Universidad de Concepción
Dirección de Postgrado
Concepción-Chile

Análisis de Error A Priori y A Posteriori de un Método de Elementos Finitos Completamente Mixto para un Problema de Interacción Sólido-Fluido Bidimensional

Tesis para optar al grado de
Doctor en Ciencias Aplicadas con mención en Ingeniería Matemática

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Marzo 2014

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Departamento de Ingeniería Matemática

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Calificación: ______________________

Concepción–Marzo de 2014

Dedicado a:

mi tío Alfredo García Andrade,

mis padres: Elena García Andrade y Daniel Domínguez,

mi tía Gloria García Andrade (Harold Domínguez),

Julián Armando Mena Zapata,

y sus padres: María Idaly Zapata de Mena y Orlando Mena.

AGRADECIMIENTOS

Con las siguientes palabras deseo expresar mis más sinceros agradecimientos, a quienes en esta etapa de mi vida me han brindado apoyo, personal y académico, hasta llevar a feliz término este trabajo de tesis.

A mis directores: profesor Salim Meddahi y profesor Gabriel Gatica; en proponer los temas a desarrollar y tutoría conducente a culminar esta Tesis. Siendo el prof. G. Gatica Director del Centro de Investigación en Ingeniería Matemática CI^{2}MA, casa de estudios donde fui acogida de manera grata.

A las instituciones: CONICYT, CI^{2}MA y CMM, el apoyo financiero otorgado para diferentes fines académicos.

Al Director del Doctorado en Ingeniería Matemática, profesor Raimund Bürger, especialmente en su diligencia y gestión. Pertenece a lo anterior los recursos dados por el Departamento de Ingeniería Matemática de la Universidad de Concepción, el programa MECESUP, la Dirección de Posgrado de la Universidad de Concepción y la Universidad de Oviedo (España).

A mis evaluadores y jurados por darse el tiempo necesario en la lectura de esta tesis; siendo los profesores evaluadores: Lucia Gastaldi, Francisco Guillén, George C. Hsiao y Ernst P. Stephan, y los profesores miembros del jurado: Ignacio Muga, Ricardo Oyarzúa y Rodolfo Rodríguez.

Especial mención doy a los académicos de la Facultad de Ciencias Físicas y Matemáticas de la Universidad de Concepción: Rodolfo Araya, Rodolfo Rodríguez, Mauricio Sepúlveda, Rommel Bustinza y Anahí Gajardo.

A funcionarios que me brindaron la mayor de sus atenciones y colaboración: José Parra, Cecilia Leiva, Cinthya Rojas, Elizabeth Sepúlveda, reconozco mi principal afecto a: Angelina Fritz, Lorena Carrasco y Eduardo López.

A mis compañeros de estudio: Bibiana y Jorge con quienes tuve mayor apego. Incluyo con gran estima a: Verónica Anaya Domínguez, Carlos Reales, Hector Torres, Abner Poza, David Mora, Pablo Venegas, Jessika Camaño, Hernán Mardones, Ricardo Ruiz, Frank Sanhueza, Fernando Betancourt, Ramiro Rebolledo, Leonardo Figueroa y Cinthya Rivas.

A egresados del Doctorado al cual pertenecí; Luis Gatica, Ramiro Acevedo y Tomas Barrios, quienes abrieron dialogo conmigo transmitiendo experiencias y opiniones del vivir.

Nombro de manera especial: Ricardo Contreras, Jairo Duque, Antonio Márquez, Walter Rudolph, Julia Margarita Silva, y Miguel Silva.

La conclusión de este trabajo ha sido en gran medida alcanzada con la participación de los anteriormente mencionados, y a quienes olvido u omito, nunca fue mi intención hacerlo.

RESUMEN

El objetivo principal de esta tesis es desarrollar un análisis de error a priori y a posteriori de un método de elementos finitos completamente mixto para un problema de interacción sólido-fluido bidimensional. Además de lo anterior, se deduce un estimador de error a posteriori residual, confiable y eficiente, para el problema de elasticidad lineal en el plano con condiciones de frontera de tracción pura.

Primero, se desarrolla un análisis de error a priori de un método de elementos finitos completamente mixto para un problema de interacción sólido-fluido bidimensional. El modelo se rige por las ecuaciones de la elastodinámica y la acústica en régimen de tiempo armónico, y las condiciones de transmisión están dadas por el equilibrio de fuerzas y la igualdad de los correspondientes desplazamientos normales. Se introduce una formulación dual mixta en ambos dominios, la cual tiene el esfuerzo y la rotación en el sólido, además del gradiente de presiones en el fluido, como las principales incógnitas. A su vez, ambas condiciones de transmisión son esenciales, las cuales se imponen débilmente por medio de multiplicadores de Lagrange. Luego, se muestra una descomposición apropiada del espacio al cual pertenecen el esfuerzo y el gradiente de presiones, y posteriormente se aplica la teoría de Babuška-Brezzi y la alternativa de Fredholm, para realizar el análisis de la formulación continua. Posteriormente, las incógnitas se aproximan por un esquema de Galerkin conforme definido en términos de los elementos de Raviart-Thomas de bajo orden en ambos dominios, y las funciones lineales a trozos continuas sobre las fronteras. Entonces, el análisis discreto se basa en una descomposición estable de los espacios de elementos finitos correspondientes y en un resultado clásico de métodos de proyección para operadores de Fredholm de índice cero.

Por otro lado, a modo de análisis preliminar y también como un subproducto de esta tesis, se considera un problema de elasticidad lineal bidimensional con condiciones de frontera de Neumann no homogéneas, y se deduce un estimador de error a posteriori residual, confiable y eficiente para su formulación variacional dual mixta, en términos del esfuerzo, el desplazamiento y la rotación. La demostración de la confiabilidad hace uso de un problema auxiliar apropiado, la condición inf-sup continua y las propiedades de aproximación local de los operadores de interpolación de Clément y Raviart-Thomas. A su vez, las desigualdades de traza discreta e inversa, y la técnica de localización basada en funciones burbuja sobre triángulos y lados, son las principales herramientas para probar la eficiencia del estimador.

Finalmente, se deduce un estimador de error a posteriori, basado en términos residuales, confiable y eficiente, para el problema de interacción estudiado en la primera parte. Las principales herramientas para probar la confiabilidad involucran una condición inf-sup global, las descomposiciones de Helmholtz continua y discreta en cada dominio, y las propiedades de aproximación local de los operadores de interpolación de Clément y Raviart-Thomas. Luego, se aplican las mismas técnicas mencionadas anteriormente para obtener la eficiencia. Finalmente, varios resultados numéricos confirman la confiabilidad y eficiencia del estimador, e ilustran el comportamiento del esquema adaptivo asociado.

ABSTRACT

The main purpose of this thesis is to develop the a priori and a posteriori error analyses of a fully-mixed finite element method for a fluid-solid interaction problem in 2D. In addition, we also derive a reliable and efficient residual-based a posteriori error estimator for the plane linear elasticity problem with pure traction boundary conditions.

First, we develop an a priori error analysis of a fully-mixed finite element method for a fluid-solid interaction problem in 2D. The media are governed by the elastodynamic and acoustic equations in time-harmonic regime, and the transmission conditions are given by the equilibrium of forces and the equality of the corresponding normal displacements. We introduce dual-mixed approaches in both domains, which yields the stress and the rotation in the solid, as well as the pressure gradient in the fluid, as the main unknowns. In turn, since both transmission conditions become essential, they are enforced weakly by means of two suitable Lagrange multipliers. Next, we show that suitable decompositions of the spaces to which the stress and the pressure gradient belong, allow the application of the Babuška-Brezzi theory and the Fredholm alternative for analyzing the solvability of the resulting continuous formulation. The unknowns are approximated by a conforming Galerkin scheme defined in terms of Raviart-Thomas element of lowest order in both domains, and continuous piecewise linear functions on the boundaries. Then, the analysis of the discrete method relies on a stable decomposition of the corresponding finite element spaces and also on a classical result on projection methods for Fredholm operators of index zero.

Next, as a preliminary analysis as well as a by product of this thesis, we consider the two-dimensional linear elasticity problem with non-homogeneous Neumann boundary conditions, and derive a reliable and efficient residual-based a posteriori error estimator for the corresponding stress-displacement-rotation dual-mixed variational formulation. The proof of reliability makes use of a suitable auxiliary problem, the continuous inf-sup conditions satisfied by the bilinear forms involved, and the local approximation properties of the Clément and Raviart-Thomas interpolation operators. In turn, inverse and discrete trace inequalities, and the localization technique based on triangle-bubble and edge-bubble functions, are the main tools yielding the efficiency of the estimator.

Finally, we derive a reliable and efficient residual-based a posteriori error estimator for the interaction problem studied in the first part. The main tools for proving the reliability of the estimator involve a global inf-sup condition, continuous and discrete Helmholtz decompositions on each domain, and the local approximation properties of the Clément and Raviart-Thomas interpolation operators. Next, we apply the above mentioned techniques to obtain the efficiency. Finally, several numerical results confirming the reliability and efficiency of the estimator, and illustrating the good performance of the associated adaptive scheme, are reported.

Contents

Chapter 1

Introducción

En esta tesis se exponen tres trabajos que abordan los temas de análisis de error a priori y a posteriori en el marco de los elementos finitos mixtos. El primero se refiere al estudio de un problema de interacción sólido-fluido bidimensional modelado por las ecuaciones de Lamé-Helmholtz. El segundo presenta un estimador de error a posteriori residual, confiable y eficiente, para el problema de elasticidad lineal con condiciones de frontera de tracción pura. Finalmente, en el tercer trabajo se deduce un estimador de error a posteriori residual, confiable y eficiente, para el problema de interacción considerado inicialmente. Cabe mencionar que el desarrollo intermedio del indicador de error a posteriori del problema de elasticidad lineal, se efectúa con el fin de adentrarnos, de una manera más sencilla, en las técnicas de demostración de los estimadores de error a posteriori de tipo residual para el cuerpo sólido (consultar en [22]), antes de abordar el problema de interacción entre el medio sólido y el fluido.

El análisis a priori del problema Lamé-Helmholtz constituye una extensión de lo trabajado en el articulo [37] (ver también [50] y [39]), dado que se desarrolla un planteamiento variacional dual mixto en ambos dominios. El concepto de extensión se refiere a la formulación variacional utilizada, la cual en vez de considerar una aproximación primal en el fluido acotado, como se aplicó en [37], emplea en cada dominio el mismo método mixto dual. Más precisamente, la formulación dual mixta presente en el fluido se origina por la introducción de la nueva incógnita $\boldsymbol{\sigma}_f = \nabla p$, esto es el gradiente de presiones, lo cual conlleva a reescribir la ecuación de Helmholtz y una de las condiciones de transmisión. La introducción de $\boldsymbol{\sigma}_f$ en la ecuación de Helmholtz es motivada por la eventual necesidad de obtener aproximaciones por elementos finitos directas y más precisas del ∇p (en vez de aplicar diferenciación numérica, con la consecuente de perdida de precisión en la aproximación de la presión p, cuando se emplea una formulación primal usual en el fluido). Lo anterior se requiere, en particular, cuando se resuelve el problema inverso relacionado con la ecuación de Helmholtz, en el que la representación integral de la frontera de un patrón de campo lejano, una variable importante en un algoritmo iterativo asociado, depende de la traza de p y la traza normal de $\boldsymbol{\sigma}_f$ (ver [25], Capítulo 2, Teorema 25).

En las formulaciones variacionales mixtas, el espacio para encontrar las soluciones $\boldsymbol{\sigma}_f$ y $\boldsymbol{\sigma}_s$ está dado por $H(\mathrm{div};\Omega)$ (donde Ω denota la región donde vive cada incógnita), siendo natural aproximar ambas con los elementos finitos de Raviart-Thomas. Lo anterior simplifica en gran medida el diseño del código computacional, dado que en ambos dominios se consideran los mismos elementos finitos. A su vez, las condiciones de transmisión son ahora esenciales, las cuales se imponen débilmente mediante multiplicadores de Lagrange apropiados. Por medio de este proceso se introducen nuevas incógnitas dadas por el desplazamiento del sólido sobre la interfase, y la presión sobre la interfase y la frontera.

Por otra parte, el indicador a posteriori del error es un estimador que, numéricamente, hace posible el proceso adaptivo, lo cual permite garantizar el buen comportamiento de la convergencia a la solución obtenida en el cálculo por elementos finitos, especialmente cuando se presentan dominios con geometrías complejas o singularidades. Dicho indicador se reperesenta usualmente por una cantidad global $\boldsymbol{\theta}$, que se expresa en términos de estimadores locales θ_T definidos sobre cada elemento T de una triangulación dada en el dominio. El estimador $\boldsymbol{\theta}$ se dice que es confiable (resp. eficiente) si existe $C_{rel} > 0$ (resp. $C_{eff} > 0$), independiente del tamaño de la malla, tal que

$$C_{eff}\boldsymbol{\theta} + h.o.t. \leq \|error\| \leq C_{rel}\boldsymbol{\theta} + h.o.t.,$$

donde h.o.t es una expresion genérica en inglés (higher order terms) que denota a uno o varios términos de orden superior. Dentro de la bibliografía existente en el ambito del análisis de error a posteriori, se puede afirmar que [35] es el único trabajo que aborda esta problemática para un problema de interación sólido-fluido modelado por las ecuaciones de la acústica y la elastodinámica en régimen de tiempo armónico. En lo referente a la cota de confiabilidad, se esbozan en esta tesis dos procedimientos diferentes. En el problema de elasticidad lineal se plantea un problema auxiliar seguidamente de la aplicación de una de las condiciones inf-sup continuas, en tanto que en el problema de Lamé-Helmholtz se hace uso de una condición inf-sup global, la cual es una consecuencia directa de la dependencia continua del problema. Ahora bien, en ambos problemas se recurre a la descomposición de Helmholtz, y a las propiedades de aproximación local de los interpolantes de Clément y Raviart-Thomas. Por otro lado, en la demostración de la eficiencia de los estimadores de error a posteriori se utilizan diferentes tipos de herramientas matemáticas, siendo las mas relevantes: las desigualdades inversas (c.f. [21] Teorema 3.2.6), las técnicas de localización basadas en funciones burbuja sobre lados y triángulos (c.f. [46] eqs. (1.5) and (1.6)), los operadores de extensión (c.f. [72]) y las desigualdades de traza discreta (c.f. [23] Lemmas 20 y 21).

El propósito de este trabajo es ampliar la gama de métodos numéricos existentes para el problema de Lamé-Helmholtz, lo cual procede de una generalización de los trabajos realizados en [35] y [37], obteniéndose de esta manera un mayor rango de elección de los elementos finitos para el esquema de Galerkin asociado a cada formulación variacional. Por otra parte, se proponen nuevas formulaciones que permiten, por un lado, la introducción de incógnitas adicionales de interés físico, y también la utilización del mismo tipo de elementos finitos en ambos dominios. Además, se extiende lo anterior a los métodos numéricos adaptivos, mediante la obtención de estimadores de error a posteriori correspondientes.

El resto de esta tesis se organiza de la manera que se indica a continuación: En el **Capítulo 2** se estudia un problema bidimensional de interacción sólido-fluido aplicando el método de los elementos finitos mixto en ambos dominios. El modelo a estudiar consiste de un cuerpo sólido con propiedades elásticas, inmerso en un fluido, sobre el cual incide una onda de sonido que se desplaza a través del fluido. El propósito es encontrar los esfuerzos generados en el sólido y el gradiente de presiones en el fluido. Se considera el dominio del fluido representado por una región anular, cuya condición de frontera exterior es de tipo Robin, la cual se impone lejos del cuerpo sólido con el fin de imitar el campo de dispersión de la onda en el infinito. Se emplean las ecuaciones elastodinámicas y acústicas en régimen de tiempo armónico, como las ecuaciones constitutivas del modelo, y se establecen las condiciones de transmisión en la frontera de acoplamiento, las cuales describen el equilibrio de fuerzas y la igualdad de los desplazamientos normales. El análisis de error a priori se basa en una formulación variacional dual mixta, tanto a nivel continuo como discreto, de donde se obtiene un sistema desacoplado a través de un operador compacto, lo cual se representa como una estructura de punto silla por bloques en la diagonal. Lo anterior permite la aplicación de la teoría de Babuška-Brezzi en cada dominio, y gracias a la perturbación compacta, se emplea la alternativa de Fredholm. Además, se utilizan varias otras herramientas matemáticas, entre las cuales se encuentran: proyectores ortogonales, descomposición ortogonal del espacio H(div), inclusiones compactas, operador de interpolación de Raviart-Thomas, levantamientos discretos estables, y propiedades de aproximación de subespacios de elementos finitos. Este capítulo está constituido por la siguiente publicación:

> Domínguez, C., Gatica, G.N., Meddahi, S. and Oyarzúa, R., *A priori error analysis of a fully-mixed finite element method for a two-dimensional fluid-solid interaction problem.* ESAIM: Mathematical Modelling and Numerical Analysis, vol. 47, 2, pp. 471-506, (2013).

En el **Capítulo 3** se deduce un estimador de error a posteriori de un problema de elasticidad lineal en el plano con condiciones de contorno de Neumann puras, cuyo planteamiento está dado por una formulación variacional dual mixta, en el cual las incógnitas respectivas están dadas por: el tensor de esfuerzos de Cauchy, el desplazamiento, la rotación y la traza del desplazamiento.

De hecho, la condición de Neumann se impone débilmente precisamente a través de la introducción de la traza del desplazamiento como un multiplicador de Lagrange. Los subespacios de elementos finitos considerados son básicamente PEERS, esto es: Raviart-Thomas + el rotacional de las funciones burbuja para el tensor de esfuerzos, las funciones constantes a trozos para el desplazamiento, y las funciones lineales a trozos y continuas para describir la rotación y el multiplicador de Lagrange sobre la frontera. En la deducción del estimador a posteriori se comienza con la cota de confiabilidad, la cual recurre en primera instancia al planteamiento de un problema auxiliar y la condición inf-sup continua. Posteriormente se introduce el rotacional de una función potencial en la variable del tensor de esfuerzos, anexándole el interpolante de Clément, y luego se aplica una descomposición de Helmholtz en la cual actúan los interpolantes de Clément y Raviart-Thomas. Por otro lado, las principales herramientas utilizadas para demostrar la eficiencia del estimador incluyen: la desigualdad inversa, las técnicas de localización basadas en funciones burbuja sobre lados y triángulos, los operadores de extensión y la desigualdad de traza. Los contenidos del Capítulo 3 constituyen la siguiente prepublicación:

> Domínguez C., Gatica G.N. and Márquez A., *A residual-based a posteriori error estimator for the plane linear elasticity problem with pure traction boundary conditions.* Preprint 2014-04, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile, (2014).

Finalmente, en el **Capítulo 4** se desarrolla un análisis de error a posteriori del método de elementos finitos completamente mixto para el problema de interacción sólido-fluido bidimensional estudiado en el Capítulo 2. El problema es modelado por las ecuaciones de la elastodinámica y la acústica en régimen de tiempo armónico, las condiciones de transmisión están dadas por el equilibrio de fuerzas y la igualdad de los desplazamientos normales correspondientes, y el fluido ocupa una región anular, la cual rodea al sólido, de modo que una condición de frontera de tipo Robin se impone sobre la frontera exterior imitando el comportamiento de las condiciones de Sommerfeld al infinito. La formulación dual mixta se aplica en ambos dominios, y las ecuaciones que rigen el modelo se emplean para eliminar el desplazamiento $\mathbf{u}$ del sólido y la presión p del fluido. Además, en este caso las condiciones de transmisión son esenciales, y por lo tanto se imponen débilmente por medio de multiplicadores de Lagrange apropiados. Las incógnitas del sólido y del fluido se aproximan por un esquema de Galerkin conforme, se definen en términos de los elementos PEERS en el sólido, los Raviart-Thomas de bajo orden en el fluido, y las funciones lineales a trozos y continuas sobre la frontera. Se obtiene un estimador de error a posteriori basado en términos residuales, confiable y eficiente, para este problema acoplado. Las principales herramientas para probar la confiabilidad del estimador involucran la condición inf-sup global continua, las descomposiciones de Helmholtz continua y discreta en cada dominio, y las propiedades de aproximación local de los interpolantes de Clément y Raviart-Thomas. Posteriormente, para demostrar la eficiencia del estimador se emplean desigualdades inversas, desigualdades de traza discreta, y técnicas de localización basadas en funciones burbuja sobre lados y triángulos. Es importante mencionar que el análisis a posteriori desarrollado en este capítulo es un complemento del Capítulo 2, en donde se realizó un análisis a priori de la misma formulación variacional. El Capítulo 4 está constituido por la siguiente prepublicación:

> Domínguez C., Gatica G.N. and Meddahi S., *A posteriori error analysis of a fully-mixed finite element method for a two-dimensional fluid-solid interaction problem.* Preprint 2014-02, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile, (2014).

Chapter 2

A priori error analysis of a fully-mixed finite element method for a two-dimensional fluid-solid interaction problem

2.1 Introduction

In this paper we focus again on the two-dimensional fluid-solid interaction problem studied recently in [37] (see also [50], which is the first paper concerning the coupling procedure of the fluid-structure interaction problems, and [39] for a version employing boundary integral equation methods). More precisely, we consider an incident acoustic wave upon a bounded elastic body (obstacle) fully surrounded by a fluid, and are interested in determining both the response of the body and the scattered wave. The obstacle is supposed to be a long cylinder parallel to the x_3-axis whose cross-section is Ω_s. The boundary of Ω_s is denoted by Σ. We assume that the incident wave and the volume force acting on the body exhibit a time-harmonic behaviour with $e^{-\imath\omega t}$ ansatz and phasors p_i and $\mathbf{f}$, respectively, so that p_i satisfies the Helmholtz equation in $\mathbb{R}^2\backslash\overline{\Omega_s}$. Hence, since the phenomenon is supposed to be invariant under a translation in the x_3-direction, we may consider a bidimensional interaction problem posed in the frequency domain. In this way, in what follows we let $\boldsymbol{\sigma}_s : \Omega_s \to \mathbb{C}^{2\times 2}$, $\mathbf{u} : \Omega_s \to \mathbb{C}^2$, and $p : \mathbb{R}^2\backslash\overline{\Omega_s} \to \mathbb{C}$ be the amplitudes of the Cauchy stress tensor, the displacement field, and the total (incident + scattered) pressure, respectively, where $\mathbb{C}$ stands for the set of complex numbers.

The fluid is assumed to be perfect, compressible, and homogeneous, with density ρ_f and wave number $\kappa_f := \dfrac{\omega}{v_0}$, where v_0 is the speed of sound in the linearized fluid, whereas the solid is supposed to be isotropic and linearly elastic with density ρ_s and Lamé constants μ and λ. The latter means, in particular, that the corresponding constitutive equation is given by Hooke's law, that is

$$\boldsymbol{\sigma}_s \,=\, \lambda \,\mathrm{tr}\, \boldsymbol{\varepsilon}(\mathbf{u})\, \mathbf{I} \,+\, 2\,\mu\, \boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in} \quad \Omega_s\,,$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\right)$ is the strain tensor of small deformations, ∇ is the gradient tensor, tr denotes the matrix trace, $^{\mathtt{t}}$ stands for the transpose of a matrix, and $\mathbf{I}$ is the identity matrix of $\mathbb{C}^{2\times 2}$. Consequently, under the hypotheses of small oscillations, both in the solid and the fluid, the unknowns $\boldsymbol{\sigma}_s$, $\mathbf{u}$, and p satisfy the elastodynamic and acoustic equations in time-harmonic regime, that is:

$$\begin{aligned} \mathbf{div}\, \boldsymbol{\sigma}_s \,+\, \kappa_s^2\, \mathbf{u} \;&=\; -\,\mathbf{f} \qquad \text{in} \quad \Omega_s\,, \\ \Delta p \,+\, \kappa_f^2\, p \;&=\; 0 \qquad\;\; \text{in} \quad \mathbb{R}^2\backslash\overline{\Omega_s}\,, \end{aligned}$$

where κ_s is defined by $\sqrt{\rho_s}\,\omega$, together with the transmission conditions:

$$\begin{aligned} \boldsymbol{\sigma}_s\, \boldsymbol{\nu} \;&=\; -\,p\,\boldsymbol{\nu} \qquad \text{on} \quad \Sigma\,, \\ \rho_f\, \omega^2\, \mathbf{u}\cdot\boldsymbol{\nu} \;&=\; \frac{\partial p}{\partial \boldsymbol{\nu}} \qquad\; \text{on} \quad \Sigma\,, \end{aligned} \tag{2.1}$$

and the behaviour at infinity given by

$$p - p_i \,=\, O(\mathtt{r}^{-1}) \tag{2.2}$$

and

$$\frac{\partial(p-p_i)}{\partial \mathtt{r}} \,-\, \imath\, \kappa_f\, (p - p_i) \,=\, o(\mathtt{r}^{-1})\,, \tag{2.3}$$

as $\mathtt{r} := \|\mathbf{x}\| \to +\infty$, uniformly for all directions $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$. Hereafter, $\mathbf{div}$ stands for the usual divergence operator div acting on each row of the tensor, $\|\mathbf{x}\|$ is the euclidean norm of a vector $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \mathbb{R}^2$, and $\boldsymbol{\nu}$ denotes the unit outward normal on Σ, that is pointing toward $\mathbb{R}^2\backslash\overline{\Omega_s}$. The transmission conditions given in (2.1) constitute the equilibrium of forces and the equality of the normal displacements of the solid and fluid. In other words, the first equation in (2.1) results from the action of pressure forces exerted by the fluid on the solid, and the second one expresses the continuity of the normal components of the acceleration, in time-harmonic, of the solid and fluid on the interface. In turn, the equation (2.3) is known as the Sommerfeld radiation condition.

Now, it is important to remark that the development of suitable numerical methods for the above described fluid-solid interaction problems has become a subject of increasing interest during the last two decades. Several approaches relying on a primal formulation in the solid, in which the displacement becomes the only unknown in this medium, were originally studied in [14], [52], [53], [54], [55], [65], and [67]. More recently, and in particular motivated by the need of obtaining direct finite element approximations of the stresses, dual-mixed formulations in the solid have begun to be considered as well (see e.g. [37] and [39]). In fact, the model is first simplified in [37] by assuming that the fluid occupies a bounded annular region Ω_f, whence a Robin boundary condition imitating the behavior of the scattered field at infinity is imposed on the exterior boundary of Ω_f, which is located far from the obstacle. Then, the method in [37] employs a dual-mixed variational formulation for plane elasticity in the solid and keeps the usual primal formulation in the linearized fluid region. In addition, the elastodynamic equation is used to eliminate the displacement unknown from the resulting formulation. Furthermore, since one of the transmission conditions becomes essential, it is enforced weakly by means of a Lagrange multiplier. As a consequence, the stress tensor in the solid and the pressure in the fluid, which solves the Helmholtz equation, constitute the main unknowns. Next, a judicious decomposition of the space of stresses renders suitable the application of the Fredholm alternative and the Babuška-Brezzi theory for the analysis of the whole coupled problem. The corresponding discrete scheme is defined with PEERS elements in the obstacle and the traditional first order Lagrange finite elements in the fluid domain. The stability and convergence of this Galerkin method also relies on a stable decomposition of the finite element space used to approximate the stress variable. On the other hand, the strategy from [37] is modified in [39] in such a way that, instead of introducing a Robin condition on the exterior boundary, a non-local absorbing boundary condition based on boundary integral equations is considered there. Consequently, the exterior boundary can be chosen as any parametrizable smooth closed curve containing the solid, which, in order to minimize the size of the computational domain, is adjusted as sharply as possible to the shape of the obstacle. The rest of the analysis for the corresponding continuous and discrete formulations follows very closely the techniques and arguments developed in [37]. We refer to [39] for further details on this modified approach.

The goal of the present paper is to additionally extend the approach from [37] and [39] by employing now dual-mixed formulations in both media. The extension concept refers here to the fact that, instead of using a primal approach in the bounded fluid domain, as in [37] and [39], we now apply in that region the same dual-mixed method that is employed in the solid. In this way, the well-posedness of the formulation that would arise from the additional use of the boundary integral equation method (BIEM) in the unbounded fluid domain, as it was done in [39], will follow straightforwardly from the analyses in that reference and the present paper. By the way, the advantages and disadvantages of using BIEM or not have to do mainly with the computational domain (smaller with BIEM) and the complexity of the resulting Galerkin system (simpler without BIEM). In any case, the above remarks emphasize that, besides $\boldsymbol{\sigma}_s$, from now on we set the additional unknown

$$\boldsymbol{\sigma}_f := \nabla p \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{\Omega_s}\,,$$

so that the Helmholtz equation and the second condition in (2.1) are rewritten, respectively, as

$$\operatorname{div} \boldsymbol{\sigma}_f + \kappa_f^2 \, p = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{\Omega_s}\,, \tag{2.4}$$

and

$$\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} = \rho_f \, \omega^2 \, \mathbf{u} \cdot \boldsymbol{\nu} \quad \text{on} \quad \Sigma\,. \tag{2.5}$$

The introduction of $\boldsymbol{\sigma}_f$ and the resulting equation (2.4) is motivated by the eventual need of obtaining direct and more accurate finite element approximations for the pressure gradient $\boldsymbol{\sigma}_f := \nabla p$ (instead of applying numerical differentiation, with the consequent loss of accuracy, to the approximation of p arising from the usual primal formulation). The above is required, for instance, to solve the inverse problem related to the Helmholtz equation, in which the boundary integral representation of the far field pattern, a crucial variable in an associated iterative algorithm, depends on both the trace of p and the normal trace of $\boldsymbol{\sigma}_f$ (see, e.g. [25, Chapter 2, Theorem 2.5], [29, 30, 31]). To this respect, a $H(\mathrm{div})$-type approximation of $\boldsymbol{\sigma}_f$ is certainly better suited for this purpose. The usefulness of the mixed formulation for the pressure p is also justified by the fact that it is locally mass conservative. Moreover, since both transmission conditions become now essential, they are enforced weakly by using the traces of the displacement and the pressure on the interface as suitable Lagrange multipliers. Hence, the fact that these variables of evident physical interest can also be approximated directly from the associated Galerkin schemes, constitute another important advantage of the fully-mixed approach proposed here. Furthermore, the use of a dual-mixed approach in the solid and the fluid simplify the corresponding computational code since Raviart-Thomas based subspaces can be used in both domains. The rest of this work is organized as follows. In Section 2.2 we redefine the fluid-solid interaction problem on an annular domain $\Omega_f \subseteq \mathbb{R}^2$ (as in [37] and [39]), and derive the associated continuous variational formulation. Then, in Section 2.3 we utilize the Fredholm and Babuška-Brezzi theories to analyze the resulting saddle point problem and provide sufficient conditions for its well-posedness. The corresponding Galerkin scheme is studied in Section 2.4. Finally, some numerical experiments illustrating the theoretical results are reported in Section 2.5.

We end this section with further notations to be used below. Since in the sequel we deal with complex valued functions, we use the symbol $\imath$ for $\sqrt{-1}$, and denote by $\overline{z}$ and $|z|$ the conjugate and modulus, respectively, of each $z \in \mathbb{C}$. Also, given $\boldsymbol{\tau}_s := (\tau_{ij})$, $\boldsymbol{\zeta}_s := (\zeta_{ij}) \in \mathbb{C}^{2\times 2}$, we define the deviator tensor $\boldsymbol{\tau}_s^{\mathrm{d}} := \boldsymbol{\tau}_s - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau}_s)\,\mathbf{I}$, the tensor product $\boldsymbol{\tau}_s : \boldsymbol{\zeta}_s := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$, and the conjugate tensor $\overline{\boldsymbol{\tau}_s} := (\overline{\tau}_{ij})$. In turn, in what follows we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, $\mathcal{S}$ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2\,, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}\,, \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^2\,.$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote H^2 and $H^{2\times 2}$, respectively. In addition, we use $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ to denote the usual duality pairings between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, and between $\mathbf{H}^{-1/2}(\mathcal{S})$ and $\mathbf{H}^{1/2}(\mathcal{S})$. Furthermore, the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \left\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\,\mathbf{w} \in L^2(\mathcal{O}) \right\},$$

is standard in the realm of mixed problems (see [19], [47]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div};\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O})$. Finally, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2.2 The continuous variational formulation

We first observe, as a consequence of (2.2) and (2.3), that the outgoing waves are absorbed by the far field. According to this fact, and in order to obtain a convenient simplification of our model problem, we now proceed similarly as in [37] and introduce a sufficiently large polyhedral surface Γ approximating a sphere centered at the origin, whose interior contains Ω_s. Then, we define Ω_f as the annular region bounded by Σ and Γ, and consider the Robin boundary condition:

$$\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} - \imath\,\kappa_f\,p = g := \nabla p_i \cdot \boldsymbol{\nu} - \imath\,\kappa_f\,p_i \quad \text{on} \quad \Gamma\,,$$

where $\boldsymbol{\nu}$ denotes also the unit outward normal on Γ. Therefore, given $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ and $g \in H^{-1/2}(\Gamma)$, we are now interested in the following fluid-solid interaction problem: Find $\boldsymbol{\sigma}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$, $\boldsymbol{\sigma}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, and $p \in H^1(\Omega_f)$, such that there hold in the distributional sense:

$$
\begin{aligned}
\boldsymbol{\sigma}_s &= \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in} \quad \Omega_s\,,\\
\mathbf{div}\,\boldsymbol{\sigma}_s + \kappa_s^2\,\mathbf{u} &= -\,\mathbf{f} && \text{in} \quad \Omega_s\,,\\
\boldsymbol{\sigma}_f &= \nabla p && \text{in} \quad \Omega_f\,,\\
\operatorname{div}\boldsymbol{\sigma}_f + \kappa_f^2\,p &= 0 && \text{in} \quad \Omega_f\,,\\
\boldsymbol{\sigma}_s\,\boldsymbol{\nu} &= -\,p\,\boldsymbol{\nu} && \text{on} \quad \Sigma\,,\\
\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu} &= \rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu} && \text{on} \quad \Sigma\,,\\
\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu} - \imath\,\kappa_f\,p &= g && \text{on} \quad \Gamma\,,
\end{aligned}
\tag{2.6}
$$

where $\mathcal{C}$ is the elasticity operator given by Hooke's law, that is

$$
\mathcal{C}\,\boldsymbol{\zeta}_s \,:=\, \lambda\,\operatorname{tr}(\boldsymbol{\zeta}_s)\,\mathbf{I} \,+\, 2\,\mu\,\boldsymbol{\zeta}_s \qquad \forall\,\boldsymbol{\zeta}_s \in \mathbb{L}^2(\Omega_s)\,. \tag{2.7}
$$

Note from (2.6) the full symmetry existing between the dual-mixed formulations in the domains and between the transmission conditions on Σ. This fact motivates later on the use of Raviart-Thomas based subspaces in both domains.

