in [40]). ∎

4.4 The finite element solution

We now introduce specific finite element subspaces, define the associated Galerkin scheme for (4.2.10) and show the solvability and quasi-optimal convergence of it. More precisely, we extend the PEERS-finite element space for linear plane elasticity given in [7] to the present nonlinear boundary value problem in hyperelasticity.

In order to define this extended space, we first let $\mathcal{T}_h$ be a regular triangulation of Ω by triangles T of diameter h_T with mesh size $h := \sup\{h_T : \; T \in \mathcal{T}_h\}$. For simplicity, in what follows we assume that the boundary Γ of Ω is a polygonal curve so that $\bar{\Omega} = \cup\{T : \; T \in \mathcal{T}_h\}$.

Next, we set the canonical triangle with vertices $\hat{P}_1 = (0,0)^{\mathbf{T}}$, $\hat{P}_2 = (1,0)^{\mathbf{T}}$ and $\hat{P}_3 = (0,1)^{\mathbf{T}}$ as a reference triangle $\hat{T}$, and introduce the family of bijective affine mappings $\{\, F_T \,\}_{T\in\mathcal{T}_h}$ such that $F_T(\hat{T}) = T$ for all $T \in \mathcal{T}_h$. It is well known that $F_T(\hat{x}) = B_T\hat{x} + d_T \;\forall\; \hat{x} := (\hat{x}_1, \hat{x}_2)^{\mathbf{T}} \in \hat{T}$, where $B_T \in \mathbb{R}^{2\times 2}$ and $d_T \in \mathbb{R}^2$ depend on the vertices of T. Also, given an integer $l \geq 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathbb{P}_l(S)$ the space of polynomials in two variables defined in S of total degree at most l.

We now consider the lowest order Raviart-Thomas elements. To this end, we let

$$RT_0(\hat{T}) := \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \right\} \subseteq [\mathbb{P}_1(\hat{T})]^2 ,$$

and for each triangle $T \in \mathcal{T}_h$, we put

$$RT_0(T) := \left\{ \boldsymbol{\tau} \in [\mathbb{P}_1(T)]^2 : \; \boldsymbol{\tau} = |B_T|^{-1} B_T \, \hat{\boldsymbol{\tau}} \circ F_T^{-1} \,, \; \hat{\boldsymbol{\tau}} \in RT_0(\hat{T}) \right\} ,$$

with which we define

$$RT_0[\mathcal{T}_h] := \left\{ \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \; (\tau_{i1}\tau_{i2})^{\mathbf{T}}|_T \in RT_0(T), \, i = 1, 2, \, \forall\, T \in \mathcal{T}_h \right\}. \tag{4.4.1}$$

On the other hand, for each triangle $T \in \mathcal{T}_h$ we take the unique polynomial $b_T \in \mathbb{P}_3(T)$ that vanishes on ∂T and is normalized by $\int_T b_T(x)\, dx = 1$. This cubic bubble function is extended by 0 onto the region $\bar{\Omega} - T$ and therefore it becomes an element of $H_0^1(\Omega)$. Hence, we define

$$\mathcal{B}[\mathcal{T}_h] := \Big\{ \boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \, (\tau_{i1}\, \tau_{i2}) \in \mathrm{span}\{\mathrm{rot}\, b_T,\, T \in \mathcal{T}_h\},\, i = 1,2 \Big\}, \tag{4.4.2}$$

where $\mathrm{rot}\, b_T := (\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1})$.

Then, we introduce the following ansatz spaces for the unknowns $\mathbf{t}$ and $\boldsymbol{\sigma}$:

$$X_{1,h} := RT_0[\mathcal{T}_h] + \mathcal{B}[\mathcal{T}_h] \tag{4.4.3}$$

and

$$M_{1,h} := RT_0[\mathcal{T}_h] \cap H_0(\mathbf{div};\Omega) + \mathcal{B}[\mathcal{T}_h] \,. \tag{4.4.4}$$

In addition, for the unknowns $(\mathbf{u}, \boldsymbol{\gamma}) \in [L^2(\Omega)]^2 \times \mathcal{R}$ we define

$$M_h := V_h \times \mathcal{R}_h \,, \tag{4.4.5}$$

where

$$V_h := \{\mathbf{v} \in [L^2(\Omega)]^2 : \quad \mathbf{v}|_T \in [\mathbb{P}_0(T)]^2 \ \forall T \in \mathcal{T}_h\} \tag{4.4.6}$$

and

$$\mathcal{R}_h := \left\{ \boldsymbol{\eta} := \begin{bmatrix} 0 & \eta \\ -\eta & 0 \end{bmatrix} \in [H^1(\Omega)]^{2\times 2} : \quad \eta \in \mathbb{P}_1[\mathcal{T}_h] \right\}, \tag{4.4.7}$$

with

$$\mathbb{P}_1[\mathcal{T}_h] := \{\eta \in H^1(\Omega) : \quad \eta|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h\} \,.$$

At this point we recall that $M_{1,h} \times M_h$ corresponds to the PEERS-space given by Arnold, Brezzi and Douglas in [7] for the approximation of the unknowns $(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma}))$.

Therefore, the Galerkin scheme for approximately solving our problem (4.2.10) is: *Find* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h)) \in X_{1,h} \times M_{1,h} \times M_h$ *such that*

$$\begin{aligned}
&[\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}_h] + [\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] &&= [\mathbf{F}_1, \mathbf{s}_h], \\
&[\mathbf{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h)] &&= [\mathbf{F}_2, \boldsymbol{\tau}_h], \\
&[\mathbf{B}(\boldsymbol{\sigma}_h), (\mathbf{v}_h, \boldsymbol{\eta}_h)] &&= [\mathbf{G}, (\mathbf{v}_h, \boldsymbol{\eta}_h)],
\end{aligned} \tag{4.4.8}$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\eta}_h)) \in X_{1,h} \times M_{1,h} \times M_h$.

The unique solvability of (4.4.8) and the corresponding quasi optimal error estimate are provided next.

Teorema 4.4.1 *There exists a unique* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h)) \in X_{1,h} \times M_{1,h} \times M_h$ *solution of the discrete scheme* (4.4.8). *Moreover, there exists* $C > 0$, *independent of the subspaces involved, such that the following Cea estimate holds*

$$||(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \,-\, (\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h))||_{X\times M}$$

$$\leq C \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\eta}_h)) \\ \in X_{1,h}\times M_{1,h}\times M_h}} ||(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) - (\mathbf{s}_h, \boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\eta}_h))||_{X\times M}\,.$$

Proof: According to Theorem 3.2 in [29] (see also Theorem 4.2 in [40]), we first need to prove that $\mathbf{B}$ and $\mathbf{B}_1$ satisfy the corresponding discrete inf-sup conditions.

Indeed, from Lemma 4.4 in [7] we have that, given $(\mathbf{v}_h, \boldsymbol{\eta}_h) \in M_h$, there exists $\boldsymbol{\tau}_h \in RT_0[\mathcal{T}_h] \cap H(\mathbf{div}; \Omega) + \mathcal{B}[\mathcal{T}_h]$, $\boldsymbol{\tau}_h \neq 0$, such that

$$[\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{v}_h, \boldsymbol{\eta}_h)] \,\geq\, \beta^* \, ||\boldsymbol{\tau}_h||_{H(\mathbf{div};\Omega)} \, ||(\mathbf{v}_h, \boldsymbol{\eta}_h)||_M \,,$$

where $\beta^* > 0$ is independent of h. Moreover, from the proof of that lemma in [7], we find that $\boldsymbol{\tau}_h$ can be chosen so that $\boldsymbol{\tau}_h \boldsymbol{\nu} = 0$ on Γ_N, whence $\boldsymbol{\tau}_h \in M_{1,h}$. This proves the discrete inf-sup condition for $\mathbf{B}$ (analogue of the continuous one given in Lemma 4.3.3).

Now, let $\tilde{M}_{1,h}$ be the discrete kernel of $\mathbf{B}$, that is

$$\tilde{M}_{1,h} := \{\boldsymbol{\tau}_h \in M_{1,h} : \;\; [\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{v}_h, \boldsymbol{\eta}_h)] = 0 \, \forall \, (\mathbf{v}_h, \boldsymbol{\eta}_h) \in M_h\}. \tag{4.4.9}$$

It follows that

$$\tilde{M}_{1,h} := \left\{ \boldsymbol{\tau}_h \in M_{1,h} : \quad \int_\Omega \mathbf{v}_h \cdot \mathbf{div}\, \boldsymbol{\tau}_h \, dx = 0 \quad \forall \, \mathbf{v}_h \in V_h \right.$$
$$\left. \text{and} \quad \int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\eta}_h \, dx = 0 \quad \forall \, \boldsymbol{\eta}_h \in \mathcal{R}_h \right\}.$$

Since $\mathbf{div}\, \boldsymbol{\tau} = 0$ in $\Omega \; \forall \, \boldsymbol{\tau} \in \mathcal{B}[\mathcal{T}_h]$, we deduce that $(\mathbf{div}\, \boldsymbol{\tau}_h)|_T$ is a constant vector $\forall \, \boldsymbol{\tau}_h \in M_{1,h}$ and $\forall \, T \in \mathcal{T}_h$. Therefore, since $\mathbf{v}_h|_T$ is also a constant vector $\forall \, \mathbf{v}_h \in V_h$ and $\forall \, T \in \mathcal{T}_h$, we conclude that $\mathbf{div}\, \boldsymbol{\tau}_h = 0$ in $\Omega \; \forall \, \boldsymbol{\tau}_h \in \tilde{M}_{1,h}$. Hence, we can write

$$\tilde{M}_{1,h} := \left\{ \boldsymbol{\tau}_h \in M_{1,h} : \quad \mathbf{div}\, \boldsymbol{\tau}_h = 0 \quad \text{in} \quad \Omega \right.$$
$$\left. \text{and} \quad \int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\eta}_h \, dx = 0 \quad \forall \, \boldsymbol{\eta}_h \in \mathcal{R}_h \right\}. \tag{4.4.10}$$

Note that the second condition in the above definition does not guarantee the symmetry of the tensors of $\tilde{M}_{1,h}$, as it was the case for the continuous kernel of $\mathbf{B}$ (cf. (4.3.11)).

Then, for any $\boldsymbol{\tau}_h \in \tilde{M}_{1,h} \subseteq X_{1,h}$ we obtain

$$\sup_{\substack{\mathbf{s}_h \in X_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{||\mathbf{s}_h||_{X_1}} = \sup_{\substack{\mathbf{s}_h \in X_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{-\int_\Omega \mathbf{s}_h : \boldsymbol{\tau}_h \, dx}{||\mathbf{s}_h||_{X_1}}$$

$$= ||\boldsymbol{\tau}_h||_{[L^2(\Omega)]^{2\times 2}} = ||\boldsymbol{\tau}_h||_{H(\mathbf{div};\Omega)},$$

which proves the discrete inf-sup condition for $\mathbf{B}_1$.

Thus, straightforward applications of the abstract Theorems 3.2 and 4.1 in [29] provide the unique solvability of the Galerkin scheme (4.4.8) and the following Strang-type error estimate

$$||(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h))||_{X\times M}$$

$$\leq C \left\{ \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\eta}_h)) \\ \in X_{1,h}\times M_{1,h}\times M_h}} ||(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) - (\mathbf{s}_h, \boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\eta}_h))||_{X\times M} \right.$$

$$\left. + \sup_{\substack{\boldsymbol{\tau}_h \in \tilde{M}_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{[\mathbf{F}_2 - \mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}_h]}{||\boldsymbol{\tau}_h||_{H(\mathbf{div};\Omega)}} \right\}, \tag{4.4.11}$$

where $C > 0$ is independent of the subspaces involved.

It remains to estimate the consistency term in (4.4.11). For this purpose, we first observe from the second equation in (4.2.10) that $\mathbf{F}_2 - \mathbf{B}_1(\mathbf{t}) = \mathbf{B}^*(\mathbf{u}, \boldsymbol{\gamma})$. Then, given $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$, we have

$$[\mathbf{F}_2 - \mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}_h] = [\mathbf{B}^*(\mathbf{u}, \boldsymbol{\gamma}), \boldsymbol{\tau}_h] = [\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{u}, \boldsymbol{\gamma})],$$

which, using from (4.4.9) that $[\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{v}_h, \boldsymbol{\eta}_h)] = 0 \ \forall \ (\mathbf{v}_h, \boldsymbol{\eta}_h) \in M_h$, yields

$$[\mathbf{F}_2 - \mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}_h] = [\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{u}, \boldsymbol{\gamma}) - (\mathbf{v}_h, \boldsymbol{\eta}_h)] \tag{4.4.12}$$

for all $(\mathbf{v}_h, \boldsymbol{\eta}_h) \in M_h$. Therefore, applying (4.4.12) and the boundedness of $\mathbf{B}$, we obtain

$$\sup_{\substack{\boldsymbol{\tau}_h \in \tilde{M}_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{[\mathbf{F}_2 - \mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}_h]}{||\boldsymbol{\tau}_h||_{H(\mathbf{div};\Omega)}} = \sup_{\substack{\boldsymbol{\tau}_h \in \tilde{M}_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{u}, \boldsymbol{\gamma}) - (\mathbf{v}_h, \boldsymbol{\eta}_h)]}{||\boldsymbol{\tau}_h||_{H(\mathbf{div};\Omega)}}$$

$$\leq ||\mathbf{B}|| \ ||(\mathbf{u}, \boldsymbol{\gamma}) - (\mathbf{v}_h, \boldsymbol{\eta}_h)||_M \quad \forall \, (\mathbf{v}_h, \boldsymbol{\eta}_h) \in M_h,$$

whence

$$\sup_{\substack{\boldsymbol{\tau}_h \in \tilde{M}_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{[\mathbf{F}_2 - \mathbf{B}_1(\mathbf{t}), \boldsymbol{\tau}_h]}{||\boldsymbol{\tau}_h||_{H(\mathbf{div};\Omega)}} \leq ||\mathbf{B}|| \inf_{(\mathbf{v}_h, \boldsymbol{\eta}_h) \in M_h} ||(\mathbf{u}, \boldsymbol{\gamma}) - (\mathbf{v}_h, \boldsymbol{\eta}_h)||_M. \tag{4.4.13}$$

Finally, (4.4.11) and (4.4.13) provide the Cea estimate and complete the proof of the theorem. ■

We observe that the approximation properties of the subspaces $X_{1,h}$, $M_{1,h}$ and M_h together with the Cea estimate provided above, imply the quasi-optimal convergence of the finite element solutions. Furthermore, appropriate regularity assumptions on the continuous solution guarantee the corresponding rates of convergence.

On the other hand, it is worth remarking that the results provided by Theorem 4.4.1 and the a-posteriori error analysis developed below in Section 6, are valid not only for the extended PEERS-space, but they also hold for any other stable finite element subspace. Here, stability means that the discrete inf-sup conditions indicated in the proof of Theorem 4.4.1 are satisfied with constants independent of the subspaces involved. In particular, if $M_{1,h} \times M_h$ is another stable mixed finite element subspace for linear elasticity, and if the elements in the discrete kernel $\tilde{M}_{1,h}$ are divergence free, then for any $X_{1,h}$ containing $\tilde{M}_{1,h}$, the triple $X_{1,h} \times M_{1,h} \times M_h$ becomes stable.

4.5 The a-posteriori error estimate

In this section we derive the a-posteriori error estimate for our Galerkin scheme (4.4.8). To this end, we follow the method proposed in [17] (which extends previous results given in [10], [2] and [3]) and combine it with the ideas from [19] and [30] (for dealing with nonlinear operators).

4.5.1 Preliminaries

In what follows, we let $\{\mathcal{T}_h\}_{h\in \mathtt{J}}$ be a regular family of triangulations of Ω, where $\mathtt{J}$ is a at most numerable set of indexes, say $\mathtt{J} := \{h_j\}_{j\in\mathbb{N}}$, with $h_j \geq h_{j+1}\ \forall\, j \in \mathbb{N}$. Here, *regular* means that the interior angles of all the triangles of all the triangulations $\mathcal{T}_h$ are uniformly bounded from below.

Now, we let $\hat{\mathbf{A}} : X \to X'$ be any linear and bounded operator such that the corresponding induced bilinear form $\hat{A} : X \times X \to \mathbb{R}$, defined by

$$\hat{A}((\mathbf{r},\boldsymbol{\zeta}),(\mathbf{s},\boldsymbol{\tau})) \,=\, [\hat{\mathbf{A}}(\mathbf{r},\boldsymbol{\zeta}),(\mathbf{s},\boldsymbol{\tau})]$$

for all $(\mathbf{r},\boldsymbol{\zeta})$, $(\mathbf{s},\boldsymbol{\tau}) \in X$, is symmetric and X-elliptic. In particular, we may consider $\hat{\mathbf{A}}$ so that $\hat{A}$ becomes the usual inner product in X, that is

$$[\hat{\mathbf{A}}(\mathbf{r},\boldsymbol{\zeta}),(\mathbf{s},\boldsymbol{\tau})] \,=\, \langle \mathbf{r},\mathbf{s}\rangle_{[L^2(\Omega)]^{2\times 2}} + \langle \boldsymbol{\zeta},\boldsymbol{\tau}\rangle_{H(\mathbf{div};\Omega)}\,. \qquad (4.5.1)$$

Also, let $\hat{\mathbf{A}}_T$ be the linear and bounded operator obtained by restricting $\hat{\mathbf{A}}$ to the finite element $T \in \mathcal{T}_h$. In other words, denoting

$$X_T := X_{1,T} \times M_{1,T} \tag{4.5.2}$$

with

$$\begin{gathered} X_{1,T} := [L^2(T)]^{2\times 2} \quad \text{and} \\ M_{1,T} = H_0(\mathbf{div};T) := \{\boldsymbol{\tau} \in H(\mathbf{div};T) : \ \boldsymbol{\tau\nu}|_{\partial T \cap \Gamma_N} = 0\}, \end{gathered} \tag{4.5.3}$$

and using (4.5.1), $\hat{\mathbf{A}}_T : X_T \to X_T'$ is defined by

$$[\hat{\mathbf{A}}_T(\mathbf{r},\boldsymbol{\zeta}),(\mathbf{s},\boldsymbol{\tau})] := \langle \mathbf{r},\mathbf{s}\rangle_{[L^2(T)]^{2\times 2}} + \langle \boldsymbol{\zeta},\boldsymbol{\tau}\rangle_{H(\mathbf{div};T)}$$

for all $(\mathbf{r},\boldsymbol{\zeta})$, $(\mathbf{s},\boldsymbol{\tau}) \in X_T$.