It is clear from (2.7) that $\mathcal{C}$ is bounded and invertible and that the operator $\mathcal{C}^{-1}$ reduces to

$$
\mathcal{C}^{-1}\,\boldsymbol{\zeta}_s \,:=\, \frac{1}{2\,\mu}\,\boldsymbol{\zeta}_s \,-\, \frac{\lambda}{4\,\mu\,(\lambda+\mu)}\,\operatorname{tr}(\boldsymbol{\zeta}_s)\,\mathbf{I} \qquad \forall\,\boldsymbol{\zeta}_s \in \mathbb{L}^2(\Omega_s)\,.
$$

In addition, the above identity and simple algebraic manipulations yield

$$
\int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\zeta}_s : \overline{\boldsymbol{\zeta}_s} \,\geq\, \frac{1}{2\mu}\,\|\boldsymbol{\zeta}_s^{\mathtt{d}}\|_{0,\Omega_s}^2 \qquad \forall\,\boldsymbol{\zeta}_s \in \mathbb{L}^2(\Omega_s)\,. \tag{2.8}
$$

We now apply dual-mixed approaches in the solid Ω_s and the fluid Ω_f to derive the fully-mixed variational formulation of (2.6). Indeed, following the usual procedure from linear elasticity (see [4], [37] and [70]), we first introduce the rotation

$$
\boldsymbol{\gamma} \,:=\, \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}) \,\in\, \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)
$$

as a further unknown, where $\mathbb{L}^2_{\mathtt{asym}}(\Omega_s)$ denotes the space of asymmetric tensors with entries in $L^2(\Omega_s)$. According to this, the constitutive equation can be rewritten in the form

$$
\mathcal{C}^{-1}\,\boldsymbol{\sigma}_s \,=\, \boldsymbol{\varepsilon}(\mathbf{u}) \,=\, \nabla\mathbf{u} \,-\, \boldsymbol{\gamma}\,,
$$

which, multiplying by a function $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$ and integrating by parts, yields

$$
\int_{\Omega_s} \mathcal{C}^{-1}\,\boldsymbol{\sigma}_s : \boldsymbol{\tau}_s + \int_{\Omega_s} \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \,-\, \langle \boldsymbol{\tau}_s\,\boldsymbol{\nu}, \mathbf{u}\rangle_\Sigma + \int_{\Omega_s} \boldsymbol{\tau}_s : \boldsymbol{\gamma} \,=\, 0\,. \tag{2.9}
$$

At this point we remark that, given $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, $\boldsymbol{\tau}_s\boldsymbol{\nu}|_\Sigma$ is the functional in $\mathbf{H}^{-1/2}(\Sigma)$ defined as

$$\langle\boldsymbol{\tau}_s\boldsymbol{\nu},\boldsymbol{\varphi}\rangle_\Sigma := \int_{\Omega_s}\boldsymbol{\tau}_s : \nabla\mathbf{w} + \int_{\Omega_s}\mathbf{w}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \qquad \forall\,\boldsymbol{\varphi}\in\mathbf{H}^{1/2}(\Sigma)\,,$$

where $\mathbf{w}$ is any function in $\mathbf{H}^1(\Omega_s)$ such that $\mathbf{w}=\boldsymbol{\varphi}$ on Σ and $\mathbf{w}=\mathbf{0}$ on Γ. Then, using the elastodynamic equation (cf. second equation of (2.6)) to eliminate $\mathbf{u}$ in Ω_s, and introducing the additional unknown

$$\boldsymbol{\varphi}_s := \mathbf{u}|_\Sigma \in \mathbf{H}^{1/2}(\Sigma)\,, \tag{2.10}$$

we find that (2.9) becomes

$$\int_{\Omega_s}\mathcal{C}^{-1}\boldsymbol{\sigma}_s:\boldsymbol{\tau}_s - \frac{1}{\kappa_s^2}\int_{\Omega_s}\mathbf{div}\,\boldsymbol{\sigma}_s\cdot\mathbf{div}\,\boldsymbol{\tau}_s - \langle\boldsymbol{\tau}_s\boldsymbol{\nu},\boldsymbol{\varphi}_s\rangle_\Sigma + \int_{\Omega_s}\boldsymbol{\tau}_s:\boldsymbol{\gamma} = \frac{1}{\kappa_s^2}\int_{\Omega_s}\mathbf{f}\cdot\mathbf{div}\,\boldsymbol{\tau}_s\,. \tag{2.11}$$

Similarly, multiplying the constitutive equation $\boldsymbol{\sigma}_f = \nabla p$ in Ω_f by $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, integrating by parts, noting that the normal vector points inward Ω_f on Σ, replacing from the Helmholtz equation $p = -\frac{1}{\kappa_f^2}\operatorname{div}\boldsymbol{\sigma}_f$ in Ω_f, and introducing the auxiliary unknown

$$\boldsymbol{\varphi}_f = (\varphi_\Sigma,\varphi_\Gamma) := (p|_\Sigma, p|_\Gamma) \in H^{1/2}(\Sigma)\times H^{1/2}(\Gamma)\,, \tag{2.12}$$

we arrive at

$$\int_{\Omega_f}\boldsymbol{\sigma}_f\cdot\boldsymbol{\tau}_f - \frac{1}{\kappa_f^2}\int_{\Omega_f}\operatorname{div}\boldsymbol{\sigma}_f\operatorname{div}\boldsymbol{\tau}_f + \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\varphi_\Sigma\rangle_\Sigma - \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\varphi_\Gamma\rangle_\Gamma = 0\,. \tag{2.13}$$

Finally, the symmetry of $\boldsymbol{\sigma}_s$, the transmission conditions on Σ, and the Robin boundary condition on Γ are imposed weakly through the relations:

$$\begin{aligned}
\int_{\Omega_s}\boldsymbol{\sigma}_s:\boldsymbol{\eta} &= 0 && \forall\,\boldsymbol{\eta}\in\mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\,,\\
-\langle\boldsymbol{\sigma}_s\boldsymbol{\nu},\boldsymbol{\psi}_s\rangle_\Sigma - \langle\varphi_\Sigma\boldsymbol{\nu},\boldsymbol{\psi}_s\rangle_\Sigma &= 0 && \forall\,\boldsymbol{\psi}_s\in\mathbf{H}^{1/2}(\Sigma)\,,\\
\langle\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu},\psi_\Sigma\rangle_\Sigma - \rho_f\,\omega^2\langle\psi_\Sigma\boldsymbol{\nu},\boldsymbol{\varphi}_s\rangle_\Sigma &= 0 && \forall\,\psi_\Sigma\in H^{1/2}(\Sigma)\,,\\
-\langle\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu},\psi_\Gamma\rangle_\Gamma + \imath\,\kappa_f\langle\varphi_\Gamma,\psi_\Gamma\rangle_\Gamma &= -\langle g,\psi_\Gamma\rangle_\Gamma && \forall\,\psi_\Gamma\in H^{1/2}(\Gamma)\,,
\end{aligned} \tag{2.14}$$

where the traces of $\mathbf{u}$ and p have been replaced by the new unknowns introduced in (2.10) and (2.12), the expression $\langle\boldsymbol{\varphi}_s\cdot\boldsymbol{\nu},\psi_\Sigma\rangle_\Sigma$ in the second transmission condition has been rewritten as $\langle\psi_\Sigma\boldsymbol{\nu},\boldsymbol{\varphi}_s\rangle_\Sigma$, and the signs of the first transmission condition and the Robin boundary condition have been changed for convenience. Note that $\boldsymbol{\varphi}_s$ and $\boldsymbol{\varphi}_f$ constitute precisely the Lagrange multipliers associated with the transmission and Robin boundary conditions.

Throughout the rest of the paper we make the identification $H^t(\partial\Omega_f) \equiv H^t(\Sigma)\times H^t(\Gamma)$ for each $t\in\mathbb{R}$, with the norm $\|\boldsymbol{\psi}_f\|_{t,\partial\Omega_f} := \|\psi_\Sigma\|_{t,\Sigma} + \|\psi_\Gamma\|_{t,\Gamma}$ for each $\boldsymbol{\psi}_f := (\psi_\Sigma,\psi_\Gamma)\in H^t(\partial\Omega_f)$.

Therefore, adding (2.11), (2.13), and (2.14), and defining the spaces

$$\mathbf{H} := \mathbb{H}(\mathbf{div};\Omega_s) \times \mathbf{H}(\mathrm{div};\Omega_f) \quad \text{and} \quad \mathbf{Q} := \mathbb{L}^2_{\mathtt{asym}}(\Omega_s) \times \mathbf{H}^{1/2}(\Sigma) \times H^{1/2}(\partial\Omega_f)\,,$$

we arrive at the following fully-mixed variational formulation of (2.6): Find $\widehat{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_f) \in \mathbf{H}$ and $\widehat{\boldsymbol{\gamma}} := (\boldsymbol{\gamma}, \boldsymbol{\varphi}_s, \boldsymbol{\varphi}_f) \in \mathbf{Q}$ such that

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\tau}}) + B(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\gamma}}) &= F(\widehat{\boldsymbol{\tau}}) \qquad \forall\, \widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}_s,\boldsymbol{\tau}_f) \in \mathbf{H}\,, \\ B(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\eta}}) + K(\widehat{\boldsymbol{\gamma}},\widehat{\boldsymbol{\eta}}) &= G(\widehat{\boldsymbol{\eta}}) \qquad \forall\, \widehat{\boldsymbol{\eta}} := (\boldsymbol{\eta},\boldsymbol{\psi}_s,\boldsymbol{\psi}_f) \in \mathbf{Q}\,, \end{aligned} \tag{2.15}$$

where $F : \mathbf{H} \to \mathbb{C}$ and $G : \mathbf{Q} \to \mathbb{C}$ are the lineal functionals

$$F(\widehat{\boldsymbol{\tau}}) := \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{f}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \quad \forall\, \widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}_s,\boldsymbol{\tau}_f) \in \mathbf{H}\,,$$

$$G(\widehat{\boldsymbol{\eta}}) := -\langle g\,,\,\psi_\Gamma\rangle_\Gamma \quad \forall\, \widehat{\boldsymbol{\eta}} := (\boldsymbol{\eta},\boldsymbol{\psi}_s,\boldsymbol{\psi}_f) := (\boldsymbol{\eta},\boldsymbol{\psi}_s,(\psi_\Sigma,\psi_\Gamma)) \in \mathbf{Q}\,,$$

and $A : \mathbf{H}\times\mathbf{H} \to \mathbb{C}$, $B : \mathbf{H}\times\mathbf{Q} \to \mathbb{C}$, and $K : \mathbf{Q}\times\mathbf{Q} \to \mathbb{C}$ are the bilinear forms defined by

$$\begin{gathered} A(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) := \int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\zeta}_s : \boldsymbol{\tau}_s - \frac{1}{\kappa_s^2}\int_{\Omega_s}\mathbf{div}\,\boldsymbol{\zeta}_s\cdot\mathbf{div}\,\boldsymbol{\tau}_s + \int_{\Omega_f}\boldsymbol{\zeta}_f\cdot\boldsymbol{\tau}_f - \frac{1}{\kappa_f^2}\int_{\Omega_f}\mathrm{div}\,\boldsymbol{\zeta}_f\,\mathrm{div}\,\boldsymbol{\tau}_f \\ \forall\,(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) := ((\boldsymbol{\zeta}_s,\boldsymbol{\zeta}_f),(\boldsymbol{\tau}_s,\boldsymbol{\tau}_f)) \in \mathbf{H}\times\mathbf{H}\,, \end{gathered} \tag{2.16}$$

$$B(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}}) := B_s(\boldsymbol{\tau}_s,(\boldsymbol{\eta},\boldsymbol{\psi}_s)) + B_f(\boldsymbol{\tau}_f,\boldsymbol{\psi}_f) \qquad \forall\,(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}}) := ((\boldsymbol{\tau}_s,\boldsymbol{\tau}_f),(\boldsymbol{\eta},\boldsymbol{\psi}_s,\boldsymbol{\psi}_f)) \in \mathbf{H}\times\mathbf{Q}\,, \tag{2.17}$$

with

$$B_s(\boldsymbol{\tau}_s,(\boldsymbol{\eta},\boldsymbol{\psi}_s)) := \int_{\Omega_s}\boldsymbol{\tau}_s : \boldsymbol{\eta} - \langle\boldsymbol{\tau}_s\boldsymbol{\nu},\boldsymbol{\psi}_s\rangle_\Sigma\,, \tag{2.18}$$

$$B_f(\boldsymbol{\tau}_f,\boldsymbol{\psi}_f) := \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\psi_\Sigma\rangle_\Sigma - \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\psi_\Gamma\rangle_\Gamma\,, \tag{2.19}$$

and

$$\begin{gathered} K(\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{\eta}}) := -\langle\xi_\Sigma\,\boldsymbol{\nu},\boldsymbol{\psi}_s\rangle_\Sigma - \rho_f\,\omega^2\,\langle\psi_\Sigma\,\boldsymbol{\nu},\boldsymbol{\xi}_s\rangle_\Sigma + \imath\,\kappa_f\,\langle\xi_\Gamma,\psi_\Gamma\rangle_\Gamma \\ \forall\,\widehat{\boldsymbol{\chi}} := (\boldsymbol{\chi},\boldsymbol{\xi}_s,\boldsymbol{\xi}_f) := (\boldsymbol{\chi},\boldsymbol{\xi}_s,(\xi_\Sigma,\xi_\Gamma)) \in \mathbf{Q}\,, \\ \forall\,\widehat{\boldsymbol{\eta}} := (\boldsymbol{\eta},\boldsymbol{\psi}_s,\boldsymbol{\psi}_f) := (\boldsymbol{\eta},\boldsymbol{\psi}_s,(\psi_\Sigma,\psi_\Gamma)) \in \mathbf{Q}\,. \end{gathered} \tag{2.20}$$

It is straightforward to see, applying the Cauchy-Schwarz inequality, the duality pairings $\langle\cdot,\cdot\rangle_\Sigma$ and $\langle\cdot,\cdot\rangle_\Gamma$, and the usual trace theorems in $\mathbb{H}(\mathbf{div};\Omega_s)$ and $\mathbf{H}(\mathrm{div};\Omega_f)$, that F, G, A, B, B_s, B_f, and K are all bounded with constants depending on κ_s, μ, κ_f, ρ_f, and ω.

2.3 Analysis of the continuous variational formulation

In this section we proceed analogously to [37] and employ suitable decompositions of $\mathbb{H}(\mathbf{div};\Omega_s)$ and $\mathbf{H}(\mathrm{div};\Omega_f)$ to show that (2.15) becomes a compact perturbation of a well-posed problem. To this end, we now need to introduce two projectors defined in terms of auxiliary Neumann boundary value problems posed in Ω_s and Ω_f, respectively.

2.3.1 The associated projectors

We begin by recalling from the analysis in [37, Section 4.1] the definition of the projector in Ω_s. In fact, let us first denote by $\mathbb{RM}(\Omega_s)$ the space of rigid body motions in Ω_s, that is

$$\mathbb{RM}(\Omega_s) := \left\{ \mathbf{v} : \Omega_s \to \mathbb{C}^2 : \quad \mathbf{v}(\mathbf{x}) = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \quad \forall\, \mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \Omega_s,\ a, b, c \in \mathbb{C} \right\},$$

and let $\mathbf{M} : \mathbf{L}^2(\Omega_s) \to \mathbb{RM}(\Omega_s)$ be the associated orthogonal projector. Then, given $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, we consider the boundary value problem

$$\begin{gathered} \tilde{\boldsymbol{\sigma}}_s = \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) \quad \text{in} \quad \Omega_s, \quad \mathbf{div}\,\tilde{\boldsymbol{\sigma}}_s = (\mathbf{I}-\mathbf{M})\big(\mathbf{div}\,\boldsymbol{\tau}_s\big) \quad \text{in} \quad \Omega_s, \\ \tilde{\boldsymbol{\sigma}}_s\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Sigma, \quad \tilde{\mathbf{u}} \in (\mathbf{I}-\mathbf{M})(\mathbf{L}^2(\Omega_s)), \end{gathered} \tag{2.21}$$

where $\mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ is defined according to (2.7). Hereafter, $\mathbf{I}$ denotes also a generic identity operator. Note that the application of the operator $\mathbf{I}-\mathbf{M}$ on the right hand side of the equilibrium equation is needed to guarantee the usual compatibility condition for the Neumann problem (2.21) (cf. [18, Theorem 9.2.30]), and that the orthogonality condition on $\tilde{\mathbf{u}}$ is required for uniqueness. Indeed, it is well known (see, e.g. [38, Section 3, Theorem 3.1]) that (2.21) is well-posed. In addition, owing to the regularity result for the elasticity problem with Neumann boundary conditions (see, e.g. [48], [49]), we know that $(\tilde{\boldsymbol{\sigma}}_s, \tilde{\mathbf{u}}) \in \mathbb{H}^{\epsilon}(\Omega_s) \times \mathbf{H}^{1+\epsilon}(\Omega_s)$, for some $\epsilon > 0$, and there holds

$$\|\tilde{\boldsymbol{\sigma}}_s\|_{\epsilon,\Omega_s} + \|\tilde{\mathbf{u}}\|_{1+\epsilon,\Omega_s} \le C\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}. \tag{2.22}$$

We now introduce the linear operator $\mathbf{P}_s : \mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{H}(\mathbf{div};\Omega_s)$ defined by

$$\mathbf{P}_s(\boldsymbol{\tau}_s) := \tilde{\boldsymbol{\sigma}}_s \qquad \forall\, \boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s), \tag{2.23}$$

where $\tilde{\boldsymbol{\sigma}}_s := \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ and $\tilde{\mathbf{u}}$ is the unique solution of (2.21). It is clear from (2.21) that

$$\mathbf{P}_s(\boldsymbol{\tau}_s)^{\mathtt{t}} = \mathbf{P}_s(\boldsymbol{\tau}_s) \quad \text{in} \quad \Omega_s, \quad \mathbf{div}\,\mathbf{P}_s(\boldsymbol{\tau}_s) = (\mathbf{I}-\mathbf{M})\big(\mathbf{div}\,\boldsymbol{\tau}_s\big) \quad \text{in} \quad \Omega_s, \tag{2.24}$$

and

$$\mathbf{P}_s(\boldsymbol{\tau}_s)\,\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Sigma. \tag{2.25}$$

Then, the continuous dependence result for (2.21) gives

$$\|\mathbf{P}_s(\boldsymbol{\tau}_s)\|_{\mathbf{div};\Omega_s} \,\le\, C\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s} \qquad \forall\,\boldsymbol{\tau}_s \,\in\, \mathbb{H}(\mathbf{div};\Omega_s)\,,$$

which shows that $\mathbf{P}_s$ is bounded. Moreover, it is easy to see from (2.21), (2.23), (2.24), and (2.25) that $\mathbf{P}_s$ is actually a projector, and hence there holds

$$\mathbb{H}(\mathbf{div};\Omega_s) \;=\; \mathbf{P}_s(\mathbb{H}(\mathbf{div};\Omega_s)) \,\oplus\, (\mathbf{I}-\mathbf{P}_s)(\mathbb{H}(\mathbf{div};\Omega_s))\,. \tag{2.26}$$

Finally, it is clear from (2.22) that $\mathbf{P}_s(\boldsymbol{\tau}_s) \,\in\, \mathbb{H}^\epsilon(\Omega_s)$ and

$$\|\mathbf{P}_s(\boldsymbol{\tau}_s)\|_{\epsilon,\Omega_s} \,\le\, C\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s} \qquad \forall\,\boldsymbol{\tau}_s \,\in\, \mathbb{H}(\mathbf{div};\Omega_s)\,. \tag{2.27}$$

We proceed analogously for the domain Ω_f. In fact, let $P_0(\Omega_f)$ be the space of constant polynomials on Ω_f, and let $\mathbf{J} : L^2(\Omega_f) \to P_0(\Omega_f)$ be the corresponding orthogonal projector. Then, given $\boldsymbol{\tau}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f)$, we consider the Neumann boundary value problem

$$\begin{gathered} \tilde{\boldsymbol{\sigma}}_f \,=\, \nabla\tilde{p} \quad \text{in} \quad \Omega_f\,, \quad \mathrm{div}\,\tilde{\boldsymbol{\sigma}}_f \,=\, (\mathbf{I}-\mathbf{J})\big(\mathrm{div}\,\boldsymbol{\tau}_f\big) \quad \text{in} \quad \Omega_f\,, \\ \tilde{\boldsymbol{\sigma}}_f\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Sigma\cup\Gamma\,, \quad \tilde{p} \,\in\, (\mathbf{I}-\mathbf{J})(L^2(\Omega_f))\,. \end{gathered} \tag{2.28}$$

Analogue remarks to those given for the compatibility condition and uniqueness of solution of (2.21) are valid here with $\mathbf{J}$ instead of $\mathbf{M}$. In addition, it is not difficult to see that (2.28) is well-posed as well. Furthermore, the classical regularity result for the Poisson problem with Neumann boundary conditions (see, e.g. [48], [49]) implies that $(\tilde{\boldsymbol{\sigma}}_f,\tilde{p}) \in \mathbf{H}^\epsilon(\Omega_f)\times H^{1+\epsilon}(\Omega_f)$, for some $\epsilon > 0$ (parameter that can be assumed, from now on, to be the same of (2.22)), and that

$$\|\tilde{\boldsymbol{\sigma}}_f\|_{\epsilon,\Omega_f} \,+\, \|\tilde{p}\|_{1+\epsilon,\Omega_f} \,\le\, C\,\|\mathrm{div}\,\boldsymbol{\tau}_f\|_{0,\Omega_f}\,. \tag{2.29}$$

We now define the linear operator $\mathbf{P}_f : \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{H}(\mathrm{div};\Omega_f)$ by

$$\mathbf{P}_f(\boldsymbol{\tau}_f) \,:=\, \tilde{\boldsymbol{\sigma}}_f \qquad \forall\,\boldsymbol{\tau}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f)\,, \tag{2.30}$$

where $\tilde{\boldsymbol{\sigma}}_f \,:=\, \nabla\tilde{p}$ and $\tilde{p}$ is the unique solution of (2.28). It follows that

$$\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_f) \,=\, (\mathbf{I}-\mathbf{J})\big(\mathrm{div}\,\boldsymbol{\tau}_f\big) \quad \text{in} \quad \Omega_f \quad \text{and} \quad \mathbf{P}_f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Sigma\cup\Gamma\,. \tag{2.31}$$

In addition, thanks to the continuous dependence result for (2.28), there holds

$$\|\mathbf{P}_f(\boldsymbol{\tau}_f)\|_{\mathrm{div};\Omega_f} \,\le\, C\,\|\mathrm{div}\,\boldsymbol{\tau}_f\|_{0,\Omega_f} \qquad \forall\,\boldsymbol{\tau}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f)\,,$$

which shows that $\mathbf{P}_f$ is bounded. Furthermore, it is straightforward from (2.28), (2.30), and (2.31) that $\mathbf{P}_f$ is a projector, and therefore

$$\mathbf{H}(\mathrm{div};\Omega_f) \,=\, \mathbf{P}_f\left(\mathbf{H}(\mathrm{div};\Omega_f)\right) \,\oplus\, (\mathbf{I}-\mathbf{P}_f)\left(\mathbf{H}(\mathrm{div};\Omega_f)\right)\,. \tag{2.32}$$

Also, it is clear from (2.29) that $\mathbf{P}_f(\boldsymbol{\tau}_f) \,\in\, \mathbf{H}^\epsilon(\Omega_f)$ and

$$\|\mathbf{P}_f(\boldsymbol{\tau}_f)\|_{\epsilon,\Omega_f} \,\le\, C\,\|\mathrm{div}\,\boldsymbol{\tau}_f\|_{0,\Omega_f} \qquad \forall\,\boldsymbol{\tau}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f)\,. \tag{2.33}$$

2.3.2 Decomposition of the bilinear form A

We begin the analysis by introducing the bilinear forms $A_s^+ : \mathbb{H}(\mathbf{div};\Omega_s)\times\mathbb{H}(\mathbf{div};\Omega_s)\to\mathbb{C}$ and $A_f^+ : \mathbf{H}(\mathrm{div};\Omega_f)\times\mathbf{H}(\mathrm{div};\Omega_f)\to\mathbb{C}$ given by

$$A_s^+(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s) := \int_{\Omega_s}\mathcal{C}^{-1}\boldsymbol{\zeta}_s:\boldsymbol{\tau}_s + \frac{1}{\kappa_s^2}\int_{\Omega_s}\mathbf{div}\,\boldsymbol{\zeta}_s\cdot\mathbf{div}\,\boldsymbol{\tau}_s \qquad \forall\,\boldsymbol{\zeta}_s,\,\boldsymbol{\tau}_s\in\mathbb{H}(\mathbf{div};\Omega_s)\,, \tag{2.34}$$

and

$$A_f^+(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f) := \int_{\Omega_f}\boldsymbol{\zeta}_f:\boldsymbol{\tau}_f + \frac{1}{\kappa_f^2}\int_{\Omega_f}\mathrm{div}\,\boldsymbol{\zeta}_f\cdot\mathrm{div}\,\boldsymbol{\tau}_f \qquad \forall\,\boldsymbol{\zeta}_f,\,\boldsymbol{\tau}_f\in\mathbf{H}(\mathrm{div};\Omega_f)\,, \tag{2.35}$$

which are clearly bounded, symmetric, and positive semi-definite. Actually, it is straightforward to see from (2.35) that A_f^+ is $\mathbf{H}(\mathrm{div};\Omega_f)$-elliptic, that is there exists $\alpha_f^+ := \min\left\{1,\frac{1}{\kappa_f^2}\right\} > 0$ such that

$$A_f^+(\boldsymbol{\tau}_f,\overline{\boldsymbol{\tau}}_f) \;\geq\; \alpha_f^+\,\|\boldsymbol{\tau}_f\|^2_{\mathrm{div};\Omega_f} \qquad \forall\,\boldsymbol{\tau}_f\in\mathbf{H}(\mathrm{div};\Omega_f)\,, \tag{2.36}$$

and we show below in Section 2.3.3 that A_s^+ is also elliptic but on a subspace of $\mathbb{H}(\mathbf{div};\Omega_s)$.

In what follows, we employ the decompositions (2.26) and (2.32) to reformulate (2.15) in a more suitable form. More precisely, the unknown $\widehat{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_s,\boldsymbol{\sigma}_f)$ and the corresponding test function $\widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}_s,\boldsymbol{\tau}_f)$, both in $\mathbf{H}$, are replaced, respectively, by the expressions

$$\boldsymbol{\sigma}_s = \mathbf{P}_s(\boldsymbol{\sigma}_s) + (\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\sigma}_s)\,,\quad \boldsymbol{\sigma}_f = \mathbf{P}_f(\boldsymbol{\sigma}_f) + (\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\sigma}_f) \tag{2.37}$$

and

$$\boldsymbol{\tau}_s = \mathbf{P}_s(\boldsymbol{\tau}_s) + (\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\tau}_s)\,,\quad \boldsymbol{\tau}_f = \mathbf{P}_f(\boldsymbol{\tau}_f) + (\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\tau}_f)\,. \tag{2.38}$$

To this respect, we observe, according to (2.24), (2.25), and the fact that $\nabla\mathbf{v}\in\mathbb{L}^2_{\mathtt{asym}}(\Omega_s)$ for all $\mathbf{v}\in\mathbb{RM}(\Omega_s)$, that for all $\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s\in\mathbb{H}(\mathbf{div};\Omega_s)$, there holds

$$\begin{aligned}\int_{\Omega_s}\mathbf{div}(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s)\cdot\mathbf{div}\,\mathbf{P}_s(\boldsymbol{\tau}_s) \;&=\; \int_{\Omega_s}\mathbf{M}(\mathbf{div}\,\boldsymbol{\zeta}_s)\cdot\mathbf{div}\,\mathbf{P}_s(\boldsymbol{\tau}_s)\\ =\quad -\int_{\Omega_s}\nabla\mathbf{M}(\mathbf{div}\,\boldsymbol{\zeta}_s):\mathbf{P}_s(\boldsymbol{\tau}_s) \;&+\; \langle\,\mathbf{P}_s(\boldsymbol{\tau}_s)\boldsymbol{\nu}\,,\,\mathbf{M}(\mathbf{div}\,\boldsymbol{\zeta}_s)\,\rangle_\Sigma = 0\,.\end{aligned} \tag{2.39}$$

Analogously, according to (2.31), we deduce that for all $\boldsymbol{\zeta}_f,\,\boldsymbol{\tau}_f\in\mathbf{H}(\mathrm{div};\Omega_f)$, there holds

$$\begin{aligned}\int_{\Omega_f}\mathrm{div}\big(\mathbf{I}-\mathbf{P}_f\big)(\boldsymbol{\zeta}_f)\,\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_f) \;&=\; \mathbf{J}(\mathrm{div}\,\boldsymbol{\zeta}_f)\int_{\Omega_f}\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_f)\\ =\; \mathbf{J}(\mathrm{div}\,\boldsymbol{\zeta}_f)\Big\{\langle\mathbf{P}_f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu},1\rangle_\Gamma \;&-\; \langle\mathbf{P}_f(\boldsymbol{\tau}_f)\cdot\boldsymbol{\nu},1\rangle_\Sigma\Big\} \;=\; 0\,.\end{aligned} \tag{2.40}$$

Hence, using the decompositions (2.26) and (2.32), and the identities (2.39) and (2.40), and adding and substracting suitable terms, we find that A (cf. (2.16)) can be decomposed as

$$A(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) \;=\; A_0(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) + K_0(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) \qquad \forall\,(\widehat{\boldsymbol{\zeta}},\widehat{\boldsymbol{\tau}}) := ((\boldsymbol{\zeta}_s,\boldsymbol{\zeta}_f),(\boldsymbol{\tau}_s,\boldsymbol{\tau}_f))\in\mathbf{H}\times\mathbf{H}\,,$$

where $A_0 : \mathbf{H} \times \mathbf{H} \to \mathbb{C}$ and $K_0 : \mathbf{H} \times \mathbf{H} \to \mathbb{C}$ are given by

$$A_0(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) \,=\, A_s(\boldsymbol{\zeta}_s, \boldsymbol{\tau}_s) \,+\, A_f(\boldsymbol{\zeta}_f, \boldsymbol{\tau}_f)\,, \tag{2.41}$$

and

$$K_0(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) \,=\, K_s(\boldsymbol{\zeta}_s, \boldsymbol{\tau}_s) \,+\, K_f(\boldsymbol{\zeta}_f, \boldsymbol{\tau}_f)\,, \tag{2.42}$$

with the bilinear forms $A_s : \mathbb{H}(\mathbf{div}; \Omega_s) \times \mathbb{H}(\mathbf{div}; \Omega_s) \to \mathbb{C}$, $A_f : \mathbf{H}(\mathrm{div}; \Omega_f) \times \mathbf{H}(\mathrm{div}; \Omega_f) \to \mathbb{C}$, $K_s : \mathbb{H}(\mathbf{div}; \Omega_s) \times \mathbb{H}(\mathbf{div}; \Omega_s) \to \mathbb{C}$, and $K_f : \mathbf{H}(\mathrm{div}; \Omega_f) \times \mathbf{H}(\mathrm{div}; \Omega_f) \to \mathbb{C}$ defined by

$$A_s\left(\boldsymbol{\zeta}_s, \boldsymbol{\tau}_s)\right) \,:=\, -\,A_s^{+}(\mathbf{P}_s(\boldsymbol{\zeta}_s), \mathbf{P}_s(\boldsymbol{\tau}_s)) \,+\, A_s^{+}((\mathbf{I} - \mathbf{P}_s)(\boldsymbol{\zeta}_s), (\mathbf{I} - \mathbf{P}_s)(\boldsymbol{\tau}_s))\,, \tag{2.43}$$

$$A_f\left(\boldsymbol{\zeta}_f, \boldsymbol{\tau}_f\right) \,:=\, -\,A_f^{+}(\mathbf{P}_f(\boldsymbol{\zeta}_f), \mathbf{P}_f(\boldsymbol{\tau}_f)) \,+\, A_f^{+}((\mathbf{I} - \mathbf{P}_f)(\boldsymbol{\zeta}_f), (\mathbf{I} - \mathbf{P}_f)(\boldsymbol{\tau}_f))\,, \tag{2.44}$$

$$\begin{aligned} K_s\left(\boldsymbol{\zeta}_s, \boldsymbol{\tau}_s)\right) \,:=\, & 2\int_{\Omega_s} \mathcal{C}^{-1}\mathbf{P}_s(\boldsymbol{\zeta}_s) : \mathbf{P}_s(\boldsymbol{\tau}_s) \;+\; \int_{\Omega_s} \mathcal{C}^{-1}\mathbf{P}_s(\boldsymbol{\zeta}_s) : (\mathbf{I} - \mathbf{P}_s)(\boldsymbol{\tau}_s) \\ + \int_{\Omega_s} \mathcal{C}^{-1}(\mathbf{I} - \mathbf{P}_s)(\boldsymbol{\zeta}_s) : \mathbf{P}_s(\boldsymbol{\tau}_s) \;-\; & \left(1 + \frac{1}{\kappa_s^2}\right) \int_{\Omega_s} \mathbf{div}(\mathbf{I} - \mathbf{P}_s)(\boldsymbol{\zeta}_s) \cdot \mathbf{div}(\mathbf{I} - \mathbf{P}_s)(\boldsymbol{\tau}_s)\;\;, \end{aligned} \tag{2.45}$$

and

$$\begin{aligned} K_f\left(\boldsymbol{\zeta}_f, \boldsymbol{\tau}_f\right) \,:=\, & 2\int_{\Omega_f} \mathbf{P}_f(\boldsymbol{\zeta}_f) \cdot \mathbf{P}_f(\boldsymbol{\tau}_f) + \int_{\Omega_f} \mathbf{P}_f(\boldsymbol{\zeta}_f) \cdot (\mathbf{I} - \mathbf{P}_f)(\boldsymbol{\tau}_f) \\ + \int_{\Omega_f} (\mathbf{I} - \mathbf{P}_f)(\boldsymbol{\zeta}_f) \cdot \mathbf{P}_f(\boldsymbol{\tau}_f) \;-\; & \left(1 + \frac{1}{\kappa_f^2}\right) \int_{\Omega_f} \mathrm{div}(\mathbf{I} - \mathbf{P}_f)(\boldsymbol{\zeta}_f) \cdot \mathrm{div}(\mathbf{I} - \mathbf{P}_f)(\boldsymbol{\tau}_f)\,. \end{aligned} \tag{2.46}$$

Next, we let $\mathbf{A}_0 : \mathbf{H} \to \mathbf{H}$, $\mathbf{K}_0 : \mathbf{H} \to \mathbf{H}$, $\mathbf{B} : \mathbf{H} \to \mathbf{Q}$ and $\mathbf{K} : \mathbf{Q} \to \mathbf{Q}$ be the linear and bounded operators induced by the bilinear forms (2.41), (2.42), (2.17), and (2.20), respectively. In addition, we let $\mathbf{B}^* : \mathbf{Q} \to \mathbf{H}$ be the adjoint of $\mathbf{B}$, and denote by $\mathbf{F}$ and $\mathbf{G}$ the Riesz representants of the functionals F and G. Hence, using these notations and taking into account the decompositions (2.37) and (2.38), the fully-mixed variational formulation (2.15) can be rewritten as the following operator equation: Find $(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \widehat{\gamma} \end{pmatrix} \;+\; \begin{pmatrix} \mathbf{K}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \widehat{\gamma} \end{pmatrix} \;=\; \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}. \tag{2.47}$$

Moreover, it is quite straightforward from the definitions of A_0 (cf. (2.41)) and B (cf. (2.17)) that (up to a permutation of rows) there holds

$$\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \widehat{\gamma} \end{pmatrix} = \left(\begin{array}{cc|cc} \mathbf{A}_s & \mathbf{B}_s^* & \mathbf{0} & \\ \mathbf{B}_s & \mathbf{0} & & \\ \hline \mathbf{0} & & \mathbf{A}_f & \mathbf{B}_f^* \\ & & \mathbf{B}_f & \mathbf{0} \end{array}\right) \left(\begin{array}{c} \boldsymbol{\sigma}_s \\ (\gamma, \boldsymbol{\varphi}_s) \\ \hline \boldsymbol{\sigma}_f \\ \boldsymbol{\varphi}_f \end{array}\right), \tag{2.48}$$

where $\mathbf{A}_s : \mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{H}(\mathbf{div};\Omega_s)$, $\mathbf{B}_s : \mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\times\mathbf{H}^{1/2}(\Sigma)$, $\mathbf{A}_f : \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{H}(\mathrm{div};\Omega_f)$, and $\mathbf{B}_f : \mathbf{H}(\mathrm{div};\Omega_f) \to H^{1/2}(\partial\Omega_f)$ are the bounded linear operators induced by A_s, B_s, A_f, and B_f, respectively.

In the following section we show that the matrix operators on the left hand side of (2.47) become bijective and compact, respectively. In particular, concerning the bijectivity issue, and because of the block-diagonal saddle point structure shown by the right-hand side of (2.48), it suffices to apply the well known Babuška-Brezzi theory independently to each one of the two blocks arising there.

2.3.3 Application of the Babuška-Brezzi and Fredholm theories

We begin with the continuous inf-sup conditions for the bilinear forms B_s and B_f, which are equivalent to the surjectivity of $\mathbf{B}_s$ and $\mathbf{B}_f$, respectively. For this purpose, we first notice from (2.18) and (2.19) that these operators are given by

$$\mathbf{B}_s(\boldsymbol{\tau}_s) \,:=\, \left(\frac{1}{2}\big(\boldsymbol{\tau}_s - \boldsymbol{\tau}_s^{\mathrm{t}}\big), -\mathcal{R}_s(\boldsymbol{\tau}_s\,\boldsymbol{\nu})\right) \qquad \forall\,\boldsymbol{\tau}_s \,\in\, \mathbb{H}(\mathbf{div};\Omega_s)\,, \tag{2.49}$$

and

$$\mathbf{B}_f(\boldsymbol{\tau}_f) \,:=\, (\mathcal{R}_\Sigma(\boldsymbol{\tau}_f\cdot\boldsymbol{\nu}), -\mathcal{R}_\Gamma(\boldsymbol{\tau}_f\cdot\boldsymbol{\nu})) \qquad \forall\,\boldsymbol{\tau}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f)\,, \tag{2.50}$$

where $\mathcal{R}_s : \mathbf{H}^{-1/2}(\Sigma) \to \mathbf{H}^{1/2}(\Sigma)$, $\mathcal{R}_\Sigma : H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)$, and $\mathcal{R}_\Gamma : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, are the respective Riesz operators. Hence, we have the following lemmas.

Lemma 2.1 *There exists $\beta_s > 0$ such that*

$$\sup_{\boldsymbol{\tau}_s\in\mathbb{H}(\mathbf{div};\Omega_s)\backslash\{\mathbf{0}\}} \frac{|\,B_s(\boldsymbol{\tau}_s,(\boldsymbol{\eta},\boldsymbol{\psi}_s)\,)\,|}{\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}} \,\geq\, \beta_s\,\|(\boldsymbol{\eta},\boldsymbol{\psi}_s)\| \qquad \forall\,(\boldsymbol{\eta},\boldsymbol{\psi}_s)\in\mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\times\mathbf{H}^{1/2}(\Sigma)\,.$$

Proof. We proceed as in the proof of [40, Lemma 4.1]. Given $(\boldsymbol{\eta},\boldsymbol{\psi}_s) \in \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\times\mathbf{H}^{1/2}(\Sigma)$ we let $\mathbf{z} \in \mathbf{H}^1(\Omega_s)$ be the unique (up to a rigid motion) solution of the variational formulation

$$\int_{\Omega_s} \boldsymbol{\varepsilon}(\mathbf{z}) : \boldsymbol{\varepsilon}(\mathbf{w}) \,=\, -\int_{\Omega_s} \mathbf{r}(\boldsymbol{\eta},\boldsymbol{\psi}_s)\cdot\mathbf{w} - \int_{\Omega_s} \boldsymbol{\eta} : \nabla\mathbf{w} + \langle\mathcal{R}_s^{-1}(\boldsymbol{\psi}_s),\mathbf{w}\rangle_\Sigma \qquad \forall\,\mathbf{w}\in\mathbf{H}^1(\Omega_s)\,, \tag{2.51}$$

where $\mathbf{r}(\boldsymbol{\eta},\boldsymbol{\psi}_s) \in \mathbb{RM}(\Omega_s)$ is characterized by

$$\int_{\Omega_s} \mathbf{r}(\boldsymbol{\eta},\boldsymbol{\psi}_s)\cdot\mathbf{w} \,=\, -\int_{\Omega_s} \boldsymbol{\eta} : \nabla\mathbf{w} \,+\, \langle\mathcal{R}_s^{-1}(\boldsymbol{\psi}_s),\mathbf{w}\rangle_\Sigma \qquad \forall\,\mathbf{w} \,\in\, \mathbb{RM}(\Omega_s)\,.$$

Then, defining $\boldsymbol{\zeta}_s := \boldsymbol{\varepsilon}(\mathbf{z}) + \boldsymbol{\eta}$, we find from (2.51) that $\mathbf{div}\,\boldsymbol{\zeta}_s = \mathbf{r}(\boldsymbol{\eta},\boldsymbol{\psi}_s)$ in Ω_s, whence $\boldsymbol{\zeta}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, and thus $\boldsymbol{\zeta}_s\,\boldsymbol{\nu} = -\mathcal{R}_s^{-1}(\boldsymbol{\psi}_s)$ on Σ. It follows that $\mathbf{B}_s(\boldsymbol{\zeta}_s) = (\boldsymbol{\eta},\boldsymbol{\psi}_s)$, which proves the surjectivity of $\mathbf{B}_s$.

□

Lemma 2.2 *There exists $\beta_f > 0$ such that*

$$\sup_{\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)\setminus\{\mathbf{0}\}} \frac{|\, B_f(\boldsymbol{\tau}_f, \boldsymbol{\psi}_f)\,|}{\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}} \,\geq\, \beta_f \,\|\boldsymbol{\psi}_f\|_{1/2,\partial\Omega_f} \qquad \forall\, \boldsymbol{\psi}_f \,:=\, (\psi_\Sigma, \psi_\Gamma) \,\in\, H^{1/2}(\partial\Omega_f)\,.$$

Proof. Given $\boldsymbol{\psi}_f := (\psi_\Sigma, \psi_\Gamma) \in H^{1/2}(\partial\Omega_f)$, we let $z \in H^1(\Omega_f)$ be the unique solution (up to a constant) of the Neumann boundary value problem

$$\begin{aligned} \Delta z \,&=\, -\frac{1}{|\Omega_f|}\Big\{\langle \mathcal{R}_\Sigma^{-1}(\psi_\Sigma), 1\rangle_\Sigma \,+\, \langle \mathcal{R}_\Gamma^{-1}(\psi_\Gamma), 1\rangle_\Gamma \Big\} \quad \text{in} \quad \Omega_f\,, \\ \nabla z \cdot \boldsymbol{\nu} \,&=\, \mathcal{R}_\Sigma^{-1}(\psi_\Sigma) \quad \text{on} \quad \Sigma\,, \quad \nabla z\cdot\boldsymbol{\nu} \,=\, -\,\mathcal{R}_\Gamma^{-1}(\psi_\Gamma) \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{2.52}$$

Then, defining $\boldsymbol{\zeta}_f := \nabla z \quad \text{in} \quad \Omega_f$, we easily see that

$$\mathbf{B}_f(\boldsymbol{\zeta}_f) \,:=\, \big(\mathcal{R}_\Sigma(\boldsymbol{\zeta}_f\cdot\boldsymbol{\nu}), -\,\mathcal{R}_\Gamma(\boldsymbol{\zeta}_f\cdot\boldsymbol{\nu})\big) \,=\, (\psi_\Sigma, \psi_\Gamma)\,,$$

which shows that $\mathbf{B}_f$ is surjective.

□

We now let $\mathbf{V}_s$ and $\mathbf{V}_f$ be the kernels of $\mathbf{B}_s$ and $\mathbf{B}_f$, respectively, that is, according to (2.49) and (2.50),

$$\mathbf{V}_s \,:=\, \Big\{ \boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s) : \quad \boldsymbol{\tau}_s \,=\, \boldsymbol{\tau}_s^{\mathtt{t}} \quad \text{in} \quad \Omega_s\,, \quad \boldsymbol{\tau}_s\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Sigma \Big\}\,, \tag{2.53}$$

$$\mathbf{V}_f \,:=\, \Big\{ \boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f) : \quad \boldsymbol{\tau}_f\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Sigma\,, \quad \boldsymbol{\tau}_f\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma \Big\}\,, \tag{2.54}$$

and aim to prove that $A_s|_{\mathbf{V}_s\times\mathbf{V}_s}$ and $A_f|_{\mathbf{V}_f\times\mathbf{V}_f}$ induce bijective operators. In particular, for A_s we proceed as in [37, Section 4.2] and make use of the decomposition

$$\mathbb{H}(\mathbf{div};\Omega_s) \,=\, \mathbb{H}_0(\mathbf{div};\Omega_s) \,\oplus\, \mathbb{C}\,\mathbf{I}\,,$$

with

$$\mathbb{H}_0(\mathbf{div};\Omega_s) \,:=\, \Big\{ \boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s) : \quad \int_{\Omega_s} \mathrm{tr}\,\boldsymbol{\tau}_s \,=\, 0 \Big\}\,, \tag{2.55}$$

and the inequalities

$$\|\boldsymbol{\tau}_s^{\mathtt{d}}\|_{0,\Omega_s}^2 \,+\, \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}^2 \,\geq\, c_1\,\|\boldsymbol{\tau}_{s,0}\|_{0,\Omega_s}^2 \qquad \forall\,\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s) \tag{2.56}$$

(cf. [19, Proposition 3.1, Chapter IV]), and

$$\|\boldsymbol{\tau}_{s,0}\|_{\mathbf{div};\Omega_s}^2 \,\geq\, c_2\,\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}^2 \qquad \forall\,\boldsymbol{\tau}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s) \tag{2.57}$$

(cf. [37, Lemma 4.5]), with

$$\tilde{\mathbb{H}}(\mathbf{div};\Omega_s) \,:=\, \Big\{ \boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s) : \quad \boldsymbol{\tau}_s\boldsymbol{\nu} \,=\, \mathbf{0} \quad \text{on} \quad \Sigma \Big\}\,, \tag{2.58}$$

where each $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$ is written as $\boldsymbol{\tau}_s = \boldsymbol{\tau}_{s,0} + d\,\mathbf{I}$, with $\boldsymbol{\tau}_{s,0} \in \mathbb{H}_0(\mathbf{div};\Omega_s)$ and $d \in \mathbb{C}$.

The following lemma establishes the $\tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$-ellipticity of A_s^+.

Lemma 2.3 *There exists $\alpha_s^+ > 0$, depending on μ, κ_s, c_1, and c_2, such that*

$$A_s^+(\boldsymbol{\tau}_s, \overline{\boldsymbol{\tau}}_s) \;\geq\; \alpha_s^+ \, \|\boldsymbol{\tau}_s\|^2_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\tau}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s)\,. \tag{2.59}$$

Proof. According to the definition of A_s^+ (cf. (2.34)), and using the inequalities (2.8), (2.56), and (2.57), we find that for each $\boldsymbol{\tau}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ there holds

$$\begin{aligned}
A_s^+(\boldsymbol{\tau}_s, \overline{\boldsymbol{\tau}}_s) &\geq \frac{1}{2\mu}\,\|\boldsymbol{\tau}_s^{\mathtt{d}}\|^2_{0,\Omega_s} + \frac{1}{\kappa_s^2}\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|^2_{0,\Omega_s} \\
&\geq \quad \min\left\{\frac{1}{2\mu}, \frac{1}{2\kappa_s^2}\right\} \left\{ \|\boldsymbol{\tau}_s^{\mathtt{d}}\|^2_{0,\Omega_s} + \|\mathbf{div}\,\boldsymbol{\tau}_s\|^2_{0,\Omega_s} \right\} + \frac{1}{2\kappa_s^2}\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|^2_{0,\Omega_s} \\
&\geq \quad \tilde{c}_1\,\|\boldsymbol{\tau}_{s,0}\|^2_{0,\Omega_s} + \frac{1}{2\kappa_s^2}\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|^2_{0,\Omega_s} \\
&\geq \quad \min\left\{\tilde{c}_1, \frac{1}{2\kappa_s^2}\right\} \|\boldsymbol{\tau}_{s,0}\|^2_{\mathbf{div};\Omega_s} \;\geq\; \alpha_s^+\,\|\boldsymbol{\tau}_s\|^2_{\mathbf{div};\Omega_s}\,,
\end{aligned}$$

with $\tilde{c}_1 := c_1 \min\left\{\frac{1}{2\mu}, \frac{1}{2\kappa_s^2}\right\}$ and $\alpha_s^+ := c_2 \min\left\{\tilde{c}_1, \frac{1}{2\kappa_s^2}\right\}$, which completes the proof. □

We are now in a position to prove that A_s and A_f satisfy the continuous inf-sup conditions required by the Babuška-Brezzi theory. To this end, we need to introduce the operators

$$\Xi_s := (\mathbf{I} - 2\,\mathbf{P}_s) : \mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{H}(\mathbf{div};\Omega_s) \tag{2.60}$$

and

$$\Xi_f := (\mathbf{I} - 2\,\mathbf{P}_f) : \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{H}(\mathrm{div};\Omega_f)\,, \tag{2.61}$$

which, recalling that $\mathbf{P}_s$ and $\mathbf{P}_f$ are projectors, are certainly bounded and satisfy

$$\mathbf{P}_s\,\Xi_s = -\,\mathbf{P}_s\,, \qquad (\mathbf{I} - \mathbf{P}_s)\,\Xi_s = \mathbf{I} - \mathbf{P}_s\,, \tag{2.62}$$

$$\mathbf{P}_f\,\Xi_f = -\,\mathbf{P}_f\,, \quad \text{and} \quad (\mathbf{I} - \mathbf{P}_f)\,\Xi_f = \mathbf{I} - \mathbf{P}_f\,. \tag{2.63}$$

Then, we can establish the following lemmas.

Lemma 2.4 *There exist α_s, $C_s > 0$ such that*

$$A_s\big(\boldsymbol{\zeta}_s, \Xi_s(\overline{\boldsymbol{\zeta}}_s)\big) \;\geq\; \alpha_s\,\|\boldsymbol{\zeta}_s\|^2_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\zeta}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s)\,, \tag{2.64}$$

and

$$\sup_{\boldsymbol{\tau}_s \in \mathbf{V}_s \setminus \{\mathbf{0}\}} \frac{|A_s(\boldsymbol{\zeta}_s, \boldsymbol{\tau}_s)|}{\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}} \;\geq\; C_s\,\|\boldsymbol{\zeta}_s\|_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\zeta}_s \in \mathbf{V}_s\,. \tag{2.65}$$

In addition, there holds

$$\sup_{\boldsymbol{\zeta}_s \in \mathbf{V}_s \setminus \{\mathbf{0}\}} |A_s(\boldsymbol{\zeta}_s, \boldsymbol{\tau}_s)| > 0 \qquad \forall\, \boldsymbol{\tau}_s \in \mathbf{V}_s\,, \quad \boldsymbol{\tau}_s \neq \mathbf{0}\,. \tag{2.66}$$

Proof. We first observe, thanks to the definitions of $\mathbf{V}_s$ and $\tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ (cf. (2.53), (2.58)), and the properties of $\mathbf{P}_s$ (cf. (2.24), (2.25)), that $\mathbf{V}_s \subseteq \tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ and $\mathbf{P}_s(\boldsymbol{\zeta}_s) \in \mathbf{V}_s$ for each $\boldsymbol{\zeta}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, and hence, in particular both $\mathbf{P}_s(\boldsymbol{\zeta}_s)$ and $(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s)$ belong to $\tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ for each $\boldsymbol{\zeta}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$. It follows, according to the definition of A_s (cf. (2.43)), the properties of Ξ_s (cf. (2.62)), and the ellipticity of A_s^+ (cf. (2.59)), that for each $\boldsymbol{\zeta}_s \in \tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ there holds

$$
\begin{aligned}
A_s\big(\boldsymbol{\zeta}_s, \Xi_s(\overline{\boldsymbol{\zeta}}_s)\big) \;&=\; A_s^+(\mathbf{P}_s(\boldsymbol{\zeta}_s), \mathbf{P}_s(\overline{\boldsymbol{\zeta}}_s)) \;+\; A_s^+((\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s), (\mathbf{I}-\mathbf{P}_s)(\overline{\boldsymbol{\zeta}}_s)) \\
&\geq\; \alpha_s^+ \Big\{ \|\mathbf{P}_s(\boldsymbol{\zeta}_s)\|^2_{\mathbf{div};\Omega_s} \;+\; \|(\mathbf{I}-\mathbf{P}_s)(\boldsymbol{\zeta}_s)\|^2_{\mathbf{div};\Omega_s} \Big\} \\
&\geq\; \frac{\alpha_s^+}{2}\, \|\boldsymbol{\zeta}_s\|^2_{\mathbf{div};\Omega_s}\,,
\end{aligned}
$$

which shows (2.64) with $\alpha_s := \alpha_s^+/2$. Next, given $\boldsymbol{\zeta}_s \in \mathbf{V}_s \setminus \{\mathbf{0}\}$, it is clear from the above analysis that $\Xi_s(\overline{\boldsymbol{\zeta}}_s) \in \mathbf{V}_s \setminus \mathbf{0}$, and therefore, applying (2.64), we deduce that

$$
\sup_{\boldsymbol{\tau}_s \in \mathbf{V}_s \setminus \{\mathbf{0}\}} \frac{|A_s(\boldsymbol{\zeta}_s, \boldsymbol{\tau}_s)|}{\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}} \;\geq\; \frac{\big|A_s(\boldsymbol{\zeta}_s, \Xi_s(\overline{\boldsymbol{\zeta}}_s))\big|}{\|\Xi_s(\overline{\boldsymbol{\zeta}}_s)\|_{\mathbf{div};\Omega_s}} \;\geq\; \alpha_s\, \frac{\|\boldsymbol{\zeta}_s\|^2_{\mathbf{div};\Omega_s}}{\|\Xi_s(\overline{\boldsymbol{\zeta}}_s)\|_{\mathbf{div};\Omega_s}}\,,
$$

which yields (2.65) with $C_s := \alpha_s/\|\Xi_s\|$. Finally, (2.66) is a straightforward consequence of (2.65) and the symmetry of A_s.

□

Lemma 2.5 *There exist $\alpha_f\,,\ C_f\, >\, 0$ such that*

$$
A_f\big(\boldsymbol{\zeta}_f, \Xi_f(\overline{\boldsymbol{\zeta}}_f)\big) \;\geq\; \alpha_f\, \|\boldsymbol{\zeta}_f\|^2_{\mathrm{div};\Omega_f} \qquad \forall\, \boldsymbol{\zeta}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f)\,, \tag{2.67}
$$

and

$$
\sup_{\boldsymbol{\tau}_f \in \mathbf{V}_f \setminus \{\mathbf{0}\}} \frac{\big|A_f(\boldsymbol{\zeta}_f, \boldsymbol{\tau}_f)\big|}{\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}} \;\geq\; C_f\, \|\boldsymbol{\zeta}_f\|_{\mathrm{div};\Omega_f} \qquad \forall\, \boldsymbol{\zeta}_f \,\in\, \mathbf{V}_f\,. \tag{2.68}
$$

In addition, there holds

$$
\sup_{\boldsymbol{\zeta}_f \in \mathbf{V}_f \setminus \{\mathbf{0}\}} \big|A_f(\boldsymbol{\zeta}_f, \boldsymbol{\tau}_f)\big| \;>\; 0 \qquad \forall\, \boldsymbol{\tau}_f \,\in\, \mathbf{V}_f\,, \quad \boldsymbol{\tau}_f \,\neq\, \mathbf{0}\,. \tag{2.69}
$$

Proof. We proceed analogously to the proof of the previous lemma. In fact, according to the definition of A_f (cf. (2.44)) and the properties of Ξ_f (cf. (2.63)), and applying the ellipticity of A_f^+ (cf. (2.36)), we find that for each $\boldsymbol{\zeta}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f)$ there holds

$$
\begin{aligned}
A_f\big(\boldsymbol{\zeta}_f, \Xi_f(\overline{\boldsymbol{\zeta}}_f)\big) \;&=\; A_f^+(\mathbf{P}_f(\boldsymbol{\zeta}_f), \mathbf{P}_f(\overline{\boldsymbol{\zeta}}_f)) \;+\; A_f^+((\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\zeta}_f), (\mathbf{I}-\mathbf{P}_f)(\overline{\boldsymbol{\zeta}}_f)) \\
&\geq\; \alpha_f^+ \Big\{ \|\mathbf{P}_f(\boldsymbol{\zeta}_f)\|^2_{\mathrm{div};\Omega_f} \;+\; \|(\mathbf{I}-\mathbf{P}_f)(\boldsymbol{\zeta}_f)\|^2_{\mathrm{div};\Omega_f} \Big\} \\
&\geq\; \frac{\alpha_f^+}{2}\, \|\boldsymbol{\zeta}_f\|^2_{\mathrm{div};\Omega_f}\,,
\end{aligned}
$$

which proves (2.67) with $\alpha_f := \alpha_f^+/2$. Next, it is clear from (2.67) that $\Xi_f(\overline{\boldsymbol{\zeta}}_f) \neq \mathbf{0}$ for each $\boldsymbol{\zeta}_f \in \mathbf{H}(\mathrm{div};\Omega_f) \setminus \{\mathbf{0}\}$. In addition, thanks to the properties of $\mathbf{P}_f$ (cf. (2.31)) and the definition of $\mathbf{V}_f$ (cf. (2.54)), we deduce that $\Xi_f(\overline{\boldsymbol{\zeta}}_f)$ belong to $\mathbf{V}_f \setminus \{\mathbf{0}\}$ for each $\boldsymbol{\zeta}_f \in \mathbf{V}_f \setminus \{\mathbf{0}\}$, and hence

$$\sup_{\boldsymbol{\tau}_f \in \mathbf{V}_f \setminus \{\mathbf{0}\}} \frac{\left|A_f(\boldsymbol{\zeta}_f, \boldsymbol{\tau}_f)\right|}{\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}} \geq \frac{\left|A_f(\boldsymbol{\zeta}_f, \Xi_f(\overline{\boldsymbol{\zeta}}_f))\right|}{\|\Xi_f(\overline{\boldsymbol{\zeta}}_f)\|_{\mathrm{div};\Omega_f}} \geq \alpha_f \frac{\|\boldsymbol{\zeta}_f\|^2_{\mathrm{div};\Omega_f}}{\|\Xi_f(\overline{\boldsymbol{\zeta}}_f)\|_{\mathrm{div};\Omega_f}},$$

which implies (2.68) with $C_f := \alpha_f/\|\Xi_f\|$. Finally, the inequality (2.69) follows directly from (2.68) and the symmetry of A_f.

$\square$

As a consequence of Lemmas 2.1, 2.2, 2.4, and 2.5, and having in mind the identity (2.48) and the classical Babuška-Brezzi theory (cf. [19, Theorem 1.1, Chapter II]), we conclude that the matrix operator $\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} : \mathbf{H} \times \mathbf{Q} \to \mathbf{H} \times \mathbf{Q}$ is an isomorphism. In turn, the compactness of $\begin{pmatrix} \mathbf{K}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix} : \mathbf{H} \times \mathbf{Q} \to \mathbf{H} \times \mathbf{Q}$ is proved by the following lemma.

Lemma 2.6 *The operators $\mathbf{K}_0 : \mathbf{H} \to \mathbf{H}$ and $\mathbf{K} : \mathbf{Q} \to \mathbf{Q}$ are compact.*

Proof. We first recall from Section 2.3.1 (cf. (2.27) and (2.33)) that there exists $\epsilon > 0$ such that $\mathbf{P}_s(\boldsymbol{\tau}_s) \in \mathbb{H}^\epsilon(\Omega_s)$ for each $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, and $\mathbf{P}_f(\boldsymbol{\tau}_f) \in \mathbf{H}^\epsilon(\Omega_f)$ for each $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, which, thanks to the compact imbeddings $\mathbb{H}^\epsilon(\Omega_s) \hookrightarrow \mathbb{L}^2(\Omega_s)$ and $\mathbf{H}^\epsilon(\Omega_f) \hookrightarrow \mathbf{L}^2(\Omega_f)$, imply the compactness of $\mathbf{P}_s : \mathbb{H}(\mathbf{div};\Omega_s) \to \mathbb{L}^2(\Omega_s)$ and $\mathbf{P}_f : \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{L}^2(\Omega_f)$. It follows that the adjoints $\mathbf{P}_s^* : \mathbb{L}^2(\Omega_s) \to \mathbb{H}(\mathbf{div};\Omega_s)$ and $\mathbf{P}_f^* : \mathbf{L}^2(\Omega_f) \to \mathbf{H}(\mathrm{div};\Omega_f)$, and hence the operators $\mathbf{P}_s^* \mathcal{C}^{-1} \mathbf{P}_s$, $(\mathbf{I} - \mathbf{P}_s)^* \mathcal{C}^{-1} \mathbf{P}_s$, $\mathbf{P}_s^* \mathcal{C}^{-1} (\mathbf{I} - \mathbf{P}_s)$, $\mathbf{P}_f^* \mathbf{P}_f$, $(\mathbf{I} - \mathbf{P}_f)^* \mathbf{P}_f$, and $\mathbf{P}_f^* (\mathbf{I} - \mathbf{P}_f)$ are all compact. This shows that the first three terms defining the bilinear forms K_s (cf. (2.45)) and K_f (cf. (2.46)) induce compact operators. In addition, it is clear from the second identity in (2.24) and the first identity in (2.31) that the fourth terms of K_s and K_f yield finite rank operators, and therefore $\mathbf{K}_0 : \mathbf{H} \to \mathbf{H}$ becomes compact.

Furthermore, the three terms defining $\mathbf{K}$ (cf. (2.20)), that is $\langle \xi_\Sigma \boldsymbol{\nu}, \boldsymbol{\psi}_s \rangle_\Sigma$, $\rho_f \omega^2 \langle \psi_\Sigma \boldsymbol{\nu}, \boldsymbol{\xi}_s \rangle_\Sigma$, and $\imath \kappa_f \langle \xi_\Gamma, \psi_\Gamma \rangle_\Gamma$ also yield compact operators because of the compactness of the composition defined by the following diagram

$$\begin{array}{ccccccc} H^{1/2}(\Sigma) & \overset{compact}{\longrightarrow} & L^2(\Sigma) & \overset{continuous}{\longrightarrow} & \mathbf{L}^2(\Sigma) & \overset{compact}{\longrightarrow} & \mathbf{H}^{-1/2}(\Sigma) \\ \psi_\Sigma & \longrightarrow & \psi_\Sigma & \longrightarrow & \psi_\Sigma \boldsymbol{\nu} & \longrightarrow & \psi_\Sigma \boldsymbol{\nu}, \end{array}$$

and thanks to the compact imbedding $H^{1/2}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)$. This completes the proof.

$\square$

We are able now to provide the main result of this section.

Theorem 2.1 *Assume that the homogeneous problem associated to* (2.15) *has only the trivial solution. Then, given* $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ *and* $g \in H^{-1/2}(\Gamma)$, *there exists a unique solution* $(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) \in \mathbf{H} \times \mathbf{Q}$ *to* (2.15) (*equivalently* (2.47)). *In addition, there exists* $C > 0$ *such that*

$$\|(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma})\|_{\mathbf{H}\times\mathbf{Q}} \;\le\; C \left\{ \|\mathbf{f}\|_{0,\Omega_s} \,+\, \|g\|_{-1/2,\Gamma} \right\}.$$

Proof. It suffices to notice, according to our previous analysis, that the left hand side of (2.47) constitutes a Fredholm operator of index zero. □

We end this section by remarking that the extension of the previous continuous analysis to the 3D version of our interaction problem is quite straightforward. However, this is not exactly the case when trying to extend to 3D the Galerkin analysis shown below in Section 2.4. In particular, the proofs of the discrete inf-sup conditions involving boundary or interface terms are rather technical and they require additional hypotheses on the triangulations of both domains. In order to circumvent these difficulties, in the recent works [40] and [42] we have developed a new approach which incorporates the exact satisfaction of the transmission conditions into the definitions of the continuous and discrete spaces.

2.4 Analysis of the Galerkin scheme

In this section we introduce a Galerkin approximation of (2.15) and show, under the same assumption of Theorem 2.1, that it is well-posed. The corresponding result is given by Theorem 2.2, whose proof is obtained as a consequence of the analysis in the following sections. In fact, we first define in Section 2.4.1 the main finite element subspaces to be employed in the definition of the Galerkin scheme (cf. (2.76)) and provide their approximation properties in Section 2.4.2. Then, in Section 2.4.3 we prove the existence of stable discrete liftings of the normal traces on Σ and Γ of the finite element subspaces approximating the stresses. These lifting operators allow us to establish certain equivalence results (cf. Lemmas 2.9 and 2.10), which later on simplify the proofs of the discrete inf-sup conditions for the bilinear forms B_f and B_s (cf. Lemmas 2.13 and 2.14). Next, in Section 2.4.4 we introduce uniformly bounded discrete operators $\mathbf{P}_{f,h}$ and $\mathbf{P}_{s,h}$ approximating $\mathbf{P}_f$ and $\mathbf{P}_s$, respectively. Recall that the latter operators were utilized in Section 2.3.3 to prove the continuous inf-sup conditions for the bilinear forms A_s and A_f (cf. Lemmas 2.4 and 2.5). Hence, the key results in Section 2.4.4 refer to the upper estimates for the errors $\|\mathbf{P}_s - \mathbf{P}_{s,h}\|$ and $\|\mathbf{P}_f - \mathbf{P}_{f,h}\|$ (cf. Lemmas 2.11 and 2.12), which are utilized in Lemma 2.16 to prove the discrete inf-sup conditions for A_s and A_f. Finally, after establishing all the above mentioned discrete inf-sup conditions in Section 2.4.5, the well-posedness of the Galerkin scheme, which follows at once, is summarized in Theorem 2.2.

2.4.1 Preliminaries

We first let $\mathcal{T}_h^s$ and $\mathcal{T}_h^f$ be triangulations, belonging to shape-regular families, of the polygonal regions $\bar{\Omega}_s$ and $\bar{\Omega}_f$, respectively, by triangles T of diameter h_T, with global mesh size

$$h := \max\Big\{ \max\big\{ h_T : \quad T \in \mathcal{T}_h^s \big\} ; \max\big\{ h_T : \quad T \in \mathcal{T}_h^f \big\} \Big\} ,$$

and such that the vertices of $\mathcal{T}_h^s$ and $\mathcal{T}_h^f$ coincide on Σ. In what follows, given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^2$, $P_\ell(S)$ denotes the space of polynomials defined in S of total degree $\leq \ell$. In addition, following the same terminology described at the end of the introduction, we denote $\mathbf{P}_\ell(S) := [P_\ell(S)]^2$. Furthermore, given $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ and $\mathbf{x} := (x_1, x_2)^{\mathbf{t}}$ a generic vector of $\mathbb{R}^2$, we let $\mathrm{RT}_0(T) := \operatorname{span}\Big\{(1,0),(0,1),(x_1,x_2)\Big\}$ be the local Raviart-Thomas space of order 0 (cf. [19], [69]), and set $\mathbf{curl}^{\mathbf{t}}\, b_T := \big(\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1}\big)$, where b_T is the usual cubic bubble function on T. Then we define

$$\mathbf{H}_h^s := \Big\{ \mathbf{v}_{s,h} \in \mathbf{H}(\mathrm{div};\Omega_s) : \quad \mathbf{v}_{s,h}|_T \in \mathrm{RT}_0(T) \oplus P_0(T)\,\mathbf{curl}^{\mathbf{t}}\, b_T \quad \forall\, T \in \mathcal{T}_h^s \Big\} ,$$

$$\mathbb{H}_h^s := \Big\{ \boldsymbol{\tau}_{s,h} \in \mathbb{H}(\mathbf{div};\Omega_s) : \quad \mathbf{c}^{\mathbf{t}}\, \boldsymbol{\tau}_{s,h} \in \mathbf{H}_h^s \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \Big\} , \tag{2.70}$$

$$\mathbf{H}_h^f := \Big\{ \boldsymbol{\tau}_{f,h} \in \mathbf{H}(\mathrm{div};\Omega_f) : \quad \boldsymbol{\tau}_{f,h}|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^f \Big\} , \tag{2.71}$$

$$\mathbb{Q}_h^s := \left\{ \boldsymbol{\eta}_h := \begin{pmatrix} 0 & \eta_h \\ -\eta_h & 0 \end{pmatrix} : \quad \eta_h \in C(\bar{\Omega}_s) , \quad \eta_h|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h^s \right\} , \tag{2.72}$$

$$\mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma) , \tag{2.73}$$

$$\mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma) , \tag{2.74}$$

where $\Lambda_h(\Sigma)$ and $\Lambda_h(\Gamma)$ are generic finite dimensional subspaces (to be specified later on) of $H^{1/2}(\Sigma)$ and $H^{1/2}(\Gamma)$, respectively, and introduce the finite element subspaces $\mathbf{H}_h \subseteq \mathbf{H}$ and $\mathbf{Q}_h \subseteq \mathbf{Q}$, given by

$$\mathbf{H}_h := \mathbb{H}_h^s \times \mathbf{H}_h^f \qquad \text{and} \qquad \mathbf{Q}_h := \mathbb{Q}_h^s \times \mathbf{Q}_h^s \times \mathbf{Q}_h^f . \tag{2.75}$$

Note that the associated generic subspaces $\mathbf{Q}_h^f$ and $\mathbf{Q}_h^s$ are employed below (cf. Lemmas 2.9 and 2.10) to establish preliminary equivalence results concerning the discrete inf-sup conditions for B_f and B_s. Explicit definitions of $\Lambda_h(\Sigma)$ and $\Lambda_h(\Gamma)$, and hence of $\mathbf{Q}_h^f$ and $\mathbf{Q}_h^s$, are given later on in Section 2.4.5 (cf. (2.117), (2.118), (2.119), and (2.120)) to finally guarantee the ocurrence of the discrete inf-sup conditions for those bilinear forms (cf. Lemmas 2.13 and 2.14).

In addition, our analysis below will also require the subspaces

$$\tilde{\mathbf{H}}_h^s := \Big\{ \mathbf{v}_{s,h} \in \mathbf{H}(\mathrm{div};\Omega_s) : \quad \mathbf{v}_{s,h}|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^s \Big\} ,$$

$$\tilde{\mathbb{H}}_h^s := \left\{ \boldsymbol{\tau}_{s,h} \in \mathbb{H}(\mathbf{div};\Omega_s) : \quad \mathbf{c}^{\mathrm{t}}\, \boldsymbol{\tau}_{s,h} \in \tilde{\mathbf{H}}_h^s \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \right\},$$

$$\mathbf{U}_h^s := \left\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega_s) : \quad \mathbf{v}_h|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^s \right\}$$

and

$$U_h^f := \left\{ v_h \in L^2(\Omega_f) : \quad v_h|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^f \right\}.$$

We recall here that $\mathbb{H}_h^s \times \mathbf{U}_h^s \times \mathbb{Q}_h^s$ constitutes the well known PEERS space introduced in [4] for a mixed finite element aproximation of the linear elasticity problem in the plane. In turn, $\mathbf{H}_h^f \times U_h^f$ is the lowest order Raviart-Thomas mixed finite element approximation of the Poisson problem for the Laplace equation (see [19], [69]). Also, it is important to notice, which will be used below, that $\tilde{\mathbf{H}}_h^s \subseteq \mathbf{H}_h^s$ and hence $\tilde{\mathbb{H}}_h^s \subseteq \mathbb{H}_h^s$.

The Galerkin scheme associated to our continuous problem (2.15) is then defined as follows: Find $\widehat{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{s,h}, \boldsymbol{\sigma}_{f,h}) \in \mathbf{H}_h$ and $\widehat{\boldsymbol{\gamma}}_h := (\boldsymbol{\gamma}_h, \boldsymbol{\varphi}_{s,h}, \boldsymbol{\varphi}_{f,h}) \in \mathbf{Q}_h$ such that

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\tau}}_h) + B(\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\gamma}}_h) &= F(\widehat{\boldsymbol{\tau}}_h) \qquad \forall\, \widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h\,, \\ B(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\eta}}_h) + K(\widehat{\boldsymbol{\gamma}}_h, \widehat{\boldsymbol{\eta}}_h) &= G(\widehat{\boldsymbol{\eta}}_h) \qquad \forall\, \widehat{\boldsymbol{\eta}}_h := (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}, \boldsymbol{\psi}_{f,h}) \in \mathbf{Q}_h\,, \end{aligned} \tag{2.76}$$

We collect next the approximation properties of the finite element subspaces introduced above.

2.4.2 Approximation properties of the subspaces

We begin with the subspaces $\mathbb{H}_h^s$ and $\mathbf{H}_h^f$. Indeed, given $\delta \in (0,1]$, we let

$$\mathcal{E}_h^s : \mathbb{H}^\delta(\Omega_s) \cap \mathbb{H}(\mathbf{div};\Omega_s) \to \tilde{\mathbb{H}}_h^s \subseteq \mathbb{H}_h^s \qquad \text{and} \qquad \mathcal{E}_h^f : \mathbf{H}^\delta(\Omega_f) \cap \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{H}_h^f$$

be the usual Raviart-Thomas interpolation operators (see [19], [69]), which, given $\boldsymbol{\tau}_s \in \mathbb{H}^\delta(\Omega_s) \cap \mathbb{H}(\mathbf{div};\Omega_s)$ and $\boldsymbol{\tau}_f \in \mathbf{H}^\delta(\Omega_f) \cap \mathbf{H}(\mathrm{div};\Omega_f)$, are characterized by the identities

$$\int_e \mathcal{E}_h^s(\boldsymbol{\tau}_s)\, \boldsymbol{\nu} \cdot \mathbf{q} = \int_e \boldsymbol{\tau}_s\, \boldsymbol{\nu} \cdot \mathbf{q} \quad \forall\, \mathbf{q} \in \mathbf{P}_0(e)\,, \quad \forall \text{ edge } e \text{ of } \mathcal{T}_h^s\,, \tag{2.77}$$

and

$$\int_e \mathcal{E}_h^f(\boldsymbol{\tau}_f) \cdot \boldsymbol{\nu}\, q = \int_e \boldsymbol{\tau}_f \cdot \boldsymbol{\nu}\, q \quad \forall\, q \in P_0(e)\,, \quad \forall \text{ edge } e \text{ of } \mathcal{T}_h^f\,. \tag{2.78}$$

In addition, the corresponding conmuting diagram properties yield

$$\mathbf{div}(\mathcal{E}_h^s(\boldsymbol{\tau}_s)) = \mathcal{P}_h^s(\mathbf{div}\, \boldsymbol{\tau}_s) \qquad \forall\, \boldsymbol{\tau}_s \in \mathbb{H}^\delta(\Omega_s) \cap \mathbb{H}(\mathbf{div};\Omega_s)\,, \tag{2.79}$$

and

$$\mathrm{div}(\mathcal{E}_h^f(\boldsymbol{\tau}_f)) = \mathcal{P}_h^f(\mathrm{div}\, \boldsymbol{\tau}_f) \qquad \forall\, \boldsymbol{\tau}_f \in \mathbf{H}^\delta(\Omega_f) \cap \mathbf{H}(\mathrm{div};\Omega_f)\,, \tag{2.80}$$

where $\mathcal{P}_h^s : \mathbf{L}^2(\Omega_s) \to \mathbf{U}_h^s$ and $\mathcal{P}_h^f : L^2(\Omega_f) \to U_h^f$ are the corresponding orthogonal projectors, which satisfy the following error estimates (see, e.g. [19])

(AP_h^s) For each $t \in (0,1]$ and for each $\mathbf{v} \in \mathbf{H}^t(\Omega_s)$, there holds

$$\|\mathbf{v} - \mathcal{P}_h^s(\mathbf{v})\|_{0,\Omega_s} \le C\, h^t\, \|\mathbf{v}\|_{t,\Omega_s} .$$

(AP_h^f) For each $t \in (0,1]$ and for each $v \in H^t(\Omega_f)$, there holds

$$\|v - \mathcal{P}_h^f(v)\|_{0,\Omega_f} \le C\, h^t\, \|v\|_{t,\Omega_f} .$$

Furthermore, it is easy to show, using the well-known Bramble-Hilbert Lemma and the boundedness of the local interpolation operators on the reference element $\widehat{T}$ (see, e.g. [51, equation (3.39)]), that there exist $\widehat{C}_s, \widehat{C}_f > 0$, independent of h, such that for each $\boldsymbol{\tau}_s \in \mathbb{H}^\delta(\Omega_s) \cap \mathbb{H}(\mathbf{div};\Omega_s)$ and for each $\boldsymbol{\tau}_f \in \mathbf{H}^\delta(\Omega_f) \cap \mathbf{H}(\mathrm{div};\Omega_f)$, there hold

$$\|\boldsymbol{\tau}_s - \mathcal{E}_h^s(\boldsymbol{\tau}_s)\|_{0,T} \le \widehat{C}_s\, h_T^\delta \left\{ |\boldsymbol{\tau}_s|_{\delta,T} + \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,T} \right\} \quad \forall\, T \in \mathcal{T}_h^s , \tag{2.81}$$

and

$$\|\boldsymbol{\tau}_f - \mathcal{E}_h^f(\boldsymbol{\tau}_f)\|_{0,T} \le \widehat{C}_f\, h_T^\delta \left\{ |\boldsymbol{\tau}_f|_{\delta,T} + \|\mathrm{div}\,\boldsymbol{\tau}_f\|_{0,T} \right\} \quad \forall\, T \in \mathcal{T}_h^f . \tag{2.82}$$

Hence, as a consequence of (2.79), (2.81), and (AP_h^s) (respectively, (2.80), (2.82), and (AP_h^f)), one can derive the following two statements

($\mathrm{AP}_h^{\boldsymbol{\sigma}_s}$) For each $\delta \in (0,1]$ and for each $\boldsymbol{\tau}_s \in \mathbb{H}^\delta(\Omega_s)$, with $\mathbf{div}\,\boldsymbol{\tau}_s \in \mathbf{H}^\delta(\Omega_s)$, there holds

$$\|\boldsymbol{\tau}_s - \mathcal{E}_h^s(\boldsymbol{\tau}_s)\|_{\mathbf{div};\Omega_s} \le C\, h^\delta \left\{ \|\boldsymbol{\tau}_s\|_{\delta,\Omega_s} + \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{\delta,\Omega_s} \right\} .$$

($\mathrm{AP}_h^{\boldsymbol{\sigma}_f}$) For each $\delta \in (0,1]$ and for each $\boldsymbol{\tau}_f \in \mathbf{H}^\delta(\Omega_f)$, with $\mathrm{div}\,\boldsymbol{\tau}_f \in H^\delta(\Omega_f)$, there holds

$$\|\boldsymbol{\tau}_f - \mathcal{E}_h^f(\boldsymbol{\tau}_f)\|_{\mathrm{div};\Omega_f} \le C\, h^\delta \left\{ \|\boldsymbol{\tau}_f\|_{\delta,\Omega_f} + \|\mathrm{div}\,\boldsymbol{\tau}_f\|_{\delta,\Omega_f} \right\} .$$

Finally, the orthogonal projector $\mathcal{R}_h : \mathbb{L}^2_{\mathtt{asym}}(\Omega_s) \to \mathbb{Q}_h^s$ satisfies the following property (see [19])

($\mathrm{AP}_h^{\boldsymbol{\gamma}}$) For each $t \in (0,1]$ and for each $\boldsymbol{\eta} \in \mathbb{H}^t(\Omega_s) \cap \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)$, there holds

$$\|\boldsymbol{\eta} - \mathcal{R}_h(\boldsymbol{\eta})\|_{0,\Omega_s} \le C\, h^t\, \|\boldsymbol{\eta}\|_{t,\Omega_s} .$$

The approximation properties of $\mathbf{Q}_h^s$ and $\mathbf{Q}_h^f$ will be provided once we introduce the specific finite element subspaces $\Lambda_h(\Sigma)$ and $\Lambda_h(\Gamma)$. In fact, as already mentioned, the choice of these discrete spaces will be indicated throughout the analysis of well-posedness of our Galerkin scheme (2.76) (see Section 2.4.5 below), particularly when proving the discrete inf-sup conditions for B_s and B_f. We previously need to define in Section 2.4.3 stable discrete liftings towards Ω_s and Ω_f of normal traces on Σ and Γ and establish its connection with those stability conditions for B_s and B_f. Then in Section 2.4.4 we introduce suitable discrete approximations of the operators $\mathbf{P}_s|_{\mathbb{H}_h^s}$ and $\mathbf{P}_f|_{\mathbf{H}_s^f}$, which will be employed in Section 2.4.5 to show the discrete inf-sup conditions for A_s and B_f.