Similarly, let

$$\begin{gathered} M_T := [L^2(T)]^2 \times \mathcal{R}_T \text{ , with} \\ \mathcal{R}_T := \{\boldsymbol{\eta} \in [L^2(T)]^{2\times 2} : \ \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathbf{T}} = 0\}. \end{gathered} \tag{4.5.4}$$

Then, we denote by $\mathbf{A}_T$ and $\mathbf{B}_T$ the corresponding restrictions to $T \in \mathcal{T}_h$ of the nonlinear operator $\mathbf{A}$ and linear operator $\mathbf{B}$, respectively. This means that $\mathbf{A}_T : X_T \to X_T'$ and $\mathbf{B}_T : M_{1,T} \to M_T'$ are defined as follows:

$$\begin{aligned} [\mathbf{A}_T(\mathbf{r},\boldsymbol{\zeta}),(\mathbf{s},\boldsymbol{\tau})] &:= \int_T [\hat{\lambda}(\mathbf{r})\,\mathrm{tr}\,(\mathbf{r})\,\mathrm{tr}\,(\mathbf{s}) + \hat{\mu}(\mathbf{r})\,\mathbf{r}:\mathbf{s}]\,dx \\ &\quad - \int_T \mathbf{r}:\boldsymbol{\tau}\,dx - \int_T \mathbf{s}:\boldsymbol{\zeta}\,dx, \\ [\mathbf{B}_T(\boldsymbol{\tau}),(\mathbf{v},\boldsymbol{\eta})] &:= -\int_T \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau}\,dx - \int_T \boldsymbol{\tau}:\boldsymbol{\eta}\,dx, \end{aligned}$$

for all $(\mathbf{r},\boldsymbol{\zeta})$, $(\mathbf{s},\boldsymbol{\tau}) \in X_T$ and for all $(\mathbf{v},\boldsymbol{\eta}) \in M_T$.

4.5.2 Main estimates

Our main goal in this section is to prove the following theorem.

Teorema 4.5.1 *Let* $(\mathbf{t},\boldsymbol{\sigma},(\mathbf{u},\boldsymbol{\gamma})) \in X_1 \times M_1 \times M$ *and* $(\mathbf{t}_h,\boldsymbol{\sigma}_h,(\mathbf{u}_h,\boldsymbol{\gamma}_h)) \in X_{1,h} \times M_{1,h} \times M_h$ *be the unique solutions of the continuous and discrete formulations* (4.2.10) *and* (4.4.8), *respectively. In addition, let* $\boldsymbol{\varphi}_h \in [H^{1/2}(\cup_{T\in\mathcal{T}_h}\partial T)]^2$ *be an approximation to the displacement* $\mathbf{u}$ *on element boundaries with* $\boldsymbol{\varphi}_h = \mathbf{g}$ *on* Γ_D, *and for each* $T \in \mathcal{T}_h$, *let* $(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T) \in X_T$ *be the unique solution of the local problem*

$$\begin{gathered} [\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T),(\mathbf{s},\boldsymbol{\tau})] \;=\; -\,[\mathbf{A}_T(\mathbf{t}_{h,T},\boldsymbol{\sigma}_{h,T}),(\mathbf{s},\boldsymbol{\tau})] \\ -\,[\mathbf{B}_T(\boldsymbol{\tau}),(\mathbf{u}_{h,T},\boldsymbol{\gamma}_{h,T})] \;-\; \int_{\partial T} \boldsymbol{\varphi}_h\cdot\boldsymbol{\tau\nu}\,ds \end{gathered} \tag{4.5.5}$$

for all $(\mathbf{s},\boldsymbol{\tau}) \in X_T$*, where* $(\mathbf{t}_{h,T},\boldsymbol{\sigma}_{h,T},(\mathbf{u}_{h,T},\boldsymbol{\gamma}_{h,T}))$ *denotes the restriction of* $(\mathbf{t}_h,\boldsymbol{\sigma}_h,(\mathbf{u}_h,\boldsymbol{\gamma}_h))$ *to* T*. Then, there exist* C*,* $h_0>0$*, such that for all* $h\le h_0$

$$||(\mathbf{t},\boldsymbol{\sigma},(\mathbf{u},\boldsymbol{\gamma}))-(\mathbf{t}_h,\boldsymbol{\sigma}_h,(\mathbf{u}_h,\boldsymbol{\gamma}_h))||_{X\times M}\;\le\; C\left\{\sum_{T\in\mathcal{T}_h}\theta_T^2\right\}^{1/2},$$

where for each triangle T *we define*

$$\begin{aligned}\theta_T^2 \;:=\; &[\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T),(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T)]\\ &+||\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h||^2_{[L^2(T)]^2}\;+\;||\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^{\mathbf{T}}||^2_{[L^2(T)]^{2\times 2}}\,.\end{aligned}$$

We remark that the unique solvability of the local problem (4.5.5) follows from the well known Lax-Milgram Theorem and the fact that the bilinear form induced by $\hat{\mathbf{A}}_T$ is bounded and X_T-elliptic (when using (4.5.1)), and that the right hand side of (4.5.5) constitutes a linear and bounded functional on X_T.

In order to prove Theorem 4.5.1 we need some preliminary results. First, we show the following theorem concerning the Ritz projection of the Galerkin error with respect to the operator $\hat{\mathbf{A}}$.

Teorema 4.5.2 *Let* $(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})$ *be the unique element in* X *such that*

$$\begin{aligned}[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\mathbf{s},\boldsymbol{\tau})] = [\mathbf{A}(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s},\boldsymbol{\tau})]-[\mathbf{A}(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s},\boldsymbol{\tau})]\\ +[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{u},\boldsymbol{\gamma})-(\mathbf{u}_h,\boldsymbol{\gamma}_h)]\end{aligned}\tag{4.5.6}$$

for all $(\mathbf{s},\boldsymbol{\tau})\in X$*. Then there holds*

$$[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})]\;\le\;\sum_{T\in\mathcal{T}_h}[\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T),(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T)]\,.$$

In particular, if $\hat{\mathbf{A}}$ *is given by (4.5.1), then we have*

$$||(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})||_X^2\;\le\;\sum_{T\in\mathcal{T}_h}||(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T)||^2_{X_T}\,.$$

Proof: We first note from (4.2.10) that

$$[\mathbf{A}(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s},\boldsymbol{\tau})]+[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{u},\boldsymbol{\gamma})] = -\int_{\Gamma_D}\mathbf{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds$$

for all $(\mathbf{s},\boldsymbol{\tau})\in X$, and hence

$$\begin{aligned}[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\mathbf{s},\boldsymbol{\tau})] = -\int_{\Gamma_D}\mathbf{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds-[\mathbf{A}(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s},\boldsymbol{\tau})]\\ -[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{u}_h,\boldsymbol{\gamma}_h)]\quad\forall\,(\mathbf{s},\boldsymbol{\tau})\in X\,.\end{aligned}\tag{4.5.7}$$

Then, since the bilinear form induced by $\hat{\mathbf{A}}$ is symmetric and X-elliptic, we have that

$$-\frac{1}{2}[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})] = \inf_{(\mathbf{s},\boldsymbol{\tau})\in X}\left\{\frac{1}{2}[\hat{\mathbf{A}}(\mathbf{s},\boldsymbol{\tau}),(\mathbf{s},\boldsymbol{\tau})] - [\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\mathbf{s},\boldsymbol{\tau})]\right\},$$

which, using (4.5.7), yields

$$-\frac{1}{2}\,[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})] \;=\; \inf_{(\mathbf{s},\boldsymbol{\tau})\in X}\,\boldsymbol{\chi}(\mathbf{s},\boldsymbol{\tau}) \tag{4.5.8}$$

with

$$\begin{aligned}\boldsymbol{\chi}(\mathbf{s},\boldsymbol{\tau}) := \frac{1}{2}\,[\hat{\mathbf{A}}(\mathbf{s},\boldsymbol{\tau}),(\mathbf{s},\boldsymbol{\tau})] + \int_{\Gamma_D}\mathbf{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds\\ +[\mathbf{A}(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s},\boldsymbol{\tau})] + [\mathbf{B}(\boldsymbol{\tau}),(\mathbf{u}_h,\boldsymbol{\gamma}_h)] \qquad \forall\,(\mathbf{s},\boldsymbol{\tau})\in X\,.\end{aligned} \tag{4.5.9}$$

Now, let $\boldsymbol{\varphi}_h \in [H^{1/2}(\cup_{T\in\mathcal{T}_h}\partial T)]^2$ be as indicated in Theorem 4.5.1. Then, since $\boldsymbol{\tau}\boldsymbol{\nu} = 0$ on Γ_N $\forall\,\boldsymbol{\tau}\in H_0(\mathbf{div};\Omega)$, we obtain

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\boldsymbol{\varphi}_h\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds = \int_{\Gamma_D}\mathbf{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds\,. \tag{4.5.10}$$

It follows from (4.5.9) and (4.5.10) that

$$\boldsymbol{\chi}(\mathbf{s},\boldsymbol{\tau}) \;=\; \sum_{T\in\mathcal{T}_h}\boldsymbol{\chi}_T(\mathbf{s}_T,\boldsymbol{\tau}_T) \quad \forall\,(\mathbf{s},\boldsymbol{\tau})\in X\,, \tag{4.5.11}$$

where $(\mathbf{s}_T,\boldsymbol{\tau}_T)\in X_T$ is the restriction of $(\mathbf{s},\boldsymbol{\tau})$ to T, and

$$\begin{aligned}\boldsymbol{\chi}_T(\mathbf{s}_T,\boldsymbol{\tau}_T) := \frac{1}{2}[\hat{\mathbf{A}}_T(\mathbf{s}_T,\boldsymbol{\tau}_T),(\mathbf{s}_T,\boldsymbol{\tau}_T)] + [\mathbf{A}_T(\mathbf{t}_{h,T},\boldsymbol{\sigma}_{h,T}),(\mathbf{s}_T,\boldsymbol{\tau}_T)]\\ +\,[\mathbf{B}_T(\boldsymbol{\tau}_T),(\mathbf{u}_{h,T},\boldsymbol{\gamma}_{h,T})] \;+\; \int_{\partial T}\boldsymbol{\varphi}_h\cdot\boldsymbol{\tau}_T\boldsymbol{\nu}\,ds\,.\end{aligned} \tag{4.5.12}$$

Let us define the *broken* space

$$X^{br} := \{(\mathbf{s},\boldsymbol{\tau})\in[L^2(\Omega)]^{2\times2}\times[L^2(\Omega)]^{2\times2}:\;(\mathbf{s}_T,\boldsymbol{\tau}_T)\in X_T\;\forall\,T\in\mathcal{T}_h\}.$$

Then, since $X\subseteq X^{br}$, we can write from (4.5.8) and (4.5.11)

$$\begin{aligned}-\frac{1}{2}\,[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})] \;&\geq\; \inf_{(\mathbf{s},\boldsymbol{\tau})\in X^{br}}\sum_{T\in\mathcal{T}_h}\boldsymbol{\chi}_T(\mathbf{s}_T,\boldsymbol{\tau}_T)\\ &= \sum_{T\in\mathcal{T}_h}\,\inf_{(\mathbf{s}_T,\boldsymbol{\tau}_T)\in X_T}\,\boldsymbol{\chi}_T(\mathbf{s}_T,\boldsymbol{\tau}_T)\,,\end{aligned} \tag{4.5.13}$$

and, according to (4.5.12),

$$\inf_{(\mathbf{s}_T,\boldsymbol{\tau}_T)\in X_T} \boldsymbol{\chi}_T(\mathbf{s}_T,\boldsymbol{\tau}_T) \;=\; -\frac{1}{2}[\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T),(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T)] \tag{4.5.14}$$

where $(\hat{\mathbf{t}}_T,\hat{\boldsymbol{\sigma}}_T) \in X_T$ is the unique solution of the local problem (4.5.5).

Therefore, (4.5.13) and (4.5.14) complete the proof of the theorem. ■

Next, we need some properties of a functional depending on the stored energy function Ψ (cf. (4.1.2)). Indeed, let $\boldsymbol{\Psi} : [L^2(\Omega)]^{2\times 2} \to \mathbb{R}$ be defined by

$$\boldsymbol{\Psi}(\mathbf{s}) := \int_\Omega \Psi(\mathbf{s})\, dx \quad \forall\, \mathbf{s} \in [L^2(\Omega)]^{2\times 2}\,.$$

For the rest of this section we follow very closely the analysis from Section 5 in [30].

Lema 4.5.1 *The functional* $\boldsymbol{\Psi}$ *has continuous second order Gâteaux derivatives and there exist* $\tilde{C}_1,\ \tilde{C}_2 > 0$ *such that*

$$\tilde{C}_1\,||\mathbf{s}||^2_{[L^2(\Omega)]^{2\times 2}} \;\le\; (D^2\boldsymbol{\Psi})(\mathbf{r})(\mathbf{s},\mathbf{s}) \;\le\; \tilde{C}_2\,||\mathbf{s}||^2_{[L^2(\Omega)]^{2\times 2}} \tag{4.5.15}$$

for all $\mathbf{r}$, $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$.

Proof: The first order Gâteaux derivative $D\boldsymbol{\Psi}$ applies $[L^2(\Omega)]^{2\times 2}$ into its dual. Hence, given $\mathbf{r}$, $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$, we have

$$D\boldsymbol{\Psi}(\mathbf{r})(\mathbf{s}) := \lim_{\epsilon\to 0}\left\{\frac{\boldsymbol{\Psi}(\mathbf{r}+\epsilon\,\mathbf{s}) - \boldsymbol{\Psi}(\mathbf{r})}{\epsilon}\right\}$$

$$= \lim_{\epsilon\to 0}\int_\Omega \left\{\frac{\Psi(\mathbf{r}+\epsilon\,\mathbf{s}) - \Psi(\mathbf{r})}{\epsilon}\right\} dx\,,$$

which gives

$$D\boldsymbol{\Psi}(\mathbf{r})(\mathbf{s}) \;=\; \sum_{i,j=1}^{2}\int_\Omega a_{ij}(\mathbf{r})\, s_{ij}\, dx\,, \tag{4.5.16}$$

where $a_{ij} : [L^2(\Omega)]^{2\times 2} \to \mathbb{R}$ are the nonlinear mappings defined in (4.3.1).

Then, the second order derivative $D^2\boldsymbol{\Psi}$ applies $[L^2(\Omega)]^{2\times 2}$ into the dual of $[L^2(\Omega)]^{2\times 2} \times [L^2(\Omega)]^{2\times 2}$, and it is defined by

$$D^2\boldsymbol{\Psi}(\mathbf{r})(\tilde{\mathbf{s}},\mathbf{s}) := \lim_{\epsilon\to 0}\left\{\frac{D\boldsymbol{\Psi}(\mathbf{r}+\epsilon\,\tilde{\mathbf{s}})(\mathbf{s}) - D\boldsymbol{\Psi}(\mathbf{r})(\mathbf{s})}{\epsilon}\right\}$$

for all $\mathbf{r}$, $\tilde{\mathbf{s}}$, $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$, which, after some computations, yields

$$D^2\mathbf{\Psi}(\mathbf{r})(\tilde{\mathbf{s}}, \mathbf{s}) = \int_\Omega \sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}} a_{ij}(\mathbf{r})\, \tilde{s}_{kl}\, s_{ij}\, dx\,. \tag{4.5.17}$$

Therefore, the inequalities (4.3.2) and (4.3.5) satisfied by a_{ij}, imply (4.5.15). Finally, the continuity of $D^2\mathbf{\Psi}$ follows from the fact that the stored energy function Ψ is of class C^2. ■

We are ready to prove our main theorem.

4.5.3 Proof of Theorem 4.5.1

For given $\mathbf{r} \in [L^2(\Omega)]^{2\times 2}$, we note from (4.5.15) and (4.5.17) that

$$D^2\mathbf{\Psi}(\mathbf{r})(\cdot,\cdot) \,:\, [L^2(\Omega)]^{2\times 2} \times [L^2(\Omega)]^{2\times 2} \to \mathbb{R}$$

is a bounded and $[L^2(\Omega)]^{2\times 2}$-elliptic bilinear form, with constants independent of $\mathbf{r}$.

Then, since the linear operators $\mathbf{B}$ and $\mathbf{B}_1$ satisfy the continuous inf-sup conditions (cf. Lemmata 4.3.3 and 4.3.4), the linear version of the abstract Theorem 2.4 in [29] implies that the whole linear operator, obtained after replacing $[\mathbf{A}_1(\cdot),\cdot]$ by $D^2\mathbf{\Psi}(\mathbf{r})(\cdot,\cdot)$, satisfies a global inf-sup condition. In other words, there exists $C_0 > 0$, independent of $\mathbf{r}$, such that

$$||(\tilde{\mathbf{s}}, \tilde{\boldsymbol{\tau}}, (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}}))||_{X\times M} \,\le\, C_0 \sup_{\substack{(\mathbf{s},\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta}))\in X\times M\\ ||(\mathbf{s},\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta}))||\le 1}} \Big\{ D^2\mathbf{\Psi}(\mathbf{r})(\tilde{\mathbf{s}}, \mathbf{s}) + [\mathbf{B}_1(\tilde{\mathbf{s}}), \boldsymbol{\tau}]$$

$$+[\mathbf{B}_1(\mathbf{s}), \tilde{\boldsymbol{\tau}}] + [\mathbf{B}(\tilde{\boldsymbol{\tau}}), (\mathbf{v}, \boldsymbol{\eta})] + [\mathbf{B}(\boldsymbol{\tau}), (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}})] \Big\}$$

for all $(\tilde{\mathbf{s}}, \tilde{\boldsymbol{\tau}}, (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}})) \in X \times M$.

In particular, for $\mathbf{r} = \mathbf{t}$ and

$$(\tilde{\mathbf{s}}, \tilde{\boldsymbol{\tau}}, (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\eta}})) = (\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \gamma)) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \gamma_h))\,,$$

we obtain

$$\frac{1}{C_0}\, ||(\mathbf{t}, \boldsymbol{\sigma}, (\mathbf{u}, \gamma)) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, (\mathbf{u}_h, \gamma_h))||_{X\times M} \,\le$$

$$\sup_{\substack{(\mathbf{s},\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta}))\in X\times M\\ ||(\mathbf{s},\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta}))||\le 1}} \Big\{ D^2\mathbf{\Psi}(\mathbf{t})(\mathbf{t} - \mathbf{t}_h, \mathbf{s}) + [\mathbf{B}_1(\mathbf{t} - \mathbf{t}_h), \boldsymbol{\tau}] + [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\sigma} - \boldsymbol{\sigma}_h]$$

$$+[\mathbf{B}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),(\mathbf{v},\boldsymbol{\eta})]+[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{u},\boldsymbol{\gamma})-(\mathbf{u}_h,\boldsymbol{\gamma}_h)]\Big\}. \qquad (4.5.18)$$

But, in virtue of the properties of the Gâteaux derivatives of $\boldsymbol{\Psi}$, and since $\lim_{h\to 0}||\mathbf{t}-\mathbf{t}_h||_{X_1}=0$ (cf. Theorem 4.4.1), we know that

$$\lim_{h\to 0}\left\{\frac{D^2\boldsymbol{\Psi}(\mathbf{t})(\mathbf{t}-\mathbf{t}_h,\mathbf{s})-D\boldsymbol{\Psi}(\mathbf{t})(\mathbf{s})+D\boldsymbol{\Psi}(\mathbf{t}_h)(\mathbf{s})}{||\mathbf{t}-\mathbf{t}_h||_{X_1}}\right\}=0$$

uniformly for all $\mathbf{s}\in X_1$ satisfying $||\mathbf{s}||_{X_1}\leq 1$. Consequently, there exists $h_0>0$ such that

$$|D^2\boldsymbol{\Psi}(\mathbf{t})(\mathbf{t}-\mathbf{t}_h,\mathbf{s})-D\boldsymbol{\Psi}(\mathbf{t})(\mathbf{s})+D\boldsymbol{\Psi}(\mathbf{t}_h)(\mathbf{s})|\;\leq\;\frac{1}{2\,C_0}\,||\mathbf{t}-\mathbf{t}_h||_{X_1} \qquad (4.5.19)$$

for all $h\leq h_0$ and for all $\mathbf{s}\in X_1$ with $||\mathbf{s}||_{X_1}\leq 1$.

Then, adding and substracting $D\boldsymbol{\Psi}(\mathbf{t})(\mathbf{s})-D\boldsymbol{\Psi}(\mathbf{t}_h)(\mathbf{s})$ in (4.5.18), and using (4.5.19) and the fact that

$$[\mathbf{A}_1(\mathbf{t}),\mathbf{s}]=D\boldsymbol{\Psi}(\mathbf{t})(\mathbf{s}) \quad \text{and} \quad [\mathbf{A}_1(\mathbf{t}_h),\mathbf{s}]=D\boldsymbol{\Psi}(\mathbf{t}_h)(\mathbf{s})$$

(cf. (4.3.3), (4.5.16)), we deduce that

$$\frac{1}{2\,C_0}\,||(\mathbf{t},\boldsymbol{\sigma},(\mathbf{u},\boldsymbol{\gamma}))-(\mathbf{t}_h,\boldsymbol{\sigma}_h,(\mathbf{u}_h,\boldsymbol{\gamma}_h))||_{X\times M}$$

$$\leq \sup_{\substack{(\mathbf{s},\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta}))\in X\times M\\ ||(\mathbf{s},\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta}))||\leq 1}}\Big\{[\mathbf{A}(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{s},\boldsymbol{\tau})]-[\mathbf{A}(\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{s},\boldsymbol{\tau})]$$

$$+[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{u},\boldsymbol{\gamma})-(\mathbf{u}_h,\boldsymbol{\gamma}_h)]+[\mathbf{B}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),(\mathbf{v},\boldsymbol{\eta})]\Big\},$$

which, in terms of the Ritz projection $(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})\in X$ defined by (4.5.6), can be rewritten as

$$\frac{1}{2\,C_0}\,||(\mathbf{t},\boldsymbol{\sigma},(\mathbf{u},\boldsymbol{\gamma}))-(\mathbf{t}_h,\boldsymbol{\sigma}_h,(\mathbf{u}_h,\boldsymbol{\gamma}_h))||_{X\times M}$$

$$\leq \sup_{\substack{(\mathbf{s},\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta}))\in X\times M\\ ||(\mathbf{s},\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta}))||\leq 1}}\Big\{\,[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\mathbf{s},\boldsymbol{\tau})]$$

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\,dx-\int_\Omega(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h):\boldsymbol{\eta}\,dx\Big\}. \qquad (4.5.20)$$

Now, the boundedness and ellipticity of $\hat{\mathbf{A}}$ together with the estimate from Theorem 4.5.2, give

$$|\,[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}}),(\mathbf{s},\boldsymbol{\tau})]\,|\;\leq\;||\hat{\mathbf{A}}||\;||(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}})||_X\,||(\mathbf{s},\boldsymbol{\tau})||_X$$

$$\leq C \left\{ \sum_{T \in \mathcal{T}_h} [\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T), (\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T)] \right\}^{1/2} . \tag{4.5.21}$$

Next, using that $\mathbf{div}\,\boldsymbol{\sigma} = -\,\mathbf{f}$ in Ω, that $\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta}\,dx = 0$ and that $\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\eta}\,dx = \frac{1}{2}\int_\Omega (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathbf{T}}) : \boldsymbol{\eta}\,dx$ for all $\boldsymbol{\eta} \in \mathcal{R}$, we get

$$\left| \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,dx + \int_\Omega (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) : \boldsymbol{\eta}\,dx \right|$$

$$\leq \;||\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h||_{[L^2(\Omega)]^2} \;+\; ||\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathbf{T}}||_{[L^2(\Omega)]^{2\times 2}} . \tag{4.5.22}$$

Finally, replacing (4.5.21) and (4.5.22) back into (4.5.20), and applying Cauchy-Schwarz's inequality, we obtain our a-posteriori error estimate and conclude the proof of Theorem 4.5.1.

4.5.4 Further remarks

We now take a closer look to the local problem (4.5.5). According to the definitions of $\mathbf{A}_T$ and $\mathbf{B}_T$, we have that

$$[\hat{\mathbf{A}}_T(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T), (\mathbf{s}, \boldsymbol{\tau})] =$$

$$\int_T \left\{ \boldsymbol{\sigma}_{h,T} - [\hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\,\mathbf{I} + \hat{\mu}(\mathbf{t}_{h,T})\,\mathbf{t}_{h,T}] \right\} : \mathbf{s}\,dx$$

$$+ \int_T (\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T}) : \boldsymbol{\tau}\,dx + \int_T \mathbf{u}_{h,T} \cdot \mathbf{div}\,\boldsymbol{\tau}\,dx - \int_{\partial T} \boldsymbol{\varphi}_h \cdot \boldsymbol{\tau}\boldsymbol{\nu}\,ds \tag{4.5.23}$$

for all $(\mathbf{s}, \boldsymbol{\tau}) \in X_T$.

We show next that the computation of $(\hat{\mathbf{t}}_T, \hat{\boldsymbol{\sigma}}_T)$ is simplified by the fact that (4.5.23) splits into two independent problems, one of them explicitly solved. Indeed, taking separately $\boldsymbol{\tau} = 0$ and $\mathbf{s} = 0$ in (4.5.23), and assuming, without loss of generality, that $\hat{\mathbf{A}}$ is given by (4.5.1), we obtain

$$\int_T \hat{\mathbf{t}}_T : \mathbf{s}\,dx \;=\;$$

$$\int_T \{\boldsymbol{\sigma}_{h,T} - [\hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\,\mathbf{I} \;+\; \hat{\mu}(\mathbf{t}_{h,T})\,\mathbf{t}_{h,T}]\} : \mathbf{s}\,dx \tag{4.5.24}$$

for all $\mathbf{s} \in [L^2(T)]^{2\times 2}$, and

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{H(\mathbf{div};T)} \;=\; \int_T (\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T}) : \boldsymbol{\tau}\,dx$$

$$+ \int_T \mathbf{u}_{h,T} \cdot \mathbf{div}\,\boldsymbol{\tau}\,dx \;-\; \int_{\partial T} \boldsymbol{\varphi}_h \cdot \boldsymbol{\tau}\boldsymbol{\nu}\,ds \tag{4.5.25}$$

for all $\boldsymbol{\tau} \in H_0(\mathbf{div};T)$.

Then, since $\boldsymbol{\sigma}_{h,T}$ and $\mathbf{t}_{h,T}$ belong to $[L^2(T)]^{2\times 2}$, we deduce from (4.5.24) that $\hat{\mathbf{t}}_T$ is given by

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - [\hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\,\mathbf{I} + \hat{\mu}(\mathbf{t}_{h,T})\,\mathbf{t}_{h,T}] \tag{4.5.26}$$

for all $T \in \mathcal{T}_h$, which corresponds to the explicit residual of the nonlinear constitutive equation.

Now, if $\boldsymbol{\varphi}_h$ was equal to the exact trace of $\mathbf{u}$ on $\cup_{T\in\mathcal{T}_h}\partial T$, then integrating by parts on T, the relation (4.5.25) would become

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau}\rangle_{H(\mathbf{div};T)} = [\mathbf{E}_{h,T}, \boldsymbol{\tau}] \qquad \forall\, \boldsymbol{\tau} \in H_0(\mathbf{div};T)\,, \tag{4.5.27}$$

where $\mathbf{E}_{h,T} \in H_0(\mathbf{div};T)'$ is defined by

$$\begin{aligned} [\mathbf{E}_{h,T}, \boldsymbol{\tau}] \quad &:= \quad \int_T (\mathbf{u}_{h,T} - \mathbf{u}) \cdot \mathbf{div}\,\boldsymbol{\tau}\,dx \\ + \int_T (\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla \mathbf{u}) \quad &: \quad \boldsymbol{\tau}\,dx \qquad \forall\, \boldsymbol{\tau} \in H_0(\mathbf{div};T)\,. \end{aligned} \tag{4.5.28}$$

It follows from (4.5.27) that $\hat{\boldsymbol{\sigma}}_T \in H_0(\mathbf{div};T)$ would be the representant, given by Riesz's representation theorem, of the linear and bounded functional $\mathbf{E}_{h,T}$, which depends directly on the actual error $(\mathbf{u}_{h,T} - \mathbf{u}, \mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\mathbf{u}) \in [L^2(T)]^2 \times [L^2(T)]^{2\times 2}$. In fact, it is easily seen from (4.5.28) that

$$||\mathbf{E}_{h,T}|| \;\leq\; \left\{ ||\mathbf{u}_{h,T} - \mathbf{u}||^2_{[L^2(T)]^2} + ||\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\mathbf{u}||^2_{[L^2(T)]^{2\times 2}} \right\}^{1/2}\,.$$

The above analysis shows that for the performance of the estimator, it is important that $\boldsymbol{\varphi}_h$ be an as good as possible approximation of $\mathbf{u}$.

In order to set an appropriate choice of $\boldsymbol{\varphi}_h$, we first define $\tilde{\boldsymbol{\varphi}}_h \in [L^2(\cup_{T\in\mathcal{T}_h}\partial T)]^2$ as follows. For each interior edge e of $\mathcal{T}_h$ we put

$$\tilde{\boldsymbol{\varphi}}_h|_e := \frac{1}{2}\,[\mathbf{u}_h|_T + \mathbf{u}_h|_{\tilde{T}}]$$

where $T,\,\tilde{T} \in \mathcal{T}_h$ are such that $e = T \cap \tilde{T}$; for each $e \subseteq \bar{\Gamma}_N$ we take $\tilde{\boldsymbol{\varphi}}_h|_e = \mathbf{u}_h|_T$ where $T \in \mathcal{T}_h$ is such that $e \subseteq T$; and for each $e \subseteq \bar{\Gamma}_D$, we set $\tilde{\boldsymbol{\varphi}}_h|_e = \mathbf{g}$.

Then, for each vertex $\bar{x}$ of $\mathcal{T}_h$ we define $\boldsymbol{\varphi}_h(\bar{x})$ as the average of the values $(\tilde{\boldsymbol{\varphi}}_h|_e)(\bar{x})$ through the edges e to which $\bar{x}$ belongs (if $\bar{x} \notin \bar{\Gamma}_D$), and $\boldsymbol{\varphi}_h(\bar{x}) = \mathbf{g}(\bar{x})$ (if $\bar{x} \in \bar{\Gamma}_D$). For the remaining vectors in $\cup_{T\in\mathcal{T}_h}\partial T$ we choose $\boldsymbol{\varphi}_h$ as the corresponding linear interpolant on the edges e of $\mathcal{T}_h$, whence $\boldsymbol{\varphi}_h \in [C(\cup_{T\in\mathcal{T}_h}\partial T)]^2 \subseteq [H^{1/2}(\cup_{T\in\mathcal{T}_h}\partial T)]^2$.

On the other hand, we remark that in practice the infinite dimensional local problem (4.5.25) (or more generally (4.5.23)) is solved approximately by replacing

$H_0(\mathbf{div};T)$ (resp. X_T) by a finite element subspace. Since the local problems are linear, independently of the fact that we are dealing with a nonlinear boundary value problem, we propose to use the p or the $h-p$ version for solving them.

Finally, for mesh refinement strategies based on the a-posteriori error estimator, we refer to [62] where suitable algorithms are described.

Capítulo 5

A Nonlinear Transmission Problem in Elasticity

The purpose of this chapter is to follow the done analysis in the fourth chapter and extend the results from [36] to provide a reliable a-posteriori error estimate for the dual-dual variational formulation of the linear-nonlinear transmission problem studied in [34]. The rest of the paper is organized as follows. In Section 2 we recall the transmission problem from [34], provide the corresponding variational formulation in the form of a two-fold saddle point operator equation, and state the unique solvability of it. Then, Section 3 deals with the a-posteriori error analysis. First, Section 3.1 gives some preliminaries results, including an upper bound for the Galerkin projection of the error. The main a-posteriori error estimate, which depends on the solution of a local Dirichlet problem and boundary residual terms in negative order Sobolev norms, is derived in Section 3.2. Finally, in Section 3.3 we consider the extended PEERS subspace from [34] and provide two fully local a-posteriori error estimates, in which the boundary residual terms are bounded by weighted local L^2-norms. Moreover, one of the error estimates does not require the explicit solution of the local problems.

Throughout the paper C denotes a generic positive constant.

5.1 The transmission problem

Before stating the transmission problem from [34], we need to provide some notations. In what follows, given any Hilbert space U, U^2 and $U^{2\times 2}$ denote, respectively, the space of vectors and square matrices of order 2 with entries in U. In particular, $\mathbb{R}^{2\times 2}$ is the space of square matrices of order 2 with real entries, $\mathbf{I}_2 := (\delta_{ij})$ is the

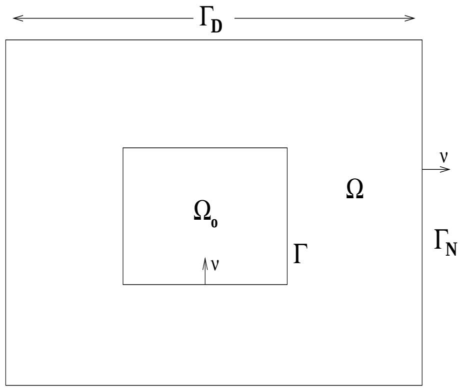

Figura 5.1: Geometry of the problem.

identity matrix of $\mathbb{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we use the notations

$$\text{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii} \quad , \quad \boldsymbol{\zeta} : \boldsymbol{\tau} := \sum_{i,j=1}^{2} \zeta_{ij}\,\tau_{ij} \quad , \quad \boldsymbol{\tau}^{\mathbf{T}} := (\tau_{ji})\,, \quad \text{and}$$

$$\text{dev}\,\boldsymbol{\tau} := \left(\boldsymbol{\tau} - \frac{1}{2}\text{tr}\,(\boldsymbol{\tau})\mathbf{I}_2\right) : \left(\boldsymbol{\tau} - \frac{1}{2}\text{tr}\,(\boldsymbol{\tau})\mathbf{I}_2\right)\,.$$

Now, let $\Omega_0 \subset \mathbb{R}^2$ be a bounded and simply connected domain with Lipschitz continuous boundary Γ. Also, let Ω be the annular region bounded by Γ and another closed, Lipschitz continuous curve $\Gamma_1 \subset \mathbb{R}^2 \setminus \bar{\Omega}_0$. Further, let Γ_D and Γ_N be two disjoint subsets of Γ_1 such that $\Gamma_1 = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \neq \emptyset$ (see Figure 5.1 below).