2.4.3 Stable discrete liftings of normal traces on Σ and Γ

In what follows we proceed as in [45, Sections 4.3 and 5.2] and assume from now on that $\{\mathcal{T}_h^s\}_{h>0}$ and $\{\mathcal{T}_h^f\}_{h>0}$ are quasi-uniform around Σ and Γ. This means that there exist Lipschitz-continuous neighborhoods Ω_Σ and Ω_Γ of Σ and Γ, respectively, such that the elements of $\mathcal{T}_h^s$ and $\mathcal{T}_h^f$ intersecting those regions are more or less of the same size. Equivalently, we define

$$\mathcal{T}_{\Sigma,h} := \Big\{ T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f : \quad T \cap \Omega_\Sigma \neq \emptyset \Big\}, \tag{2.83}$$

$$\mathcal{T}_{\Gamma,h} := \Big\{ T \in \mathcal{T}_h^f : \quad T \cap \Omega_\Gamma \neq \emptyset \Big\}, \tag{2.84}$$

and assume that there exist $c > 0$, independent of h, such that

$$\max \left\{ \max_{T \in \mathcal{T}_{\Sigma,h}} h_T \,;\, \max_{T \in \mathcal{T}_{\Gamma,h}} h_T \right\} \,\leq\, c \,\min \left\{ \min_{T \in \mathcal{T}_{\Sigma,h}} h_T \,;\, \min_{T \in \mathcal{T}_{\Gamma,h}} h_T \right\} \qquad \forall\, h > 0\,. \tag{2.85}$$

Note that the above assumption and the shape-regularity property of the meshes imply that Σ_h, the partition on Σ inherited from $\mathcal{T}_h^s$ (or from $\mathcal{T}_h^f$), and Γ_h, the partition on Γ inherited from $\mathcal{T}_h^f$, are also quasi-uniform, which means that there exist C_Σ, $C_\Gamma > 0$, independent of h, such that

$$h_\Sigma \,:=\, \max\Big\{ |e| : \; e \text{ edge of } \Sigma_h \Big\} \,\leq\, C_\Sigma \,\min\Big\{ |e| : \; e \text{ edge of } \Sigma_h \Big\}$$

and

$$h_\Gamma \,:=\, \max\Big\{ |e| : \; e \text{ edge of } \Gamma_h \Big\} \,\leq\, C_\Gamma \,\min\Big\{ |e| : \; e \text{ edge of } \Gamma_h \Big\}.$$

Also, it is easy to see that there exist c, $C > 0$, independent of h, such that

$$c\, h_\Sigma \,\leq\, h_\Gamma \,\leq\, C\, h_\Sigma\,. \tag{2.86}$$

In addition, the quasi-uniformity of Σ_h and Γ_h guarantees the inverse inequality on the spaces

$$\Phi_h(\Sigma) \,:=\, \Big\{ \phi_h \,\in\, L^2(\Sigma) : \quad \phi_h|_e \,\in\, P_0(e) \quad \forall\, e \text{ edge of } \Sigma_h \Big\}$$

and

$$\Phi_h(\Gamma) \,:=\, \Big\{ \phi_h \,\in\, L^2(\Gamma) : \quad \phi_h|_e \,\in\, P_0(e) \quad \forall\, e \text{ edge of } \Gamma_h \Big\},$$

which means that

$$\|\phi_h\|_{-1/2+\delta,\Sigma} \;\leq\; C\, h_\Sigma^{-\delta}\, \|\phi_h\|_{-1/2,\Sigma} \quad \forall\, \phi_h \in \Phi_h(\Sigma)\,, \quad \forall\, \delta \in [0,1/2] \tag{2.87}$$

and

$$\|\phi_h\|_{-1/2+\delta,\Gamma} \;\leq\; C\, h_\Gamma^{-\delta}\, \|\phi_h\|_{-1/2,\Gamma} \quad \forall\, \phi_h \in \Phi_h(\Gamma)\,, \quad \forall\, \delta \in [0,1/2]\,. \tag{2.88}$$

The following two lemmas establish our results on the existence of stable discrete liftings. These lifting operators will then be employed to prove the equivalence results given by Lemmas 2.9 and 2.10, which later on simplify the proofs of the discrete inf-sup conditions for B_f and B_s.

Lemma 2.7 *There exist uniformly bounded linear operators $\mathcal{L}_h^f : \Phi_h(\Sigma) \times \Phi_h(\Gamma) \to \mathbf{H}_h^f$ such that*

$$\mathcal{L}_h^f(\boldsymbol{\phi}_h) \cdot \boldsymbol{\nu} = \phi_{h,\Sigma} \;\text{ on } \Sigma \qquad \text{and} \qquad \mathcal{L}_h^f(\boldsymbol{\phi}_h) \cdot \boldsymbol{\nu} = -\,\phi_{h,\Gamma} \;\text{ on } \Gamma \tag{2.89}$$

for each $\boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \in \Phi_h(\Sigma) \times \Phi_h(\Gamma)$.

Proof. Given $\boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \in \Phi_h(\Sigma) \times \Phi_h(\Gamma)$, we let $z \in H^1(\Omega_f)$ be the unique solution (up to a constant) of the Neumann boundary value problem

$$\begin{aligned} \Delta z \;&=\; -\,\frac{1}{|\Omega_f|}\Big\{\langle \phi_{h,\Sigma}, 1\rangle_\Sigma \,+\, \langle \phi_{h,\Gamma}, 1\rangle_\Gamma\Big\} \quad \text{in} \quad \Omega_f\,, \\ \nabla z \cdot \boldsymbol{\nu} \;&=\; \phi_{h,\Sigma} \quad \text{on} \quad \Sigma\,, \quad \nabla z \cdot \boldsymbol{\nu} \;=\; -\,\phi_{h,\Gamma} \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{2.90}$$

which can be seen as a discrete version of (2.52), and whose corresponding continuous dependence result says that

$$\|z\|_{1,\Omega_f} \;\leq\; C\,\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f} \;:=\; C\Big\{\,\|\phi_{h,\Sigma}\|_{-1/2,\Sigma} \,+\, \|\phi_{h,\Gamma}\|_{-1/2,\Gamma}\,\Big\}\,. \tag{2.91}$$

Furthermore, since the Neumann datum $\boldsymbol{\phi}_h$ belongs to $H^\delta(\Sigma) \times H^\delta(\Gamma)$ for any $\delta \in [-1/2, 1/2)$, the classical regularity result for mixed boundary value problems on polygonal domains (see, e.g. [49]) implies that $z \in H^{5/4}(\Omega_f)$ and

$$\|z\|_{5/4,\Omega_f} \;\leq\; C\,\|\boldsymbol{\phi}_h\|_{-1/4,\partial\Omega_f} \;:=\; C\Big\{\,\|\phi_{h,\Sigma}\|_{-1/4,\Sigma} \,+\, \|\phi_{h,\Gamma}\|_{-1/4,\Gamma}\,\Big\}\,. \tag{2.92}$$

In addition, since $\Omega_f^{\texttt{int}} := \Omega_f \backslash \big(\Omega_\Sigma \cup \Omega_\Gamma\big)$ is strictly contained in Ω_f, the interior elliptic regularity estimate (see, e.g. [66, Theorem 4.16]) yields

$$\|z\|_{2,\Omega_f^{\texttt{int}}} \;\leq\; C\,\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}\,. \tag{2.93}$$

According to the above, we now let $\boldsymbol{\zeta}_f := \nabla z$ in Ω_f, whence $\boldsymbol{\zeta}_f$ belongs to $\mathbf{H}^{1/4}(\Omega_f)$, and notice from the first equation in (2.90) that

$$\operatorname{div} \boldsymbol{\zeta}_f \;=\; -\,\frac{1}{|\Omega_f|}\Big\{\langle \phi_{h,\Sigma}, 1\rangle_\Sigma \,+\, \langle \phi_{h,\Gamma}, 1\rangle_\Gamma\Big\} \quad \text{in} \quad \Omega_f\,, \tag{2.94}$$

thus showing that $\boldsymbol{\zeta}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$. Then we can define

$$\mathcal{L}_h^f(\boldsymbol{\phi}_h) \;:=\; \mathcal{E}_h^f(\boldsymbol{\zeta}_f) \in \mathbf{H}_h^f\,,$$

which, in virtue of the conmuting diagram property (2.80) and the characterization (2.78), and having in mind (2.94) and the boundary conditions in (2.90), clearly satisfies

$$\mathrm{div}\,\mathcal{L}_h^f(\boldsymbol{\phi}_h) \;=\; -\,\frac{1}{|\Omega_f|}\left\{\langle\phi_{h,\Sigma},1\rangle_\Sigma \,+\, \langle\phi_{h,\Gamma},1\rangle_\Gamma\right\} \quad \text{in} \quad \Omega_f\,, \tag{2.95}$$

and the identities required by (2.89).

It remains to show that $\mathcal{L}_h^f$ is uniformly bounded. We first deduce, using (2.95), that there exists $C>0$, independent of h, such that

$$\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{\mathrm{div};\Omega_f} \;\le\; C\left\{\,\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f} \,+\, \|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{0,\Omega_f}\right\}. \tag{2.96}$$

Next, in order to estimate $\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{0,\Omega_f}$, we divide Ω_f into three regions by defining (cf. (2.83), (2.84))

$$\begin{aligned}
\Omega_{\Sigma,h}^f \;&:=\; \cup\left\{T: \quad T \in \mathcal{T}_h^f \cap \mathcal{T}_{\Sigma,h}\right\},\\
\Omega_{\Gamma,h} \;&:=\; \cup\left\{T: \quad T \in \mathcal{T}_{\Gamma,h}\right\},
\end{aligned}$$

and

$$\Omega_{f,h}^{\mathtt{int}} \;:=\; \Omega_f \setminus \left(\Omega_{\Sigma,h}^f \cup \Omega_{\Gamma,h}\right).$$

It follows, using the stability of $\mathcal{E}_h^f$ in $\mathbf{H}^1(\Omega_{f,h}^{\mathtt{int}})$, the fact that $\boldsymbol{\zeta}_f|_{\Omega_{f,h}^{\mathtt{int}}} \in \mathbf{H}^1(\Omega_{f,h}^{\mathtt{int}})$, the inclusion $\Omega_{f,h}^{\mathtt{int}} \subseteq \Omega_f^{\mathtt{int}}$, and the estimate (2.93), that

$$\begin{aligned}
\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{0,\Omega_f} \;&=\; \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_f} \;\le\; \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{f,h}^{\mathtt{int}}} \,+\, \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f} \,+\, \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Gamma,h}}\\
&\le\quad C\,\|z\|_{2,\Omega_f^{\mathtt{int}}} \,+\, \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f} \,+\, \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Gamma,h}}\\
&\le\quad C\,\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f} \,+\, \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Sigma,h}^f} \,+\, \|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|_{0,\Omega_{\Gamma,h}}\,.
\end{aligned} \tag{2.97}$$

Now, adding and substracting $\boldsymbol{\zeta}_f \;=\; \nabla z$ in $\Omega_{\Sigma,h}^f \subseteq \Omega_f$, noting that $\|\boldsymbol{\zeta}_f\|_{0,\Omega_{\Sigma,h}^f} \le \|z\|_{1,\Omega_f}$, and employing the estimates (2.91), (2.82) (with $\delta = 1/4$) and (2.92), together with the identity (2.95), the quasi-uniformity bound (2.85), the inverse inequalities (2.87) and (2.88), and the equivalence between h_Σ and h_Γ (cf. (2.86)), we arrive at

$$\begin{aligned}
\|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|^2_{0,\Omega_{\Sigma,h}^f} \;&\le\; C\left\{\|\boldsymbol{\zeta}_f - \mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|^2_{0,\Omega_{\Sigma,h}^f} \,+\, \|\boldsymbol{\zeta}_f\|^2_{0,\Omega_{\Sigma,h}^f}\right\}\\
&\le\; C\left\{\sum_{T\in\mathcal{T}_{\Sigma,h}^f} h_T^{1/2}\,\|z\|^2_{5/4,T} \,+\, \|\boldsymbol{\phi}_h\|^2_{-1/2,\partial\Omega_f}\right\}\\
&\le\; C\left\{h_\Sigma^{1/2}\,\|\boldsymbol{\phi}_h\|^2_{-1/4,\partial\Omega_f} \,+\, \|\boldsymbol{\phi}_h\|^2_{-1/2,\partial\Omega_f}\right\}\\
&\le\; C\,\|\boldsymbol{\phi}_h\|^2_{-1/2,\partial\Omega_f}\,.
\end{aligned} \tag{2.98}$$

The estimate for $\|\mathcal{E}_h^f(\boldsymbol{\zeta}_f)\|^2_{0,\Omega_{\Gamma,h}}$ proceeds similarly and yields the same upper bound. In this way, (2.96), (2.97), and (2.98) provide the uniform boundedness of $\mathcal{L}_h^f$, which completes the proof.

$\square$

Lemma 2.8 *There exist uniformly bounded linear operators* $\mathcal{L}_h^s : \Phi_h(\Sigma) \times \Phi(\Sigma) \to \mathbb{H}_h^s$ *such that*

$$\mathcal{L}_h^s(\boldsymbol{\phi}_h)\,\boldsymbol{\nu} = \boldsymbol{\phi}_h \text{ on } \Sigma \qquad \forall\, \boldsymbol{\phi}_h \in \Phi_h(\Sigma) \times \Phi_h(\Sigma)\,. \tag{2.99}$$

Proof. Given $\boldsymbol{\phi}_h \in \Phi_h(\Sigma) \times \Phi_h(\Sigma)$ we let $\mathbf{z} \in \mathbf{H}^1(\Omega_s)$ be the unique solution (up to a constant vector) of the Neumann boundary value problem (in vectorial form)

$$\Delta \mathbf{z} \;=\; \frac{1}{|\Omega_s|}\int_\Sigma \boldsymbol{\phi}_h \quad \text{in} \quad \Omega_s\,, \qquad \nabla \mathbf{z}\,\boldsymbol{\nu} \;=\; \boldsymbol{\phi}_h \quad \text{on} \quad \Sigma\,,$$

whose corresponding continuous dependence result states that

$$\|\mathbf{z}\|_{1,\Omega_s} \;\leq\; C\,\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}\,.$$

Since the Neumann datum $\boldsymbol{\phi}_h$ belongs to $\mathbf{H}^\delta(\Sigma)$ for any $\delta \in [0,1/2)$, we know that we have at least $\mathbf{H}^{3/2}(\Omega_s)$-regularity for $\mathbf{z}$ and

$$\|\mathbf{z}\|_{3/2,\Omega_s} \;\leq\; C\,\|\boldsymbol{\phi}_h\|_{0,\Sigma}\,.$$

In addition, noting that $\Omega_s^{\mathtt{int}} := \Omega_s \setminus \Omega_\Sigma$ is an interior region of Ω_s, the interior elliptic regularity estimate again (see, e.g. [66, Theorem 4.16]) yields

$$\|\mathbf{z}\|_{2,\Omega_s^{\mathtt{int}}} \;\leq\; C\,\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}\,.$$

Next, we set $\boldsymbol{\zeta}_s := \nabla \mathbf{z}$ in Ω_s, which belongs to $\mathbb{H}^{1/2}(\Omega_s) \cap \mathbb{H}(\mathbf{div};\Omega_s)$, define $\mathcal{L}_h^s(\boldsymbol{\phi}_h) := \mathcal{E}_h^s(\boldsymbol{\zeta}_s)$, and proceed analogously to the proof of the previous lemma, by using now the conmuting diagram property (2.79), the characterization (2.77), the error estimate (2.81), the quasi-uniformity bound (2.85), and the inverse inequality (2.87). We omit further details.

$\square$

As a first consequence of Lemmas 2.7 and 2.8, and noting from the definitions of $\mathbf{H}_h^f$ (cf. (2.71)) and $\mathbb{H}_h^s$ (cf. (2.70)) that

$$\boldsymbol{\tau}_{f,h}\cdot\boldsymbol{\nu}|_{\partial\Omega_f} \;\equiv\; \big(\boldsymbol{\tau}_{f,h}\cdot\boldsymbol{\nu}|_\Sigma, \boldsymbol{\tau}_{f,h}\cdot\boldsymbol{\nu}|_\Gamma\big) \;\in\; \Phi_h(\Sigma)\times\Phi_h(\Gamma) \qquad \forall\,\boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f\,,$$

and

$$\boldsymbol{\tau}_{s,h}\,\boldsymbol{\nu}|_\Sigma \;\in\; \Phi_h(\Sigma)\times\Phi_h(\Sigma) \qquad \forall\,\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\,,$$

we deduce that actually there hold

$$\Phi_h(\Sigma) \times \Phi_h(\Gamma) \,=\, \Big\{ \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}|_{\partial\Omega_f} : \quad \boldsymbol{\tau}_{f,h} \,\in\, \mathbf{H}_h^f \Big\} , \tag{2.100}$$

and

$$\Phi_h(\Sigma) \times \Phi_h(\Sigma) \,=\, \Big\{ \boldsymbol{\tau}_{s,h} \, \boldsymbol{\nu}|_{\Sigma} : \quad \boldsymbol{\tau}_{s,h} \,\in\, \mathbb{H}_h^s \Big\} . \tag{2.101}$$

Hence, the stable discrete liftings $\mathcal{L}_h^f$ and $\mathcal{L}_h^s$, and the identities (2.100) and (2.101) allow to show equivalence results concerning the discrete inf-sup conditions for B_f (cf. (2.19)) and for the second term defining B_s (cf. (2.18)). More precisely, we have the following lemmas.

Lemma 2.9 *Let us define, for each* $\boldsymbol{\psi}_{f,h} \,:=\, (\psi_{h,\Sigma}, \psi_{h,\Gamma}) \,\in\, \mathbf{Q}_h^f \,:=\, \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)$,

$$S(\boldsymbol{\psi}_{f,h}) \,:=\, \sup_{\boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f \setminus \{\mathbf{0}\}} \frac{|\, B_f(\boldsymbol{\tau}_{f,h}, \boldsymbol{\psi}_{f,h}) \,|}{\|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}}$$

and

$$\widetilde{S}(\boldsymbol{\psi}_{f,h}) \,:=\, \sup_{\substack{\boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \\ \in \Phi_h(\Sigma) \times \Phi_h(\Gamma) \setminus \{\mathbf{0}\}}} \frac{|\, \langle \phi_{h,\Sigma}, \psi_{h,\Sigma} \rangle_\Sigma \,+\, \langle \phi_{h,\Gamma}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}} .$$

Then there exist C_1, $C_2 > 0$, *independent of* h, *such that*

$$C_1 \, \widetilde{S}(\boldsymbol{\psi}_{f,h}) \,\le\, S(\boldsymbol{\psi}_{f,h}) \,\le\, C_2 \, \widetilde{S}(\boldsymbol{\psi}_{f,h}) \qquad \forall \, \boldsymbol{\psi}_{f,h} \,\in\, \mathbf{Q}_h^f . \tag{2.102}$$

Proof. Let $c_f > 0$, independent of h, whose existence is provided by Lemma 2.7, such that

$$\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{\mathrm{div};\Omega_f} \,\le\, c_f \, \|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f} \qquad \forall \, \boldsymbol{\phi}_h \,:=\, (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \in \Phi_h(\Sigma) \times \Phi_h(\Gamma) .$$

Then, for each $\boldsymbol{\phi}_h \,:=\, (\phi_{h,\Sigma}, \phi_{h,\Gamma}) \in \Phi_h(\Sigma) \times \Phi_h(\Gamma) \setminus \{\mathbf{0}\}$ there holds, using (2.89),

$$\begin{aligned} \frac{|\, \langle \phi_{h,\Sigma}, \psi_{h,\Sigma} \rangle_\Sigma \,+\, \langle \phi_{h,\Gamma}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}} \,&\le\, c_f \, \frac{|\, \langle \phi_{h,\Sigma}, \psi_{h,\Sigma} \rangle_\Sigma \,+\, \langle \phi_{h,\Gamma}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{\mathrm{div};\Omega_f}} \\ =\, c_f \, \frac{|\, \langle \mathcal{L}_h^f(\boldsymbol{\phi}_h) \cdot \boldsymbol{\nu}, \psi_{h,\Sigma} \rangle_\Sigma \,-\, \langle \mathcal{L}_h^f(\boldsymbol{\phi}_h) \cdot \boldsymbol{\nu}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\mathcal{L}_h^f(\boldsymbol{\phi}_h)\|_{\mathrm{div};\Omega_f}} \,&\le\, c_f \, S(\boldsymbol{\psi}_{f,h}) , \end{aligned}$$

which implies the left-hand side of (2.102) with $C_1 \,=\, c_f^{-1}$. Similarly, for each $\boldsymbol{\tau}_{f,h} \,\in\, \mathbf{H}_h^f$ we find, using that $\|\boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}\|_{-1/2,\partial\Omega_f} \,:=\, \|\boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}\|_{-1/2,\Sigma} \,+\, \|\boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}\|_{-1/2,\Gamma} \,\le\, C \, \|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}$ and (2.100), that

$$\begin{aligned} \frac{|\, B_f(\boldsymbol{\tau}_{f,h}, \boldsymbol{\psi}_{f,h}) \,|}{\|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}} \,&=\, \frac{|\, \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Sigma} \rangle_\Sigma \,-\, \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}} \\ &\le\, C \, \frac{|\, \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Sigma} \rangle_\Sigma \,-\, \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Gamma} \rangle_\Gamma \,|}{\|\boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}\|_{-1/2,\partial\Omega_f}} \,\le\, C \, \widetilde{S}(\boldsymbol{\psi}_{f,h}) , \end{aligned}$$

which yields the right-hand side of (2.102) with $C_2 \,=\, C$.

$\square$

Lemma 2.10 *Let us define for each* $\boldsymbol{\psi}_{s,h} \in \mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma)$

$$T(\boldsymbol{\psi}_{s,h}) := \sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s \setminus \{\mathbf{0}\}} \frac{|\langle \boldsymbol{\tau}_{s,h}\,\boldsymbol{\nu}, \boldsymbol{\psi}_{s,h}\rangle_\Sigma|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}}$$

and

$$\widetilde{T}(\boldsymbol{\psi}_{s,h}) := \sup_{\substack{\boldsymbol{\phi}_h \in \Phi_h(\Sigma)\times\Phi_h(\Sigma) \\ \boldsymbol{\phi}_h \neq \mathbf{0}}} \frac{|\langle \boldsymbol{\phi}_h, \boldsymbol{\psi}_{s,h}\rangle_\Sigma|}{\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}}\,.$$

Then there exist C_3, $C_4 > 0$, *independent of* h, *such that*

$$C_3\,\widetilde{T}(\boldsymbol{\psi}_{s,h}) \,\le\, T(\boldsymbol{\psi}_{s,h}) \,\le\, C_4\,\widetilde{T}(\boldsymbol{\psi}_{s,h}) \qquad \forall\,\boldsymbol{\psi}_{s,h} \in \mathbf{Q}_h^s\,. \tag{2.103}$$

Proof. It follows analogously to the proof of Lemma 2.9 by using now, thanks to Lemma 2.8, that there exists $c_s > 0$, independent of h, such that $\|\mathcal{L}_h^s(\boldsymbol{\phi}_h)\|_{\mathbf{div};\Omega_s} \le c_s\,\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}$ $\forall\,\boldsymbol{\phi}_h \in \Phi_h(\Sigma)\times\Phi_h(\Sigma)$, and noting that $\|\boldsymbol{\tau}_{s,h}\,\boldsymbol{\nu}\|_{-1/2,\Sigma} \le C\,\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}$. We omit further details.

□

The previous two lemmas, more precisely the left-hand sides of the equivalences (2.102) and (2.103), will be employed below in Section 2.4.5 to show that the bilinear forms B_f and B_s satisfy the discrete inf-sup conditions on the corresponding finite element subspaces.

2.4.4 Discrete approximations of $\mathbf{P}_s|_{\mathbb{H}_h^s}$ and $\mathbf{P}_f|_{\mathbf{H}_h^f}$

In what follows we introduce uniformly bounded linear operators $\mathbf{P}_{s,h} : \mathbb{H}_h^s \to \mathbb{H}_h^s$ and $\mathbf{P}_{f,h} : \mathbf{H}_h^f \to \mathbf{H}_h^f$ approximating $\mathbf{P}_s|_{\mathbb{H}_h^s} : \mathbb{H}_h^s \to \mathbb{H}(\mathbf{div};\Omega_s)$ and $\mathbf{P}_f|_{\mathbf{H}_h^f} : \mathbf{H}_s^f \to \mathbf{H}(\mathrm{div};\Omega_f)$, respectively, and derive upper bounds for the associated errors given by $\|\mathbf{P}_s(\boldsymbol{\tau}_{s,h}) - \mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h})\|_{\mathbf{div};\Omega_s}$ (cf. Lema 2.11) and $\|\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) - \mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h})\|_{\mathrm{div};\Omega_f}$ (cf. Lemma 2.12) for each $(\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h := \mathbb{H}_h^s \times \mathbf{H}_h^f$. These are the key estimates utilized below in Section 2.4.5 to prove the discrete inf-sup conditions for the bilinear forms A_s and A_f (cf. Lemma 2.16).

Indeed, given $(\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h$, we first recall from (2.23) and (2.21) that $\mathbf{P}_s(\boldsymbol{\tau}_{s,h}) := \tilde{\boldsymbol{\sigma}}_s$, where $\tilde{\boldsymbol{\sigma}}_s = \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})$ and $\tilde{\mathbf{u}}$ is the unique solution of

$$\begin{gathered} \tilde{\boldsymbol{\sigma}}_s = \mathcal{C}\,\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) \quad \text{in} \quad \Omega_s\,, \quad \mathbf{div}\,\tilde{\boldsymbol{\sigma}}_s = (\mathbf{I}-\mathbf{M})\big(\mathbf{div}\,\boldsymbol{\tau}_{s,h}\big) \quad \text{in} \quad \Omega_s\,, \\ \tilde{\boldsymbol{\sigma}}_s\,\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Sigma\,, \quad \tilde{\mathbf{u}} \in (\mathbf{I}-\mathbf{M})(\mathbf{L}^2(\Omega_s))\,, \end{gathered} \tag{2.104}$$

In turn, we know from (2.30) and (2.28) that $\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) := \tilde{\boldsymbol{\sigma}}_f$, where $\tilde{\boldsymbol{\sigma}}_f := \nabla\tilde{p}$ and $\tilde{p}$ is the unique solution of

$$\begin{gathered} \tilde{\boldsymbol{\sigma}}_f = \nabla\tilde{p} \quad \text{in} \quad \Omega_f\,, \quad \mathrm{div}\,\tilde{\boldsymbol{\sigma}}_f = (\mathbf{I}-\mathbf{J})\big(\mathrm{div}\,\boldsymbol{\tau}_{f,h}\big) \quad \text{in} \quad \Omega_f\,, \\ \tilde{\boldsymbol{\sigma}}_f\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Sigma\cup\Gamma\,, \quad \tilde{p} \in (\mathbf{I}-\mathbf{J})(L^2(\Omega_f))\,. \end{gathered} \tag{2.105}$$

We now let $(\tilde{\boldsymbol{\sigma}}_{s,h}, \tilde{\mathbf{u}}_h, \tilde{\gamma}_h) \in \mathbb{H}_h^s \times (\mathbf{I}-\mathbf{M})(\mathbf{U}_h^s) \times \mathbb{Q}_h^s$ be the mixed finite element approximation of (2.104), which was introduced and analyzed in [37, Section 5.2], and define

$$\mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h}) \,:=\, \tilde{\boldsymbol{\sigma}}_{s,h}\,. \tag{2.106}$$

Hence, we know from [37, Section 5.2] that there hold

$$\|\mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h})\|_{\mathbf{div};\Omega_s} \;\leq\; C\,\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}\,, \tag{2.107}$$

$$\mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h})\,\boldsymbol{\nu} \,=\, \mathbf{0} \quad \text{on} \quad \Sigma \qquad \text{and} \qquad \int_{\Omega_s} \mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h}) : \tilde{\boldsymbol{\eta}}_h \,=\, 0 \quad \forall\, \tilde{\boldsymbol{\eta}}_h \in \mathbb{Q}_h^s\,. \tag{2.108}$$

The uniform boundedness of $\mathbf{P}_{s,h}$ is obvious from (2.107), whereas the first equation of (2.108) says that $\mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h})$ belongs to $\tilde{\mathbb{H}}(\mathbf{div};\Omega_s)$ (cf. (2.58)). Furthermore, in virtue of [37, Lemma 5.4], whose proof makes use of the definition (2.106), the conmuting diagram identity (2.79), the approximation properties (2.81), (AP_h^s), and ($\mathrm{AP}_h^{\boldsymbol{\gamma}}$), and the regularity estimate for (2.104) (cf. (2.22), (2.27)), we have the following error estimate.

Lemma 2.11 *Let $\epsilon > 0$ be the parameter defining the regularity of the solution of* (2.104)*. Then, there exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s$ there holds*

$$\|\mathbf{P}_s(\boldsymbol{\tau}_{s,h}) \,-\, \mathbf{P}_{s,h}(\boldsymbol{\tau}_{s,h})\|_{\mathbf{div};\Omega_s} \;\leq\; C\,h^{\epsilon}\,\|\mathbf{div}\,\boldsymbol{\tau}_{s,h}\|_{0,\Omega_s}\,. \tag{2.109}$$

We now turn to the definition and properties of $\mathbf{P}_{f,h}$. According to the regularity estimates given by (2.29) and (2.33), we know that $\mathbf{P}_f(\boldsymbol{\tau}_{f,h})$ belongs to $\mathbf{H}^{\epsilon}(\Omega_f)$ and

$$\|\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\|_{\epsilon,\Omega_f} \;\leq\; C\,\|\mathrm{div}\,\boldsymbol{\tau}_{f,h}\|_{0,\Omega_f}\,, \tag{2.110}$$

which suggests to consider the Raviart-Thomas interpolation operator $\mathcal{E}_h^f$ and define

$$\mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h}) \,:=\, \mathcal{E}_h^f\left(\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\right). \tag{2.111}$$

It follows, employing the conmuting diagram property (2.80), the second equation in (2.105) (which says that $\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) \,=\, \big(\mathbf{I}-\mathbf{J}\big)(\mathrm{div}\,\boldsymbol{\tau}_{f,h})$), and the fact that $\mathrm{div}\,\boldsymbol{\tau}_{f,h}$ is piecewise constant, that

$$\mathrm{div}\,\mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h}) \,=\, \mathcal{P}_h^f\big(\mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\big) \,=\, \mathcal{P}_h^f\big((\mathbf{I}-\mathbf{J})(\mathrm{div}\,\boldsymbol{\tau}_{f,h})\big) \,=\, \mathrm{div}\,\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\,. \tag{2.112}$$

Also, it is easy to see that the uniform boundedness of $\mathcal{E}_h^f : \mathbf{H}^{\epsilon}(\Omega_f) \cap \mathbf{H}(\mathrm{div};\Omega_f) \to \mathbf{H}_h^f$ (which follows from (2.82) and (2.80)), together with the estimate (2.110) and the identity (2.112), imply that $\mathbf{P}_{f,h}$ is uniformly bounded as well. In addition, using the characterization property (2.78) and the third equation in (2.105) (which says that $\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Sigma\cup\Gamma$), we easily deduce that

$$\mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h})\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Sigma\cup\Gamma\,. \tag{2.113}$$

We are now in a position to establish our second error estimate.

Lemma 2.12 *Let $\epsilon > 0$ be the parameter defining the regularity of the solution of* (2.105). *Then, there exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f$ there holds*

$$\|\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) - \mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h})\|_{\mathrm{div};\Omega_f} \le C\, h^\epsilon \, \|\mathrm{div}\, \boldsymbol{\tau}_{f,h}\|_{0,\Omega_f} . \tag{2.114}$$

Proof. We proceed as in the proof of [37, Lemma 5.4], though the present one becomes simpler. Let us first notice, in virtue of (2.111) and (2.112), that

$$\|\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) - \mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h})\|_{\mathrm{div};\Omega_f} = \|\mathbf{P}_f(\boldsymbol{\tau}_{f,h}) - \mathbf{P}_{f,h}(\boldsymbol{\tau}_{f,h})\|_{0,\Omega_f} = \|(\mathbf{I} - \mathcal{E}_h^f)\big(\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\big)\|_{0,\Omega_f} .$$

Hence, applying the approximation property (2.82) and the identity (2.112), we find that

$$\begin{aligned}
\|(\mathbf{I} - \mathcal{E}_h^f)\big(\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\big)\|_{0,\Omega_f}^2 &= \sum_{T\in\mathcal{T}_h^f} \|(\mathbf{I} - \mathcal{E}_h^f)\big(\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\big)\|_{0,T}^2 \\
&\le C \sum_{T\in\mathcal{T}_h^f} h_T^{2\epsilon} \Big\{ |\mathbf{P}_f(\boldsymbol{\tau}_{f,h})|_{\epsilon,T}^2 + \|\mathrm{div}\, \mathbf{P}_f(\boldsymbol{\tau}_{f,h})\|_{0,T}^2 \Big\} \\
&\le C\, h^{2\epsilon} \Big\{ \|\mathbf{P}_f(\boldsymbol{\tau}_{f,h})\|_{\epsilon,\Omega_f}^2 + \|\big(\mathbf{I} - \mathbf{J}\big)(\mathrm{div}\, \boldsymbol{\tau}_{f,h})\|_{0,\Omega_f}^2 \Big\} ,
\end{aligned}$$

which, together with the estimate (2.110) and the fact that $\|\mathbf{I} - \mathbf{J}\| \le 1$, completes the proof.

$\square$

2.4.5 Well-posedness of the Galerkin scheme

We now aim to show the well-posedness of the mixed finite element scheme (2.76). For this purpose, as established by a classical result on projection methods for Fredholm operators of index zero (see, e.g. [59, Theorem 13.7]), one just needs to prove that the Galerkin scheme associated to the isomorphism $\begin{pmatrix} \mathbf{A}_0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ is well-posed. Equivalently, in virtue of the identity (2.48), it suffices to apply the discrete Babuška-Brezzi theory to each one of the blocks $\begin{pmatrix} \mathbf{A}_s & \mathbf{B}_s^* \\ \mathbf{B}_s & \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{A}_f & \mathbf{B}_f^* \\ \mathbf{B}_f & \mathbf{0} \end{pmatrix}$. According to the above, in what follows we show that the bilinear forms A_s, B_s, A_f, and B_f (not necessarily in this order) satisfy the discrete inf-sup conditions on the corresponding finite element subspaces.

We begin our analysis with the derivation of the discrete inf-sup condition for B_f. To this end, and in order to apply Lemma 2.9, we first notice that for each $\boldsymbol{\psi}_{f,h} := (\psi_{h,\Sigma}, \psi_{h,\Gamma}) \in \mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)$ there holds

$$\begin{aligned}
\widetilde{S}(\boldsymbol{\psi}_h) &:= \sup_{\substack{\boldsymbol{\phi}_h := (\phi_{h,\Sigma},\phi_{h,\Gamma}) \\ \in \Phi_h(\Sigma)\times\Phi_h(\Gamma) \setminus\{\mathbf{0}\}}} \frac{|\langle \phi_{h,\Sigma}, \psi_{h,\Sigma}\rangle_\Sigma + \langle \phi_{h,\Gamma}, \psi_{h,\Gamma}\rangle_\Gamma|}{\|\boldsymbol{\phi}_h\|_{-1/2,\partial\Omega_f}} \\
&\ge \frac{1}{2} \left\{ \sup_{\phi_{h,\Sigma} \in \Phi_h(\Sigma)\setminus\{\mathbf{0}\}} \frac{|\langle \phi_{h,\Sigma}, \psi_{h,\Sigma}\rangle_\Sigma|}{\|\phi_{h,\Sigma}\|_{-1/2,\Sigma}} + \sup_{\phi_{h,\Gamma} \in \Phi_h(\Gamma)\setminus\{\mathbf{0}\}} \frac{|\langle \phi_{h,\Gamma}, \psi_{h,\Gamma}\rangle_\Gamma|}{\|\phi_{h,\Gamma}\|_{-1/2,\Gamma}} \right\} .
\end{aligned}$$

It follows, in virtue also of the left-hand side of (2.102), that a sufficient condition for the required inequality concerning B_f is the existence of $\tilde{\beta}_{f,\Sigma},\, \tilde{\beta}_{f,\Gamma} > 0$, independent of h, such that

$$\sup_{\phi_{h,\Sigma} \in \Phi_h(\Sigma)\backslash\{\mathbf{0}\}} \frac{|\langle \phi_{h,\Sigma}, \psi_{h,\Sigma}\rangle_\Sigma|}{\|\phi_{h,\Sigma}\|_{-1/2,\Sigma}} \geq \tilde{\beta}_{f,\Sigma}\, \|\psi_{h,\Sigma}\|_{1/2,\Sigma} \qquad \forall\, \psi_{h,\Sigma} \in \Lambda_h(\Sigma)\,, \tag{2.115}$$

and

$$\sup_{\phi_{h,\Gamma} \in \Phi_h(\Gamma)\backslash\{\mathbf{0}\}} \frac{|\langle \phi_{h,\Gamma}, \psi_{h,\Gamma}\rangle_\Gamma|}{\|\phi_{h,\Gamma}\|_{-1/2,\Gamma}} \geq \tilde{\beta}_{f,\Gamma}\, \|\psi_{h,\Gamma}\|_{1/2,\Gamma} \qquad \forall\, \psi_{h,\Gamma} \in \Lambda_h(\Gamma)\,. \tag{2.116}$$

Note that (2.115) and (2.116) constitute two independent discrete inf-sup conditions holding between subspaces living in Σ and Γ, respectively. Then, we recall from [45, Lemma 5.2] that a suitable choice of the subspaces $\Lambda_h(\Sigma)$ and $\Lambda_h(\Gamma)$ guarantees the ocurrence of the above. More precisely, let us assume, without loss of generality, that the number of edges of Σ_h and Γ_h are even numbers. Then, we let Σ_{2h} (resp. Γ_{2h}) be the partition of Σ (resp. Γ) arising by joining pairs of adjacent elements, and define

$$\Lambda_h(\Sigma) := \Big\{ \psi_h \in C(\Sigma): \quad \psi_h|_e \in P_1(e) \quad \forall\, e \text{ edge of } \Sigma_{2h} \Big\}\,, \tag{2.117}$$

$$\Lambda_h(\Gamma) := \Big\{ \psi_h \in C(\Gamma): \quad \psi_h|_e \in P_1(e) \quad \forall\, e \text{ edge of } \Gamma_{2h} \Big\}\,, \tag{2.118}$$

$$\mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)\,. \tag{2.119}$$

and

$$\mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma)\,. \tag{2.120}$$

In this way, we are in a position to establish the following result.