We are interested in determining the displacements and stresses of a material occupying the region $\bar{\Omega}_0 \cup \bar{\Omega}$ which is linear elastic in Ω_0 and hyperelastic in the surrounding annular domain Ω. As usual, the linear elasticity is defined in terms of the well known Hooke law. In addition, for the description of the hyperelasticity we assume the validity of the Hencky-von Mises stress-strain relation as discussed in [56] (see, also, [57] and [63]). Therefore, given displacement fields $\mathbf{u}_0$ and $\mathbf{u}$ in Ω_0 and Ω, respectively, strain fields $\mathbf{e}(\mathbf{u}_0)$, $\mathbf{e}(\mathbf{u})$, and stress fields $\boldsymbol{\sigma}_0$, $\boldsymbol{\sigma}$, there hold the following constitutive equations:

$$\begin{aligned} \boldsymbol{\sigma}_0 &= \lambda_0\,\text{tr}\,\mathbf{e}(\mathbf{u}_0)\mathbf{I}_2 + 2\mu_0\,\mathbf{e}(\mathbf{u}_0) && \text{in } \Omega_0\,, \\ \boldsymbol{\sigma} &= [\kappa - \mu(\text{dev}\,\mathbf{e}(u))]\text{tr}\,\mathbf{e}(\mathbf{u})\,\mathbf{I}_2 + 2\mu(\text{dev}\,\mathbf{e}(u))\,\mathbf{e}(\mathbf{u}) && \text{in } \Omega\,, \end{aligned} \tag{5.1.1}$$

where λ_0 and μ_0 are the Lamé coefficients for the elastic material, which satisfy $\mu_0 > 0$ and $\lambda_0 + \mu_0 > 0$, κ is a positive constant, called the *bulk modulus*, and $\mu : \mathbb{R}^+ \to \mathbb{R}$ is a nonlinear Lamé function. We assume that $\mu \in C^1(\mathbb{R}^+)$ and that there exist constants μ_1, μ_2 such that

$$\begin{gathered} 0 < \mu_1 \leq \mu(t) \;<\; \kappa \quad \text{and} \\ 0 < \mu_1 \leq \mu(t) + 2\,t\,\mu'(t) \leq \mu_2 \end{gathered} \tag{5.1.2}$$

for all $t \in \mathbb{R}^+$.

Then, given a body load $\mathbf{f} \in [L^2(\Omega)]^2$ and a traction $\mathbf{g} \in [H^{-1/2}(\Gamma_N)]^2$, our linear-nonlinear transmission problem reads: *Find* $\mathbf{u}_0 \in [H^1(\Omega_0)]^2$ *and* $\mathbf{u} \in [H^1(\Omega)]^2$ *such that*

$$\begin{gathered} \mathbf{div}\, \boldsymbol{\sigma}_0 = 0 \quad \text{in} \quad \Omega_0 \,, \\ \mathbf{u}_0 - \mathbf{u} = 0 \quad \text{and} \quad \boldsymbol{\sigma}_0 \boldsymbol{\nu} - \boldsymbol{\sigma} \boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma \,, \\ \mathbf{div}\, \boldsymbol{\sigma} = -\,\mathbf{f} \quad \text{in} \quad \Omega \,, \\ \mathbf{u} = 0 \quad \text{on} \quad \Gamma_D \,, \quad \boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{g} \quad \text{on} \quad \Gamma_N \,, \end{gathered} \tag{5.1.3}$$

where $\boldsymbol{\nu}$ is the unit outward normal to $\partial\Omega := \Gamma \cup \Gamma_1$, and **div** denotes the usual divergence operator acting along each row of the corresponding tensor.

Problem (5.1.3) was solved in [34] by using the boundary integral equation method in Ω_0 and a dual-mixed finite element approach in Ω. In order to set the corresponding variational formulation (see eqs. (4.2) and (4.3) in [34]) we first let $\mathcal{N}_\Gamma(\mathbf{e})$ be the restriction to Γ of $\mathcal{N}(\mathbf{e})$, the null space of the strain tensor $\mathbf{e}$, also known as the space of rigid body displacements, and let $\langle \cdot, \cdot \rangle_\Gamma$ stand for the duality between $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$ with respect to the $[L^2(\Gamma)]^2$-inner product. Then we define the spaces

$$\mathbf{H}(\mathbf{div}\,;\Omega) := \left\{ \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}\ \boldsymbol{\tau} \in [L^2(\Omega)]^2 \right\} \,,$$

$$\mathbf{H_0}(\mathbf{div}\,;\Omega) := \{ \boldsymbol{\tau} \in \mathbf{H}(\mathbf{div}\,;\Omega) : \quad \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\omega} \rangle_\Gamma = 0 \quad \forall\, \boldsymbol{\omega} \in \mathcal{N}_\Gamma(\mathbf{e}) \, \} \,,$$

$$\mathcal{R} := \left\{ \boldsymbol{\delta} \in [L^2(\Omega)]^{2\times 2} : \quad \boldsymbol{\delta} + \boldsymbol{\delta}^{\mathbf{T}} = 0 \, \right\} \,,$$

$$[\tilde{H}^{1/2}(\Gamma_N)]^2 := \left\{ \mathbf{z}|_{\Gamma_N} : \quad \mathbf{z} \in [H^1(\Omega)]^2 \quad \text{and} \quad \mathbf{z} = 0 \quad \text{on} \quad \Gamma_D \, \right\} \,,$$

and

$$[H_0^{1/2}(\Gamma)]^2 \; := \; [H^{1/2}(\Gamma)]^2 / \mathcal{N}_\Gamma(\mathbf{e}) \,.$$

Next, we introduce the auxiliary unknowns

$$\mathbf{t} := \mathbf{e}(\mathbf{u}) \in [L^2(\Omega)]^{2\times 2} \,,$$

$$\boldsymbol{\gamma} := \frac{1}{2} \,(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathbf{T}}) \in \mathcal{R} \,,$$

$$\boldsymbol{\phi} := \mathbf{u}|_\Gamma \in [H^{1/2}(\Gamma)]^2 \,, \quad \text{and} \quad \boldsymbol{\eta} := -\mathbf{u}|_{\Gamma_N} \in [\tilde{H}^{1/2}(\Gamma_N)]^2 \,,$$

the nonlinear functions $\hat{\lambda}$, $\hat{\mu} : \mathbb{R}^{2\times 2} \to \mathbb{R}$, with

$$\hat{\mu}(\mathbf{s}) := \mu(\operatorname{dev} \mathbf{s}) \quad \text{and} \quad \hat{\lambda}(\mathbf{s}) := \kappa - \hat{\mu}(\mathbf{s}) \quad \forall\, \mathbf{s} \in \mathbb{R}^{2\times 2} \,,$$

and the boundary integral operators $\mathbf{V}$, $\mathbf{K}$, $\mathbf{K}'$ and $\mathbf{W}$ denoting, respectively, the single, double, adjoint of the double, and hypersingular layer potentials of the Lamé system.

Thus, we observe that

$$\mathbf{t} + \boldsymbol{\gamma} \, = \, \nabla \mathbf{u} \quad \text{in} \quad \Omega \,,$$
$$\boldsymbol{\sigma} \, = \, \hat{\lambda}(\mathbf{t}) \, \mathrm{tr}\,(\mathbf{t}) \, \mathbf{I}_2 + 2 \, \hat{\mu}(\mathbf{t}) \, \mathbf{t} \quad \text{in} \quad \Omega \,,$$

and recall from [34] that there hold

$$\boldsymbol{\phi} = \left(\frac{1}{2} \mathbf{I} + \mathbf{K} \right) \boldsymbol{\phi} \, - \, \mathbf{V}(\boldsymbol{\sigma\nu}) \quad \text{on} \quad \Gamma$$

and

$$\mathbf{W} \boldsymbol{\phi} + \left(\frac{1}{2} \mathbf{I} + \mathbf{K}' \right) (\boldsymbol{\sigma\nu}) = 0 \quad \text{on} \quad \Gamma \,,$$

where $\mathbf{I}$ is a generic identity operator.

It is well known that $\mathbf{V} : [H^{-1/2+\epsilon}(\Gamma)]^2 \to [H^{1/2+\epsilon}(\Gamma)]^2$, $\mathbf{K} : [H^{1/2+\epsilon}(\Gamma)]^2 \to [H^{1/2+\epsilon}(\Gamma)]^2$, $\mathbf{K}' : [H^{-1/2+\epsilon}(\Gamma)]^2 \to [H^{-1/2+\epsilon}(\Gamma)]^2$ and $\mathbf{W} : [H^{1/2+\epsilon}(\Gamma)]^2 \to [H^{-1/2+\epsilon}(\Gamma)]^2$ are linear and continuous $\forall \, \epsilon \in [-\frac{1}{2}, \frac{1}{2}]$ (see, e.g., [24]). Also, $\mathbf{K}(\boldsymbol{\omega}) = \frac{\boldsymbol{\omega}}{2}$ and $\mathbf{W}(\boldsymbol{\omega}) = 0 \; \forall \, \boldsymbol{\omega} \in \mathcal{N}_\Gamma(\mathbf{e})$, and $\mathbf{W}$ and $\mathbf{V}$ are symmetric positive definite and positive semi-definite on $[H_0^{1/2}(\Gamma)]^2$ and $[H^{-1/2}(\Gamma)]^2$, respectively (see, e.g., [25]).

Then, denoting

$$X_1 := [L^2(\Omega)]^{2\times 2} \times [H_0^{1/2}(\Gamma)]^2 \,, \quad M_1 := \mathbf{H_0}(\mathbf{div}\,; \Omega) \,,$$
$$\text{and} \quad M := [L^2(\Omega)]^2 \times \mathcal{R} \times [\tilde{H}^{1/2}(\Gamma_N)]^2 \,,$$

the variational formulation of (5.1.3) can be written as the following two-fold saddle point operator equation (see [35]): *Find* $((\mathbf{t}, \boldsymbol{\phi}), \boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta})) \in \mathbf{H} := X_1 \times M_1 \times M$ *such that*

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1' & \mathbf{O} \\ \mathbf{B}_1 & -\mathbf{S} & \mathbf{B}' \\ \mathbf{O} & \mathbf{B} & \mathbf{O} \end{bmatrix} \begin{bmatrix} (\mathbf{t}, \boldsymbol{\phi}) \\ \boldsymbol{\sigma} \\ (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta}) \end{bmatrix} = \begin{bmatrix} \mathbf{O} \\ \mathbf{O} \\ \mathbf{F} \end{bmatrix} , \qquad (5.1.4)$$

where $\mathbf{O}$ denotes a generic null operator/functional, and $\mathbf{A}_1 : X_1 \to X_1'$, $\mathbf{B}_1 : X_1 \to M_1'$, $\mathbf{B}_1' : M_1 \to X_1'$, $\mathbf{B} : M_1 \to M'$, $\mathbf{B}' : M \to M_1'$, $\mathbf{S} : M_1 \to M_1'$, and $\mathbf{F} \in M'$, are defined as follows

$$[\mathbf{A}_1(\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{s}, \boldsymbol{\psi})] := \int_\Omega \left[\hat{\lambda}(\mathbf{r}) \, \mathrm{tr}\,(\mathbf{r}) \, \mathrm{tr}\,(\mathbf{s}) + 2 \, \hat{\mu}(\mathbf{r}) \, \mathbf{r} : \mathbf{s} \right] \, dx + \langle \mathbf{W} \boldsymbol{\varrho}, \boldsymbol{\psi} \rangle_\Gamma \,,$$

$$[\mathbf{B}_1(\mathbf{r}, \boldsymbol{\varrho}), \boldsymbol{\tau}] := - \int_\Omega \mathbf{r} : \boldsymbol{\tau} \, dx + \langle \boldsymbol{\tau\nu}, \left(\frac{1}{2} \mathbf{I} + \mathbf{K} \right) \boldsymbol{\varrho} \rangle_\Gamma \,,$$
$$[\mathbf{B}_1'(\boldsymbol{\zeta}), (\mathbf{s}, \boldsymbol{\psi})] := [\mathbf{B}_1(\mathbf{s}, \boldsymbol{\psi}), \boldsymbol{\zeta}] \,,$$

$$[\mathbf{B}(\boldsymbol{\zeta}), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi})] := -\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\zeta}\, dx - \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\delta}\, dx - \langle \boldsymbol{\zeta}\boldsymbol{\nu}, \boldsymbol{\xi} \rangle_{\Gamma_N},$$

$$[\mathbf{B}'(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi}), \boldsymbol{\tau}] := [\mathbf{B}(\boldsymbol{\tau}), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi})],$$

$$[\mathbf{S}(\boldsymbol{\zeta}), \boldsymbol{\tau}] := \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{V}(\boldsymbol{\zeta}\boldsymbol{\nu}) \rangle_\Gamma,$$

and

$$[\mathbf{F}, (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi})] := \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx - \langle \mathbf{g}, \boldsymbol{\xi} \rangle_{\Gamma_N},$$

for all $(\mathbf{r}, \boldsymbol{\varrho})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$, $\boldsymbol{\zeta}$, $\boldsymbol{\tau} \in M_1$, and $(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi}) \in M$. Here and in the following $[\cdot, \cdot]$ denotes the duality induced by the operators appearing in each case, and $\langle \cdot, \cdot \rangle_{\Gamma_N}$ denotes the duality between $[H^{-1/2}(\Gamma_N)]^2$ and $[\tilde{H}^{1/2}(\Gamma_N)]^2$ with respect to the $[L^2(\Gamma_N)]^2$-inner product.

Now, let $X := X_1 \times M_1 := [L^2(\Omega)]^{2\times 2} \times [H_0^{1/2}(\Gamma)]^2 \times \mathbf{H_0}(\mathbf{div}\,; \Omega)$ and define $\mathbf{A} : X \to X'$ by

$$[\mathbf{A}(\mathbf{r}, \boldsymbol{\varrho}, \boldsymbol{\zeta}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] := [\mathbf{A}_1(\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{s}, \boldsymbol{\psi})] + [\mathbf{B}_1(\mathbf{r}, \boldsymbol{\varrho}), \boldsymbol{\tau}]$$
$$+ [\mathbf{B}_1(\mathbf{s}, \boldsymbol{\psi}), \boldsymbol{\zeta}] - [\mathbf{S}(\boldsymbol{\zeta}), \boldsymbol{\tau}],$$

for all $(\mathbf{r}, \boldsymbol{\varrho}, \boldsymbol{\zeta})$, $(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}) \in X$. It is important to remark that $\mathbf{A}_1$, and hence $\mathbf{A}$, is a nonlinear operator.

Then (5.1.4) can be reformulated as: *Find* $((\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta})) \in \mathbf{H} := X \times M$ *such that*

$$\begin{aligned} [\mathbf{A}(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] \quad + [\mathbf{B}(\boldsymbol{\tau}), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta})] \quad &= 0, \\ [\mathbf{B}(\boldsymbol{\sigma}), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi})] \quad &= [\mathbf{F}, (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi})], \end{aligned} \tag{5.1.5}$$

for all $((\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi})) \in \mathbf{H}$.

Concerning the solvability of (5.1.4) (equivalently (5.1.5)), we have the following result.

Teorema 5.1.1 *There exists a unique* $((\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta})) \in \mathbf{H}$ *solution of the nonlinear two-fold saddle point formulation* (5.1.4). *Moreover, there exists* $C > 0$, *independent of the solution, such that*

$$||((\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta}))||_{\mathbf{H}} \leq C \left\{ ||\mathbf{f}||_{[L^2(\Omega)]^2} + ||\mathbf{g}||_{[H^{-1/2}(\Gamma_N)]^2} \right\}.$$

Proof: It reduces to show that $\mathbf{A}_1$ is strongly monotone and Lipschitz continuous, that $\mathbf{B}_1$ and $\mathbf{B}$ satisfy appropriate inf-sup conditions, and that $\mathbf{S}$ is positive semi-definite. For details we refer to Section 4 in [34], and to the abstract results provided in [35]. ∎

5.2 The a-posteriori error analysis

5.2.1 Preliminaries

Throughout this section we assume that Γ and Γ_1 are polygonal curves. Then we let $\{\mathcal{T}_h\}_{h\in\mathtt{I}}$ be a regular family of triangulations of Ω, made up of triangles T of diameter h_T, such that $h := \sup_{T\in\mathcal{T}_h} h_T$ and $\bar{\Omega} = \cup\{T : \quad T \in \mathcal{T}_h\}$. Here $\mathtt{I}$ is a set of indexes, say $\mathtt{I} := \{h_j\}_{j\in\mathbb{N}}$, with $h_j \geq h_{j+1}\ \forall j \in \mathbb{N}$. In addition, let $\mathbf{H}_h := X_{1,h} \times M_{1,h} \times M_h$ be a finite element subspace of $\mathbf{H} := X_1 \times M_1 \times M$, and let $(\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, u_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h$ be an approximation to the exact solution $(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, u, \boldsymbol{\gamma}, \boldsymbol{\eta}) \in \mathbf{H}$ of the variational formulation (5.1.4).

We now apply the approach from [17] (see also [18] and [36]) to derive an a-posteriori estimate for the error $||(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, u, \boldsymbol{\gamma}, \boldsymbol{\eta}) - (\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, u_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)||_{\mathbf{H}}$. For this purpose, we need to choose any linear and bounded operator $\hat{\mathbf{A}} : X \to X'$ such that the corresponding induced bilinear form $\hat{A} : X \times X \to \mathbb{R}$, defined by

$$\hat{A}((\mathbf{r}, \boldsymbol{\varrho}, \boldsymbol{\zeta}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})) = [\hat{\mathbf{A}}(\mathbf{r}, \boldsymbol{\varrho}, \boldsymbol{\zeta}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})]$$

for all $(\mathbf{r}, \boldsymbol{\varrho}, \boldsymbol{\zeta})$, $(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}) \in X$, is symmetric and X-elliptic.

In the following, for simplicity, we take

$$[\hat{\mathbf{A}}(\mathbf{r}, \boldsymbol{\varrho}, \boldsymbol{\zeta}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] := \langle \mathbf{r}, \mathbf{s}\rangle_{[L^2(\Omega)]^{2\times 2}} + \langle \mathbf{W}\boldsymbol{\varrho}, \boldsymbol{\psi}\rangle_\Gamma + \langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{\mathbf{H}(\mathbf{div}\,;\Omega)}\,, \tag{5.2.6}$$

where

$$\langle \mathbf{r}, \mathbf{s}\rangle_{[L^2(\Omega)]^{2\times 2}} := \int_\Omega \mathbf{r} : \mathbf{s}\, dx \tag{5.2.7}$$

and

$$\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{\mathbf{H}(\mathbf{div}\,;\Omega)} := \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau}\, dx + \int_\Omega \mathbf{div}\,\boldsymbol{\zeta} \cdot \mathbf{div}\,\boldsymbol{\tau}\, dx \tag{5.2.8}$$

are the usual inner products of $[L^2(\Omega)]^{2\times 2}$ and $\mathbf{H}(\mathbf{div}\,;\Omega)$, respectively.

Then, given $T \in \mathcal{T}_h$ we denote by $\langle\cdot,\cdot\rangle_{[L^2(T)]^{2\times 2}}$ and $\langle\cdot,\cdot\rangle_{\mathbf{H}(\mathbf{div}\,;T)}$ the corresponding restrictions of (5.2.7) and (5.2.8) to $[L^2(T)]^{2\times 2}$ and $\mathbf{H}(\mathbf{div}\,;T)$. Further, we let $(\mathbf{t}_{h,T}, \boldsymbol{\sigma}_{h,T}, \mathbf{u}_{h,T}, \boldsymbol{\gamma}_{h,T})$ be the restriction of $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)$ to T.

The following theorem provides an important upper bound for the Ritz projection of the Galerkin error with respect to the operator $\hat{\mathbf{A}}$.

Teorema 5.2.1 *Let $(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}})$ be the unique element in X such that*

$$\begin{aligned}[\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] &= [\mathbf{A}(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] \\ &-[\mathbf{A}(\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] + [\mathbf{B}(\boldsymbol{\tau}), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta}) - (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)]\end{aligned} \tag{5.2.9}$$

for all $(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau}) \in X$, *and let* $\boldsymbol{\varphi}_h \in [H^{1/2}(\cup_{T\in\mathcal{T}_h}\partial T)]^2$ *be an approximation to* $\mathbf{u}$ *on element boundaries such that* $\boldsymbol{\varphi}_h = 0$ *on* Γ_D, $\boldsymbol{\varphi}_h(\bar{\mathbf{x}}) = -\boldsymbol{\eta}_h(\bar{\mathbf{x}})$ *for each vertex* $\bar{\mathbf{x}}$ *of* $\mathcal{T}_h$ *on* Γ_N, *and* $\boldsymbol{\varphi}_h(\bar{\mathbf{x}}) = -\mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu})(\bar{\mathbf{x}}) + \left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)\boldsymbol{\phi}_h(\bar{\mathbf{x}})$ *for each vertex* $\bar{\mathbf{x}}$ *of* $\mathcal{T}_h$ *on* Γ. *In addition, for each* $T\in\mathcal{T}_h$ *define*

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\,\mathbf{I}_2 - 2\,\hat{\mu}(\mathbf{t}_{h,T})\,\mathbf{t}_{h,T}\,,$$

and let $\hat{\boldsymbol{\sigma}}_T \in \mathbf{H}(\mathbf{div}\,;T)$ *be the unique solution of the local problem*

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau}\rangle_{\mathbf{H}(\mathbf{div}\,;T)} \;=\; F_{h,T}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbf{H}(\mathbf{div}\,;T)\,, \tag{5.2.10}$$

where

$$\begin{aligned} F_{h,T}(\boldsymbol{\tau}) := &\int_T (\mathbf{t}_{h,T}+\boldsymbol{\gamma}_{h,T}) : \boldsymbol{\tau}\,dx + \int_T \mathbf{u}_{h,T}\cdot\mathbf{div}\,\boldsymbol{\tau}\,dx - \int_{\partial T}\boldsymbol{\varphi}_h\cdot\boldsymbol{\tau}\boldsymbol{\nu}_T\,ds \\ &+\langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{\varphi}_h+\boldsymbol{\eta}_h\rangle_{\partial T\cap\Gamma_N} + \langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{\varphi}_h+\mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu}) - \left(\frac{1}{2}\mathbf{I}+\mathbf{K}\right)\boldsymbol{\phi}_h\rangle_{\partial T\cap\Gamma}\,, \end{aligned} \tag{5.2.11}$$

with $\boldsymbol{\nu}_T$ *being the unit outward normal to* ∂T. *Then there holds*

$$\begin{aligned} [\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\sigma}})] \;\le\; &\sum_{T\in\mathcal{T}_h} ||\hat{\mathbf{t}}_T||^2_{[L^2(T)]^{2\times 2}} \;+\; \sum_{T\in\mathcal{T}_h} ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} \\ &+||\,\mathbf{W}^{-1}\,||\;||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I}+\mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})||^2_{[H^{-1/2}(\Gamma)]^2}\,. \end{aligned} \tag{5.2.12}$$

Proof: We first observe from (5.1.5) that

$$[\mathbf{A}(\mathbf{t},\boldsymbol{\phi},\boldsymbol{\sigma}),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})] + [\mathbf{B}(\boldsymbol{\tau}),(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\eta})] \;=\; 0 \quad \forall\,(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})\in X\,,$$

and hence

$$\begin{aligned} [\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\sigma}}),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})] = &-[\mathbf{A}(\mathbf{t}_h,\boldsymbol{\phi}_h,\boldsymbol{\sigma}_h),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})] \\ &-[\mathbf{B}(\boldsymbol{\tau}),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\boldsymbol{\eta}_h)] \quad \forall\,(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})\in X\,. \end{aligned} \tag{5.2.13}$$

Since the bilinear form induced by $\hat{\mathbf{A}}$ is symmetric and X-elliptic, we have that

$$\begin{aligned} &-\frac{1}{2}\,[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\sigma}})] \\ &= \min_{(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})\in X} \left\{ \frac{1}{2}\,[\hat{\mathbf{A}}(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau}),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})] \,-\, [\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\sigma}}),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})] \right\}\,, \end{aligned}$$

which, according to (5.2.13), yields

$$-\frac{1}{2}\,[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\sigma}})] \;=\; \min_{(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})\in X} \mathbf{J}(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})\,, \tag{5.2.14}$$

with

$$\mathbf{J}(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau}) := \frac{1}{2}\,[\hat{\mathbf{A}}(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau}),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})]$$
$$+\,[\mathbf{A}(\mathbf{t}_h,\boldsymbol{\phi}_h,\boldsymbol{\sigma}_h),(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau})] \,+\, [\mathbf{B}(\boldsymbol{\tau}),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\boldsymbol{\eta}_h)]\,. \tag{5.2.15}$$

Now, since $\boldsymbol{\varphi}_h = 0$ on Γ_D, we get

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\boldsymbol{\varphi}_h\cdot\boldsymbol{\tau}\boldsymbol{\nu}_T\,ds \,-\, \int_{\Gamma_N\cup\Gamma}\boldsymbol{\varphi}_h\cdot\boldsymbol{\tau}\boldsymbol{\nu}\,ds \,=\, 0\,. \tag{5.2.16}$$

Then, adding (5.2.16) to (5.2.15) and using the definitions of $\hat{\mathbf{A}}$, $\mathbf{A}$ and $\mathbf{B}$, we obtain

$$\mathbf{J}(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau}) \,=\, \sum_{T\in\mathcal{T}_h}\mathbf{J}_{1,T}(\mathbf{s}_T) \,+\, \mathbf{J}_2(\boldsymbol{\psi}) \,+\, \sum_{T\in\mathcal{T}_h}\mathbf{J}_{3,T}(\boldsymbol{\tau}_T)\,, \tag{5.2.17}$$

where $(\mathbf{s}_T,\boldsymbol{\tau}_T)$ is the restriction of $(\mathbf{s},\boldsymbol{\tau})$ to T,

$$\mathbf{J}_{1,T}(\mathbf{s}_T) := \frac{1}{2}\,||\mathbf{s}_T||^2_{[L^2(T)]^{2\times 2}} \,-\, \langle\hat{\mathbf{t}}_T,\mathbf{s}_T\rangle_{[L^2(T)]^{2\times 2}}\,, \tag{5.2.18}$$

$$\mathbf{J}_2(\boldsymbol{\psi}) := \frac{1}{2}\,\langle\mathbf{W}\boldsymbol{\psi},\boldsymbol{\psi}\rangle_\Gamma + \langle\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I}+\mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu}),\boldsymbol{\psi}\rangle_\Gamma\,, \tag{5.2.19}$$

and

$$\mathbf{J}_{3,T}(\boldsymbol{\tau}_T) := \frac{1}{2}\,||\boldsymbol{\tau}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} \,-\, F_{h,T}(\boldsymbol{\tau}_T)\,. \tag{5.2.20}$$

By replacing (5.2.17) back into (5.2.14), and using that $\mathbf{s}\in[L^2(\Omega)]^{2\times 2}$ if and only if $\mathbf{s}_T\in[L^2(T)]^{2\times 2}$ $\forall\,T\in\mathcal{T}_h$, we can write

$$-\,\frac{1}{2}\,[\hat{\mathbf{A}}(\bar{\mathbf{t}},\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\sigma}}),(\bar{\mathbf{t}},\bar{\boldsymbol{\phi}},\bar{\boldsymbol{\sigma}})] \,=\, \sum_{T\in\mathcal{T}_h}\left\{\min_{\mathbf{s}_T\in[L^2(T)]^{2\times 2}}\mathbf{J}_{1,T}(\mathbf{s}_T)\right\}$$
$$+\,\min_{\boldsymbol{\psi}\in[H_0^{1/2}(\Gamma)]^2}\mathbf{J}_2(\boldsymbol{\psi}) \,+\, \min_{\boldsymbol{\tau}\in\mathbf{H_0}(\mathbf{div}\,;\Omega)}\left\{\sum_{T\in\mathcal{T}_h}\mathbf{J}_{3,T}(\boldsymbol{\tau}_T)\right\}. \tag{5.2.21}$$

Since $\boldsymbol{\sigma}_{h,T}$, $\mathbf{t}_{h,T}\in[L^2(T)]^{2\times 2}$, it follows from (5.2.18) that

$$\min_{\mathbf{s}_T\in[L^2(T)]^{2\times 2}}\mathbf{J}_{1,T}(\mathbf{s}_T) \,=\, -\,\frac{1}{2}\,||\hat{\mathbf{t}}_T||^2_{[L^2(T)]^{2\times 2}}\,. \tag{5.2.22}$$

Note that $\hat{\mathbf{t}}_T$ corresponds to the explicit local residual of the nonlinear constitutive equation in Ω (cf. (5.1.1)).

Now, let us introduce the *broken* space

$$\mathbf{H}(\mathbf{div}\,;\Omega)^{br} := \{\boldsymbol{\tau}\in[L^2(\Omega)]^{2\times 2}: \quad \boldsymbol{\tau}_T\in\mathbf{H}(\mathbf{div}\,;T) \quad \forall\,T\in\mathcal{T}_h\}\,.$$

Then, since $\mathbf{H_0}(\mathbf{div}\,;\Omega) \subseteq \mathbf{H}(\mathbf{div}\,;\Omega)^{br}$, we get

$$\min_{\boldsymbol{\tau}\in\mathbf{H_0}(\mathbf{div}\,;\Omega)} \left\{ \sum_{T\in\mathcal{T}_h} \mathbf{J}_{3,T}(\boldsymbol{\tau}_T) \right\} \geq \min_{\boldsymbol{\tau}\in\mathbf{H}(\mathbf{div}\,;\Omega)^{br}} \left\{ \sum_{T\in\mathcal{T}_h} \mathbf{J}_{3,T}(\boldsymbol{\tau}_T) \right\}$$

$$= \sum_{T\in\mathcal{T}_h} \min_{\boldsymbol{\tau}_T\in\mathbf{H}(\mathbf{div}\,;T)} \mathbf{J}_{3,T}(\boldsymbol{\tau}_T) = -\frac{1}{2} \sum_{T\in\mathcal{T}_h} ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)}\,, \tag{5.2.23}$$

where the last equation has made use of (5.2.20) and the definition of the local problem (5.2.10). At this point we remark that the unique solvability of (5.2.10) follows from the Lax-Milgram Theorem and the fact that $F_{h,T}$ becomes a linear and bounded functional on $\mathbf{H}(\mathbf{div}\,;T)$.

On the other hand, the minimizer to $\mathbf{J}_2$ is the unique $\tilde{\boldsymbol{\psi}} \in [H_0^{1/2}(\Gamma)]^2$ (guaranteed by the positive definiteness of $\mathbf{W}$) solution to the equation

$$\langle \mathbf{W}\tilde{\boldsymbol{\psi}}, \boldsymbol{\psi}\rangle_\Gamma = -\langle \mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu}), \boldsymbol{\psi}\rangle_\Gamma$$

for all $\boldsymbol{\psi} \in [H_0^{1/2}(\Gamma)]^2$, which yields

$$\min_{\boldsymbol{\psi}\in[H_0^{1/2}(\Gamma)]^2} \mathbf{J}_2(\boldsymbol{\psi}) = -\frac{1}{2}\langle \mathbf{W}\tilde{\boldsymbol{\psi}}, \tilde{\boldsymbol{\psi}}\rangle_\Gamma$$

$$\geq -\frac{1}{2}||\mathbf{W}^{-1}||\;||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})||^2_{[H^{-1/2}(\Gamma)]^2}\,. \tag{5.2.24}$$

Finally, (5.2.21), (5.2.22), (5.2.23) and (5.2.24) imply (5.2.12) and complete the proof of the theorem. ■

The following technical result will also be needed.

Lema 5.2.1 *The operator $\mathbf{A}_1$ has continuous first order Gâteaux derivative $D\mathbf{A}_1$, and there exist $\tilde{C}_1$, $\tilde{C}_2 > 0$ such that*

$$|\,D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{s}, \boldsymbol{\psi}))\,| \leq \tilde{C}_1\,||(\mathbf{r}, \boldsymbol{\varrho})||_{X_1}\,||(\mathbf{s}, \boldsymbol{\psi})||_{X_1}$$

and

$$D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\mathbf{s}, \boldsymbol{\psi}), (\mathbf{s}, \boldsymbol{\psi})) \geq \tilde{C}_2\,||(\mathbf{s}, \boldsymbol{\psi})||^2_{X_1}\,,$$

for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$, $(\mathbf{r}, \boldsymbol{\varrho})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1 := [L^2(\Omega)]^{2\times 2} \times [H_0^{1/2}(\Gamma)]^2$.

Proof: Given $i, j \in \{1, 2\}$, we let $a_{ij} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be the nonlinear mapping defined by

$$a_{ij}(\mathbf{r}) := \hat{\lambda}(\mathbf{r})\,\mathrm{tr}\,(\mathbf{r})\,\delta_{ij} + 2\,\hat{\mu}(\mathbf{r})\,r_{ij}$$

for all $\mathbf{r} := (r_{ij}) \in \mathbb{R}^{2\times 2}$. According to the assumptions on κ and μ (cf. (5.1.2)), one can prove (see [37] or Lemmas 4.1, 4.2, and 4.3 in [6]) that a_{ij} is of class C^1, and that there exist C_1, $C_2 > 0$ such that

$$\left| \frac{\partial}{\partial \tilde{r}_{kl}} a_{ij}(\tilde{\mathbf{r}}) \right| \leq C_1 \tag{5.2.25}$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial \tilde{r}_{kl}} a_{ij}(\tilde{\mathbf{r}})\, s_{kl}\, s_{ij} \geq C_2 \sum_{i,j=1}^{2} s_{ij}^2 , \tag{5.2.26}$$

for all $\tilde{\mathbf{r}} := (\tilde{r}_{ij})$, $\mathbf{s} := (s_{ij}) \in \mathbb{R}^{2\times 2}$.