Lemma 2.13 *Let $\mathbf{Q}_h^f$ be given by* (2.119)*. Then there exists $\tilde{\beta}_f > 0$, independent of h, such that*

$$\sup_{\boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f\backslash\{\mathbf{0}\}} \frac{|B_f(\boldsymbol{\tau}_{f,h}, \boldsymbol{\psi}_{f,h})|}{\|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}} \geq \tilde{\beta}_f\, \|\boldsymbol{\psi}_{f,h}\|_{1/2,\partial\Omega_f} \qquad \forall\, \boldsymbol{\psi}_{f,h} \in \mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)\,.$$

Proof. A straightforward application of [45, Lemma 5.2] to the pairs of subspaces $(\Phi_h(\Sigma), \Lambda_h(\Sigma))$ and $(\Phi_h(\Gamma), \Lambda_h(\Gamma))$ imply (2.115) and (2.116), and hence the previous discussion completes the proof with the constant $\tilde{\beta}_f = \frac{C_1}{2} \min\Big\{ \tilde{\beta}_{f,\Sigma}, \tilde{\beta}_{f,\Gamma} \Big\}$.

□

Before continuing the analysis, we let $\Pi_\Sigma : H^{1/2}(\Sigma) \to \Lambda_h(\Sigma)$ and $\Pi_\Gamma : H^{1/2}(\Gamma) \to \Lambda_h(\Gamma)$ be the orthogonal projectors, and recall from [10] that the approximation properties of $\Lambda_h(\Sigma)$ and $\Lambda_h(\Gamma)$ are given as follows:

($\mathrm{AP}_{\Sigma,h}$) For each $\delta \in (0,1]$ and for each $\psi \in H^{1/2+\delta}(\Sigma)$, there holds

$$\|\psi - \Pi_\Sigma(\psi)\|_{1/2,\Sigma} \leq C\, h_\Sigma^\delta\, \|\psi\|_{1/2+\delta,\Sigma}\,.$$

$(\mathrm{AP}_{\Gamma,h})$ For each $\delta \in (0,1]$ and for each $\psi \in H^{1/2+\delta}(\Gamma)$, there holds

$$\|\psi - \Pi_\Gamma(\psi)\|_{1/2,\Gamma} \leq C\, h_\Gamma^\delta \, \|\psi\|_{1/2+\delta,\Gamma} \,.$$

Note that $(\mathrm{AP}_{\Sigma,h})$ and $(\mathrm{AP}_{\Gamma,h})$ yield the approximation properties of $\mathbf{Q}_h^s$ and $\mathbf{Q}_h^f$ (cf. (2.73), (2.74)).

We now turn to the connection between Lemma 2.10 and the discrete inf-sup condition for the bilinear form B_s (cf. (2.18)) with $\mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma)$ and $\Lambda_h(\Sigma)$ given by (2.117). We first notice that for each $\boldsymbol{\psi}_{s,h} := (\psi_{h,\Sigma}, \widetilde{\psi}_{h,\Sigma}) \in \mathbf{Q}_h^s$ there holds, denoting $\boldsymbol{\phi}_h := (\phi_{h,\Sigma}, \widetilde{\phi}_{h,\Sigma}) \in \Phi_h(\Sigma) \times \Phi_h(\Sigma)$,

$$\begin{aligned} \widetilde{T}(\boldsymbol{\psi}_{s,h}) \,&:=\, \sup_{\substack{\boldsymbol{\phi}_h \in \Phi_h(\Sigma)\times\Phi_h(\Sigma) \\ \boldsymbol{\phi}_h \neq \mathbf{0}}} \frac{|\,\langle \boldsymbol{\phi}_h, \boldsymbol{\psi}_{s,h}\rangle_\Sigma\,|}{\|\boldsymbol{\phi}_h\|_{-1/2,\Sigma}} \\ &\geq\, \frac{1}{2} \left\{ \sup_{\phi_{h,\Sigma} \in \Phi_h(\Sigma)\backslash\{\mathbf{0}\}} \frac{|\,\langle \phi_{h,\Sigma}, \psi_{h,\Sigma}\rangle_\Sigma\,|}{\|\phi_{h,\Sigma}\|_{-1/2,\Sigma}} \,+\, \sup_{\widetilde{\phi}_{h,\Sigma} \in \Phi_h(\Sigma)\backslash\{\mathbf{0}\}} \frac{|\,\langle \widetilde{\phi}_{h,\Sigma}, \widetilde{\psi}_{h,\Sigma}\rangle_\Sigma\,|}{\|\widetilde{\phi}_{h,\Sigma}\|_{-1/2,\Sigma}} \right\}. \end{aligned}$$

Hence, since [45, Lemma 5.2] guarantees (2.115), we deduce from the above inequality that

$$\widetilde{T}(\boldsymbol{\psi}_{s,h}) \,\geq\, \tilde{\beta}_{f,\Sigma} \left\{ \|\psi_{h,\Sigma}\|_{1/2,\Sigma} + \|\widetilde{\psi}_{h,\Sigma}\|_{1/2,\Sigma} \right\} \qquad \forall\, \boldsymbol{\psi}_{s,h} := (\psi_{h,\Sigma}, \widetilde{\psi}_{h,\Sigma}) \in \mathbf{Q}_h^s \,,$$

which, combined with the left-hand side of (2.103), yields

$$T(\boldsymbol{\psi}_{s,h}) \,:=\, \sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\backslash\{\mathbf{0}\}} \frac{|\,\langle \boldsymbol{\tau}_{s,h}\,\boldsymbol{\nu}, \boldsymbol{\psi}_{s,h}\rangle_\Sigma\,|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \,\geq\, C_3\, \tilde{\beta}_{f,\Sigma}\, \|\boldsymbol{\psi}_{s,h}\|_{1/2,\Sigma} \qquad \forall\, \boldsymbol{\psi}_{s,h} \in \mathbf{Q}_h^s \,. \tag{2.121}$$

Consequently, we are now able to prove the following lemma.

Lemma 2.14 *Let $\mathbf{Q}_h^s$ be given by* (2.120)*. Then there exists $\tilde{\beta}_s > 0$, independent of h, such that*

$$\sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\backslash\{\mathbf{0}\}} \frac{|\,B_s(\boldsymbol{\tau}_{s,h}, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}))\,|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \,\geq\, \tilde{\beta}_s\, \|(\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h})\| \qquad \forall\, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}) \in \mathbb{Q}_h^s \times \mathbf{Q}_h^s \,.$$

Proof. Given $(\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}) \in \mathbb{Q}_h^s \times \mathbf{Q}_h^s$ we have, according to the definition of B_s (cf. (2.18)), that

$$\sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\backslash\{\mathbf{0}\}} \frac{|\,B_s(\boldsymbol{\tau}_{s,h}, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}))\,|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \,\geq\, \sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\backslash\{\mathbf{0}\}} \frac{|\,\langle \boldsymbol{\tau}_{s,h}\,\boldsymbol{\nu}, \boldsymbol{\psi}_{s,h}\rangle_\Sigma\,|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \,-\, \|\boldsymbol{\eta}_h\|_{0,\Omega_s} \,,$$

which, thanks to (2.121), implies that

$$\sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s\backslash\{\mathbf{0}\}} \frac{|\,B_s(\boldsymbol{\tau}_{s,h}, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}))\,|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \,\geq\, C_3\, \tilde{\beta}_{f,\Sigma}\, \|\boldsymbol{\psi}_{s,h}\|_{1/2,\Sigma} \,-\, \|\boldsymbol{\eta}_h\|_{0,\Omega_s} \,. \tag{2.122}$$

Furthermore, we know from [62, Theorem 4.5] (see also [4, Lemma 4.4]) that there exists $\boldsymbol{\zeta}_{s,h} \in \mathbb{H}_h^s$ such that $\boldsymbol{\zeta}_{s,h}\,\boldsymbol{\nu} = \mathbf{0}$ on Σ, $\mathbf{div}\,\boldsymbol{\zeta}_{s,h} = \mathbf{0}$ in Ω_s, and

$$| \, B_s(\boldsymbol{\zeta}_{s,h}, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h})) \, | \,\geq\, C \, \|\boldsymbol{\zeta}_{s,h}\|_{0,\Omega_s} \, \|\boldsymbol{\eta}\|_{0,\Omega_s} \,=\, C \, \|\boldsymbol{\zeta}_{s,h}\|_{\mathbf{div};\Omega_s} \, \|\boldsymbol{\eta}\|_{0,\Omega_s} \,,$$

which yields

$$\sup_{\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s \backslash \{\mathbf{0}\}} \frac{| \, B_s(\boldsymbol{\tau}_{s,h}, (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h})) \, |}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \,\geq\, C \, \|\boldsymbol{\eta}_h\|_{0,\Omega_s} \,. \tag{2.123}$$

Finally, a suitable linear combination of (2.122) and (2.123) gives the required inequality.

□

We now let $\mathbf{V}_{s,h}$ and $\mathbf{V}_{f,h}$ be the discrete kernels of B_s (cf. (2.18)) and B_f (cf. (2.19)), that is,

$$\mathbf{V}_{s,h} := \left\{ \boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s : \quad \int_{\Omega_s} \boldsymbol{\tau}_{s,h} : \boldsymbol{\eta}_h = 0 \quad \forall \, \boldsymbol{\eta}_h \in \mathbb{Q}_h^s \,, \quad \langle \boldsymbol{\tau}_{s,h} \, \boldsymbol{\nu}, \boldsymbol{\psi}_{s,h} \rangle_\Sigma = 0 \quad \forall \, \boldsymbol{\psi}_{s,h} \in \mathbf{Q}_h^s \right\}, \tag{2.124}$$

$$\mathbf{V}_{f,h} \,:=\, \left\{ \boldsymbol{\tau}_{f,h} \in \mathbf{H}_h^f : \quad \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Sigma} \rangle_\Sigma \,=\, \langle \boldsymbol{\tau}_{f,h} \cdot \boldsymbol{\nu}, \psi_{h,\Gamma} \rangle_\Gamma = 0 \quad \forall \, (\psi_{h,\Sigma}, \psi_{h,\Gamma}) \in \mathbf{Q}_h^f \right\}, \tag{2.125}$$

and aim to prove that the bilinear forms A_s and A_f satisfy the discrete inf-sup conditions on $\mathbf{V}_{s,h} \times \mathbf{V}_{s,h}$ and $\mathbf{V}_{f,h} \times \mathbf{V}_{f,h}$, respectively.

We begin by observing that $\mathbf{V}_{s,h}$ is certainly contained in

$$\tilde{\mathbf{V}}_{s,h} \,:=\, \left\{ \boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div}; \Omega_s) : \quad \langle \boldsymbol{\tau}_s \, \boldsymbol{\nu}, \boldsymbol{\psi}_{s,h} \rangle_\Sigma = 0 \quad \forall \, \boldsymbol{\psi}_{s,h} \in \mathbf{Q}_h^s \right\},$$

which is not a subspace of $\tilde{\mathbb{H}}(\mathbf{div}; \Omega_s)$ (cf. (2.58)) but on the contrary contains it. While this latter fact prevent us of applying directly (2.57) (and hence the ellipticity estimates (2.59) and (2.64)) to the whole $\tilde{\mathbf{V}}_{s,h}$, we show next that actually (2.57) does also hold in this bigger space. In fact, let us first pick one corner point of Σ and define a function v that is continuous, linear on each side of Σ, equal to one in the chosen vertex and zero on all other ones. Then, it is easy to check that, if $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ are the normal vectors on the two sides of Σ that meet at the corner point, the function $\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Sigma)$ given by $\boldsymbol{\psi} \,:=\, v\,(\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2)$ belongs to $\mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma)$ for each $h > 0$, and satisfies

$$\langle \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma \,\neq 0 \,.$$

This function $\boldsymbol{\psi}$ in $\mathbf{Q}_h^s$ is employed next to prove the validity of (2.57) in $\tilde{\mathbf{V}}_{s,h}$.

Lemma 2.15 *There exists $\tilde{c}_2 > 0$, independent of h, such that*

$$\|\boldsymbol{\tau}_{s,0}\|^2_{\mathbf{div};\Omega_s} \,\geq\, \tilde{c}_2 \, \|\boldsymbol{\tau}_s\|^2_{\mathbf{div};\Omega_s} \qquad \forall \, \boldsymbol{\tau}_s \,\in\, \tilde{\mathbf{V}}_{s,h} \,, \tag{2.126}$$

where $\boldsymbol{\tau}_s \,=\, \boldsymbol{\tau}_{s,0} \,+\, d\,\mathbf{I}$, with $\boldsymbol{\tau}_{s,0} \,\in\, \mathbb{H}_0(\mathbf{div}; \Omega_s)$ (cf. (2.55)) and $d \in \mathbb{C}$.

Proof. Given $\boldsymbol{\tau}_s \in \tilde{\mathbf{V}}_{s,h}$ we clearly have, using that $\boldsymbol{\psi} \in \mathbf{Q}_h^s$ for each $h > 0$, that

$$0 \,=\, \langle \boldsymbol{\tau}_s \,\boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma \,=\, \langle \boldsymbol{\tau}_{s,0} \,\boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma \,+\, d \,\langle \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma \,,$$

which gives

$$d \,=\, - \,\frac{\langle \boldsymbol{\tau}_{s,0} \,\boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma}{\langle \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma} \,,$$

and hence

$$|d| \,\leq\, C \,\frac{\|\boldsymbol{\psi}\|_{1/2,\Sigma}}{|\,\langle \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Sigma \,|} \,\|\boldsymbol{\tau}_{s,0}\|_{\mathbf{div};\Omega_s} \,.$$

This inequality and the fact that $\|\boldsymbol{\tau}_s\|^2_{\mathbf{div};\Omega_s} \,=\, \|\boldsymbol{\tau}_{s,0}\|^2_{\mathbf{div};\Omega_s} \,+\, 2\,d^2\,|\Omega_s|$ imply (2.126).

□

As a consequence of Lemma 2.15, and following basically the same arguments employed in the proofs of Lemmas 2.3 and 2.4, we deduce that the inequalities (2.59) and (2.64) also hold in $\tilde{\mathbf{V}}_{s,h}$. In particular, the latter says that there exists $\tilde{\alpha}_s > 0$, independent of h, such that

$$A_s\big(\boldsymbol{\tau}_s, \Xi_s(\overline{\boldsymbol{\tau}}_s)\big) \,\geq\, \tilde{\alpha}_s \,\|\boldsymbol{\tau}_s\|^2_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\tau}_s \,\in\, \tilde{\mathbf{V}}_{s,h} \,. \tag{2.127}$$

We are now ready to prove the discrete analogues of (2.65) (cf. Lemma 2.4) and (2.68) (cf. Lemma 2.5), which constitute the required discrete inf-sup conditions for A_s and A_f.

Lemma 2.16 *There exist $\widetilde{C}_s$, $\widetilde{C}_f$, $h_0 \,>\, 0$, independent of h, such that for each $h \leq h_0$ there holds*

$$\sup_{\boldsymbol{\tau}_{s,h} \in \mathbf{V}_{s,h} \backslash \{\mathbf{0}\}} \frac{\big|A_s(\boldsymbol{\zeta}_{s,h}, \boldsymbol{\tau}_{s,h})\big|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \,\geq\, \widetilde{C}_s \,\|\boldsymbol{\zeta}_{s,h}\|_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\zeta}_{s,h} \,\in\, \mathbf{V}_{s,h} \,. \tag{2.128}$$

and

$$\sup_{\boldsymbol{\tau}_{f,h} \in \mathbf{V}_{f,h} \backslash \{\mathbf{0}\}} \frac{\big|A_f(\boldsymbol{\zeta}_{f,h}, \boldsymbol{\tau}_{f,h})\big|}{\|\boldsymbol{\tau}_{f,h}\|_{\mathrm{div};\Omega_f}} \,\geq\, \widetilde{C}_f \,\|\boldsymbol{\zeta}_{f,h}\|_{\mathrm{div};\Omega_f} \qquad \forall\, \boldsymbol{\zeta}_{f,h} \,\in\, \mathbf{V}_{f,h} \,. \tag{2.129}$$

Proof. In order to prove (2.128) we introduce the natural discrete approximation of the operator Ξ_s (cf. (2.60)) given by $\Xi_{s,h} \,:=\, \big(\mathbf{I} \,-\, 2\,\mathbf{P}_{s,h}\big) : \mathbb{H}_h^s \to \mathbb{H}_h^s$, with $\mathbf{P}_{s,h}$ defined by (2.106). In this way, it follows directly from (2.109) (cf. Lemma 2.11) that

$$\|\Xi_s(\boldsymbol{\zeta}_{s,h}) \,-\, \Xi_{s,h}(\boldsymbol{\zeta}_{s,h})\|_{\mathbf{div};\Omega_s} \,\leq\, C\,h^\epsilon\,\|\boldsymbol{\zeta}_{s,h}\|_{\mathbf{div};\Omega_s} \qquad \forall\, \boldsymbol{\zeta}_{s,h} \in \mathbb{H}_h^s \,.$$

Hence, taking in particular $\boldsymbol{\zeta}_{s,h} \,\in\, \mathbf{V}_{s,h}$, adding and substracting $\Xi_s\big(\overline{\boldsymbol{\zeta}}_{s,h}\big)$, using the boundedness of A_s, and applying the inequality (2.127) (having in mind that $\mathbf{V}_{s,h} \,\subseteq\, \tilde{\mathbf{V}}_{s,h}$), we find that

$$\big|\,A_s(\boldsymbol{\zeta}_{s,h}, \Xi_{s,h}(\overline{\boldsymbol{\zeta}}_{s,h}))\,\big| \,\geq\, \big|\,A_s(\boldsymbol{\zeta}_{s,h}, \Xi_s(\overline{\boldsymbol{\zeta}}_{s,h})\,\big| \,-\, \tilde{C}\,h^\epsilon\,\|\boldsymbol{\zeta}_{s,h}\|^2_{\mathbf{div};\Omega_s} \,\geq\, \Big\{\tilde{\alpha}_s \,-\, \tilde{C}\,h^\epsilon\Big\}\,\|\boldsymbol{\zeta}_{s,h}\|^2_{\mathbf{div};\Omega_s} \,,$$

from which we deduce the existence of c, $h_0 > 0$, independent of h, such that

$$\big| A_s(\boldsymbol{\zeta}_{s,h}, \Xi_{s,h}(\overline{\boldsymbol{\zeta}}_{s,h})) \big| \,\geq\, c\,\|\boldsymbol{\zeta}_{s,h}\|^2_{\mathbf{div};\Omega_s} \qquad \forall\,\boldsymbol{\zeta}_{s,h} \in \mathbf{V}_{s,h}\,, \quad \forall\, h \leq h_0\,. \tag{2.130}$$

Note from this inequality that $\Xi_{s,h}(\boldsymbol{\zeta}_{s,h}) \neq \mathbf{0}$ for each $\boldsymbol{\zeta}_{s,h} \neq \mathbf{0}$. Also, it is clear from (2.108) and the characterization of $\mathbf{V}_{s,h}$ (cf. (2.124)) that $\mathbf{P}_{s,h}(\boldsymbol{\zeta}_{s,h})$, and hence $\Xi_{s,h}(\boldsymbol{\zeta}_{s,h})$, belong to $\mathbf{V}_{s,h}$ for each $\boldsymbol{\zeta}_{s,h} \in \mathbf{V}_{s,h}$. Consequently, we employ (2.130) to bound the supremum on $\mathbf{V}_{s,h}\backslash\{\mathbf{0}\}$ as follows

$$\sup_{\boldsymbol{\tau}_{s,h}\in\mathbf{V}_{s,h}\backslash\{\mathbf{0}\}} \frac{\big|A_s(\boldsymbol{\zeta}_{s,h}, \boldsymbol{\tau}_{s,h})\big|}{\|\boldsymbol{\tau}_{s,h}\|_{\mathbf{div};\Omega_s}} \,\geq\, \frac{\big|A_s(\boldsymbol{\zeta}_{s,h}, \Xi_{s,h}(\overline{\boldsymbol{\zeta}}_{s,h}))\big|}{\|\Xi_{s,h}(\overline{\boldsymbol{\zeta}}_{s,h})\|_{\mathbf{div};\Omega_s}} \,\geq\, c\,\frac{\|\boldsymbol{\zeta}_{s,h}\|^2_{\mathbf{div};\Omega_s}}{\|\Xi_{s,h}(\overline{\boldsymbol{\zeta}}_{s,h})\|_{\mathbf{div};\Omega_s}}$$

for each $\boldsymbol{\zeta}_{s,h} \in \mathbf{V}_{s,h}$ and for each $h \leq h_0$, which, thanks to the uniform boundedness of $\|\Xi_{s,h}\|$, say by a constant $\bar{C} > 0$, imply (2.128) with $\widetilde{C}_s \,=\, c/\bar{C}$.

The proof of (2.129) proceeds analogously by considering now $\Xi_{f,h} \,:=\, \big(\mathbf{I} \,-\, 2\,\mathbf{P}_{f,h}\big) : \mathbf{H}^f_h \to \mathbf{H}^f_h$, with $\mathbf{P}_{f,h}$ defined by (2.111), applying the inequality (2.67) (cf. Lemma 2.5), using, thanks to (2.114) (cf. Lemma 2.12), that

$$\|\Xi_f(\boldsymbol{\zeta}_{f,h}) \,-\, \Xi_{f,h}(\boldsymbol{\zeta}_{f,h})\|_{\mathrm{div};\Omega_f} \,\leq\, C\,h^{\epsilon}\,\|\boldsymbol{\zeta}_{f,h}\|_{\mathrm{div};\Omega_f} \qquad \forall\,\boldsymbol{\zeta}_{f,h} \in \mathbb{H}^f_h\,,$$

and noting, in virtue of (2.113), that $\Xi_{f,h}(\boldsymbol{\zeta}_{f,h}) \,\in\, \mathbf{V}_{f,h}$ (cf. (2.125)) for each $\boldsymbol{\zeta}_{f,h} \in \mathbf{V}_{f,h}$.

$\square$

The following theorem establishes the well-posedness and convergence of the discrete scheme (2.76) with the finite element subspaces $\mathbb{H}^s_h$, $\mathbf{H}^f_s$, $\mathbb{Q}^s_h$, $\mathbf{Q}^s_h$, $\mathbf{Q}^f_h$, $\Lambda_h(\Sigma)$, and $\Lambda_h(\Gamma)$, given, respectively, by (2.70), (2.71), (2.72), (2.73), (2.74), (2.117), and (2.118).

Theorem 2.2 *Assume that the homogeneous problem associated to* (2.15) *has only the trivial solution, and let $h_0 > 0$ be the constant provided by Lemma* 2.16*. Then there exists $h_1 \,\in\, (0, h_0]$ such that for each $h \,\in\, (0, h_1]$, the fully-mixed finite element scheme* (2.76) *has a unique solution $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h) \,:=\, ((\boldsymbol{\sigma}_{s,h}, \boldsymbol{\sigma}_{f,h}), (\gamma_h, \boldsymbol{\varphi}_{s,h}, \boldsymbol{\varphi}_{f,h})) \,\in\, \mathbf{H}_h \times \mathbf{Q}_h$. In addition, there exist C_1, $C_2 \,>\, 0$, independent of h, such that for each $h \,\in\, (0, h_1]$ there hold*

$$\|(\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leq\, C_1\left\{\sup_{\widehat{\boldsymbol{\tau}}_h\,\in\mathbf{H}_h\backslash\{\mathbf{0}\}} \frac{|F(\widehat{\boldsymbol{\tau}}_h)|}{\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} \,+\, \sup_{\widehat{\boldsymbol{\eta}}_h\,\in\mathbf{Q}_h\backslash\{\mathbf{0}\}} \frac{|G(\widehat{\boldsymbol{\eta}}_h)|}{\|\widehat{\boldsymbol{\eta}}_h\|_{\mathbf{Q}}}\right\} \,\leq\, C_1\left\{\|\mathbf{f}\|_{0,\Omega_s} + \|g\|_{-1/2,\Gamma}\right\}$$

and

$$\|(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) \,-\, (\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leq\, C_2 \inf_{(\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\eta}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h} \|(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) \,-\, (\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\eta}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\,, \tag{2.131}$$

where $(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\gamma}}) := ((\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_f), (\boldsymbol{\gamma}, \boldsymbol{\varphi}_s, \boldsymbol{\varphi}_f)) \in \mathbf{H} \times \mathbf{Q}$ *is the unique solution of* (2.15). *Furthermore, if there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_s \in \mathbb{H}^\delta(\Omega_s)$, $\mathbf{div}\, \boldsymbol{\sigma}_s \in \mathbf{H}^\delta(\Omega_s)$, $\boldsymbol{\sigma}_f \in \mathbf{H}^\delta(\Omega_f)$, $\operatorname{div} \boldsymbol{\sigma}_f \in H^\delta(\Omega_f)$, $\boldsymbol{\gamma} \in \mathbb{H}^\delta(\Omega_s)$, $\boldsymbol{\varphi}_s \in \mathbf{H}^{1/2+\delta}(\Sigma)$, *and* $\boldsymbol{\varphi}_f \in H^{1/2+\delta}(\partial\Omega_f)$, *then for each* $h \in (0, h_1]$ *there holds*

$$\|(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\gamma}}) - (\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\gamma}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C_3\, h^\delta \Big\{ \|\boldsymbol{\sigma}_s\|_{\delta,\Omega_s} + \|\mathbf{div}\, \boldsymbol{\sigma}_s\|_{\delta,\Omega_s} + \|\boldsymbol{\sigma}_f\|_{\delta,\Omega_f}$$
$$+ \quad \|\operatorname{div} \boldsymbol{\sigma}_f\|_{\delta,\Omega_f} + \|\boldsymbol{\gamma}\|_{\delta,\Omega_s} + \|\boldsymbol{\varphi}_s\|_{1/2+\delta,\Sigma} + \|\boldsymbol{\varphi}_f\|_{1/2+\delta,\partial\Omega_f} \Big\},$$

with a constant $C_3 > 0$, *independent of* h.

Proof. Because of Lemmas 2.13, 2.14, and 2.16, the proof of the first part is a straightforward application of [59, Theorem 13.7]. In turn, the rate of convergence follows directly from the Cea estimate (2.131) and the approximation properties of the finite element subspaces involved (see ($\mathrm{AP}_h^{\boldsymbol{\sigma}_s}$), ($\mathrm{AP}_h^{\boldsymbol{\sigma}_f}$), ($\mathrm{AP}_h^{\boldsymbol{\gamma}}$) in Section 2.4.2, and ($\mathrm{AP}_{\Sigma,h}$) and ($\mathrm{AP}_{\Gamma,h}$) above in the present section).

□

2.5 Numerical results

In this section we present three examples showing the performance of our fully-mixed finite element scheme (2.76). Examples 1 and 2 consider smooth exact solutions, whereas Example 3, whose exact solution is singular, is utilized to illustrate the regularity dependence of the rate of convergence (cf. Theorem 2.2). We begin by introducing additional notations. The variable N stands for the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$ (cf. (2.75)), and the individual errors are denoted by

$$\mathsf{e}(\boldsymbol{\sigma}_s) := \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{\mathbf{div};\Omega_s}, \quad \mathsf{e}(\boldsymbol{\sigma}_f) := \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{\operatorname{div};\Omega_f}, \quad \mathsf{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega_s},$$

$$\mathsf{e}(\boldsymbol{\varphi}_s) := \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{1/2,\Sigma}, \quad \mathsf{e}(\varphi_\Sigma) := \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{1/2,\Sigma} \quad \text{and} \quad \mathsf{e}(\varphi_\Gamma) := \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{1/2,\Gamma},$$

where $\boldsymbol{\varphi}_f := (\varphi_\Sigma, \varphi_\Gamma) \in H^{1/2}(\Sigma) \times H^{1/2}(\Gamma)$ and $\boldsymbol{\varphi}_{f,h} := (\varphi_{\Sigma,h}, \varphi_{\Gamma,h}) \in \mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)$. Also, we let $r(\boldsymbol{\sigma}_s)$, $r(\boldsymbol{\sigma}_f)$, $r(\boldsymbol{\gamma})$, $r(\boldsymbol{\varphi}_s)$, $r(\varphi_\Sigma)$ and $r(\varphi_\Gamma)$ be the experimental rates of convergence given by

$$r(\boldsymbol{\sigma}_s) := \frac{\log\big(\mathsf{e}(\boldsymbol{\sigma}_s)/\mathsf{e}'(\boldsymbol{\sigma}_s)\big)}{\log(h/h')}, \quad r(\boldsymbol{\sigma}_f) := \frac{\log\big((\mathsf{e}(\boldsymbol{\sigma}_f)/\mathsf{e}'(\boldsymbol{\sigma}_f)\big)}{\log(h/h')},$$

$$r(\boldsymbol{\gamma}) := \frac{\log\big(\mathsf{e}(\boldsymbol{\gamma})/\mathsf{e}'(\boldsymbol{\gamma})\big)}{\log(h/h')}, \quad r(\boldsymbol{\varphi}_s) := \frac{\log\big(\mathsf{e}(\boldsymbol{\varphi}_s)/\mathsf{e}'(\boldsymbol{\varphi}_s)\big)}{\log(h/h')},$$

$$r(\varphi_\Sigma) := \frac{\log\big(\mathsf{e}(\varphi_\Sigma)/\mathsf{e}'(\varphi_\Sigma)\big)}{\log(h/h')} \quad \text{and} \quad r(\varphi_\Gamma) := \frac{\log\big(\mathsf{e}(\varphi_\Gamma)/\mathsf{e}'(\varphi_\Gamma)\big)}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with corresponding errors e and e'.

We first consider $\Omega_s := (-0.2, 0.2) \times (-0.4, 0.4)$ and let the artificial boundary Γ be the ellipse centered at the origin with minor and major semiaxis given by 0.4 and 0.6, respectively, that is $\Omega_f := \left\{ (x_1, x_2)^{\mathtt{t}} \in \mathbb{R}^2 : \quad \frac{x_1^2}{0.4^2} + \frac{x_2^2}{0.6^2} < 1 \right\} \setminus \overline{\Omega}_s$. We take $\rho_s = \rho_f = \lambda = \mu = 1$, and the rest of parameters are given by the sets

$$\left\{ v_0 = 1;\, \omega = 5;\, \kappa_s = 5;\, \kappa_f = 5 \right\} \quad \text{and} \quad \left\{ v_0 = 0.7;\, \omega = 7;\, \kappa_s = 7;\, \kappa_f = 10 \right\},$$

which define Examples 1 and 2, respectively. Furthermore, let K_0, K_1 and K_2 be the modified Bessel functions of the second kind and order 0, 1, and 2, respectively, and let $H_0^{(1)}$ be the Hankel function of the first kind and order zero. Then, we choose the data in such a way that the exact solution of (2.6) (or (2.15)) is determined by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \dfrac{1}{2\pi}\, \psi(\mathbf{x}) \; - \; \dfrac{(x_1 - 1)^2}{r_1^2}\, \chi(\mathbf{x}) \\[2ex] - \, \dfrac{(x_1 - 1)\, x_2}{r_1^2}\, \chi(\mathbf{x}) \end{pmatrix} \quad \forall\, \mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega_s\,, \text{ and } \; p(\mathbf{x}) = H_0^{(1)}(\omega\, |\mathbf{x}|) \;\; \forall\, \mathbf{x} \in \Omega_f\,,$$

where $\quad r_1 := \sqrt{(x_1 - 1)^2 + x_2^2}, \quad \psi(\mathbf{x}) := K_0(\imath\, \omega\, r_1) + \frac{1}{\imath\, \omega\, r_1} \left\{ K_1(\imath\, \omega\, r_1) - \frac{1}{\sqrt{3}}\, K_1\left(\frac{\imath\, \omega\, r_1}{\sqrt{3}} \right) \right\}$, and $\quad \chi(\mathbf{x}) := K_2(\imath\, \omega\, r_1) - \frac{1}{3}\, K_2\left(\frac{\imath\, \omega\, r_1}{\sqrt{3}} \right)$. Actually, $\mathbf{u}$ is the fundamental solution, centered at $(1, 0)^{\mathtt{t}}$, of the elastodynamic equation, which yields $\mathbf{f} = \mathbf{0}$ in Ω_s, and p is the fundamental solution, centered at the origin, of the Helmholtz equation in Ω_f.

Then, for Example 3 we let Ω_s be the L-shaped domain $(-0.3, 0.3)^2 \setminus (0, 0.3)^2$ and consider Γ as the boundary of the unit circle $B(\mathbf{0}, 1)$. In addition, we take $\rho_s = \rho_f = \lambda = \mu = 1$, $v_0 = 6/11$, and $\omega = 6$, so that $\kappa_s = 6$ and $\kappa_f = 11$. Then, we choose the data in such a way that the exact solution of (2.6) (or (2.15)) is given by

$$\mathbf{u}(\mathbf{r}, \theta) \; := \; \mathbf{r}^{5/3}\, \sin\big((2\,\theta - \pi)/3 \big) \begin{pmatrix} 1 + \imath \\[1ex] 1 + \imath \end{pmatrix} \qquad \forall\, (\mathbf{r}, \theta) \, \in \, \Omega_s\,,$$

and

$$p(\mathbf{x}) = H_0^{(1)}(\omega\, |\mathbf{x} + (0.15, 0)|) \qquad \forall\, \mathbf{x} \in \Omega_f\,,$$

Note that $\mathbf{u}$ becomes singular at the origin, the corner of the L. More precisely, it is not difficult to see that around this singularity $\mathbf{div}\, \boldsymbol{\sigma}_s$ behaves of order $\mathbf{r}^{-1/3}$. It follows that $\mathbf{div}\, \boldsymbol{\sigma}_s$ belongs to $\mathbf{H}^{2/3 - \epsilon}(\Omega_s)$ for each $\epsilon > 0$, and hence, according to Theorem 2.2, we expect experimental rates of convergence, particularly $r(\boldsymbol{\sigma}_s)$, close to $2/3$.

In Tables 2.1 to 2.4 we present the convergence history of Examples 1 and 2 for finite sequences of quasi-uniform triangulations of the computational domain $\overline{\Omega}_s \cup \overline{\Omega}_f$. We remark that the rate of convergence $O(h)$ predicted by Theorem 2.2 (when $\delta = 1$) is attained for all the unknowns in both cases. In particular, we observe that the errors $\mathsf{e}(\boldsymbol{\varphi}_s)$, $\mathsf{e}(\varphi_\Sigma)$, and $\mathsf{e}(\varphi_\Gamma)$ converge a bit faster than expected. On the other hand, in Table 2.5 we display the convergence history of some unknowns of Example 3 for finite sequences of quasi-uniform triangulations of the computational domain $\overline{\Omega}_s \cup \overline{\Omega}_f$. We notice here, as already announced, that $r(\boldsymbol{\sigma}_s)$ oscillates in fact around 2/3. However, the other rates of convergence shown there are not affected by the lack of regularity of $\boldsymbol{\sigma}_s$. Finally, in Figures 2.1 to 2.8 we display real and imaginary parts of some components of the approximate and exact solutions of Examples 1 and 2 for $N = 13666$. The fact that they do not distinguish from each other illustrates the accurateness of the proposed fully-mixed method. Note that in the case of the unknowns on the boundaries, they are depicted along straight lines beginning at the points $(0.2, 0.4)$ and $(0.4, 0.0)$ for Σ and Γ, respectively, and then continuing counterclockwise.