Now, for all $(\mathbf{r}, \boldsymbol{\varrho})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$, with $\mathbf{r}(\cdot) = (r_{ij}(\cdot))$ and $\mathbf{s}(\cdot) = (s_{ij}(\cdot))$, we can write

$$[\mathbf{A}_1(\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{s}, \boldsymbol{\psi})] = [\tilde{\mathbf{A}}_1(\mathbf{r}), \mathbf{s}] + \langle \mathbf{W} \boldsymbol{\varrho}, \boldsymbol{\psi} \rangle_\Gamma ,$$

where $\tilde{\mathbf{A}}_1 : [L^2(\Omega)]^{2\times 2} \to [L^2(\Omega)]^{2\times 2\prime}$, the pure nonlinear part of $\mathbf{A}_1$, is defined by

$$[\tilde{\mathbf{A}}_1(\mathbf{r}), \mathbf{s}] := \sum_{i,j=1}^{2} \int_\Omega a_{ij}(\mathbf{r})\, s_{ij}\, dx .$$

Then, because of the linearity of $\mathbf{W}$, we obtain

$$D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{s}, \boldsymbol{\psi})) = D\tilde{\mathbf{A}}_1(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s}) + \langle \mathbf{W} \boldsymbol{\varrho}, \boldsymbol{\psi} \rangle_\Gamma ,$$

with

$$D\tilde{\mathbf{A}}_1(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s}) := \int_\Omega \sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial \tilde{r}_{kl}} a_{ij}(\tilde{\mathbf{r}})\, r_{kl}\, s_{ij}\, dx \tag{5.2.27}$$

for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$, $(\mathbf{r}, \boldsymbol{\varrho})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$.

Therefore, (5.2.25) and the Cauchy-Schwarz inequality imply

$$\left| D\tilde{\mathbf{A}}_1(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s}) \right| \leq C_1 \int_\Omega \sum_{i,j,k,l=1}^{2} |r_{kl}|\, |s_{ij}|\, dx \leq 4\, C_1\, ||\mathbf{r}||_{[L^2(\Omega)]^{2\times 2}}\, ||\mathbf{s}||_{[L^2(\Omega)]^{2\times 2}} ,$$

and hence, using the boundedness of $\mathbf{W}$, we get

$$|D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\mathbf{r}, \boldsymbol{\varrho}), (\mathbf{s}, \boldsymbol{\psi}))| \leq \tilde{C}_1\, ||(\mathbf{r}, \boldsymbol{\varrho})||_{X_1}\, ||(\mathbf{s}, \boldsymbol{\psi})||_{X_1} \tag{5.2.28}$$

for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$, $(\mathbf{r}, \boldsymbol{\varrho})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$, where $\tilde{C}_1 := \max\{4C_1, ||\mathbf{W}||\}$.

On the other hand, since $\mathbf{W}$ is positive definite on $[H_0^{1/2}(\Gamma)]^2$, there exists $\alpha_0 > 0$ such that

$$\langle \mathbf{W} \boldsymbol{\psi}, \boldsymbol{\psi} \rangle_\Gamma \geq \alpha_0\, ||\boldsymbol{\psi}||^2_{[H_0^{1/2}(\Gamma)]^2} \qquad \forall\, \boldsymbol{\psi} \in [H_0^{1/2}(\Gamma)]^2 .$$

Then, using (5.2.26) and (5.2.27), we deduce that

$$\begin{aligned} D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\mathbf{s}, \boldsymbol{\psi}), (\mathbf{s}, \boldsymbol{\psi})) &= D\tilde{\mathbf{A}}_1(\tilde{\mathbf{r}})(\mathbf{s}, \mathbf{s}) + \langle \mathbf{W}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle_\Gamma \\ &\geq \tilde{C}_2 \, ||(\mathbf{s}, \boldsymbol{\psi})||^2_{X_1} \end{aligned} \tag{5.2.29}$$

for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$, $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$, where $\tilde{C}_2 := \min\{C_2, \alpha_0\}$. Finally, the continuity of $D\mathbf{A}_1$ is a consequence of (5.2.27) and the fact that the nonlinear mappings a_{ij} are of class C^1. ∎

We remark from the previous lemma that, for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}}) \in X_1$, $D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$ can be identified with the bilinear form on $X_1 \times X_1$ defined by (5.2.27), which, in virtue of (5.2.28) and (5.2.29), is uniformly bounded and X_1-elliptic, respectively. In addition, using the conditions satisfied by $D\mathbf{A}_1$ and $D\tilde{\mathbf{A}}_1$ one can easily show that $\mathbf{A}_1$ and $\tilde{\mathbf{A}}_1$ are strongly monotone and Lipschitz continuous. Moreover, (5.2.25) and (5.2.26) also yield these properties for $\tilde{\mathbf{A}}_{1,T}$, the restriction of $\tilde{\mathbf{A}}_1$ to a triangle $T \in \mathcal{T}_h$, with constants independent of T.

5.2.2 The main a-posteriori error estimate

The results given in the previous subsection allow us to prove now the following main theorem.

Teorema 5.2.2 *Let $\boldsymbol{\varphi}_h \in [H^{1/2}(\cup_{T \in \mathcal{T}_h} \partial T)]^2$ be as stated in Theorem 5.2.1. In addition, for each $T \in \mathcal{T}_h$ define*

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T}) \, \mathrm{tr}\,(\mathbf{t}_{h,T}) \, \mathbf{I}_2 - 2\,\hat{\mu}(\mathbf{t}_{h,T}) \, \mathbf{t}_{h,T} \,,$$

and let $\hat{\boldsymbol{\sigma}}_T \in \mathbf{H}(\mathbf{div}\,; T)$ be the unique solution of the local problem (5.2.10). Then there exists $C > 0$, independent of h, such that

$$||(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta}) - (\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)||_{\mathbf{H}} \leq C \left\{ \sum_{T \in \mathcal{T}_h} \boldsymbol{\theta}_T^2 + R_\Gamma^2 + R_{\Gamma_N}^2 \right\}^{1/2},$$

where

$$\begin{aligned} \boldsymbol{\theta}_T^2 \;:=\; \Big\{ \, ||\hat{\mathbf{t}}_T||^2_{[L^2(T)]^{2\times 2}} \,+\, ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} \\ + \, ||\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_{h,T}||^2_{[L^2(T)]^2} \,+\, ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}^{\mathbf{T}}_{h,T}||^2_{[L^2(T)]^{2\times 2}} \Big\}, \end{aligned}$$

$$R_\Gamma := ||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}' \right) (\boldsymbol{\sigma}_h \boldsymbol{\nu})||_{[H^{-1/2}(\Gamma)]^2} \,,$$

and

$$R_{\Gamma_N} := ||\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}||_{[H^{-1/2}(\Gamma_N)]^2} \,.$$

Proof: Let $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}}) \in X_1$ and let $\mathbf{L}_{(\tilde{\mathbf{r}},\tilde{\boldsymbol{\varrho}})} : \mathbf{H} \to \mathbf{H}'$ be the linear operator arising from the variational formulation (5.1.4) (or (5.1.5)) after replacing $[\mathbf{A}_1(\cdot,\cdot),(\cdot,\cdot)]$ by $D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\cdot,\cdot),(\cdot,\cdot))$.

Using that $D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$ is a uniformly bounded and X_1-elliptic bilinear form, that $\mathbf{B}$ and $\mathbf{B}_1$ satisfy the continuous inf-sup conditions, and that $\mathbf{S}$ is positive semi-definite, we conclude, by applying Theorem 2.3 from [35], that $\mathbf{L}_{(\tilde{\mathbf{r}},\tilde{\boldsymbol{\varrho}})}$ satisfies a uniform global inf-sup condition. This means that there exists $C_0 > 0$, independent of $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})$, such that

$$
\begin{aligned}
||(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}}, \tilde{\gamma}, \tilde{\boldsymbol{\eta}})||_{\mathbf{H}} \le C_0 \sup_{\substack{(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau},\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})\in\mathbf{H} \\ ||(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau},\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})||\le 1}} & \Big\{ D\mathbf{A}_1(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}})((\tilde{\mathbf{t}}, \tilde{\boldsymbol{\phi}}),(\mathbf{s},\boldsymbol{\psi})) \\
& + [\mathbf{B}_1(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\phi}}), \boldsymbol{\tau}] + [\mathbf{B}_1(\mathbf{s},\boldsymbol{\psi}), \tilde{\boldsymbol{\sigma}}] - [\mathbf{S}(\tilde{\boldsymbol{\sigma}}), \boldsymbol{\tau}] \\
& + [\mathbf{B}(\tilde{\boldsymbol{\sigma}}), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\xi})] + [\mathbf{B}(\boldsymbol{\tau}), (\tilde{\mathbf{u}}, \tilde{\gamma}, \tilde{\boldsymbol{\eta}})] \Big\}
\end{aligned}
\tag{5.2.30}
$$

for all $(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}}, \tilde{\gamma}, \tilde{\boldsymbol{\eta}}) \in \mathbf{H}$ and for all $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}}) \in X_1$.

Now, since $D\mathbf{A}_1$ is continuous, we deduce that there exists $(\tilde{\mathbf{r}}_h, \tilde{\boldsymbol{\varrho}}_h) \in X_1$, which is a convex combination of $(\mathbf{t}, \boldsymbol{\phi})$ and $(\mathbf{t}_h, \boldsymbol{\phi}_h)$, such that

$$
[\mathbf{A}_1(\mathbf{t}, \boldsymbol{\phi}) - \mathbf{A}_1(\mathbf{t}_h, \boldsymbol{\phi}_h), (\mathbf{s}, \boldsymbol{\psi})] = D\mathbf{A}_1(\tilde{\mathbf{r}}_h, \tilde{\boldsymbol{\varrho}}_h)((\mathbf{t} - \mathbf{t}_h, \boldsymbol{\phi} - \boldsymbol{\phi}_h), (\mathbf{s}, \boldsymbol{\psi})) \tag{5.2.31}
$$

for all $(\mathbf{s}, \boldsymbol{\psi}) \in X_1$.

Then, applying (5.2.30) with $(\tilde{\mathbf{r}}, \tilde{\boldsymbol{\varrho}}) = (\tilde{\mathbf{r}}_h, \tilde{\boldsymbol{\varrho}}_h)$ and $(\tilde{\mathbf{t}}, \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}}, \tilde{\gamma}, \tilde{\boldsymbol{\eta}})$ given by the error, and using (5.2.31), the definition of the operator $\mathbf{A}$, the Ritz projection $(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}})$ introduced in Theorem 5.2.1, the second equation of (5.1.5), and the definitions of $\mathbf{B}$ and $\mathbf{F}$, we arrive to

$$
\begin{aligned}
& \frac{1}{C_0} ||(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta}) - (\mathbf{t}_h, \boldsymbol{\phi}_{,h}\, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)||_{\mathbf{H}} \\
\le & \sup_{\substack{(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau},\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})\in\mathbf{H} \\ ||(\mathbf{s},\boldsymbol{\psi},\boldsymbol{\tau},\mathbf{v},\boldsymbol{\delta},\boldsymbol{\xi})||\le 1}} \Big\{ [\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})] + \int_\Omega (\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_h) \cdot \mathbf{v}\, dx \\
& + \langle \boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}, \boldsymbol{\xi} \rangle_{\Gamma_N} + \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\delta}\, dx \Big\}.
\end{aligned}
\tag{5.2.32}
$$

Next, the boundedness and ellipticity of $\hat{\mathbf{A}}$, and the estimate from Theorem 5.2.1, imply

$$
|[\hat{\mathbf{A}}(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}}), (\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})]| \le ||\hat{\mathbf{A}}||\; ||(\bar{\mathbf{t}}, \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\sigma}})||_X\, ||(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})||_X
$$

$$\leq C \left\{ \sum_{T \in \mathcal{T}_h} \left(||\hat{\mathbf{t}}_T||^2_{[L^2(T)]^{2\times 2}} + ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} \right) + R_\Gamma^2 \right\}^{1/2} ||(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau})||_X \qquad (5.2.33)$$

for all $(\mathbf{s}, \boldsymbol{\psi}, \boldsymbol{\tau}) \in X$.

On the other hand, it is easily seen that

$$\left| \int_\Omega (\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_h) \cdot \mathbf{v}\, dx \right|$$

$$\leq \left\{ \sum_{T \in \mathcal{T}_h} ||\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_{h,T}||^2_{[L^2(T)]^2} \right\}^{1/2} ||\mathbf{v}||_{[L^2(\Omega)]^2}\,, \qquad (5.2.34)$$

$$|\, \langle \boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}, \boldsymbol{\xi} \rangle_{\Gamma_N} \,| \;\leq\; ||\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}||_{[H^{-1/2}(\Gamma_N)]^2}\, ||\boldsymbol{\xi}||_{[\tilde{H}^{1/2}(\Gamma_N)]^2}\,, \qquad (5.2.35)$$

and

$$\left| \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\delta}\, dx \right| = \left| \frac{1}{2} \int_\Omega (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathbf{T}}) : \boldsymbol{\delta}\, dx \right|$$

$$\leq \frac{1}{2} \left\{ \sum_{T \in \mathcal{T}_h} ||\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathbf{T}}||^2_{[L^2(T)]^{2\times 2}} \right\}^{1/2} ||\boldsymbol{\delta}||_{[L^2(\Omega)]^{2\times 2}}\,, \qquad (5.2.36)$$

for all $\mathbf{v} \in [L^2(\Omega)]^2$, $\boldsymbol{\xi} \in [\tilde{H}^{1/2}(\Gamma_N)]^2$, $\boldsymbol{\delta} \in \mathcal{R}$. Consequently, replacing (5.2.33), (5.2.34), (5.2.35) and (5.2.36) back into (5.2.32), we obtain the a-posteriori error estimate and conclude the proof of the theorem. ■

We now comment on the importance of choosing $\boldsymbol{\varphi}_h$ as close as possible to the trace of $\mathbf{u}$ on $\cup_{T \in \mathcal{T}_h} \partial T$. In fact, if $\mathbf{u} \in [H^1(\Omega)]^2$ and $\boldsymbol{\varphi}_h|_{\partial T} = \mathbf{u}|_{\partial T}$ $\forall\, T \in \mathcal{T}_h$, we get

$$\int_{\partial T} \boldsymbol{\varphi}_h \cdot \boldsymbol{\tau} \boldsymbol{\nu}_T\, ds = \int_{\partial T} \mathbf{u} \cdot \boldsymbol{\tau} \boldsymbol{\nu}_T\, ds = \int_T \nabla \mathbf{u} : \boldsymbol{\tau}\, dx + \int_T \mathbf{u} \cdot \mathbf{div}\, \boldsymbol{\tau}\, dx\,,$$

and hence the local problem (5.2.10) becomes

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{\mathbf{H}(\mathbf{div}\,;T)} \;=\; F_{h,T}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathbf{div}\,;T)\,, \qquad (5.2.37)$$

with

$$F_{h,T}(\boldsymbol{\tau}) = \int_T (\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla \mathbf{u}) : \boldsymbol{\tau}\, dx + \int_T (\mathbf{u}_{h,T} - \mathbf{u}) \cdot \mathbf{div}\, \boldsymbol{\tau}\, dx$$

$$+ \langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u} + \boldsymbol{\eta}_h \rangle_{\partial T \cap \Gamma_N} \;+\; \langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u} + \mathbf{V}(\boldsymbol{\sigma}_h \boldsymbol{\nu}) - \left(\frac{1}{2}\mathbf{I} + \mathbf{K} \right) \boldsymbol{\phi}_h \rangle_{\partial T \cap \Gamma}\,. \qquad (5.2.38)$$

This shows that $\hat{\boldsymbol{\sigma}}_T \in \mathbf{H}(\mathbf{div}\,;T)$, the Riesz representant of the functional $F_{h,T} \in \mathbf{H}(\mathbf{div}\,;T)'$, depends directly on the actual errors

$$(\mathbf{t}_{h,T} \;+\; \boldsymbol{\gamma}_{h,T} - \nabla \mathbf{u}) \in [L^2(T)]^{2\times 2}\,,$$

$$(\mathbf{u}_{h,T} - \mathbf{u}) \in [L^2(T)]^2 \,,$$
$$(\mathbf{u} + \boldsymbol{\eta}_h) \in [\tilde{H}^{1/2}(\partial T \cap \Gamma_N)]^2 \,,$$

and

$$\left(\mathbf{u} + \mathbf{V}(\boldsymbol{\sigma}_h \boldsymbol{\nu}) - \left(\frac{1}{2}\mathbf{I} + \mathbf{K} \right) \boldsymbol{\phi}_h \right) \in [H^{1/2}(\partial T \cap \Gamma)]^2 \,,$$

where $[\tilde{H}^{1/2}(\partial T \cap \Gamma_N)]^2 := \{ \boldsymbol{\delta}|_{\partial T \cap \Gamma_N} : \quad \boldsymbol{\delta} \in [\tilde{H}^{1/2}(\Gamma_N)]^2 \}$.

Moreover, it follows from (5.2.37) and (5.2.38) that

$$\begin{aligned} ||\hat{\boldsymbol{\sigma}}_T||_{\mathbf{H}(\mathbf{div}\,;T)} \,\leq\, \Big\{ & ||\mathbf{t}_{h,T} \,+\, \boldsymbol{\gamma}_{h,T} - \nabla \mathbf{u}||^2_{[L^2(T)]^{2\times 2}} \\ & + ||\mathbf{u}_{h,T} - \mathbf{u}||^2_{[L^2(T)]^2} \,+\, ||\mathbf{u} + \boldsymbol{\eta}_h||^2_{[\tilde{H}^{1/2}(\partial T \cap \Gamma_N)]^2} \\ & + ||\mathbf{u} + \mathbf{V}(\boldsymbol{\sigma}_h \boldsymbol{\nu}) - \left(\frac{1}{2}\mathbf{I} + \mathbf{K} \right) \boldsymbol{\phi}_h||^2_{[H^{1/2}(\partial T \cap \Gamma)]^2} \Big\}^{1/2} . \end{aligned} \qquad (5.2.39)$$

The above analysis motivates the following lemma, which shows that the efficiency of the a-posteriori error estimate given by Theorem 5.2.2 depends directly on how well $\boldsymbol{\varphi}_h$ approximates the trace of $\mathbf{u}$ on the element boundaries.

Lema 5.2.2 *Assume that $\mathbf{u} \in [H^1(\Omega)]^2$ and define $\boldsymbol{\varphi}_h|_{\partial T} = \mathbf{u}|_{\partial T}$ $\forall\, T \in \mathcal{T}_h$. Then there exists $C > 0$, independent of h, such that for all $T \in \mathcal{T}_h$*

$$\begin{aligned} \boldsymbol{\theta}_T^2 \,\leq\, C \Big\{ & ||\mathbf{t}_{h,T} - \mathbf{t}||^2_{[L^2(T)]^{2\times 2}} + ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}||^2_{\mathbf{H}(\mathbf{div}\,;T)} + ||\boldsymbol{\gamma}_{h,T} - \boldsymbol{\gamma}||^2_{[L^2(T)]^{2\times 2}} \\ & + ||\mathbf{u}_{h,T} - \mathbf{u}||^2_{[L^2(T)]^2} \,+\, ||\boldsymbol{\eta}_h - \boldsymbol{\eta}||^2_{[\tilde{H}^{1/2}(\partial T \cap \Gamma_N)]^2} \\ & + \left|\left| \left(\frac{1}{2}\mathbf{I} + \mathbf{K} \right) (\boldsymbol{\phi}_h - \boldsymbol{\phi}) \right|\right|^2_{[H^{1/2}(\partial T \cap \Gamma)]^2} + ||\mathbf{V}(\boldsymbol{\sigma}_h \boldsymbol{\nu} - \boldsymbol{\sigma} \boldsymbol{\nu})||^2_{[H^{1/2}(\partial T \cap \Gamma)]^2} \Big\} . \end{aligned} \qquad (5.2.40)$$

In addition, there exists $\tilde{C} > 0$, independent of h, such that

$$\left\{ \sum_{T \in \mathcal{T}_h} \boldsymbol{\theta}_T^2 + R_\Gamma^2 + R_{\Gamma_N}^2 \right\}^{1/2} \,\leq\, \tilde{C}\, ||\vec{\mathbf{t}} - \vec{\mathbf{t}}_h||_{\mathbf{H}} \,, \qquad (5.2.41)$$

where $\vec{\mathbf{t}} := (\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta})$, $\vec{\mathbf{t}}_h := (\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)$, and $\boldsymbol{\theta}_T$, R_Γ, R_{Γ_N} are defined in Theorem 5.2.2.

Proof: By taking $\mathbf{s} = \boldsymbol{\psi} = 0$ in the first equation of (5.1.5), and then integrating by parts in Ω, we get

$$\int_\Omega [\nabla \mathbf{u} - (\mathbf{t} + \boldsymbol{\gamma})] : \boldsymbol{\tau}\, dx \,-\, \langle \boldsymbol{\tau} \boldsymbol{\nu}, \mathbf{u} + \boldsymbol{\eta} \rangle_{\Gamma_N}$$

$$+ \left\langle \boldsymbol{\tau\nu}, \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right) \boldsymbol{\phi} - \mathbf{V}(\boldsymbol{\sigma\nu}) - \mathbf{u} \right\rangle_\Gamma = 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H_0}(\mathbf{div}\,;\Omega)\,.$$

It follows that

$$\nabla \mathbf{u} = \mathbf{t} + \boldsymbol{\gamma} \quad \text{in} \quad \Omega\,, \qquad \mathbf{u} = -\boldsymbol{\eta} \quad \text{on} \quad \Gamma_N\,,$$

and

$$\mathbf{u} = \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right) \boldsymbol{\phi} - \mathbf{V}(\boldsymbol{\sigma\nu}) \quad \text{on} \quad \Gamma\,,$$

whence (5.2.39) yields

$$\begin{aligned} ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} \,\leq\, 2 \Big\{ & ||\mathbf{t}_{h,T} - \mathbf{t}||^2_{[L^2(T)]^{2\times 2}} \,+\, ||\boldsymbol{\gamma}_{h,T} - \boldsymbol{\gamma}||^2_{[L^2(T)]^{2\times 2}} \\ & + ||\mathbf{u}_{h,T} - \mathbf{u}||^2_{[L^2(T)]^2} \,+\, ||\boldsymbol{\eta}_h - \boldsymbol{\eta}||^2_{[\tilde{H}^{1/2}(\partial T \cap \Gamma_N)]^2} \\ + \left|\left|\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)(\boldsymbol{\phi}_h - \boldsymbol{\phi})\right|\right|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} & + ||\mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu} - \boldsymbol{\sigma\nu})||^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} \Big\}\,. \end{aligned} \tag{5.2.42}$$

Now, taking $\boldsymbol{\delta} = \boldsymbol{\xi} = 0$ in the second equation of (5.1.5), and $\boldsymbol{\psi} = \boldsymbol{\tau} = 0$ in the first one, we obtain, respectively,

$$-\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{and} \quad \boldsymbol{\sigma} = \hat{\lambda}(\mathbf{t})\,\mathrm{tr}\,(\mathbf{t})\,\mathbf{I}_2 + 2\,\hat{\mu}(\mathbf{t})\,\mathbf{t} \quad \text{in} \quad \Omega\,.$$

Then we have

$$||\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{h,T}||_{[L^2(T)]^2} \,=\, ||\mathbf{div}\,(\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma})||_{[L^2(T)]^2}\,, \tag{5.2.43}$$

and also

$$||\hat{\mathbf{t}}_T||_{[L^2(T)]^{2\times 2}} \,\leq\, ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}||_{[L^2(T)]^{2\times 2}} \,+\, ||\tilde{\mathbf{A}}_{1,T}(\mathbf{t}_{h,T}) \,-\, \tilde{\mathbf{A}}_{1,T}(\mathbf{t})||_{[L^2(T)]^{2\times 2'}}\,,$$

where $\tilde{\mathbf{A}}_{1,T} : [L^2(T)]^{2\times 2} \to [L^2(T)]^{2\times 2'}$ is the restriction to $T \in \mathcal{T}_h$ of $\tilde{\mathbf{A}}_1$ (cf. proof of Lemma 5.2.1). For the definition of $\hat{\mathbf{t}}_T$ see Theorem 5.2.2.

Since $\tilde{\mathbf{A}}_{1,T}$ is Lipschitz continuous, we deduce that there exists $C > 0$, independent of h and T, such that

$$||\hat{\mathbf{t}}_T||_{[L^2(T)]^{2\times 2}} \,\leq\, C\,\Big\{\, ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}||_{[L^2(T)]^{2\times 2}} \,+\, ||\mathbf{t}_{h,T} - \mathbf{t}||_{[L^2(T)]^{2\times 2}} \Big\}\,. \tag{5.2.44}$$

Next, taking $\mathbf{v} = \boldsymbol{\xi} = 0$ in the second equation of (5.1.5) we get $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathbf{T}}$ in Ω, with which one can prove that

$$||\boldsymbol{\sigma}_{h,T} \,-\, \boldsymbol{\sigma}^{\mathbf{T}}_{h,T}||_{[L^2(T)]^{2\times 2}} \,\leq\, 2\,||\boldsymbol{\sigma}_{h,T} \,-\, \boldsymbol{\sigma}||_{[L^2(T)]^{2\times 2}}\,. \tag{5.2.45}$$

Thus, (5.2.42), (5.2.43), (5.2.44), (5.2.45), and the definition of $\boldsymbol{\theta}^2_T$, yield (5.2.40).

On the other hand, we also find from the variational formulation (5.1.5) that

$$\mathbf{W}\boldsymbol{\phi} + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}\boldsymbol{\nu}) = 0 \quad \text{on} \quad \Gamma, \quad \text{and} \quad \boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{g} \quad \text{on} \quad \Gamma_N\,,$$

which, using the continuity of the boundary integral operators $\mathbf{W}$ and $\mathbf{K}'$, imply that

$$R_{\Gamma}^2 \,+\, R_{\Gamma_N}^2 \,\leq\, C\,\left\{\, ||\boldsymbol{\phi}_h - \boldsymbol{\phi}||^2_{[H^{1/2}(\Gamma)]^2} \,+\, ||\boldsymbol{\sigma}_h - \boldsymbol{\sigma}||^2_{\mathbf{H}(\mathbf{div}\,;\Omega)} \right\}. \tag{5.2.46}$$

Next, according to the definition of the Sobolev norm of order 1/2, we see that

$$\sum_{T\in\mathcal{T}_h} ||\boldsymbol{\eta}_h - \boldsymbol{\eta}||^2_{[\tilde{H}^{1/2}(\partial T\cap\Gamma_N)]^2} \,\leq\, C\,||\boldsymbol{\eta}_h - \boldsymbol{\eta}||^2_{[\tilde{H}^{1/2}(\Gamma_N)]^2}\,, \tag{5.2.47}$$

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \left|\left|\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)(\boldsymbol{\phi}_h - \boldsymbol{\phi})\right|\right|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} &\leq C\,\left|\left|\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)(\boldsymbol{\phi}_h - \boldsymbol{\phi})\right|\right|^2_{[H^{1/2}(\Gamma)]^2} \\ &\leq\, C\,||\boldsymbol{\phi}_h - \boldsymbol{\phi}||^2_{[H^{1/2}(\Gamma)]^2}\,,\end{aligned} \tag{5.2.48}$$

and

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} ||\mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu} - \boldsymbol{\sigma}\boldsymbol{\nu})||^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} &\leq\, C\,||\mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu} - \boldsymbol{\sigma}\boldsymbol{\nu})||^2_{[H^{1/2}(\Gamma)]^2} \\ &\leq\, C\,||\boldsymbol{\sigma}_h \,-\, \boldsymbol{\sigma}||^2_{\mathbf{H}(\mathbf{div}\,;\Omega)}\,,\end{aligned} \tag{5.2.49}$$

where the boundedness of $\mathbf{K}$ and $\mathbf{V}$ has also been used for the derivation of (5.2.48) and (5.2.49).

Finally, the efficiency estimate (5.2.41) is a simple consequence of (5.2.40), (5.2.46), (5.2.47), (5.2.48) and (5.2.49). This finishes the proof. ■

Before ending this subsection we remark that the a-posteriori error estimate provided by Theorem 5.2.2 includes the non-local terms R_Γ and R_{Γ_N}, which, though of explicit residual type, are defined in terms of negative order Sobolev norms. In addition, it is also important to observe that the local problem (5.2.10) holds in a space of infinite dimension, and hence it could only be solved approximately by replacing $\mathbf{H}(\mathbf{div}\,;T)$ by a finite dimensional subspace. In particular, since that problem is linear, we suggest to apply the p or the $h-p$ version to approximate $\hat{\boldsymbol{\sigma}}_T$ (see, e.g., [2]).

According to the above, the purpose of the next subsection is to bound the contributions R_Γ, R_{Γ_N}, and $||\hat{\boldsymbol{\sigma}}_T||_{\mathbf{H}(\mathbf{div}\,;T)}$ by easily computable expressions.

5.2.3 Fully local a-posteriori error estimates

We first need to specify the finite dimensional subspace $\mathbf{H}_h$ to which the approximate solution $(\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)$ belongs. To this end we consider the same finite element and boundary element spaces from [34], which are a natural extension of the PEERS space from [7]. In fact, let $X_{1,h} := X_{1,h}^{\mathbf{t}} \times X_{1,h}^{\boldsymbol{\phi}}$, $M_{1,h}$, and $M_h := M_h^{\mathbf{u}} \times \mathcal{R}_h \times M_h^{\boldsymbol{\eta}}$ denote the corresponding subspaces to approximate $(\mathbf{t}, \boldsymbol{\phi})$, $\boldsymbol{\sigma}$, and $(\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta})$, respectively. Further, let $\{\Gamma_1, ..., \Gamma_m\}$ and $\{\tilde{\Gamma}_1, ..., \tilde{\Gamma}_n\}$ be the partitions on Γ and Γ_N, respectively, induced by the triangulation $\mathcal{T}_h$. Then we define

$$X_{1,h}^{\mathbf{t}} := \{\text{piecewise RT of order } 0 + \text{rotations of cubic bubble functions}\}\,,$$

$$X_{1,h}^{\boldsymbol{\phi}} := [H^{1/2}(\Gamma)]_h^2 / \mathcal{N}_\Gamma(\mathbf{e})$$

or, equivalently

$$X_{1,h}^{\boldsymbol{\phi}} := \{\boldsymbol{\psi}_h \in [H^{1/2}(\Gamma)]_h^2 : \quad \langle \boldsymbol{\omega}, \boldsymbol{\psi}_h \rangle_\Gamma = 0 \quad \forall\, \boldsymbol{\omega} \in \mathcal{N}_\Gamma(\mathbf{e})\,\}\,,$$

where

$$[H^{1/2}(\Gamma)]_h^2 := \{\boldsymbol{\psi}_h \in [C(\Gamma)]^2 : \quad \boldsymbol{\psi}_h|_{\Gamma_j} \in [\mathbf{P}_1(\Gamma_j)]^2 \quad \forall\, j \in \{1, ..., m\}\,\}\,,$$

$$M_{1,h} := \{\text{RT of order } 0 + \text{rotations of cubic bubble functions}\} \cap \mathbf{H_0}(\mathbf{div}\,; \Omega)\,,$$

$$M_h^{\mathbf{u}} := \{\mathbf{v}_h \in [L^2(\Omega)]^2 : \quad \mathbf{v}_h|_T \in [\mathbf{P}_0(T)]^2 \quad \forall\, T \in \mathcal{T}_h\,\}\,,$$

$$\mathcal{R}_h := \left\{ \begin{bmatrix} 0 & \delta \\ -\delta & 0 \end{bmatrix} \in [H^1(\Omega)]^{2\times 2} : \quad \delta|_T \in \mathbf{P}_1(T)\ \forall\, T \in \mathcal{T}_h \right\}\,,$$

and

$$M_h^{\boldsymbol{\eta}} := \{\boldsymbol{\xi}_h \in [\tilde{H}^{1/2}(\Gamma_N)]^2 : \quad \boldsymbol{\xi}_h|_{\tilde{\Gamma}_j} \in [\mathbf{P}_1(\tilde{\Gamma}_j)]^2 \quad \forall\, j \in \{1, ..., n\}\,\}\,.$$

Hereafter, given integers $l \geq 0$, $k \in \{1, 2\}$ and a subset S of $\mathbb{R}^k$, $\mathbf{P}_l(S)$ stands for the space of polynomials in k variables defined in S of total degree at most l.

Therefore, the Galerkin scheme associated to the variational formulation (5.1.5) reads: *Find* $((\mathbf{t}_h, \boldsymbol{\phi}_h), \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)) \in \mathbf{H}_h := X_{1,h} \times M_{1,h} \times M_h$ *such that*

$$\begin{aligned} [\mathbf{A}(\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h), (\mathbf{s}_h, \boldsymbol{\psi}_h, \boldsymbol{\tau}_h)] \,+\, [\mathbf{B}(\boldsymbol{\tau}_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)] &= 0\,, \\ [\mathbf{B}(\boldsymbol{\sigma}_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h)] \,=\, [\mathbf{F}, (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h)]&\,, \end{aligned} \tag{5.2.50}$$

for all $((\mathbf{s}_h, \boldsymbol{\psi}_h), \boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\xi}_h)) \in \mathbf{H}_h$.

We remark that the existence of a unique $((\mathbf{t}_h, \boldsymbol{\phi}_h), \boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)) \in \mathbf{H}_h$ solution to (5.2.50), and the corresponding error analysis, were provided in [34] for the particular case $\Gamma_N = \emptyset$. Hence, we do not give any further details here and just assume, from now on, unique solvability of the discrete scheme (5.2.50).