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	1117	6.150E−02	–	8.865E−01	–	6.642E−03	–
$2\pi/96$	2090	4.264E−02	0.903	5.996E−01	0.964	3.975E−03	1.266
$2\pi/128$	3686	3.112E−02	1.095	4.414E−01	1.065	2.570E−03	1.516
$2\pi/192$	7869	2.107E−02	0.962	3.044E−01	0.917	1.530E−03	1.279
$2\pi/256$	13666	1.586E−02	0.987	2.249E−01	1.053	1.018E−03	1.415
$2\pi/384$	31282	1.038E−02	1.046	1.489E−01	1.017	6.623E−04	1.061
$2\pi/512$	55438	7.784E−03	1.000	1.106E−01	1.035	4.324E−04	1.482
$2\pi/768$	125069	5.152E−03	1.017	7.397E−02	0.991	2.745E−04	1.121
$2\pi/1024$	221848	3.871E−03	0.994	5.540E−02	1.005	2.034E−04	1.041
$2\pi/1536$	498545	2.579E−03	1.001	3.670E−02	1.016	1.298E−04	1.109
$2\pi/2048$	887629	1.927E−03	1.014	2.770E−02	0.978	9.678E−05	1.019

Table 2.1: Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (Example 1)

h	N	$\mathsf{e}(\boldsymbol{\varphi}_s)$	$r(\boldsymbol{\varphi}_s)$	$\mathsf{e}(\varphi_\Sigma)$	$r(\varphi_\Sigma)$	$\mathsf{e}(\varphi_\Gamma)$	$r(\varphi_\Gamma)$
$2\pi/64$	1117	9.684E−03	–	1.689E−01	–	4.819E−02	–
$2\pi/96$	2090	4.899E−03	1.681	7.439E−02	2.022	2.030E−02	2.133
$2\pi/128$	3686	2.727E−03	2.037	4.415E−02	1.813	1.226E−02	1.752
$2\pi/192$	7869	1.427E−03	1.598	2.362E−02	1.542	5.610E−03	1.928
$2\pi/256$	13666	8.446E−04	1.822	1.348E−02	1.951	3.850E−03	1.308
$2\pi/384$	31282	4.023E−04	1.829	6.741E−03	1.708	1.834E−03	1.830
$2\pi/512$	55438	2.521E−04	1.625	3.849E−03	1.948	1.187E−03	1.511
$2\pi/768$	125069	1.266E−04	1.699	1.896E−03	1.746	6.280E−04	1.571
$2\pi/1024$	221848	8.236E−05	1.494	1.290E−03	1.339	4.437E−04	1.208
$2\pi/1536$	498545	4.112E−05	1.713	6.765E−04	1.592	2.231E−04	1.695
$2\pi/2048$	887629	2.633E−05	1.550	4.455E−04	1.452	1.533E−04	1.305

Table 2.2: Convergence history for $\boldsymbol{\varphi}_s$, φ_Σ, and φ_Γ (Example 1)

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	1117	1.260E−01	–	9.166E−01	–	1.166E−02	–
$2\pi/96$	2090	7.827E−02	1.174	6.046E−01	1.026	5.671E−03	1.777
$2\pi/128$	3686	5.687E−02	1.111	4.434E−01	1.077	3.591E−03	1.588
$2\pi/192$	7869	3.851E−02	0.962	3.052E−01	0.921	2.119E−03	1.301
$2\pi/256$	13666	2.880E−02	1.009	2.252E−01	1.057	1.414E−03	1.406
$2\pi/384$	31282	1.880E−02	1.052	1.490E−01	1.019	8.978E−04	1.121
$2\pi/512$	55438	1.410E−02	1.001	1.106E−01	1.036	5.736E−04	1.557
$2\pi/768$	125069	9.319E−03	1.021	7.398E−02	0.992	3.624E−04	1.133
$2\pi/1024$	221848	6.999E−03	0.995	5.541E−02	1.005	2.665E−04	1.069
$2\pi/1536$	498545	4.662E−03	1.002	3.670E−02	1.016	1.682E−04	1.135
$2\pi/2048$	887629	3.485E−03	1.012	2.770E−02	0.978	1.247E−04	1.040

Table 2.3: Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (Example 2)

h	N	$\mathsf{e}(\boldsymbol{\varphi}_s)$	$r(\boldsymbol{\varphi}_s)$	$\mathsf{e}(\varphi_\Sigma)$	$r(\varphi_\Sigma)$	$\mathsf{e}(\varphi_\Gamma)$	$r(\varphi_\Gamma)$
$2\pi/64$	1117	2.051E−02	–	2.498E−01	–	7.683E−02	–
$2\pi/96$	2090	8.132E−03	2.281	9.442E−02	2.399	2.670E−02	2.607
$2\pi/128$	3686	4.515E−03	2.045	5.483E−02	1.890	1.581E−02	1.820
$2\pi/192$	7869	2.478E−03	1.480	2.897E−02	1.573	7.554E−03	1.822
$2\pi/256$	13666	1.438E−03	1.892	1.611E−02	2.041	4.685E−03	1.660
$2\pi/384$	31282	7.075E−04	1.749	7.925E−03	1.749	2.200E−03	1.865
$2\pi/512$	55438	4.504E−04	1.570	4.488E−03	1.976	1.393E−03	1.587
$2\pi/768$	125069	2.114E−04	1.865	2.162E−03	1.802	7.204E−04	1.627
$2\pi/1024$	221848	1.435E−04	1.346	1.448E−03	1.393	5.041E−04	1.241
$2\pi/1536$	498545	7.019E−05	1.764	7.478E−04	1.629	2.517E−04	1.713
$2\pi/2048$	887629	4.461E−05	1.575	4.897E−04	1.472	1.728E−04	1.307

Table 2.4: Convergence history for $\boldsymbol{\varphi}_s$, φ_Σ, and φ_Γ (EXAMPLE 2)

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	2215	9.938E−01	–	1.375E+01	–	1.115E−01	–
$2\pi/96$	4767	6.768E−01	0.947	8.337E−00	1.235	2.291E−02	3.903
$2\pi/128$	8495	5.373E−01	0.802	5.973E−00	1.159	1.020E−02	2.814
$2\pi/192$	19067	4.468E−01	0.455	3.971E−00	1.007	5.789E−03	1.396
$2\pi/256$	33331	3.899E−01	0.474	3.001E−00	0.974	3.776E−03	1.485
$2\pi/384$	75077	2.800E−01	0.817	1.973E−00	1.034	1.680E−03	1.998
$2\pi/512$	133497	2.351E−01	0.607	1.488E−00	0.981	1.154E−03	1.303
$2\pi/768$	299000	1.883E−01	0.547	9.898E−01	1.006	6.706E−04	1.340
$2\pi/1024$	534105	1.493E−01	0.807	7.408E−01	1.007	4.519E−04	1.372
$2\pi/1536$	1199275	1.109E−01	0.735	4.947E−01	0.996	2.701E−04	1.270

Table 2.5: Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (EXAMPLE 3)

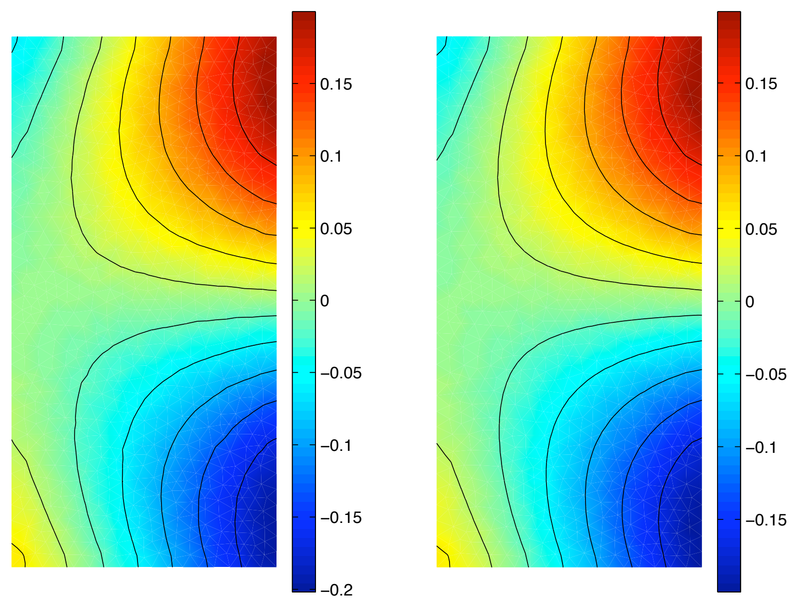

Figure 2.1: Approximate and exact imaginary part of $\boldsymbol{\sigma}_{s,12}$ (EXAMPLE 1)

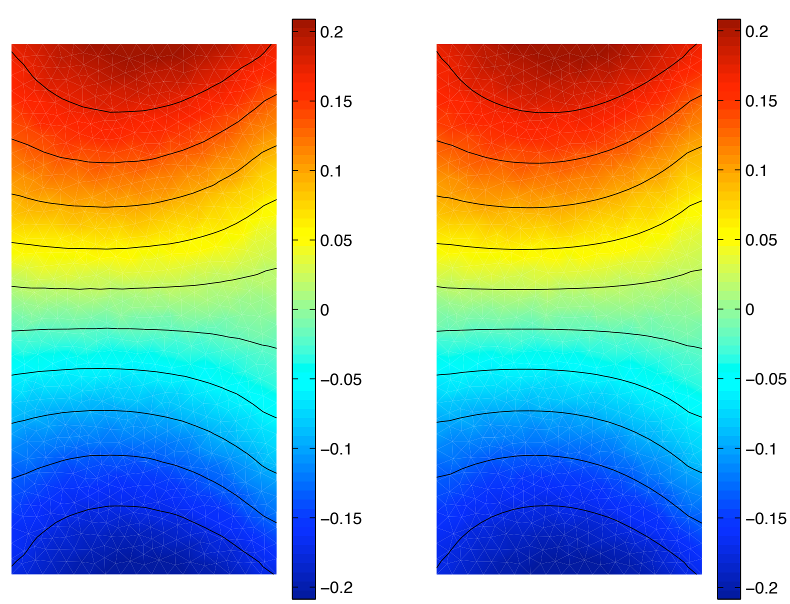

Figure 2.2: Approximate and exact real part of $\boldsymbol{\sigma}_{s,21}$ (EXAMPLE 1)

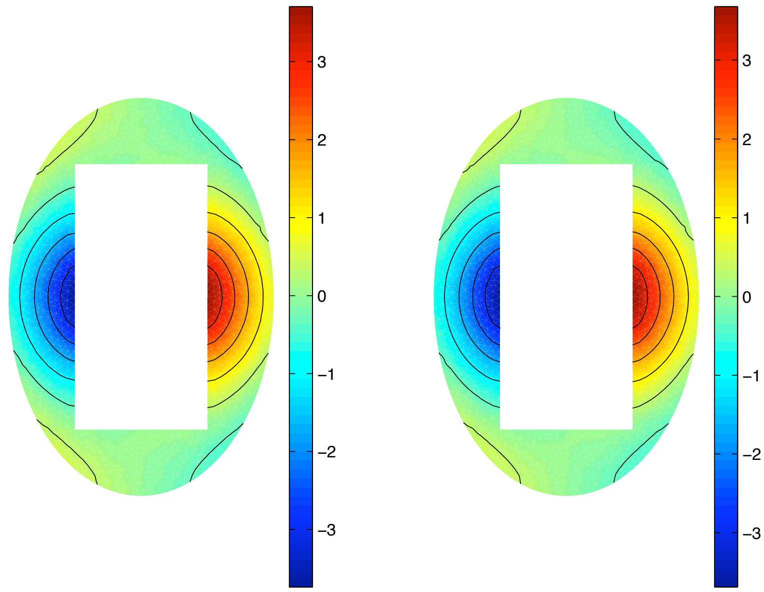

Figure 2.3: Approximate and exact imaginary part of $\boldsymbol{\sigma}_{f,1}$ (Example 1)

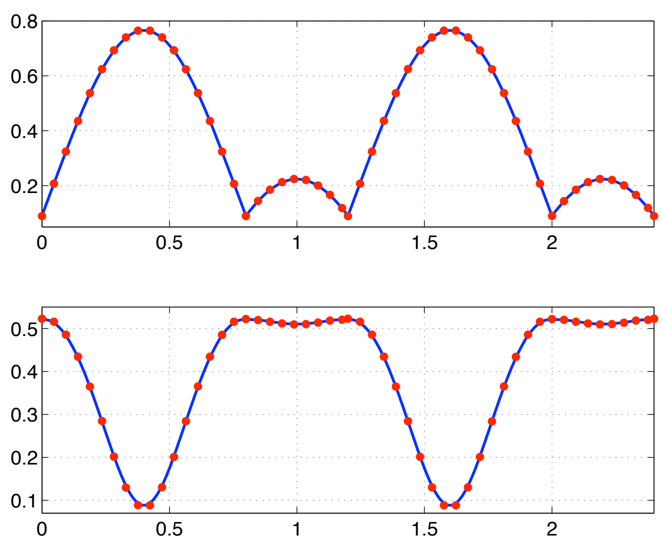

Figure 2.4: Approximate (red) and exact (blue) real and imaginary parts of φ_{Σ} (Example 1)

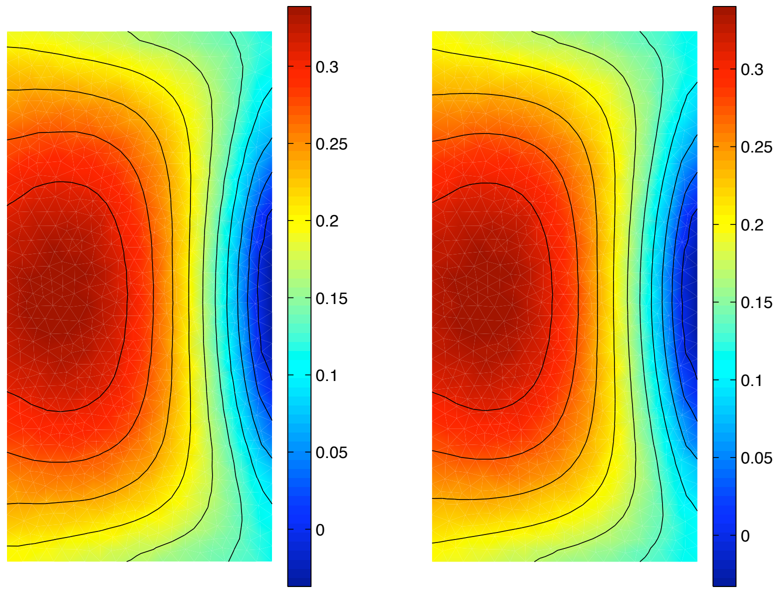

Figure 2.5: Approximate and exact imaginary part of $\boldsymbol{\sigma}_{s,11}$ (Example 2)

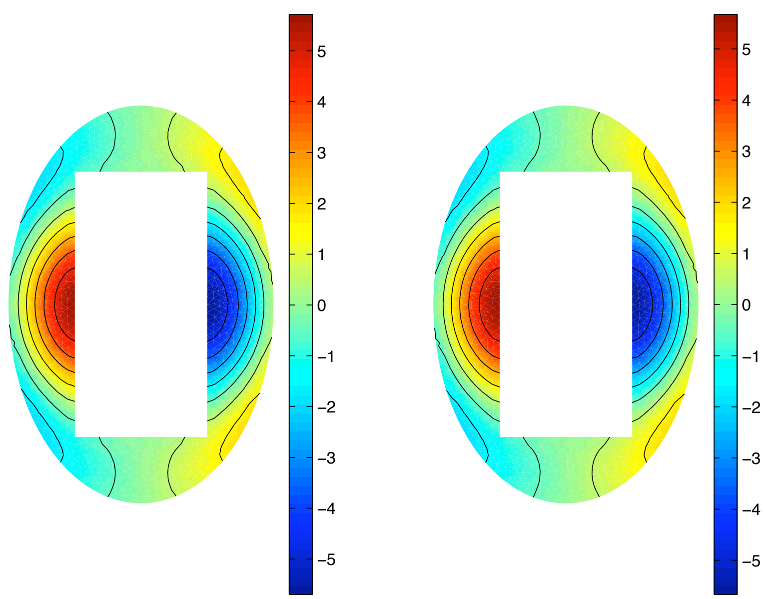

Figure 2.6: Approximate and exact real part of $\boldsymbol{\sigma}_{f,1}$ (Example 2)

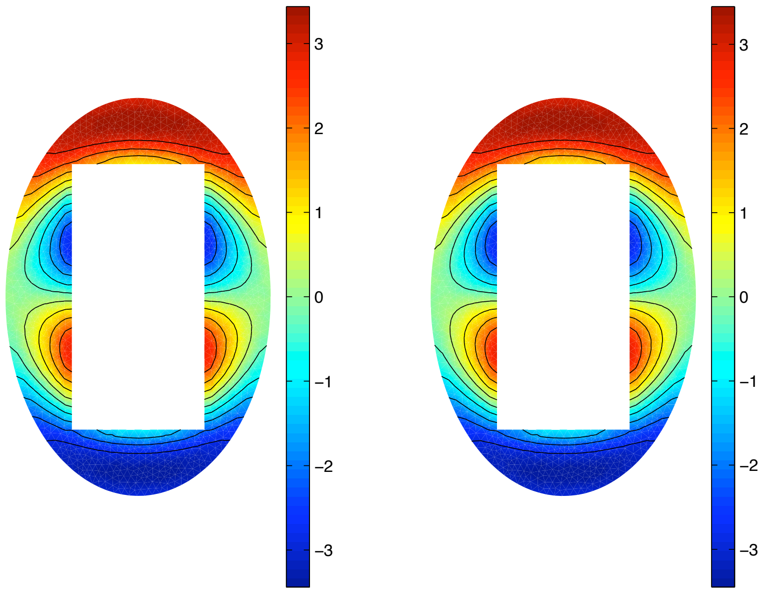

Figure 2.7: Approximate and exact real part of $\boldsymbol{\sigma}_{f,2}$ (EXAMPLE 2)

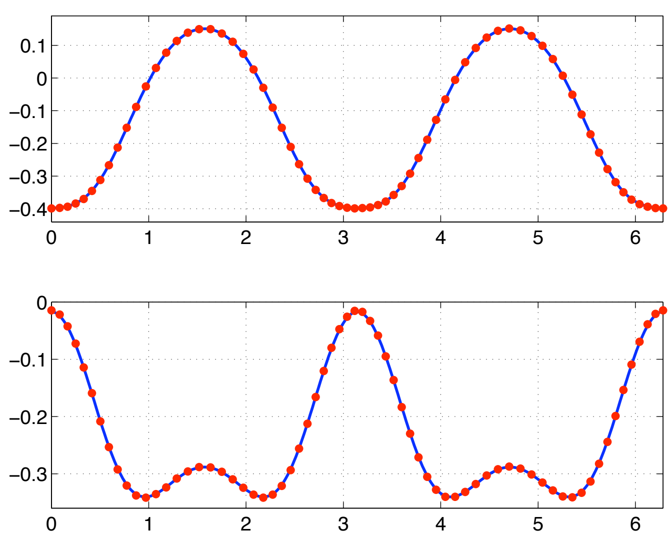

Figure 2.8: Approximate (red) and exact (blue) real and imaginary parts of φ_{Γ} (EXAMPLE 2)

Chapter 3

A residual-based a posteriori error estimator for the plane linear elasticity problem with pure traction boundary conditions

3.1 Introduction

The possibility of introducing further unknowns of physical interest, such as stresses and rotations, and the need of locking-free numerical schemes when the corresponding Poisson ratio approaches 1/2, constitute the main reasons for the utilization of dual-mixed variational formulations and the associated mixed finite element methods to solve elasticity problems. Consequently, the derivation of appropriate finite element subspaces yielding well posed Galerkin schemes has been extensively studied and several choices, including the classical PEERS element and recent approaches, are already available in the literature (see, e.g. [4], [5], [6], [7], [8], [19], [58], [70], and [73]). It is also well known that, within the framework of dual-mixed formulations, and on the contrary to the usual primal ones, the Dirichlet and Neumann data exchange their roles and become now natural and essential boundary conditions, respectively. In particular, non-homogeneous Neumann data usually lead to non-conforming Galerkin schemes and respective consistency terms, which need to be suitably estimated to be able to prove stability and convergence of the discrete methods. These facts explain why most of the works dealing with dual-mixed finite element methods in continuum mechanics consider either pure Dirichlet or mixed boundary conditions with homogeneous Neumann datum, thus avoiding the additional difficulties arising from the presence of non-homogeneous essential boundary conditions. Nevertheless, one way of successfully handling these conditions consists of the introduction of appropriate Lagrange multipliers enforcing them weakly, as done originally in [9] for the primal finite element method with non-homogeneous Dirichlet boundary conditions. The extension of the method from [9] to a large class of dual-mixed variational formulations was studied in [11], where a second order elliptic equation in divergence form with mixed boundary conditions and non-homogeneous Neumann datum was considered as a model problem.

In turn, the extension of the results from [11] to the dual-mixed variational formulation of the linear elasticity problem in the plane was performed in [38]. More precisely, the stress-displacement-rotation formulation for the case of non-homogeneous pure traction boundary conditions was considered in [38], and a new dual-mixed finite element method for approximating its solution was developed there. The main novelty of the approach in [38] lies on the weak enforcement of the non-homogeneous Neumann boundary condition, similarly as done in [11], through the introduction of the boundary trace of the displacement as a Lagrange multiplier. In addition, since the rigid body motions solve the associated homogeneous boundary value problem, the displacements lie in the respective orthogonal complement and are computed through the introduction of an artificial unknown as an additional Lagrange multiplier. A suitable combination of PEERS and continuous piecewise linear functions on the boundary are employed to define the dual-mixed finite element scheme, and the classical Babuška-Brezzi theory is applied to show the well-posedness of the continuous and discrete formulations. A priori rates of convergence of the method, including an estimate for the global error when the stresses are measured with the L^2-norm, are also derived in [38]. It is important to remark that this work is actually the first one dealing with the dual-mixed finite element method for the above mentioned boundary value problem, in which the stress-displacement-rotation formulation and triangular elements are employed. Moreover, the analysis of the corresponding continuous variational formulation, which is also provided there, was not available before. On the contrary, the analysis of the continuous and discrete primal variational formulations for the linear elasticity problem with pure Neumann boundary conditions is nowadays very well established (see, e.g. [18, Chapter 9], [17], [32], and [60] for detailed analyses).

On the other hand, in order to guarantee a good convergence behaviour of the finite element solutions, particularly under the presence of singularities, one usually needs to apply an adaptive strategy based on a posteriori error estimates. These are usually represented by global quantities $\boldsymbol{\theta}$ that are expressed in terms of local estimators θ_T defined on each element T of a given triangulation of the domain. The estimator $\boldsymbol{\theta}$ is said to be reliable (resp. efficient) if there exists

$C_{\mathtt{rel}} > 0$ (resp. $C_{\mathtt{eff}} > 0$), independent of the meshsizes, such that

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} + \text{h.o.t.} \;\leq\; \|error\| \;\leq\; C_{\mathtt{rel}}\,\boldsymbol{\theta} + \text{h.o.t.}\,,$$

where h.o.t. is a generic expression denoting one or several terms of higher order. Most of the a posteriori error estimators for the mixed finite element formulation of the linear elasticity problem are derived similarly as those for elliptic partial differential equations of second order in divergence form (see. e.g. [2] where estimators based on residuals and on the solution of local problems, using Raviart-Thomas and Brezzi-Douglas-Marini spaces, are provided). In connection with Raviart-Thomas spaces, one may also refers to [16], [21], and [36], where reliable and efficient residual-based a posteriori error estimators for the Poisson problem are obtained. The main tools of the corresponding analyses include Helmholtz decompositions, the localization technique based on bubble functions, discrete trace and inverse inequalities, and the approximation properties of the Clément interpolant. The extension of the results in [21] to the linear elasticity problem is developed in [22] and [62]. In addition, energy norm a posteriori error estimates based on postprocessing are obtained in [63], and functional-type error estimates are presented in [68].

Motivated by the preceding remarks, the main purpose of the present paper is to consider the plane linear elasticity problem with pure traction boundary conditions and derive a reliable and efficient residual-based a posteriori error estimator for the corresponding dual-mixed finite element method introduced and analyzed in [38]. The rest of this work is organized as follows. In Section 3.2 we recall from [38] the boundary value problem of interest and its dual-mixed variational formulation. In Section 3.3 we reconsider the mixed finite element scheme from [38] and introduce some improvements in its definition and solvability analysis that have arisen in recent related works. The core of the present work is Section 3.4, where we develop the announced a posteriori error analysis. The reliability and efficiency of the proposed estimator are proved in Sections 3.4.1 and 3.4.2, respectively. Finally, several numerical examples confirming these properties and showing the good performance of the associated adaptive algorithm, are provided in Section 3.5.

We end this section with further notations to be used below. In what follows, $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$, tr denotes the matrix trace, $^{\mathtt{t}}$ stands for the transpose of a matrix, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta}_s := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we define the deviator tensor $\boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I}$, and the tensor product $\boldsymbol{\tau} : \boldsymbol{\zeta}_s := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$. Also, we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, $\mathcal{S}$ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2\,, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}\,, \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^2\,.$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote H^2 and $H^{2\times 2}$, respectively. In addition, we use $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ to denote the usual duality pairings between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, and between $\mathbf{H}^{-1/2}(\mathcal{S})$ and $\mathbf{H}^{1/2}(\mathcal{S})$. Furthermore, the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) \;:=\; \left\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\,\mathbf{w} \in L^2(\mathcal{O})\right\},$$

is standard in the realm of mixed problems (see [19], [47]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O})$, where $\mathbf{div}$ stands for the usual divergence operator div acting on each row of the tensor, The Hilbert norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div},\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div},\mathcal{O}}$, respectively. Finally, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

3.2 The boundary value problem

In this section we recall from [38] the boundary value problem of interest, its associated dual-mixed variational formulation, and the corresponding well-posedness result. To this end, we let Ω be a bounded and simply connected polygonal domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary Γ. Our goal is to determine the displacement $\mathbf{u}$ and stress tensor $\boldsymbol{\sigma}$ of a linear elastic material occupying the region Ω and which is subject to a volume force and pure traction boundary conditions. In other words, given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{-1/2}(\Gamma)$, we seek a symmetric tensor field $\boldsymbol{\sigma}$ and a vector field $\mathbf{u}$ such that

$$\boldsymbol{\sigma} \;=\; \mathcal{C}\,\varepsilon(\mathbf{u})\,, \quad \mathbf{div}\,\boldsymbol{\sigma} \;=\; -\,\mathbf{f} \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \boldsymbol{\sigma}\boldsymbol{\nu} \;=\; \mathbf{g} \quad \text{on} \quad \Gamma\,, \tag{3.1}$$

where $\mathcal{C}$ is the elasticity operator determined by Hooke's law, that is, given Lamé constants $\lambda,\, \mu \,>\, 0$,

$$\mathcal{C}\,\boldsymbol{\zeta}_s \;:=\; \lambda\,\mathrm{tr}(\boldsymbol{\zeta}_s)\,\mathbf{I} \;+\; 2\,\mu\,\boldsymbol{\zeta}_s \qquad \forall\,\boldsymbol{\zeta}_s \,\in\, \mathbb{L}^2(\Omega)\,, \tag{3.2}$$

$\varepsilon(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\right)$ is the strain tensor of small deformations, and $\boldsymbol{\nu}$ is the unit outward normal to Γ. Concerning the existence of solution of (3.1), we first recall (see, e.g. [18, Theorem 9.2.30]) that this problem is solvable if and only if

$$\int_\Omega \mathbf{f}\cdot\boldsymbol{\chi} \;+\; \langle\mathbf{g},\boldsymbol{\chi}\rangle_\Gamma \;=\; 0 \qquad \forall\,\boldsymbol{\chi} \,\in\, \mathbb{RM}(\Omega)\,, \tag{3.3}$$

where $\mathbb{RM}(\Omega)$, the space of rigid body motions in Ω, is defined as

$$\mathbb{RM}(\Omega) := \left\{ \boldsymbol{\chi} : \Omega \to \mathbb{R}^2 \; : \; \boldsymbol{\chi}(\mathbf{x}) = \left(\begin{array}{c} a \\ b \end{array} \right) + c \left(\begin{array}{c} x_2 \\ -x_1 \end{array} \right) \; \forall \, \mathbf{x} := \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \Omega \,, \; a, \, b \,, \, c \in \mathbb{R} \right\}.$$

Hence, throughout the rest of the paper we assume that the compatibility condition (3.3) holds.

Next, following the usual procedure for the stress-displacement-rotation formulation of the elasticity problem (see, e.g. [4], [19], [70]), that is defining the rotation $\boldsymbol{\gamma} := \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}) \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)$ as an auxiliary unknown, where

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega) \; := \; \left\{ \boldsymbol{\tau} \, \in \, \mathbb{L}^2(\Omega) : \quad \boldsymbol{\tau} \, + \, \boldsymbol{\tau}^{\mathtt{t}} \, = \, \mathbf{0} \right\}$$

is the space of skew-symmetric tensors, and introducing the trace $\boldsymbol{\varphi} \, := - \, \mathbf{u} \, \in \, \mathbf{H}^{1/2}(\Gamma)$ as an additional Lagrange multiplier, we obtain, at first instance, the dual-mixed variational formulation: Find $(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma})) \in \mathbb{H}(\mathbf{div}; \Omega) \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \, + \, \mathbf{b}(\boldsymbol{\tau}, (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma})) \quad &= \quad 0 \qquad \forall \, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; \Omega) \,, \\ \mathbf{b}(\boldsymbol{\sigma}, (\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta})) \quad &= \quad - \int_\Omega \mathbf{f} \cdot \mathbf{v} \, + \, \langle \mathbf{g}, \boldsymbol{\psi} \rangle_\Gamma \qquad \forall \, (\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta}) \in \mathbf{Q} \,, \end{aligned} \tag{3.4}$$

where

$$\mathbf{Q} \; := \; \mathbf{L}^2(\Omega) \times \mathbf{H}^{1/2}(\Gamma) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \,,$$

and $\mathbf{a} : \mathbb{H}(\mathbf{div}; \Omega) \times \mathbb{H}(\mathbf{div}; \Omega) \to \mathbb{R}$ and $\mathbf{b} : \mathbb{H}(\mathbf{div}; \Omega) \times \mathbf{Q} \to \mathbb{R}$ are the bilinear forms given by

$$\mathbf{a}(\boldsymbol{\zeta}_s, \boldsymbol{\tau}) \; := \; \int_\Omega \mathcal{C}^{-1} \boldsymbol{\zeta}_s : \boldsymbol{\tau} \qquad \forall \, (\boldsymbol{\zeta}_s, \boldsymbol{\tau}) \in \mathbb{H}(\mathbf{div}; \Omega) \times \mathbb{H}(\mathbf{div}; \Omega) \,, \tag{3.5}$$

and

$$\mathbf{b}(\boldsymbol{\tau}, (\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta})) \; := \; \int_\Omega \mathbf{v} \cdot \mathbf{div} \, \boldsymbol{\tau} + \langle \boldsymbol{\tau \nu}, \boldsymbol{\psi} \rangle_\Gamma + \int_\Omega \boldsymbol{\tau} : \boldsymbol{\eta} \qquad \forall \, (\boldsymbol{\tau}, (\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta})) \in \mathbb{H}(\mathbf{div}; \Omega) \times \mathbf{Q} \,. \tag{3.6}$$

However, it is easy to see that, given any $\boldsymbol{\chi} \in \mathbb{RM}(\Omega)$, $(\mathbf{0}, (\boldsymbol{\chi}, -\boldsymbol{\chi}|_\Gamma, \nabla \boldsymbol{\chi}))$ solves the homogeneous system associated to (3.4), and therefore, in order to avoid these spurious solutions, we now look for displacements $\mathbf{u}$ in the orthogonal complement of the rigid body motions. According to the foregoing analysis, we arrive at the following dual-mixed variational formulation of (3.1): Find $((\boldsymbol{\sigma}, \boldsymbol{\rho}), (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma})) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{A}((\boldsymbol{\sigma}, \boldsymbol{\rho}), (\boldsymbol{\tau}, \boldsymbol{\chi})) \, + \, \mathbf{B}((\boldsymbol{\tau}, \boldsymbol{\chi}), (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma})) \quad &= \quad 0 \qquad \forall \, (\boldsymbol{\tau}, \boldsymbol{\chi}) \in \mathbf{H} \,, \\ \mathbf{B}((\boldsymbol{\sigma}, \boldsymbol{\rho}), (\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta})) \quad &= \quad - \int_\Omega \mathbf{f} \cdot \mathbf{v} \, + \, \langle \mathbf{g}, \boldsymbol{\psi} \rangle_\Gamma \qquad \forall \, (\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta}) \in \mathbf{Q} \,, \end{aligned} \tag{3.7}$$

where

$$\mathbf{H} := \mathbb{H}(\mathbf{div};\Omega)\times\mathbb{RM}(\Omega)\,,$$

and $\mathbf{A}:\mathbf{H}\times\mathbf{H}\to\mathbb{R}$ and $\mathbf{B}:\mathbf{H}\times\mathbf{Q}\to\mathbb{R}$ are the bilinear forms given by

$$\mathbf{A}((\boldsymbol{\zeta}_s,\boldsymbol{\varrho}),(\boldsymbol{\tau},\boldsymbol{\chi})) := \mathbf{a}(\boldsymbol{\zeta}_s,\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\varrho}\cdot\boldsymbol{\chi} \qquad \forall\,(\boldsymbol{\zeta}_s,\boldsymbol{\varrho}),\,(\boldsymbol{\tau},\boldsymbol{\chi})\in\mathbf{H}\,, \tag{3.8}$$

and

$$\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) := \mathbf{b}(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) + \int_\Omega \boldsymbol{\chi}\cdot\mathbf{v} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\chi})\in\mathbf{H}\,,\quad \forall\,(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})\in\mathbf{Q}\,. \tag{3.9}$$

The following lemmas are needed to establish the well-posedness of (3.7) and also to carry on the announced a posteriori error analysis in Section 3.4.

Lemma 3.1 *Let* $\mathbf{V} := \{\,(\boldsymbol{\tau},\boldsymbol{\chi})\in\mathbf{H}: \quad \mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})) = 0 \quad \forall\,(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})\in\mathbf{Q}\,\}$. *Then there holds*

$$\mathbf{V} = V\times\{\mathbf{0}\}\,, \tag{3.10}$$

with

$$V := \Big\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega): \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad in \quad \Omega\,, \quad \boldsymbol{\tau}\boldsymbol{\nu} = \mathbf{0} \quad on \quad \Gamma\,, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \quad in \quad \Omega\Big\}\,, \tag{3.11}$$

and there exists $\alpha>0$, *independent of* λ, *such that*

$$\mathbf{A}((\boldsymbol{\tau},\boldsymbol{\chi}),(\boldsymbol{\tau},\boldsymbol{\chi})) \;\geq\; \alpha\,\|(\boldsymbol{\tau},\boldsymbol{\chi})\|^2_{\mathbf{H}} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{\chi})\,\in\,\mathbf{V}\,.$$

Proof. See [38, Lemma 3.3]. □

Lemma 3.2 *There exists* $\beta>0$, *independent of* λ, *such that*

$$\sup_{\substack{(\boldsymbol{\tau},\boldsymbol{\chi})\in\mathbf{H}\\(\boldsymbol{\tau},\boldsymbol{\chi})\neq\mathbf{0}}} \frac{|\,\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta}))\,|}{\|(\boldsymbol{\tau},\boldsymbol{\chi})\|_{\mathbf{H}}} \;\geq\; \beta\,\|(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})\|_{\mathbf{Q}} \qquad \forall\,(\mathbf{v},\boldsymbol{\psi},\boldsymbol{\eta})\,\in\,\mathbf{Q}\,.$$

Proof. See [38, Lemma 3.4]. □

The well-posedness of the variational formulation (3.7) is stated as follows.

Theorem 3.1 *There exists a unique solution* $((\boldsymbol{\sigma},\boldsymbol{\rho}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}))\in\mathbf{H}\times\mathbf{Q}$ *to* (3.7). *In addition,* $\boldsymbol{\rho} = \mathbf{0}$ *and there exists* $C>0$, *independent of* λ, *such that*

$$\|(\boldsymbol{\sigma},(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}))\|_{\mathbb{H}(\mathbf{div};\Omega)\times\mathbf{Q}} \;\leq\; C\Big\{\,\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{-1/2,\Gamma}\,\Big\}\,. \tag{3.12}$$

Proof. See [38, Theorem 3.1]. □

Actually, thanks to Lemmas 3.1 and 3.2, we can establish the following more general result.

Theorem 3.2 *Given $\bar{F} \in \mathbf{H}'$ and $\bar{G} \in \mathbf{Q}'$, there exists a unique $((\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}}), (\bar{\mathbf{u}}, \bar{\boldsymbol{\varphi}}, \bar{\gamma})) \in \mathbf{H} \times \mathbf{Q}$ such that*

$$\begin{aligned} \mathbf{A}((\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}}), (\boldsymbol{\tau}, \boldsymbol{\chi})) + \mathbf{B}((\boldsymbol{\tau}, \boldsymbol{\chi}), (\bar{\mathbf{u}}, \bar{\boldsymbol{\varphi}}, \bar{\gamma})) &= \bar{F}\big((\boldsymbol{\tau}, \boldsymbol{\chi})\big) \qquad \forall\, (\boldsymbol{\tau}, \boldsymbol{\chi}) \in \mathbf{H}\,, \\ \mathbf{B}((\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}}), (\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta})) &= \bar{G}\big((\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta})\big) \qquad \forall\, (\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta}) \in \mathbf{Q}\,. \end{aligned} \tag{3.13}$$

In addition, there exists $C > 0$, depending only on β, α, $\|\mathbf{a}\|$, and $\|\mathbf{b}\|$, such that

$$\|(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}} + \|(\bar{\mathbf{u}}, \bar{\boldsymbol{\varphi}}, \bar{\gamma})\|_{\mathbf{Q}} \leq C \left\{ \|\bar{F}\|_{\mathbf{H}'} + \|\bar{G}\|_{\mathbf{Q}'} \right\}. \tag{3.14}$$

We end this section with the converse of the derivation of (3.7). Indeed, the following theorem establishes that the unique solution of (3.7) solves the original boundary value problem (3.1). This result will be used later on in Section 3.4.2 to prove the efficiency of the a posteriori error estimator.