Lema 5.2.3 *Assume that the Neumann data $\mathbf{g} \in [L^2(\Gamma_N)]^2$. Then there exists $C > 0$, independent of h, such that*

$$R_\Gamma^2 \;\leq\; C\,\log[1 + C_h(\Gamma)] \sum_{j=1}^{m} h_j\,||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})||^2_{[L^2(\Gamma_j)]^2}\,, \tag{5.2.51}$$

and

$$R_{\Gamma_N}^2 \;\leq\; C\,\log[1 + C_h(\Gamma_N)] \sum_{j=1}^{n} \tilde{h}_j\,||\boldsymbol{\sigma}_h\boldsymbol{\nu} \,-\, \mathbf{g}||^2_{[L^2(\tilde{\Gamma}_j)]^2}\,, \tag{5.2.52}$$

where h_j = length of Γ_j, $\tilde{h}_j$ = length of $\tilde{\Gamma}_j$,

$$C_h(\Gamma) := \max\left\{\frac{h_i}{h_j}: \quad |i-j| = 1\,, \quad i,j \in \{1,...,m\}\right\},$$

and

$$C_h(\Gamma_N) := \max\left\{\frac{\tilde{h}_i}{\tilde{h}_j}: \quad |i-j| = 1\,, \quad i,j \in \{1,...,n\}\right\}.$$

Proof: We proceed similarly as in Corollary 1 of [36]. First, the definitions of the subspaces $X_{1,h}^{\boldsymbol{\phi}}$ and $M_{1,h}$ guarantee that $\boldsymbol{\phi}_h \in [H^1(\Gamma)]^2$, $(\boldsymbol{\sigma}_h\boldsymbol{\nu})|_\Gamma \in [L^2(\Gamma)]^2$, and $(\boldsymbol{\sigma}_h\boldsymbol{\nu})|_{\Gamma_N} \in [L^2(\Gamma_N)]^2$. Hence, the mapping properties of $\mathbf{W}$ and $\mathbf{K}'$ imply that $\left(\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})\right) \in [L^2(\Gamma)]^2$. Also, taking $\mathbf{s}_h = \boldsymbol{\tau}_h = 0$ in the first equation of (5.2.50), we get

$$\langle \mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu}), \boldsymbol{\psi}_h\rangle_\Gamma = 0 \quad \forall\, \boldsymbol{\psi}_h \in X_{1,h}^{\boldsymbol{\phi}}\,.$$

Next, given $\tilde{\boldsymbol{\psi}}_h \in [H^{1/2}(\Gamma)]_h^2$ we let $\boldsymbol{\psi}_h \in X_{1,h}^{\boldsymbol{\phi}}$ be such that $(\tilde{\boldsymbol{\psi}}_h - \boldsymbol{\psi}_h) \in \mathcal{N}_\Gamma(\mathbf{e})$. Since $\langle \mathbf{W}\boldsymbol{\phi}_h, \boldsymbol{\omega}\rangle_\Gamma = \langle \mathbf{W}(\boldsymbol{\omega}), \boldsymbol{\phi}_h\rangle_\Gamma = 0$ and $\langle \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu}), \boldsymbol{\omega}\rangle_\Gamma = \langle \boldsymbol{\sigma}_h\boldsymbol{\nu}, \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)(\boldsymbol{\omega})\rangle_\Gamma$ $= \langle \boldsymbol{\sigma}_h\boldsymbol{\nu}, \boldsymbol{\omega}\rangle_\Gamma = 0$ for all $\boldsymbol{\omega} \in \mathcal{N}_\Gamma(\mathbf{e})$, we deduce that $\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})$ is $[L^2(\Gamma)]^2$-orthogonal to $[H^{1/2}(\Gamma)]_h^2$. Thus, the estimate (5.2.51) follows from a straightforward application of Theorem 2 in [21].

On the other hand, taking $\mathbf{v}_h = \boldsymbol{\delta}_h = 0$ in the second equation of (5.2.50), we find that $(\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g})$ is $[L^2(\Gamma_N)]^2$-orthogonal to $M_h^{\boldsymbol{\eta}}$, and therefore, the estimate (5.2.52) is also a consequence of Theorem 2 in [21]. ∎

As a corollary of Theorem 5.2.2 and Lemma 5.2.3 we can establish the following a-posteriori error estimate.

Teorema 5.2.3 *Let $\boldsymbol{\varphi}_h \in [H^{1/2}(\cup_{T\in\mathcal{T}_h}\partial T)]^2$ be as stated in Theorem 5.2.1, and assume that $\mathbf{g} \in [L^2(\Gamma_N)]^2$. In addition, for each $T \in \mathcal{T}_h$ define*

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\,\mathbf{I}_2 - 2\,\hat{\mu}(\mathbf{t}_{h,T})\,\mathbf{t}_{h,T}\,,$$

and let $\hat{\boldsymbol{\sigma}}_T \in \mathbf{H}(\mathbf{div}\,;T)$ *be the unique solution of the local problem* (5.2.10). *Then there exists* $C > 0$*, independent of* h*, such that*

$$||(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \gamma, \boldsymbol{\eta}) - (\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \gamma_h, \boldsymbol{\eta}_h)||_{\mathbf{H}} \leq C \left\{ \sum_{T \in \mathcal{T}_h} \hat{\boldsymbol{\theta}}_T^2 \right\}^{1/2},$$

where

$$\hat{\boldsymbol{\theta}}_T^2 := ||\hat{\mathbf{t}}_T||^2_{[L^2(T)]^{2\times 2}} + ||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} + ||\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_{h,T}||^2_{[L^2(T)]^2}$$
$$+ ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}^{\mathbf{T}}_{h,T}||^2_{[L^2(T)]^{2\times 2}} + \log[1 + C_h(\Gamma_N)]\, h_T\, ||\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}||^2_{[L^2(\partial T \cap \Gamma_N)]^2}$$
$$+ \log[1 + C_h(\Gamma)]\, h_T\, ||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h \boldsymbol{\nu})||^2_{[L^2(\partial T \cap \Gamma)]^2}.$$

Proof: It follows from Theorem 5.2.2 and Lemma 5.2.3 by noting that h_j (resp. $\tilde{h}_j$) is smaller than h_T, where $T \in \mathcal{T}_h$ is the triangle having Γ_j (resp. $\tilde{\Gamma}_j$) as a boundary edge. ∎

We remark that the upper bounds for R_Γ and R_{Γ_N} given by Lemma 5.2.3 are also lower bounds (with different constants) as long as the partitions on Γ and Γ_N are quasi-uniform (see Theorem 3 in [21]).

To end this section, we set an appropriate choice of $\boldsymbol{\varphi}_h$. Indeed, we first define $\tilde{\boldsymbol{\varphi}}_h \in [C(\bar{\Omega})]^2$ as the unique function satisfying the following conditions:

(a) $\tilde{\boldsymbol{\varphi}}_{h,T} := \tilde{\boldsymbol{\varphi}}_h|_T \in [\mathrm{P}_1(T)]^2$ for all $T \in \mathcal{T}_h$.

(b) $\tilde{\boldsymbol{\varphi}}_h(\bar{\mathbf{x}}) = (0,0)^{\mathbf{T}}$ for each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ lying on $\bar{\Gamma}_D$.

(c) $\boldsymbol{\varphi}_h(\bar{\mathbf{x}}) = -\boldsymbol{\eta}_h(\bar{\mathbf{x}})$ for each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ on Γ_N.

(d) $\boldsymbol{\varphi}_h(\bar{\mathbf{x}}) = -\mathbf{V}(\boldsymbol{\sigma}_h \boldsymbol{\nu})(\bar{\mathbf{x}}) + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right) \boldsymbol{\phi}_h(\bar{\mathbf{x}})$ for each vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$ on Γ.

(e) $\tilde{\boldsymbol{\varphi}}_h(\bar{\mathbf{x}})$ is the average of the constant values of $\mathbf{u}_h$ on all the triangles $T \in \mathcal{T}_h$ to which $\bar{\mathbf{x}}$ belongs, for any other vertex $\bar{\mathbf{x}}$ of $\mathcal{T}_h$.

Then, since $\tilde{\boldsymbol{\varphi}}_h \in [H^1(\Omega)]^2$ we just put

$$\boldsymbol{\varphi}_h|_{\partial T} = \tilde{\boldsymbol{\varphi}}_h|_{\partial T} \qquad \forall\, T \in \mathcal{T}_h\,. \tag{5.2.53}$$

Furthermore, we can also use $\tilde{\boldsymbol{\varphi}}_h$ to derive a simpler fully local a-posteriori error estimate, which does not require the explicit solution of the problem (5.2.10). More precisely, we have the following result.

Teorema 5.2.4 *Let $\tilde{\boldsymbol{\varphi}}_h \in [H^1(\Omega)]^2$ be as indicated above, assume that $\mathbf{g} \in [L^2(\Gamma_N)]^2$, and for each $T \in \mathcal{T}_h$ define*

$$\hat{\mathbf{t}}_T := \boldsymbol{\sigma}_{h,T} - \hat{\lambda}(\mathbf{t}_{h,T})\,\mathrm{tr}\,(\mathbf{t}_{h,T})\,\mathbf{I}_2 - 2\,\hat{\mu}(\mathbf{t}_{h,T})\,\mathbf{t}_{h,T}\,.$$

In addition, let $\{\bar{\mathbf{x}}_{k,T}\}_{k=1}^3$ be the vertices of $T \in \mathcal{T}_h$, and assume, without loss of generality, that $\bar{\mathbf{x}}_{1,T}$, $\bar{\mathbf{x}}_{2,T} \in \Gamma_N$ when $\partial T \cap \Gamma_N$ is not empty. Then there exists $C > 0$, independent of h, such that

$$\begin{aligned} & ||(\mathbf{t}, \boldsymbol{\phi}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\eta}) - (\mathbf{t}_h, \boldsymbol{\phi}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\eta}_h)||_{\mathbf{H}} \\ & \leq C \left\{ \sum_{T\in\mathcal{T}_h} \left(\tilde{\boldsymbol{\theta}}_T^2 + \tilde{\boldsymbol{\theta}}_{T,\Gamma}^2 + \tilde{\boldsymbol{\theta}}_{T,\Gamma_N}^2 \right) \right\}^{1/2}, \end{aligned} \tag{5.2.54}$$

where

$$\begin{aligned} \tilde{\boldsymbol{\theta}}_T^2 := \Bigg\{ & ||\hat{\mathbf{t}}_T||_{[L^2(T)]^{2\times 2}}^2 + \left|\left| \mathbf{t}_{h,T} - \frac{1}{2}\left[\nabla \tilde{\boldsymbol{\varphi}}_{h,T} + (\nabla \tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}\right] \right|\right|_{[L^2(T)]^{2\times 2}}^2 \\ & + h_T^2 \sum_{k=1}^{3} \left|\left| \boldsymbol{\gamma}_{h,T}(\bar{\mathbf{x}}_{k,T}) - \frac{1}{2}\left[\nabla \tilde{\boldsymbol{\varphi}}_{h,T} - (\nabla \tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}\right] \right|\right|_{\mathbb{R}^{2\times 2}}^2 \\ & + h_T^2 \sum_{k=1}^{3} ||\mathbf{u}_{h,T} - \tilde{\boldsymbol{\varphi}}_{h,T}(\bar{\mathbf{x}}_{k,T})||_{\mathbb{R}^2}^2 \\ & + ||\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{h,T}||_{[L^2(T)]^2}^2 + ||\boldsymbol{\sigma}_{h,T} - \boldsymbol{\sigma}_{h,T}^{\mathbf{T}}||_{[L^2(T)]^{2\times 2}}^2 \Bigg\}, \end{aligned}$$

$$\begin{aligned} \tilde{\boldsymbol{\theta}}_{T,\Gamma}^2 := \Bigg\{ & \left|\left| \tilde{\boldsymbol{\varphi}}_{h,T} + \mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu}) - \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\phi}_h \right|\right|_{[H^{1/2}(\partial T\cap\Gamma)]^2}^2 \\ & + \log[1 + C_h(\Gamma)]\, h_T\, ||\mathbf{W}\boldsymbol{\phi}_h + \left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)(\boldsymbol{\sigma}_h\boldsymbol{\nu})||_{[L^2(\partial T\cap\Gamma)]^2}^2 \Bigg\}, \end{aligned}$$

and

$$\begin{aligned} \tilde{\boldsymbol{\theta}}_{T,\Gamma_N}^2 := p_N(T) \Bigg\{ & h_T \sum_{k=1}^{2} ||(\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h)(\bar{\mathbf{x}}_{k,T})||_{\mathbb{R}^2}^2 + ||(\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h)(\bar{\mathbf{x}}_{2,T} - \bar{\mathbf{x}}_{1,T})||_{\mathbb{R}^2}^2 \\ & + \log[1 + C_h(\Gamma_N)]\, h_T\, ||\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}||_{[L^2(\partial T\cap\Gamma_N)]^2}^2 \Bigg\}, \end{aligned}$$

with $p_N(T) = 1$ if $\partial T \cap \Gamma_N$ is not empty, and 0 otherwise.

Proof: Because of Theorem 3.6, it suffices to bound $||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)}$ for the present choice of $\boldsymbol{\varphi}_h$. In fact, according to the analysis before Lemma 5.2.2, and replacing $\mathbf{u}$ by $\tilde{\boldsymbol{\varphi}}_h$ in (5.2.39), we obtain

$$\begin{aligned}||\hat{\boldsymbol{\sigma}}_T||^2_{\mathbf{H}(\mathbf{div}\,;T)} \;\leq\; & ||\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\tilde{\boldsymbol{\varphi}}_{h,T}||^2_{[L^2(T)]^{2\times 2}} + ||\mathbf{u}_{h,T} - \tilde{\boldsymbol{\varphi}}_{h,T}||^2_{[L^2(T)]^2} \\ & + ||\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h||^2_{[\tilde{H}^{1/2}(\partial T\cap\Gamma_N)]^2} \\ & + \left|\left|\tilde{\boldsymbol{\varphi}}_{h,T} \,+\, \mathbf{V}(\boldsymbol{\sigma}_h\boldsymbol{\nu}) \,-\, \left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\boldsymbol{\phi}_h\right|\right|^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} . \qquad (5.2.55)\end{aligned}$$

Also, it is easy to see that

$$\begin{aligned}||\mathbf{t}_{h,T} + \boldsymbol{\gamma}_{h,T} - \nabla\tilde{\boldsymbol{\varphi}}_{h,T}||^2_{[L^2(T)]^{2\times 2}} \leq 2\Big\{ & \left|\left|\mathbf{t}_{h,T} - \frac{1}{2}[\nabla\tilde{\boldsymbol{\varphi}}_{h,T} + (\nabla\tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}]\right|\right|^2_{[L^2(T)]^{2\times 2}} \\ & + \left|\left|\boldsymbol{\gamma}_{h,T} - \frac{1}{2}[\nabla\tilde{\boldsymbol{\varphi}}_{h,T} - (\nabla\tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}]\right|\right|^2_{[L^2(T)]^{2\times 2}} \Big\}. \qquad (5.2.56)\end{aligned}$$

Next, since $\tilde{\boldsymbol{\varphi}}_{h,T} \in [\mathbf{P}_1(T)]^2$, $\nabla\tilde{\boldsymbol{\varphi}}_{h,T} \in [\mathbf{P}_0(T)]^{2\times 2}$, $\boldsymbol{\gamma}_{h,T} \in [\mathbf{P}_1(T)]^{2\times 2}$, and the area of T is bounded by h_T^2, we deduce that

$$||\mathbf{u}_{h,T} - \tilde{\boldsymbol{\varphi}}_{h,T}||^2_{[L^2(T)]^2} \;\leq\; h_T^2 \sum_{k=1}^{3} ||\mathbf{u}_{h,T} - \tilde{\boldsymbol{\varphi}}_{h,T}(\bar{\mathbf{x}}_{k,T})||^2_{\mathbb{R}^2}\,, \qquad (5.2.57)$$

and

$$\begin{aligned}& \left|\left|\boldsymbol{\gamma}_{h,T} - \frac{1}{2}[\nabla\tilde{\boldsymbol{\varphi}}_{h,T} - (\nabla\tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}]\right|\right|^2_{[L^2(T)]^{2\times 2}} \\ \leq\; & h_T^2 \sum_{k=1}^{3} \left|\left|\boldsymbol{\gamma}_{h,T}(\bar{\mathbf{x}}_{k,T}) - \frac{1}{2}[\nabla\tilde{\boldsymbol{\varphi}}_{h,T} - (\nabla\tilde{\boldsymbol{\varphi}}_{h,T})^{\mathbf{T}}]\right|\right|^2_{\mathbb{R}^{2\times 2}} . \qquad (5.2.58)\end{aligned}$$

On the other hand, since $(\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h)|_{\partial T\cap\Gamma_N} \in [\mathbf{P}_1(\partial T\cap\Gamma_N)]^2$, we find that

$$\begin{aligned}||\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h||^2_{[\tilde{H}^{1/2}(\partial T\cap\Gamma_N)]^2} \;\leq\; & h_T \sum_{k=1}^{2} ||(\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h)(\bar{\mathbf{x}}_{k,T})||^2_{\mathbb{R}^2} \\ & + ||(\tilde{\boldsymbol{\varphi}}_{h,T} + \boldsymbol{\eta}_h)(\bar{\mathbf{x}}_{2,T} - \bar{\mathbf{x}}_{1,T})||^2_{\mathbb{R}^2} . \qquad (5.2.59)\end{aligned}$$

In this way, (5.2.55), (5.2.56), (5.2.57), (5.2.58), (5.2.59) and the estimate from Theorem 3.6 yield (5.2.54) and complete the proof. ■

Remark: For implementations, the $[H^{1/2}(\partial T\cap\Gamma)]^2$-norms in the terms $\tilde{\boldsymbol{\theta}}_{T,\Gamma}$ of the previous theorem can be bounded by using the interpolation theorem:

$$||\cdot||^2_{[H^{1/2}(\partial T\cap\Gamma)]^2} \;\leq\; ||\cdot||_{[L^2(\partial T\cap\Gamma)]^2}\, ||\cdot||_{[H^1(\partial T\cap\Gamma)]^2} .$$

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