Theorem 3.3 *Let $((\boldsymbol{\sigma}, \boldsymbol{\rho}), (\mathbf{u}, \boldsymbol{\varphi}, \gamma)) \in \mathbf{H} \times \mathbf{Q}$ be the unique solution of (3.7). Then $\boldsymbol{\rho} = \mathbf{0}$ in Ω, $\mathbf{div}\, \boldsymbol{\sigma} = -\mathbf{f}$ in Ω, $\nabla \mathbf{u} = \mathcal{C}^{-1} \boldsymbol{\sigma} + \boldsymbol{\gamma}$ in Ω (which yields $\mathbf{u} \in \mathbf{H}^1(\Omega)$), $\mathbf{u} = -\boldsymbol{\varphi}$ on Γ, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathtt{t}}$ in Ω, $\boldsymbol{\gamma} = \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}\right)$ in Ω (which yields $\boldsymbol{\sigma} = \mathcal{C}\, \boldsymbol{\varepsilon}(\mathbf{u})$), and $\boldsymbol{\sigma}\, \boldsymbol{\nu} = \mathbf{g}$ on Γ.*

Proof. It suffices to apply integration by parts backwardly in (3.7) and then use suitable test functions. Further details are omitted. □

3.3 The mixed finite element scheme

We now recall from [38] the mixed finite element scheme for (3.7). As said there, we could define this discrete scheme by utilizing any of the classical finite element subspaces available in the literature (see, e.g. [19] and the references therein), or those that have emerged recently from the finite element exterior calculus (see, e.g. [6], [7]). However, for simplicity of the presentation, we consider in what follows the well known PEERS elements. To this end, we first let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the polygonal region $\bar{\Omega}$ by triangles T of diameter h_T with global mesh size $h := \max\{\, h_T : \quad T \in \mathcal{T}_h \,\}$, such that they are quasi-uniform around Γ. In what follows, given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^2$, $P_\ell(S)$ denotes the space of polynomials defined in S of total degree $\leq \ell$. Recall that, according to the notation convention explained in the introduction, we denote $\mathbf{P}_\ell(S) := [P_\ell(S)]^2$. Furthermore, given $T \in \mathcal{T}_h$ and $\mathbf{x} := (x_1, x_2)^{\mathtt{t}}$ a generic vector of $\mathbb{R}^2$, we let $\mathrm{RT}_0(T) := \text{span}\left\{(1,0), (0,1), (x_1, x_2)\right\}$ be the local Raviart-Thomas space of order 0 (cf. [19], [69]), and let $\mathbf{curl}^{\mathtt{t}}\, b_T := \left(\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1}\right)$, where b_T is the usual cubic bubble function on T. Then we define the finite element subspaces $H_h^{\boldsymbol{\sigma}}$,

$Q_h^{\mathbf{u}}$, and $Q_h^{\boldsymbol{\gamma}}$, associated with the unknowns $\boldsymbol{\sigma}$, $\mathbf{u}$, and $\boldsymbol{\gamma}$, respectively, as follows:

$$H_h^{\boldsymbol{\sigma}} := \left\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega): \ \mathbf{c}^{\mathrm{t}}\,\boldsymbol{\tau}_h|_T \in \mathrm{RT}_0(T) \oplus P_0(T)\,\mathbf{curl}^{\mathrm{t}}\, b_T \quad \forall\, T \in \mathcal{T}_h\,, \quad \forall\, \mathbf{c} \in \mathbb{R}^2\right\}, \tag{3.15}$$

$$Q_h^{\mathbf{u}} := \left\{\mathbf{v}_h \in \mathbf{L}^2(\Omega): \quad \mathbf{v}_h|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h\right\}, \tag{3.16}$$

and

$$Q_h^{\boldsymbol{\gamma}} := \left\{\begin{pmatrix} 0 & \eta_h \\ -\eta_h & 0 \end{pmatrix}: \quad \eta_h \in C(\bar{\Omega})\,, \quad \eta_h|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h\right\}. \tag{3.17}$$

Note here that $H_h^{\boldsymbol{\sigma}} \times Q_h^{\mathbf{u}} \times Q_h^{\boldsymbol{\gamma}}$ constitutes the classical PEERS introduced in [4] for a mixed finite element aproximation of the linear elasticity problem with Dirichlet boundary conditions. Next, in order to set the finite dimensional subspace $Q_h^{\boldsymbol{\varphi}}$ associated with the unknown $\boldsymbol{\varphi}$, we let Γ_h be the partition of Γ inherited from the triangulation $\mathcal{T}_h$, and suppose, without loss generality, that the numbers of edges of Γ_h is even. The case of an odd number of edges is easily reduced to the even case (see [45, remark at the end of Section 5.3] for details). Then, we let Γ_{2h} be the partition of Γ arising by joining pairs of adjacent edges of Γ_h. Because of the assumptions on the triangulations, Γ_h is automatically of bounded variation, and, therefore, so is Γ_{2h}. Hence, we now define

$$Q_h^{\boldsymbol{\varphi}} := \left\{\boldsymbol{\psi}_h \in \mathbf{C}(\Gamma): \quad \boldsymbol{\psi}_h|_e \in \mathbf{P}_1(e) \quad \forall e \text{ edge of } \Gamma_{2h}\right\}. \tag{3.18}$$

It is important to remark at this point that the above choice of $Q_h^{\boldsymbol{\varphi}}$, using the "*double*" partition Γ_{2h} instead of an independent partition $\Gamma_{\hat{h}}$ of Γ as in the original work [38], constitutes a clear simplification of the discrete analysis of our problem. In fact, thanks to the recent results obtained in [45, Section 5.3, particularly Lemma 5.2] (see also [34, Section 4.4]), the restriction on the mesh sizes given by $h \le C_0\,\hat{h}$, with an unknown constant C_0, which is required in [38, Lemmas 4.2 and 4.3] to prove the discrete inf-sup condition for $\mathbf{B}$, is not needed any more. Moreover, the aforementioned requirement of quasi-uniformity of the triangulations around Γ, which is a key ingredient in [45], was removed recently in [64, Sections 4 and 5] for the 2D case. However, we prefer to keep it here since the a posteriori error analysis to be developed below can also be extended to three-dimensional problems, for which that assumption is still necessary.

According to the foregoing analysis, we introduce the product spaces

$$\mathbf{H}_h := H_h^{\boldsymbol{\sigma}} \times \mathbb{RM}(\Omega) \quad \text{and} \quad \mathbf{Q}_h := Q_h^{\mathbf{u}} \times Q_h^{\boldsymbol{\varphi}} \times Q_h^{\boldsymbol{\gamma}}\,,$$

and consider the following Galerkin approximation of (3.7): Find $((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$\begin{aligned} \mathbf{A}((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau}_h,\boldsymbol{\chi}_h)) + \mathbf{B}((\boldsymbol{\tau}_h,\boldsymbol{\chi}_h),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) &= 0\,, \\ \mathbf{B}((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{v}_h,\boldsymbol{\psi}_h,\boldsymbol{\eta}_h)) &= -\int_\Omega \mathbf{f}\cdot\mathbf{v}_h + \langle \mathbf{g},\boldsymbol{\psi}_h\rangle_\Gamma\,, \end{aligned} \tag{3.19}$$

for all $((\boldsymbol{\tau}_h, \boldsymbol{\chi}_h), (\mathbf{v}_h, \boldsymbol{\psi}_h, \boldsymbol{\eta}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$. Concerning the analysis of (3.19) we remark that, besides the advances arising from the results in [45, Section 5.3], the asymptotic equivalence of norms given in [38, Lemma 4.4], which is actually taken from [33, Lemma 4.4], has also been improved lately to the case of arbitrary mesh sizes (see [28, Lemma 4.9]). Consequently, instead of the original result provided in [38, Theorem 4.1], the well-posedness of the Galerkin scheme (3.19) is now stated as follows.

Theorem 3.4 *There exists a unique* $((\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *solution of* (3.19). *Moreover, there exist* $C\,,\ \widetilde{C} > 0$, *independent of* h *and* λ, *such that*

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} + \|\boldsymbol{\rho}_h\|_{0,\Omega} + \|\mathbf{u}_h\|_{0,\Omega} + \|\boldsymbol{\varphi}_h\|_{1/2,\Gamma} + \|\gamma_h\|_{0,\Omega} \le C \Big\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{-1/2,\Gamma} \Big\}, \tag{3.20}$$

and

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} + \|\boldsymbol{\rho}_h\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Gamma} + \|\gamma - \gamma_h\|_{0,\Omega} \\ &\quad \le \widetilde{C} \left\{ \operatorname{dist}(\boldsymbol{\sigma}, H_h^{\boldsymbol{\sigma}}) + \operatorname{dist}(\mathbf{u}, Q_h^{\mathbf{u}}) + \operatorname{dist}(\boldsymbol{\varphi}, Q_h^{\boldsymbol{\varphi}}) + \operatorname{dist}(\gamma, Q_h^{\gamma}) \right\}, \end{aligned} \tag{3.21}$$

where $((\boldsymbol{\sigma}, \mathbf{0}), (\mathbf{u}, \boldsymbol{\varphi}, \gamma)) \in \mathbf{H} \times \mathbf{Q}$ *is the unique solution of* (3.7).

3.4 A residual-based a posteriori error estimator

In this section we derive reliable and efficient residual based a posteriori error estimators for (3.19). We begin by introducing several notations. We let $\mathcal{E}_h$ be the set of all edges of the triangulation $\mathcal{T}_h$, and given $T \in \mathcal{T}_h$, we let $\mathcal{E}(T)$ be the set of its edges. Then we write $\mathcal{E}_h = \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$, where $\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h : \quad e \subseteq \Omega\}$ and $\mathcal{E}_h(\Gamma) := \{e \in \mathcal{E}_h : \quad e \subseteq \Gamma\}$. In what follows, h_e stands for the length of a given edge e. Also, for each edge $e \in \mathcal{E}_h$ we fix a unit normal vector $\boldsymbol{\nu}_e := (\nu_1, \nu_2)^{\mathbf{t}}$, and let $\mathbf{s}_e := (-\nu_2, \nu_1)^{\mathbf{t}}$ be the corresponding fixed unit tangential vector along e. However, when no confusion arises, we simple write $\boldsymbol{\nu}$ and $\mathbf{s}$ instead of $\boldsymbol{\nu}_e$ and $\mathbf{s}_e$, respectively. Now, let $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$ such that $\boldsymbol{\tau}|_T \in \mathbb{C}(T)$ on each $T \in \mathcal{T}_h$. Then, given $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)$, we denote by $[\boldsymbol{\tau}\,\mathbf{s}]$ the tangential jump of $\boldsymbol{\tau}$ across e, that is $[\boldsymbol{\tau}\,\mathbf{s}] := (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e\,\mathbf{s}$, where T and T' are the triangles of $\mathcal{T}_h$ having e as a common edge. Similar definitions hold for the tangential jumps of scalar fields $v \in L^2(\Omega)$ such that $v|_T \in C(T)$ on each $T \in \mathcal{T}_h$. Finally, given scalar, vector and tensor valued fields v, $\boldsymbol{\varphi} := (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ and $\boldsymbol{\tau} := (\tau_{ij})$, respectively we let

$$\mathbf{curl}(v) := \begin{pmatrix} \frac{\partial v}{\partial x_2} \\ \\ -\frac{\partial v}{\partial x_1} \end{pmatrix}, \quad \underline{\mathbf{curl}}(\boldsymbol{\varphi}) := \begin{pmatrix} \frac{\partial \boldsymbol{\varphi}_1}{\partial x_2} & -\frac{\partial \boldsymbol{\varphi}_1}{\partial x_1} \\ \\ \frac{\partial \boldsymbol{\varphi}_2}{\partial x_2} & -\frac{\partial \boldsymbol{\varphi}_2}{\partial x_1} \end{pmatrix}, \quad \text{and} \quad \operatorname{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \\ \\ \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

Next, letting $\big((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ be the unique solution of (3.19), we define for each $T \in \mathcal{T}_h$ the a posteriori error indicator:

$$\begin{aligned}
\theta_T^2 \,:=\, & \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T}^2 + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,T}^2 + \|\boldsymbol{\rho}_h\|_{0,T}^2 + h_T^2\,\|\mathrm{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\|_{0,T}^2 \\
& + \; h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega)} h_e\,\|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2 \\
& + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\,\left\|(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\,\mathbf{s} + \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}}\right\|_{0,e}^2 \\
& + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\,\|\mathbf{g} - \boldsymbol{\sigma}_h\,\boldsymbol{\nu}\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\,\|\boldsymbol{\varphi}_h + \mathbf{u}_h\|_{0,e}^2\,,
\end{aligned} \tag{3.22}$$

and introduce the global a posteriori error estimator

$$\boldsymbol{\theta} \,:=\, \left\{\sum_{T\in\mathcal{T}_h} \theta_T^2\right\}^{1/2}.$$

Then, the following theorem constitutes the main result of this paper.

Theorem 3.5 *Assume that* $\mathbf{g} \in \mathbf{L}^2(\Gamma)$. *Let* $((\boldsymbol{\sigma},\boldsymbol{\rho}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \in \mathbf{H}\times\mathbf{Q}$ *and* $((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the unique solutions of* (3.7) *and* (3.19), *respectively. Then, there exists constants* $C_{\mathtt{rel}} > 0$ *and* $C_{\mathtt{eff}} > 0$, *independent of* h, *such that*

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} \,\le\, \|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}} + \|(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma}) - (\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)\|_{\mathbf{Q}} \,\le\, C_{\mathtt{rel}}\,\boldsymbol{\theta}\,. \tag{3.23}$$

The efficiency of the global a posteriori error estimator (lower bound in (3.23)) is proved below in Subsection 3.4.2, whereas the corresponding reliability (upper bound in (3.23)) is derived next.

3.4.1 Reliability of the a posteriori error estimator

We begin with the following preliminary estimate for the partial error $\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}}$.

Lemma 3.3 *Let* $S_h : \mathbb{H}(\mathbf{div};\Omega) \to \mathbb{R}$ *be the functional defined by*

$$S_h(\boldsymbol{\tau}) \,:=\, \mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)\,,$$

and let $S_h|_V$ *be its restriction to* V, *the first component of the kernel* $\mathbf{V}$ *of* $\mathbf{B}$ (*cf.* (3.11)). *Then, there exists* $C > 0$, *independent of* h, *such that*

$$\begin{aligned}
\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}} \,\le\, C\Big\{ & \|S_h|_V\|_{V'} + \|\mathbf{f} + \mathbf{div}\boldsymbol{\sigma}_h\|_{0,\Omega} \\
+ \; & \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,\Omega} + \|\boldsymbol{\rho}_h\|_{0,\Omega} + \|\mathbf{g} - \boldsymbol{\sigma}_h\,\boldsymbol{\nu}\|_{-1/2,\Gamma}\Big\}\,,
\end{aligned} \tag{3.24}$$

and there holds $S_h(\boldsymbol{\tau}_h) \,=\, 0$ *for each* $\boldsymbol{\tau}_h \in H_h^{\boldsymbol{\sigma}}$.

Proof. We make use of a particular problem of the form (3.13). More precisely, let $((\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}}), (\bar{\mathbf{u}}, \bar{\boldsymbol{\varphi}}, \bar{\boldsymbol{\gamma}})) \in \mathbf{H} \times \mathbf{Q}$ be the unique solution of problem (3.13) with $\bar{F} \in \mathbf{H}'$ and $\bar{G} \in \mathbf{Q}'$ defined by

$$\bar{F}(\boldsymbol{\tau}, \boldsymbol{\chi}) := 0 \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{\chi}) \in \mathbf{H} \quad \text{and} \quad \bar{G}(\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta}) := B((\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta})) \quad \forall\, (\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta}) \in \mathbf{Q}\,.$$

According to the second equation of (3.7) and the definition of $\mathbf{B}$ (cf. (3.9)), we easily find that

$$\bar{G}(\mathbf{v}, \boldsymbol{\psi}, \boldsymbol{\eta}) \;=\; -\int_\Omega \mathbf{v} \cdot \big(\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_h\big) \;-\; \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\eta} \;-\; \int_\Omega \boldsymbol{\rho}_h \cdot \mathbf{v} \;+\; \langle \mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_\Gamma\,,$$

which, noting that $\displaystyle\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\eta} = \tfrac{1}{2} \int_\Omega (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}) : \boldsymbol{\eta}$, yields

$$\|\bar{G}\|_{\mathbf{Q}'} \;\le\; C \left\{ \|\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_h\|_{0,\Omega} \;+\; \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,\Omega} \;+\; \|\boldsymbol{\rho}_h\|_{0,\Omega} \;+\; \|\mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}\|_{-1/2,\Gamma} \right\}.$$

Then, the continuous dependence result (3.14) and the above estimate for $\|\bar{G}\|_{\mathbf{Q}'}$ imply

$$\|(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}} \;\le\; C \left\{ \|\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_h\|_{0,\Omega} \;+\; \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,\Omega} \;+\; \|\boldsymbol{\rho}_h\|_{0,\Omega} \;+\; \|\mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}\|_{-1/2,\Gamma} \right\}. \tag{3.25}$$

On the other hand, a straightforward application of the triangle inequality gives

$$\|(\boldsymbol{\sigma}, \boldsymbol{\rho}) \,-\, (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h)\|_{\mathbf{H}} \;\le\; \|(\boldsymbol{\sigma}, \boldsymbol{\rho}) \,-\, (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h) \,-\, (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}} \,+\, \|(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}}\,, \tag{3.26}$$

and hence, thanks to (3.25), it only remains to estimate $\|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}}$. To this end, we first observe from the second equation of (3.13) that $(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})$ belongs to $\mathbf{V}$, the kernel of operator $\mathbf{B}$ (cf. (3.10)). Hence, applying the ellipticity of $\mathbf{A}$ on $\mathbf{V}$ (cf. Lemma 3.1), we obtain that

$$\begin{aligned} \alpha\, \|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}}^2 \;&\le\; \mathbf{A}\big((\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}}), (\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\big) \\ &\le\; \mathbf{A}\big((\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\big) \\ &\quad +\, \|\mathbf{A}\|\, \|(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}}\, \|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}}\,, \end{aligned}$$

which, dividing by $\|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}}$, taking supremum on $\mathbf{V}$, and then recalling from (3.11) (cf. Lemma 3.1) that $\mathbf{V} = V \times \{\mathbf{0}\}$, gives

$$\alpha\, \|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h) - (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}} \;\le\; \sup_{\substack{\boldsymbol{\tau} \in V \\ \boldsymbol{\tau} \neq 0}} \frac{\mathbf{A}((\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\boldsymbol{\tau}, \mathbf{0}))}{\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}} \;+\; \|\mathbf{A}\|\, \|(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\rho}})\|_{\mathbf{H}}\,. \tag{3.27}$$

Next, from the first equation of (3.7) we have

$$\mathbf{A}((\boldsymbol{\sigma}, \boldsymbol{\rho}), (\boldsymbol{\tau}, \mathbf{0})) \;=\; -\,\mathbf{B}((\boldsymbol{\tau}, \mathbf{0}), (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma})) \;=\; 0 \qquad \forall\, \boldsymbol{\tau} \,\in\, V\,,$$

and then, bearing in mind the definition of $\mathbf{A}$ (cf. (3.8)), we get

$$\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\mathbf{0})) \,=\, -\,\mathbf{A}((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\mathbf{0})) \,=\, -\,\mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in V\,,$$

which, together with the fact that $\mathbf{b}(\boldsymbol{\tau},(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h))$ certainly vanishes for each $\boldsymbol{\tau}\in V$, yields

$$\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\mathbf{0})) \,=\, -\,S_h(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in V\,. \tag{3.28}$$

In this way, (3.24) follows directly from (3.25), (3.26), (3.27), and (3.28). Finally, it is quite clear from the first equation of (3.19) that

$$\begin{aligned} 0 \;&=\; \mathbf{A}((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau}_h,\mathbf{0})) \,+\, \mathbf{B}((\boldsymbol{\tau}_h,\mathbf{0}),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \\ &=\; \mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) \,+\, \mathbf{b}(\boldsymbol{\tau}_h,(\mathbf{u}_h,\boldsymbol{\varphi}_h,\boldsymbol{\gamma}_h)) \,=\, S_h(\boldsymbol{\tau}_h) \qquad \forall\,\boldsymbol{\tau}_h\in H_h^{\boldsymbol{\sigma}}\,, \end{aligned}$$

which completes the proof. □

We now aim to estimate

$$\|S_h|_V\|_{V'} \,:=\, \sup_{\substack{\boldsymbol{\tau}\in V\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{S_h(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}}$$

in (3.24), for which, according to the null property of S_h provided by the previous theorem, we will replace $S_h(\boldsymbol{\tau})$ by $S_h(\boldsymbol{\tau}-\boldsymbol{\tau}_h)$ with a suitably chosen $\boldsymbol{\tau}_h\in H_h^{\boldsymbol{\sigma}}$ depending each time on the given $\boldsymbol{\tau}\in V$. To this end, we now let $I_h : H^1(\Omega)\to X_h$ be the Clément interpolation operator (cf. [23]), where

$$X_h \,:=\, \Big\{ v_h\in C(\bar{\Omega}) : \quad v_h|_T \,\in\, P_1(T) \quad \forall\,T\in\mathcal{T}_h \Big\}\,. \tag{3.29}$$

A vectorial version of I_h, say $\mathbf{I}_h : \mathbf{H}^1(\Omega)\to\mathbf{X}_h \,:=\, X_h\times X_h$, which is defined componentwise by I_h, is also required. The following lemma provides the local approximation properties of I_h. Analogue estimates hold for the operator $\mathbf{I}_h$.

Lemma 3.4 *There exist c_1, $c_2>0$, independent of h, such that for all $v\in H^1(\Omega)$ there holds*

$$\|v - I_h(v)\|_{0,T} \,\le\, c_1\; h_T\,\|v\|_{1,\Delta(T)} \qquad \forall\,T\in\mathcal{T}_h$$

and

$$\|v - I_h(v)\|_{0,e} \,\le\, c_2\; h_e^{1/2}\,\|v\|_{1,\Delta(e)} \qquad \forall\,e\in\mathcal{E}_h(\Omega)\cup\mathcal{E}_h(\Gamma)\,,$$

where $\Delta(T) \,:=\, \cup\{T'\in\mathcal{T}_h : \quad T'\cap T\neq\emptyset\}$ and $\Delta(e) \,:=\, \cup\{T'\in\mathcal{T}_h : \quad T'\cap e\neq\emptyset\}$.

Proof. See [23]. □

The estimate for $\|S_h|_V\|_{V'}$ is established as follows.

Lemma 3.5 *Let* $((\boldsymbol{\sigma}, \boldsymbol{\rho}), (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma})) \in \mathbf{H} \times \mathbf{Q}$ *and* $((\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the unique solutions of* (3.7) *and* (3.19)*, respectively. Then, there exists* $C > 0$*, independent of* h*, such that*

$$\|S_h|_V\|_{V'} \;\le\; C \left\{ \sum_{T\in\mathcal{T}_h} \widetilde{\theta}_T^2 \right\}^{1/2}, \tag{3.30}$$

where

$$\begin{aligned}\widetilde{\theta}_T^2 \;:=\; h_T^2\, \|\mathrm{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\|_{0,T}^2 \;+ \sum_{e\in\mathcal{E}(T)\,\cap\,\mathcal{E}_h(\Omega)} h_e\, \|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2 \\ +\; \sum_{e\in\mathcal{E}(T)\,\cap\,\mathcal{E}_h(\Gamma)} h_e \left\|(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\,\mathbf{s} + \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}}\right\|_{0,e}^2 .\end{aligned} \tag{3.31}$$

Proof. Given $\boldsymbol{\tau} \in V$ (cf. (3.11)) we clearly have $\mathbf{div}\,\boldsymbol{\tau} = \mathbf{0}$ in Ω, and hence there exists $\boldsymbol{\phi} := (\phi_1, \phi_2) \in \mathbf{H}^1(\Omega)$ such that $\int_\Omega \phi_1 = \int_\Omega \phi_2 = 0$ and $\boldsymbol{\tau} = \underline{\mathbf{curl}}\,\boldsymbol{\phi}$. Note that the conditions satisfied by the components of $\boldsymbol{\phi}$ guarantee that $\|\boldsymbol{\phi}\|_{1,\Omega}$ and $|\boldsymbol{\phi}|_{1,\Omega}$ are equivalent. Then, we let $\boldsymbol{\phi}_h \in \mathbf{X}_h$ be the Clément interpolant of $\boldsymbol{\phi}$, that is $\boldsymbol{\phi}_h := \mathbf{I}_h(\boldsymbol{\phi})$, and define $\boldsymbol{\tau}_h := \underline{\mathbf{curl}}\,\boldsymbol{\phi}_h$ so that $\boldsymbol{\tau} - \boldsymbol{\tau}_h = \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)$. In turn, it is easy to see that $\boldsymbol{\tau}_h$ belongs to $H_h^{\boldsymbol{\sigma}}$, and therefore the null property satisfied by S_h (cf. Lemma 3.3) implies that

$$S_h(\boldsymbol{\tau}) \;=\; S_h(\boldsymbol{\tau} - \boldsymbol{\tau}_h) \;=\; \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau} - \boldsymbol{\tau}_h) \;+\; \mathbf{b}(\boldsymbol{\tau} - \boldsymbol{\tau}_h, (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h))\,,$$

which, in virtue of the definitions of $\mathbf{a}$ and $\mathbf{b}$ (cf. (3.5), (3.6)), gives

$$S_h(\boldsymbol{\tau}) \;=\; \int_\Omega \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big) : \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \;+\; \big\langle \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,\boldsymbol{\nu}, \boldsymbol{\varphi}_h \big\rangle_\Gamma\,. \tag{3.32}$$

Next, since

$$\underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,\boldsymbol{\nu} \;=\; -\frac{d}{d\mathbf{s}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \quad \text{and} \quad \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}} \in \mathbf{L}^2(\Gamma)\,,$$

we find, integrating by parts on Γ, that

$$\langle \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,\boldsymbol{\nu}, \boldsymbol{\varphi}_h \rangle_\Gamma \;=\; -\left\langle \frac{d}{d\mathbf{s}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h), \boldsymbol{\varphi}_h \right\rangle_\Gamma \;=\; \int_\Gamma \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}} \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,. \tag{3.33}$$

On the other hand, integrating by parts on each $T \in \mathcal{T}_h$, we obtain that

$$\begin{aligned}\int_\Omega \big\{\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big\} : \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \;&=\; \sum_{T\in\mathcal{T}_h} \int_T \big\{\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big\} : \underline{\mathbf{curl}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) \\ = \sum_{T\in\mathcal{T}_h} \Big\{ -\int_T \mathrm{curl}\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \;&+\; \int_{\partial T} \big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s} \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \Big\} \\ = -\sum_{T\in\mathcal{T}_h} \int_T \mathrm{curl}\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \;&+\; \sum_{e\in\mathcal{E}_h(\Omega)} \int_e \big[\big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\big] \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \\ +\; \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \big(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s} \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h)\,,\end{aligned}$$

which, together with (3.33), yields

$$S_h(\boldsymbol{\tau}) \,=\, -\sum_{T\in\mathcal{T}_h}\int_T \operatorname{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_h) + \sum_{e\in\mathcal{E}_h(\Omega)}\int_e \big[\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\big]\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_h)$$
$$+ \sum_{e\in\mathcal{E}_h(\Gamma)}\int_e \left\{\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s} \,+\, \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}}\right\}\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\,. \tag{3.34}$$

Then, applying Cauchy-Schwarz inequality and the approximation properties of the Clément interpolation operator $\mathbf{I}_h$ (cf. Lemma 3.4), and then using that the number of elements of $\Delta(T)$ is bounded independently of $T \in \mathcal{T}_h$, it follows that

$$\left|\sum_{T\in\mathcal{T}_h}\int_T \operatorname{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\right| \,\le\, c_1\sum_{T\in\mathcal{T}_h} h_T\,\big\|\operatorname{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\big\|_{0,T}\,\|\boldsymbol{\phi}\|_{1,\Delta(T)}$$
$$\le\, C\left\{\sum_{T\in\mathcal{T}_h} h_T^2\,\big\|\operatorname{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\big\|_{0,T}^2\right\}^{1/2}\|\boldsymbol{\phi}\|_{1,\Omega}\,. \tag{3.35}$$

Proceeding analogously, and now employing that the number of elements of $\Delta(e)$ is bounded independently of $e \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$, we find that

$$\left|\sum_{e\in\mathcal{E}_h(\Omega)}\int_e \big[\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\big]\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\right| \le C\left\{\sum_{e\in\mathcal{E}_h(\Omega)} h_e\,\big\|\big[\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\big]\big\|_{0,e}^2\right\}^{1/2}\|\boldsymbol{\phi}\|_{1,\Omega}\,, \tag{3.36}$$

and

$$\left|\sum_{e\in\mathcal{E}_h(\Gamma)}\int_e \left\{\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s} \,+\, \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}}\right\}\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\right|$$
$$\le\, C\left\{\sum_{e\in\mathcal{E}_h(\Gamma)} h_e\,\left\|\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big)\,\mathbf{s} \,+\, \frac{d\boldsymbol{\varphi}_h}{d\mathbf{s}}\right\|_{0,e}^2\right\}^{1/2}\|\boldsymbol{\phi}\|_{1,\Omega}\,. \tag{3.37}$$

Finally, (3.34), (3.35), (3.36), and (3.37), together with the fact that

$$\|\boldsymbol{\phi}\|_{1,\Omega} \,\le\, c\,|\boldsymbol{\phi}|_{1,\Omega} \,=\, \|\underline{\mathbf{curl}}\,\boldsymbol{\phi}\|_{0,\Omega} \,=\, \|\boldsymbol{\tau}\|_{0,\Omega} \,=\, \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,,$$

imply (3.30) and complete the proof. □

Besides Lemmas 3.3 and 3.5, and in order to complete the upper bound for $\|(\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}}$ in terms of local quantities, we need to estimate the boundary term $\|\mathbf{g} - \boldsymbol{\sigma}_h\,\boldsymbol{\nu}\|_{-1/2,\Gamma}$. In fact, we first observe that taking $(\mathbf{v}_h,\boldsymbol{\eta}_h) = (\mathbf{0},\mathbf{0})$ in (3.19), we arrive at

$$\langle\boldsymbol{\sigma}_h\,\boldsymbol{\nu} - \mathbf{g},\boldsymbol{\psi}_h\rangle_\Gamma \,=\, 0 \qquad \forall\boldsymbol{\psi}_h \in Q_h^{\boldsymbol{\varphi}}\,,$$

which says, having in mind that $\mathbf{g} \in \mathbf{L}^2(\Gamma)$, that each component of $(\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g})$ is $\mathbf{L}^2(\Gamma)$-orthogonal to the continuous piecewise linear functions on the double partition Γ_{2h} of Γ. Consequently, applying [20, Theorem 2] and recalling that Γ_h and Γ_{2h} are of bounded variation, we obtain

$$\|\mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}\|^2_{-1/2,\Gamma} \le c \sum_{e \in \mathcal{E}_h(\Gamma)} h_e \, \|\mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}\|^2_{0,e} \,. \tag{3.38}$$

In this way, the a posteriori error estimate for $\|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h)\|_{\mathbf{H}}$ follows straightforwardly from Lemmas 3.3 and 3.5, and (3.38). More precisely, we have the following result.

Lemma 3.6 *Let* $((\boldsymbol{\sigma}, \boldsymbol{\rho}), (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma})) \in \mathbf{H} \times \mathbf{Q}$ *and* $((\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the unique solutions of* (3.7) *and* (3.19)*, respectively. Then, there exists a constant* $C > 0$ *independent of* h*, such that*

$$\|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h)\|_{\mathbf{H}} \le C \left\{ \sum_{T \in \mathcal{T}_h} \widehat{\theta}_T^2 \right\}^{1/2} , \tag{3.39}$$

where

$$\widehat{\theta}_T^2 := \widetilde{\theta}_T^2 + \|\mathbf{f} + \mathbf{div}\boldsymbol{\sigma}_h\|^2_{0,T} + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|^2_{0,T} + \|\boldsymbol{\rho}_h\|^2_{0,T} + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \, \|\mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}\|^2_{0,e} \tag{3.40}$$

for each $T \in \mathcal{T}_h$*, with* $\widetilde{\theta}_T^2$ *defined by* (3.31)*.*

We proceed next to obtain the corresponding upper bound for $\|(\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma}) - (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)\|$. For this purpose, we need some additional preliminary results concerning the Helmholtz decomposition of $\mathbb{H}(\mathbf{div}; \Omega)$ and the approximation properties of the Raviart-Thomas interpolation operator. We begin with the following lemma.

Lemma 3.7 *For each* $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; \Omega)$ *there exist* $\boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega)$ *and* $\boldsymbol{\phi} := (\phi_1, \phi_2)^{\mathtt{t}} \in \mathbf{H}^1(\Omega)$*, with* $\int_\Omega \phi_1 = \int_\Omega \phi_2 = 0$*, such that* $\boldsymbol{\tau} = \boldsymbol{\zeta}_s + \underline{\mathbf{curl}}\, \boldsymbol{\phi}$ *in* Ω *and*

$$\|\boldsymbol{\zeta}_s\|_{1,\Omega} + \|\boldsymbol{\phi}\|_{1,\Omega} \le C \, \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega} \,, \tag{3.41}$$

where C *is a positive constant independent of* $\boldsymbol{\tau}$*.*

Proof. It is an adaptation of the analysis from [35, Section 3.2.2]. See also [27, Lemma 3.4] for full details. □

On the other hand, we also need to introduce the space of pure Raviart-Thomas tensors of order 0, that is

$$\mathrm{RT}_h := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\, \Omega) : \ \mathbf{c}^{\mathtt{t}} \, \boldsymbol{\tau}_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h \,, \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \right\} ,$$

which is clearly contained in $H_h^{\boldsymbol{\sigma}}$ (cf. (3.15)). Then, we let $\Pi_h : \mathbb{H}^1(\Omega) \to \mathrm{RT}_h$ be the usual Raviart-Thomas interpolation operator, which is characterized by the identity

$$\int_e \Pi_h(\boldsymbol{\zeta}_s)\,\boldsymbol{\nu} = \int_e \boldsymbol{\zeta}_s\,\boldsymbol{\nu} \qquad \forall\, e \in \mathcal{T}_h\,, \quad \forall\, \boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega)\,. \tag{3.42}$$

It is easy to show, using (3.42), that

$$\mathbf{div}(\Pi_h(\boldsymbol{\zeta}_s)) \;=\; \mathcal{P}_h(\mathbf{div}\,\boldsymbol{\zeta}_s) \qquad \forall\, \boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega)\,, \tag{3.43}$$

where $\mathcal{P}_h$ is the $\mathbf{L}^2(\Omega)$-orthogonal projector onto $Q_h^{\mathbf{u}}$ (cf. (3.16)). In addition, it is well known (see, e.g. [19], [34, Lemmas 3.16 and 3.18], and [69]) that Π_h satisfies the following approximation properties

$$\|\boldsymbol{\zeta}_s - \Pi_h(\boldsymbol{\zeta}_s)\|_{0,T} \;\le\; C\,h_T\,\|\boldsymbol{\zeta}_s\|_{1,T} \quad \forall\, T \in \mathcal{T}_h\,, \quad \forall\, \boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega)\,, \tag{3.44}$$

and

$$\|(\boldsymbol{\zeta}_s - \Pi_h(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}\|_{0,e} \;\le\; C\,h_e^{1/2}\,\|\boldsymbol{\zeta}_s\|_{1,T_e} \quad \forall\, e \in \mathcal{T}_h, \quad \forall\, \boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega)\,, \tag{3.45}$$

where T_e in (3.45) is a triangle of $\mathcal{T}_h$ containing e on its boundary.

We are now in a position to establish the remaining a posteriori error estimate.

Lemma 3.8 *Let* $((\boldsymbol{\sigma},\boldsymbol{\rho}),(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\gamma})) \in \mathbf{H} \times \mathbf{Q}$ *and* $((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\gamma_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the unique solutions of* (3.7) *and* (3.19)*, respectively. Then, there exists a constant* $C > 0$*, independent of* h*, such that*

$$\|(\mathbf{u},\boldsymbol{\varphi},\gamma) - (\mathbf{u}_h,\boldsymbol{\varphi}_h,\gamma_h)\|_{\mathbf{Q}} \;\le\; C\,\left\{\sum_{T\in\mathcal{T}_h} \theta_T^2\right\}^{1/2}\,, \tag{3.46}$$

where θ_T^2 *is the complete a posteriori error indicator defined by* (3.22)*.*

Proof. We begin by applying the continuous inf-sup condition for $\mathbf{B}$ (cf. Lemma 3.2), which yields

$$\beta\,\|(\mathbf{u},\boldsymbol{\varphi},\gamma) - (\mathbf{u}_h,\boldsymbol{\varphi}_h,\gamma_h)\|_{\mathbf{Q}} \;\le\; \sup_{\substack{(\boldsymbol{\tau},\boldsymbol{\chi})\in\mathbf{H}\\ (\boldsymbol{\tau},\boldsymbol{\chi})\neq 0}} \frac{\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u},\boldsymbol{\varphi},\gamma) - (\mathbf{u}_h,\boldsymbol{\varphi}_h,\gamma_h))}{\|(\boldsymbol{\tau},\boldsymbol{\chi})\|_{\mathbf{H}}}\,. \tag{3.47}$$

Next, using from the first equation of (3.7) that $\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u},\boldsymbol{\varphi},\gamma)) \;=\; -\,\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}),(\boldsymbol{\tau},\boldsymbol{\chi}))$, and then substracting and adding $(\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)$ in the first component, we find that for each $(\boldsymbol{\tau},\boldsymbol{\chi}) \in \mathbf{H}$ there holds

$$\begin{aligned}\mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u},\boldsymbol{\varphi},\gamma) - (\mathbf{u}_h,\boldsymbol{\varphi}_h,\gamma_h)) \;&=\; -\,\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{\rho}) - (\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\boldsymbol{\chi}))\\ &\quad -\,\mathbf{A}((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau},\boldsymbol{\chi})) \;-\; \mathbf{B}((\boldsymbol{\tau},\boldsymbol{\chi}),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\gamma_h))\,.\end{aligned} \tag{3.48}$$

Then, noting that $\int_\Omega (\mathbf{u}_h + \boldsymbol{\rho}_h) \cdot \boldsymbol{\chi} = 0$, which follows from the first equation of (3.19) when taking $\boldsymbol{\chi}_h = \boldsymbol{\chi}$ and $\boldsymbol{\tau}_h = \mathbf{0}$, and bearing in mind the functional $S_h : \mathbb{H}(\mathbf{div};\Omega) \to \mathbb{R}$ defined in the statement of Lemma 3.3, we obtain that for each $(\boldsymbol{\tau}, \boldsymbol{\chi}) \in \mathbf{H}$ there holds

$$- \mathbf{A}((\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\boldsymbol{\tau}, \boldsymbol{\chi})) - \mathbf{B}((\boldsymbol{\tau}, \boldsymbol{\chi}), (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)) \,=\, - \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - \mathbf{b}(\boldsymbol{\tau}, (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)) \,=:\, - S_h(\boldsymbol{\tau})\,,$$

whence (3.48) becomes

$$\mathbf{B}((\boldsymbol{\tau}, \boldsymbol{\chi}), (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma}) - (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)) \,=\, - \mathbf{A}((\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h), (\boldsymbol{\tau}, \boldsymbol{\chi})) \,-\, S_h(\boldsymbol{\tau}) \qquad \forall\, (\boldsymbol{\tau}, \boldsymbol{\chi}) \in \mathbf{H}\,.$$

Thus, replacing the above expression back into (3.47) and using the boundedness of $\mathbf{A}$ (with constant $\|\mathbf{A}\|$), we easily deduce that

$$\beta\, \|(\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma}) - (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{Q}} \,\le\, \|\mathbf{A}\|\, \|(\boldsymbol{\sigma}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{\rho}_h)\|_{\mathbf{H}} \,+\, \|S_h\|_{\mathbb{H}(\mathbf{div};\,\Omega)'} \tag{3.49}$$

It remains to bound $\|S_h\|_{\mathbb{H}(\mathbf{div};\,\Omega)'}$ on the right hand side of (3.49), for which we appeal to the Helmholtz decompositions from Lemma 3.7. In other words, given $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$, we let $\boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega)$, $\boldsymbol{\phi} \in \mathbf{H}^1(\Omega)$, and C a positive constant independent of $\boldsymbol{\tau}$, such that $\boldsymbol{\tau} \,=\, \boldsymbol{\zeta}_s + \underline{\mathbf{curl}}\, \boldsymbol{\phi}$ in Ω and

$$\|\boldsymbol{\zeta}_s\|_{1,\Omega} + \|\boldsymbol{\phi}\|_{1,\Omega} \,\le\, C\, \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,. \tag{3.50}$$

Then, we introduce

$$\boldsymbol{\phi}_h \,:=\, \mathbf{I}_h(\boldsymbol{\phi}) \in \mathbf{X}_h \quad \text{and} \quad \boldsymbol{\tau}_h \,:=\, \Pi_h(\boldsymbol{\zeta}_s) \,+\, \underline{\mathbf{curl}}(\boldsymbol{\phi}_h) \,\in\, \mathrm{RT}_h \,\subseteq\, H_h^{\boldsymbol{\sigma}}\,,$$

which yields

$$\boldsymbol{\tau} \,-\, \boldsymbol{\tau}_h \,=\, \boldsymbol{\zeta}_s \,-\, \Pi_h(\boldsymbol{\zeta}_s) \,+\, \underline{\mathbf{curl}}\big(\boldsymbol{\phi} - \boldsymbol{\phi}_h\big)\,.$$

It follows using (3.43) that

$$\mathbf{div}\big(\boldsymbol{\tau} \,-\, \boldsymbol{\tau}_h\big) \,=\, \big(\mathbf{I} - \mathcal{P}_h\big)(\mathbf{div}\, \boldsymbol{\zeta}_s) \,=\, \big(\mathbf{I} - \mathcal{P}_h\big)(\mathbf{div}\, \boldsymbol{\tau})\,,$$

which is $\mathbf{L}^2(\Omega)$-orthogonal to $Q_h^{\mathbf{u}}$, and hence, taking into account from Lemma 3.3 the null property satisfied by S_h, we can write that

$$S_h(\boldsymbol{\tau}) = S_h(\boldsymbol{\tau} - \boldsymbol{\tau}_h) \,=\, S_h\big(\boldsymbol{\zeta}_s \,-\, \Pi_h(\boldsymbol{\zeta}_s)\big) \,+\, S_h\big(\underline{\mathbf{curl}}\big(\boldsymbol{\phi} - \boldsymbol{\phi}_h\big)\big)\,, \tag{3.51}$$

where, recalling that $S_h(\boldsymbol{\tau}) = \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau}, (\mathbf{u}_h, \boldsymbol{\varphi}_h, \boldsymbol{\gamma}_h))$, we have

$$S_h\big(\boldsymbol{\zeta}_s \,-\, \Pi_h(\boldsymbol{\zeta}_s)\big) \,=\, \int_\Omega \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\big) : \big(\boldsymbol{\zeta}_s \,-\, \Pi_h(\boldsymbol{\zeta}_s)\big) \,+\, \langle \big(\boldsymbol{\zeta}_s \,-\, \Pi_h(\boldsymbol{\zeta}_s)\big)\, \boldsymbol{\nu}, \boldsymbol{\varphi}_h \rangle_\Gamma\,,$$

and

$$S_h\big(\underline{\mathbf{curl}}(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\big) \;=\; \int_\Omega \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h+\boldsymbol{\gamma}_h\big) : \underline{\mathbf{curl}}(\boldsymbol{\phi}-\boldsymbol{\phi}_h) \;+\; \langle \underline{\mathbf{curl}}(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\,\boldsymbol{\nu}, \boldsymbol{\varphi}_h\rangle_\Gamma\,.$$

The estimate for the latter term proceeds exactly as in the proof of Lemma 3.5, which gives, using now (3.50), that

$$\big|S_h\big(\underline{\mathbf{curl}}(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\big)\big| \;\le\; C\left\{\sum_{T\in\mathcal{T}_h}\widetilde{\theta}_T^2\right\}^{1/2} \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,, \tag{3.52}$$

with $\widetilde{\theta}_T^2$ defined by (3.31). In turn, for the former term we first notice that the fact that $\boldsymbol{\zeta}_s$ belongs to $\mathbb{H}^1(\Omega)$ guarantees that $\big(\boldsymbol{\zeta}_s - \Pi_h(\boldsymbol{\zeta}_s)\big)\,\boldsymbol{\nu}\in\mathbf{L}^2(\Gamma)$, and then, utilizing additionally the characterization (3.42), we get

$$S_h\big(\boldsymbol{\zeta}_s - \Pi_h(\boldsymbol{\zeta}_s)\big) \;=\; \int_\Omega \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_h+\boldsymbol{\gamma}_h\big) : \big(\boldsymbol{\zeta}_s - \Pi_h(\boldsymbol{\zeta}_s)\big) + \sum_{e\in\mathcal{E}_h(\Gamma)}\int_e \big(\boldsymbol{\zeta}_s - \Pi_h(\boldsymbol{\zeta}_s)\big)\,\boldsymbol{\nu}\cdot\big(\boldsymbol{\varphi}_h+\mathbf{u}_h\big)\,. \tag{3.53}$$

In this way, employing the Cauchy-Schwarz inequality, the approximation properties (3.44) and (3.45), and the estimate (3.50), we deduce from (3.53) that

$$\big|S_h\big(\boldsymbol{\zeta}_s - \Pi_h(\boldsymbol{\zeta}_s)\big)\big| \;\le\; C\left\{\sum_{T\in\mathcal{T}_h} h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_h+\boldsymbol{\gamma}_h\|_{0,T}^2 \;+\; \sum_{e\in\mathcal{E}_h(\Gamma)} h_e\,\|\boldsymbol{\varphi}_h+\mathbf{u}_h\|_{0,e}^2\right\}^{1/2}\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,. \tag{3.54}$$

Finally, it follows from (3.51), (3.52), and (3.54) that

$$\|S_h\|_{\mathbb{H}(\mathbf{div};\,\Omega)'} \;\le\; C\left\{\sum_{T\in\mathcal{T}_h}\Big(\widetilde{\theta}_T^2 \;+\; h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_h+\boldsymbol{\gamma}_h\|_{0,T}^2\Big) \;+\; \sum_{e\in\mathcal{E}_h(\Gamma)} h_e\,\|\boldsymbol{\varphi}_h+\mathbf{u}_h\|_{0,e}^2\right\}^{1/2},$$

which, together with (3.49) and the estimate for $\|(\boldsymbol{\sigma},\boldsymbol{\rho})-(\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h)\|_{\mathbf{H}}$ given by Lemma 3.6, yields (3.46) and completes the proof. □

We end this section by remarking that the reliability of $\boldsymbol{\theta}$, that is the upper bound in (3.23), is a straightforward consequence of Lemmas 3.6 and 3.8.

3.4.2 Efficiency of the a posteriori error estimators

The goal of this section is to show the efficiency of our a posteriori error estimator $\boldsymbol{\theta}$. In other words, we provide upper bounds depending on the actual errors for the nine terms defining the local indicator θ_T^2 (cf. (3.22)). We begin with the first three ones appearing there. In fact, since $\mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f}$ in Ω, we easily see that

$$\|\mathbf{f} \,+\, \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,T}^2 \;=\; \|\mathbf{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,T}^2 \;\le\; \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div},T}^2\,. \tag{3.55}$$

Next, adding and substracting $\boldsymbol{\sigma}$, and using that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathtt{t}}$ in Ω, we obtain

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,T}^2 \;\leq\; 4\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2\,. \tag{3.56}$$

Finally, since actually $\boldsymbol{\rho} = \mathbf{0}$ (cf. Theorem 3.1), it is clear that

$$\|\boldsymbol{\rho}_h\|_{0,T}^2 \;=\; \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,T}^2\,. \tag{3.57}$$

In what follows we give the corresponding upper bounds for the remaining terms in (3.22). Since most of these estimates are already available in the literature or can be easily derived from related ones (see, e.g. [21], [22], [27], [35], and [43]), we either refer to the corresponding proofs or sketch them. The main techniques involved include the localization technique based on triangle-bubble and edge-bubble functions, together with extension operators, discrete trace and inverse inequalities. For a better understanding of them, we now introduce further notations and preliminary results. Given $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T)$, we let ψ_T and ψ_e be the usual triangle-bubble and edge-bubble functions, respectively (see [72, eqs. (1.5) and (1.6)]), which satisfy:

ii) $\psi_T \in P_3(T)$, $\psi_T = 0$ on ∂T, $\mathrm{supp}(\psi_T) \subseteq T$, and $0 \leq \psi_T \leq 1$ in T.

ii) $\psi_e|_T \in P_2(T)$, $\psi_e = 0$ on $\partial T \setminus e$, $\mathrm{supp}(\psi_e) \subseteq w_e := \cup\{T' \in \mathcal{T}_h :\ e \in \mathcal{E}(T')\}$, and $0 \leq \psi_e \leq 1$ in w_e.

We also know from [71] that, given $k \in \mathbb{N} \cup \{0\}$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p) \in P_k(T)$ and $L(p)|_e = p$ for all $p \in P_k(e)$. Additional properties of ψ_T, ψ_e and L are collected in the following lemma.

Lemma 3.9 *Given $k \in \mathbb{N} \cup \{0\}$, there exist positive constants c_1, c_2 and c_3, depending only on k and the shape regularity of the triangulations (minimun angle condition), such that for each $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T)$, there hold*

$$\|q\|_{0,T}^2 \leq c_1 \|\psi_T^{1/2} q\|_{0,T}^2 \qquad \forall q \in P_k(T) \tag{3.58}$$

$$\|p\|_{0,e}^2 \leq c_2 \|\psi_e^{1/2} p\|_{0,e}^2 \qquad \forall p \in P_k(e) \tag{3.59}$$

and

$$\|\psi_e^{1/2} L(p)\|_{0,T}^2 \leq c_3\, h_e \|p\|_{0,e}^2 \qquad \forall p \in P_k(e) \tag{3.60}$$

Proof. See [71, Lemma 1.3]. □

The following inverse and discrete trace inequalities are also employed.

Lemma 3.10 *Let k, l, $m \in \mathbb{N} \cup \{0\}$ such that $l \leq m$. Then there exists $c > 0$, depending only on k, l, m and the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h$ there holds*

$$|q|_{m,T} \,\leq\, c\, h_T^{l-m}\, |q|_{l,T} \qquad \forall\, q \in P_k(T)\,. \tag{3.61}$$

Proof. See [24, Theorem 3.2.6]. □

Lemma 3.11 *There exists $C > 0$, depending only on the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h$ and $e \in \mathcal{E}(T)$, there holds*

$$\|v\|_{0,e}^2 \,\leq\, C \left\{ h_e^{-1}\, \|v\|_{0,T}^2 \,+\, h_e\, |v|_{1,T}^2 \right\} \quad \forall\, v \in H^1(T)\,. \tag{3.62}$$

Proof. See [1, Theorem 3.10] or [3, eq. (2.4)]. □

The upper bounds for the terms involving only the tensor $\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h$, whose proofs make use of the techniques and results described above, are given next.

Lemma 3.12 *There exists $C > 0$, independent of h and λ, such that for each $T \in \mathcal{T}_h$ there holds*

$$h_T^2\, \|\mathrm{curl}\,(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\|_{0,T}^2 \,\leq\, C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\}.$$

Proof. See [22, Lemma 6.3] or [13, Lemma 4.7]. □

Lemma 3.13 *There exists $C > 0$, independent of h and λ, such that for each $T \in \mathcal{T}_h$ there holds*

$$h_T^2\, \|\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h\|_{0,T}^2 \,\leq\, C \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \,+\, h_T^2\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, h_T^2\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\}.$$

Proof. See [22, Lemma 6.6]. □

Lemma 3.14 *There exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Omega)$ there holds*

$$h_e\, \|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\mathbf{s}]\|_{0,e}^2 \,\leq\, C \sum_{T \subseteq \omega_e} \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\},$$

where $\omega_e := \cup\big\{T' \in \mathcal{T}_h : \quad e \in \mathcal{E}(T')\big\}$.

Proof. See [22, Lemma 6.4]. □

The upper bound for the term involving the tensor $\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h$ and the tangential derivative of $\boldsymbol{\varphi}_h$ is given now.

Lemma 3.15 *There exists $C > 0$, independent of h and λ, such that*

$$\sum_{e \in \mathcal{E}_h(\Gamma)} h_e \left\| (\mathcal{C}^{-1} \boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\, \mathbf{s} + \frac{d\,\boldsymbol{\varphi}_h}{d\mathbf{s}} \right\|_{0,e}^2$$
$$\leq C \left\{ \sum_{e \in \mathcal{E}_h(\Gamma)} \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T_e}^2 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T_e}^2 \right\} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Gamma}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Gamma)$, T_e is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. We refer to [35, Lemma 20] where this result was established and proved. The proof makes use of the vector version of the extension operator $L : C(e) \to C(T)$, the fact that $\nabla \mathbf{u} = \mathcal{C}^{-1} \boldsymbol{\sigma} + \boldsymbol{\gamma}$ in Ω, the boundedness of the tangential derivative $\frac{d}{d\mathbf{s}} : \mathbf{H}^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}(\Gamma)$, the inverse and the Cauchy-Schwarz inequalities, and the bound for $h_{T_e}^2 \|\mathrm{curl}\,(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\|_{0,T_e}^2$ provided by Lemma 3.12. $\square$

While the estimate given by Lemma 3.15 is of non local character, it certainly suffices to conclude the efficiency of $\boldsymbol{\theta}$. However, the following lemma establishes that, under an additional regularity assumption on $\boldsymbol{\varphi}$, a corresponding local estimate can also be obtained.

Lemma 3.16 *Assume that $\boldsymbol{\varphi}|_e \in \mathbf{H}^1(e)$ for each $e \in \mathcal{E}_h(\Gamma)$. Then there exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Gamma)$ there holds*

$$h_e \left\| (\mathcal{C}^{-1} \boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\, \mathbf{s} + \frac{d\,\boldsymbol{\varphi}_h}{d\mathbf{s}} \right\|_{0,e}^2$$
$$\leq C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T_e}^2 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T_e}^2 + h_e \left\| \frac{d}{d\mathbf{s}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \right\|_{0,e}^2 \right\},$$

where T_e is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. See [35, Lemma 21]. $\square$

We derive now the upper bound for the term concerning the Neumann boundary condition on Γ. To this end, and for simplicity, we assume that $\mathbf{g}$ is piecewise polynomial on Γ. Otherwise, one would proceed as in the proof of related results by adding and substracting a suitable projection of $\mathbf{g}$ onto a polynomial space (see, e.g. [35, Lemma 23]).

Lemma 3.17 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Gamma)$ there holds*

$$h_e \|\mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}\|_{0,e}^2 \leq C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 + h_T^2 \|\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,T}^2 \right\}, \tag{3.63}$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. Given $e \in \mathcal{E}_h(\Gamma)$, we let T be the triangle of $\mathcal{T}_h$ having e as an edge, define $\mathbf{v}_e := \mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}$ on e, and consider the vector version $\mathbf{L}$ of the extension operator $L : C(e) \to C(T)$. Then, applying (3.59), recalling that $\psi_e = 0$ on $\partial T \backslash e$, extending $\psi_e \mathbf{L}(v_e)$ by zero in $\Omega \backslash T$ so that the resulting function belongs to $\mathbf{H}^1(\Omega)$, and replacing the datum $\mathbf{g}$ by $\boldsymbol{\sigma}\boldsymbol{\nu}$ on Γ, we get

$$\|\mathbf{v}_e\|_{0,e}^2 \,\le\, c_2 \int_e \psi_e \, \mathbf{v}_e \cdot \big(\mathbf{g} - \boldsymbol{\sigma}_h \boldsymbol{\nu}\big) \,=\, c_2 \langle (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\, \boldsymbol{\nu}, \psi_e \, \mathbf{L}(\mathbf{v}_e) \rangle_\Gamma \,.$$

Hence, integrating by parts in Ω, and then employing the Cauchy-Schwarz inequality, the inverse estimate (3.61), and the bound given by (3.60), we get

$$\begin{aligned} \|\mathbf{v}_e\|_{0,e}^2 \,&\le\, c_2 \left\{ \int_T \psi_e \, \mathbf{L}(\mathbf{v}_e) \cdot \mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \,+\, \int_T (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) : \nabla\big(\psi_e \, \mathbf{L}(\mathbf{v}_e)\big) \right\} \\ &\le\, C \left\{ \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,T} \,+\, h_T^{-1} \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} \right\} \|\psi_e \, \mathbf{L}(\mathbf{v}_e)\|_{0,T} \\ &\le\, C \, h_e^{1/2} \left\{ \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,T} \,+\, h_T^{-1} \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} \right\} \|\mathbf{v}_e\|_{0,e} \,, \end{aligned}$$

which, after minor manipulations and using that $h_e \le h_T$, yields (3.63) and completes the proof. □

The proof of efficiency of $\boldsymbol{\theta}$ is completed with the following result.

Lemma 3.18 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Gamma)$ there holds*

$$h_e \, \|\boldsymbol{\varphi}_h + \mathbf{u}_h\|_{0,e}^2 \,\le\, C \left\{ h_T^2 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, h_T^2 \, \|\gamma - \gamma_h\|_{0,T}^2 \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \,+\, h_e \, \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. Adding and substracting $\boldsymbol{\varphi} \,=\, -\,\mathbf{u}$ on Γ, and then employing the discrete trace inequality (3.62) (cf. Lemma 3.11), we obtain for each $e \in \mathcal{E}_h(\Gamma)$

$$\begin{aligned} h_e \, \|\boldsymbol{\varphi}_h + \mathbf{u}_h\|_{0,e}^2 \,&\le\, 2 \, h_e \left\{ \|\boldsymbol{\varphi}_h - \boldsymbol{\varphi}\|_{0,e}^2 \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,e}^2 \right\} \\ &\le\, C \left\{ h_e \, \|\boldsymbol{\varphi}_h - \boldsymbol{\varphi}\|_{0,e}^2 \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \,+\, h_T^2 \, |\mathbf{u}|_{1,T}^2 \right\}, \end{aligned} \tag{3.64}$$

where the last term uses that $h_e \le h_T$ and that $\mathbf{u}_h$ is piecewise constant (cf. (3.16)). Then, using that $\nabla \mathbf{u} = \mathcal{C}^{-1}\boldsymbol{\sigma} + \gamma$ in Ω, adding and substracting $\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \gamma_h$, and employing the upper bound from Lemma 3.13, we find that

$$\begin{aligned} h_T^2 \, |\mathbf{u}|_{1,T}^2 \,=\, h_T^2 \, \|\mathcal{C}^{-1}\boldsymbol{\sigma} + \gamma\|_{0,T}^2 \,&\le\, 2 \, h_T^2 \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, \|\gamma - \gamma_h\|_{0,T}^2 \,+\, \|\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \gamma_h\|_{0,T}^2 \right\} \\ &\le\, C \left\{ h_T^2 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, h_T^2 \, \|\gamma - \gamma_h\|_{0,T}^2 \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \right\}. \end{aligned} \tag{3.65}$$

Finally, (3.64) and (3.65) yield the required estimate and finish the proof. □

3.5 Numerical results

In this section we present some numerical results confirming the reliability and efficiency of the a posteriori error estimator $\boldsymbol{\theta}$ analyzed in Section 3.4, and illustrating the performance of the associated adaptive algorithm. We begin by introducing additional notations. The variable N stands for the number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$ (equivalently, the number of unknowns of (3.19)), and the individual and global errors are denoted by:

$$\mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega}\,, \qquad \mathsf{e}(\mathbf{u}) := \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\,,$$

$$\mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,\Gamma}\,, \qquad \mathsf{e}(\gamma) := \|\gamma-\gamma_h\|_{0,\Omega}\,, \quad \text{and}$$

$$\mathsf{e} := \Big\{[\mathsf{e}(\boldsymbol{\sigma})]^2 + [\mathsf{e}(\boldsymbol{\rho})]^2 + [\mathsf{e}(\mathbf{u})]^2 + [\mathsf{e}(\boldsymbol{\varphi})]^2 + [\mathsf{e}(\gamma)]^2\Big\}^{1/2},$$

where $((\boldsymbol{\sigma},\mathbf{0}),(\mathbf{u},\boldsymbol{\varphi},\gamma)) \in \mathbf{H}\times\mathbf{Q}$ and $((\boldsymbol{\sigma}_h,\boldsymbol{\rho}_h),(\mathbf{u}_h,\boldsymbol{\varphi}_h,\gamma_h)) \in \mathbf{H}_h\times\mathbf{Q}_h$ are the unique solutions of (3.7) and (3.19), respectively. Also, we define the effectivity index

$$\mathtt{eff}(\boldsymbol{\theta}) := \mathsf{e}/\boldsymbol{\theta}\,.$$

In turn, we let $r(\boldsymbol{\sigma})$, $r(\mathbf{u})$, $r(\boldsymbol{\varphi})$, $r(\gamma)$, and r be the experimental rates of convergence given by

$$r(\boldsymbol{\sigma}) := \frac{\log(\mathsf{e}(\boldsymbol{\sigma})/\mathsf{e}'(\boldsymbol{\sigma}))}{\log(h/h')}\,, \quad r(\mathbf{u}) := \frac{\log(\mathsf{e}(\mathbf{u})/\mathsf{e}'(\mathbf{u}))}{\log(h/h')}\,, \quad r(\boldsymbol{\varphi}) := \frac{\log(\mathsf{e}(\boldsymbol{\varphi})/\mathsf{e}'(\boldsymbol{\varphi}))}{\log(h/h')}\,,$$

$$r(\gamma) := \frac{\log(\mathsf{e}(\gamma)/\mathsf{e}'(\gamma))}{\log(h/h')}\,, \quad \text{and} \quad r := \frac{\log\big(\mathsf{e}/\mathsf{e}'\big)}{\log(h/h')}\,,$$

where h and h' denote two consecutive meshsizes with errors e and e', respectively. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

In what follows we describe the examples to be considered, which are basically the same ones employed in [38]. In Example 1 we consider the Young modulus $E = 1$ and the Poisson ratio $\nu = 0.4999$, which yields the Lamé parameters $\lambda := \frac{E\,\nu}{(1+\nu)\,(1-2\,\nu)} = 1666.4444$ and $\mu := \frac{E}{2(1+\nu)} = 0.3333$. Then, we take the square domain $\Omega :=]-1/2,1/2[^2$ and choose $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution $\mathbf{u}$ is given by the first column of the fundamental solution at $\mathbf{x}_0 := (1,0)^{\mathtt{t}}$, that is

$$\mathbf{u}(\mathbf{x}) := \left\{-\frac{(\lambda+3\mu)}{4\pi\mu\,(\lambda+2\,\mu)}\,\log\|\mathbf{x}-\mathbf{x}_0\|\,\mathbf{I} + \frac{(\lambda+\mu)}{4\pi\mu\,(\lambda+2\mu)}\,\frac{(\mathbf{x}-\mathbf{x}_0)\,(\mathbf{x}-\mathbf{x}_0)^{\mathtt{t}}}{\|\mathbf{x}-\mathbf{x}_0\|^2}\right\}\begin{pmatrix}1\\0\end{pmatrix} \quad \forall\,\mathbf{x}\in\Omega\,.$$

In particular, $\mathbf{f} = \mathbf{0}$ and $\mathbf{u}$ is smooth in a neighborhood of $\bar{\Omega}$, whence [38, Theorem 4.2] yields an a priori rate of convergence of $O(h)$. This fact was confirmed by the numerical results provided in [38].

Next, in Example 2 we consider the same Lamé parameters from Example 1, take the L-shaped domain $\Omega :=]-1,1[^2 \setminus [0,1]^2$, and choose $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution $\mathbf{u}$ is given, in polar coordinates, by

$$\mathbf{u}(\mathtt{r},\theta) = \mathtt{r}^{5/3}\sin((2\theta-\pi)/3)\begin{pmatrix}1\\1\end{pmatrix} \qquad \forall\,(\mathtt{r},\theta)\,\in\,\Omega\,.$$

Note in this case that the partial derivatives of $\mathbf{u}$, of order ≥ 2, are singular at the origin. Moreover, because of the power of $\mathtt{r}$, we find that $\mathbf{f} := -\,\mathbf{div}\,\boldsymbol{\sigma}$ belongs to $\mathbf{H}^{2/3}(\Omega)$, whence [38, Theorem 4.2] yields in this case an a priori rate of convergence of $O(h^{2/3})$. This fact was also confirmed by the numerical results provided in [38]. According to the preceding remarks, this example is utilized to illustrate the behavior of the adaptive algorithm associated with $\boldsymbol{\theta}$, which applies the following procedure from [72]:

1) Start with a coarse mesh $\mathcal{T}_h$.
2) Solve the discrete problem (3.19) for the actual mesh $\mathcal{T}_h$.
3) Compute the error indicators θ_T on each triangle $T \in \mathcal{T}_h$.
4) Evaluate stopping criterion and decide to finish or go to next step.
5) Use *blue-green* procedure to refine each $T' \in \mathcal{T}_h$ whose local error indicator $\theta_{T'}$ satisfies
$$\theta_{T'} \,\geq\, \frac{1}{2}\max\left\{\theta_T : \quad T\,\in\,\mathcal{T}_h\right\}.$$
6) Define resulting mesh as actual mesh $\mathcal{T}_h$, and go to step 2.

The numerical results shown below were obtained using a MATLAB code. In Tables 3.1 and 3.2 we display the convergence history of our mixed finite element scheme (3.19) as applied to Example 1 for a finite sequence of quasi-uniform triangulations of $\overline{\Omega}$. While this example was already considered in [38, Section 6] (though with different Lamé constants), the novelty now is certainly the computation of the effectivity indexes. Indeed, we notice from the last column of Table 3.2 that the effectivity indexes $\mathtt{eff}(\boldsymbol{\theta})$ remain always in a neighborhood of 0.15, which illustrates the reliability and efficiency of $\boldsymbol{\theta}$ in the case of a regular solution. In turn, as previously observed in [38, Section 6], it is clear from the experimental rates of convergence shown in these tables that the $O(h)$ predicted by [38, Theorem 4.2] when $\delta = 1$ is attained in all the unknowns of this example.

Next, in Tables 3.3 up to 3.6 we provide the convergence history of the quasi-uniform and adaptive refinements, as applied to Example 2. We notice in the quasi-uniform case that $r(\boldsymbol{\sigma})$ oscillates around 2/3, whence, being $\mathsf{e}(\boldsymbol{\sigma})$ the dominant component of the total error e, this oscillation is also reflected in the global rate of convergence r. In addition, it is clear from these tables that the total errors of the adaptive scheme decrease faster than those obtained by the quasi-uniform one, which is confirmed by the global experimental rates of convergence provided in Table 3.6. This fact becomes also evident from Figure 3.1, mainly from $N \cong 1E+04$ on, where we display e vs. N for both refinements. Furthermore, it is quite straightforward from the values of r in Table 3.6 that the adaptive method is able to recover the quasi-optimal rate of convergence $O(h)$ for e. In turn, the reliability and efficiency of $\boldsymbol{\theta}$ is clearly confirmed by the effectivity indexes from Table 3.6 (most of them around 0.30) for this example with a non-smooth solution. Intermediate meshes obtained with the adaptive refinement are displayed in Figure 3.2. As expected, the method is able to recognize the origin as a singularity of the solution of this example. Finally, in order to illustrate the accurateness of the proposed mixed method and the adaptive scheme induced by $\boldsymbol{\theta}$, several components of the approximate (left) and exact (right) solutions of Example 2 are displayed in Figures 3.3 up to 3.5. Note here that the values of $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}_h$ on Γ are depicted along a straight line beginning at the point $(-1,-1)$ and then continuing counterclockwise.

h	N	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$
1/8	1044	3.364E−02	–	1.087E−02	–
1/12	2284	2.159E−02	1.094	7.206E−03	1.014
1/16	4004	1.595E−02	1.052	5.396E−03	1.005
1/24	8884	1.051E−02	1.029	3.594E−03	1.002
1/32	15684	7.845E−03	1.017	2.695E−03	1.001
1/48	35044	5.208E−03	1.010	1.796E−03	1.000
1/64	62084	3.899E−03	1.007	1.347E−03	1.000
1/96	139204	2.595E−03	1.004	8.980E−04	1.000
1/128	247044	1.944E−03	1.003	6.735E−04	1.000
1/192	554884	1.295E−03	1.002	4.490E−04	1.000
1/256	985604	9.711E−04	1.001	3.367E−04	1.000

Table 3.1: Convergence history for $\boldsymbol{\sigma}$ and $\mathbf{u}$ (Example 1)

N	$\mathtt{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathtt{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
1044	3.642E−02	–	2.387E−02	–	5.609E−02	–	0.2271
2284	1.605E−02	2.021	1.234E−02	1.628	3.046E−02	1.506	0.1916
4004	9.025E−03	2.000	7.851E−03	1.570	2.066E−02	1.350	0.1765
8884	4.126E−03	1.930	4.220E−03	1.531	1.258E−02	1.223	0.1642
15684	2.404E−03	1.877	2.731E−03	1.513	9.058E−03	1.142	0.1592
35044	1.140E−03	1.841	1.482E−03	1.508	5.818E−03	1.092	0.1548
62084	6.765E−04	1.814	9.610E−04	1.506	4.289E−03	1.060	0.1529
139204	3.267E−04	1.795	5.220E−04	1.505	2.814E−03	1.040	0.1512
247044	1.957E−04	1.782	3.386E−04	1.505	2.095E−03	1.026	0.1504
554884	9.536E−05	1.773	1.840E−04	1.504	1.386E−03	1.018	0.1497
985604	5.737E−05	1.766	1.194E−04	1.503	1.036E−03	1.012	0.1494

Table 3.2: Convergence history for $\boldsymbol{\varphi}$, $\boldsymbol{\gamma}$, $\mathtt{e}$, and effectivity index (EXAMPLE 1)

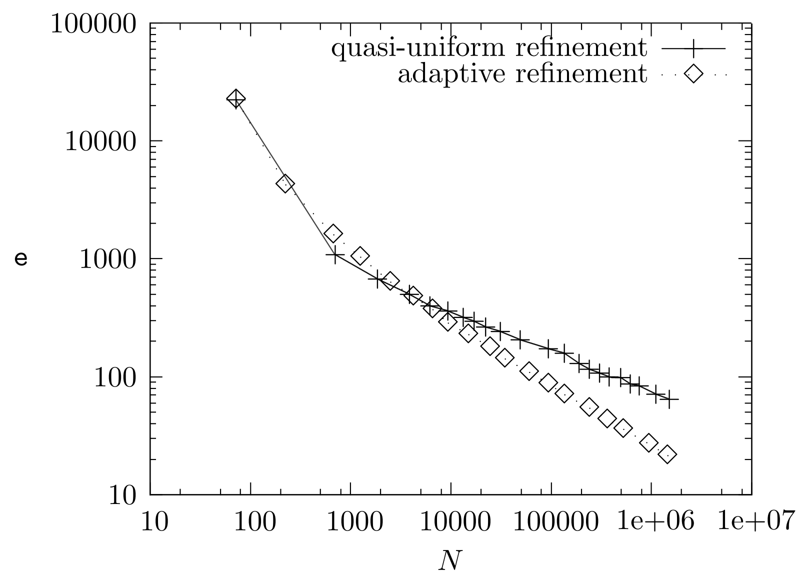

Figure 3.1: EXAMPLE 2, total error $\mathtt{e}$ vs. N for the quasi-uniform and adaptive schemes

h	N	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$
1/1	73	3.107E+03	–	3.966E+03	–
1/3	713	8.641E+02	1.165	1.615E+02	2.914
1/5	1878	6.384E+02	0.593	5.721E+01	2.031
1/7	3898	4.873E+02	0.803	2.527E+01	2.429
1/9	6218	3.920E+02	0.866	1.593E+01	1.835
1/11	9408	3.599E+02	0.425	1.042E+01	2.114
1/13	13333	3.172E+02	0.756	7.628E+00	1.869
1/15	17318	2.958E+02	0.487	5.740E+00	1.988
1/17	22338	2.633E+02	0.931	4.638E+00	1.702
1/20	31364	2.427E+02	0.501	3.404E+00	1.904
1/25	49273	2.049E+02	0.759	2.034E+00	2.308
1/35	94938	1.720E+02	0.520	1.126E+00	1.758
1/42	137179	1.583E+02	0.457	7.667E−01	2.106
1/50	191774	1.294E+02	1.152	5.588E−01	1.814
1/56	242234	1.152E+02	1.026	4.354E−01	2.202
1/63	308798	1.077E+02	0.571	3.342E−01	2.246
1/70	381354	9.951E+01	0.753	2.785E−01	1.731
1/80	497729	9.789E+01	0.127	2.201E−01	1.763
1/90	624949	8.678E+01	1.022	1.745E−01	1.970
1/100	768804	8.348E+01	0.368	1.427E−01	1.913
1/120	1124779	7.142E+01	0.855	9.926E−02	1.989
1/140	1518284	6.433E+01	0.678	6.918E−02	2.342

Table 3.3: Convergence history for $\boldsymbol{\sigma}$ and $\mathbf{u}$ (quasi-uniform scheme, EXAMPLE 2)

N	$\mathtt{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathtt{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
73	2.020E+04	–	7.724E+03	–	2.220E+04	–	0.8936
713	5.520E+02	3.277	2.970E+02	2.966	1.080E+03	2.752	0.3984
1878	1.955E+02	2.032	1.054E+02	2.028	6.783E+02	0.910	0.3591
3898	9.219E+01	2.234	5.730E+01	1.811	4.999E+02	0.907	0.3496
6218	5.910E+01	1.769	3.552E+01	1.903	3.983E+02	0.904	0.3287
9408	3.866E+01	2.114	2.645E+01	1.469	3.631E+02	0.461	0.3610
13333	2.924E+01	1.673	1.982E+01	1.728	3.193E+02	0.770	0.3488
17318	2.284E+01	1.726	1.546E+01	1.734	2.972E+02	0.501	0.3636
22338	1.896E+01	1.485	1.335E+01	1.175	2.644E+02	0.935	0.3585
31364	1.249E+01	2.568	1.043E+01	1.518	2.433E+02	0.511	0.4102
49273	8.879E+00	1.530	7.007E+00	1.783	2.052E+02	0.763	0.3686
94938	5.090E+00	1.654	4.420E+00	1.369	1.721E+02	0.522	0.3812
137179	3.418E+00	2.184	3.478E+00	1.314	1.583E+02	0.458	0.4279
191774	2.687E+00	1.380	2.692E+00	1.470	1.295E+02	1.153	0.3675
242234	2.324E+00	1.280	2.335E+00	1.254	1.153E+02	1.026	0.3607
308798	2.106E+00	0.838	2.006E+00	1.290	1.078E+02	0.572	0.3805
381354	1.487E+00	3.300	1.734E+00	1.383	9.954E+01	0.754	0.3749
497729	1.202E+00	1.595	1.490E+00	1.137	9.790E+01	0.124	0.3842
624949	9.922E−01	1.629	1.264E+00	1.394	8.680E+01	1.022	0.3922
768804	8.616E−01	1.340	1.137E+00	1.009	8.349E+01	0.369	0.3949
1124779	6.768E−01	1.324	9.039E−01	1.257	7.143E+01	0.855	0.3939
1518284	5.304E−01	1.580	7.456E−01	1.249	6.434E+01	0.678	0.4148

Table 3.4: Convergence history for $\boldsymbol{\varphi}$, $\boldsymbol{\gamma}$, $\mathtt{e}$, and effectivity index (quasi-uniform scheme, Example 2)

N	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\mathbf{u})$	$r(\mathbf{u})$
73	3.107E+03	–	3.966E+03	–
224	1.852E+03	0.923	9.058E+02	2.634
676	1.135E+03	0.887	2.911E+02	2.056
1264	8.404E+02	0.959	1.351E+02	2.452
2495	5.810E+02	1.086	5.425E+01	2.685
4286	4.479E+02	0.962	2.962E+01	2.237
6636	3.496E+02	1.133	1.905E+01	2.020
9471	2.752E+02	1.346	1.310E+01	2.106
15034	2.187E+02	0.994	8.178E+00	2.039
24695	1.730E+02	0.945	4.884E+00	2.078
34829	1.377E+02	1.329	3.172E+00	2.511
60717	1.063E+02	0.931	1.918E+00	1.810
94831	8.485E+01	1.010	1.198E+00	2.112
136714	6.844E+01	1.175	8.121E−01	2.125
241371	5.276E+01	0.915	4.664E−01	1.951
368348	4.248E+01	1.025	3.090E−01	1.948
525936	3.511E+01	1.070	2.143E−01	2.055
955896	2.628E+01	0.970	1.164E−01	2.042
1453383	2.106E+01	1.057	7.816E−02	1.902

Table 3.5: Convergence history for $\boldsymbol{\sigma}$ and $\mathbf{u}$ (adaptive scheme Example 2)

N	$\mathtt{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathtt{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
73	2.220E+04	–	7.724E+03	–	2.220E+04	–	0.8936
224	3.345E+03	3.205	1.556E+03	2.858	4.226E+03	2.959	0.5302
676	9.327E+02	2.312	5.313E+02	1.946	1.589E+03	1.771	0.4089
1264	4.642E+02	2.230	3.424E+02	1.404	1.028E+03	1.391	0.3802
2495	2.054E+02	2.398	1.495E+02	2.437	6.364E+02	1.411	0.3557
4286	1.192E+02	2.011	1.073E+02	1.225	4.766E+02	1.069	0.3402
6636	8.668E+01	1.457	8.520E+01	1.057	3.706E+02	1.151	0.3430
9471	5.086E+01	2.998	5.997E+01	1.974	2.865E+02	1.447	0.3211
15034	3.557E+01	1.548	4.537E+01	1.208	2.263E+02	1.020	0.3203
24695	2.223E+01	1.893	3.464E+01	1.088	1.779E+02	0.970	0.3248
34829	1.693E+01	1.586	2.752E+01	1.338	1.414E+02	1.334	0.3066
60717	1.081E+01	1.615	2.049E+01	1.062	1.088E+02	0.945	0.3103
94831	7.247E+00	1.793	1.602E+01	1.102	8.666E+01	1.020	0.3117
136714	5.505E+00	1.503	1.306E+01	1.130	6.989E+01	1.175	0.2990
241371	3.940E+00	1.177	9.571E+00	1.093	5.377E+01	0.923	0.3057
368348	2.654E+00	1.870	7.464E+00	1.176	4.322E+01	1.034	0.3066
525936	1.779E+00	2.247	6.778E+00	0.542	3.580E+01	1.057	0.2994
955896	1.241E+00	1.205	4.713E+00	1.216	2.673E+01	0.979	0.3029
1453383	8.444E−01	1.838	3.852E+00	0.963	2.143E+01	1.055	0.3022

Table 3.6: Convergence history for $\boldsymbol{\varphi}$, $\boldsymbol{\gamma}$, $\mathtt{e}$, and effectivity index (adaptive scheme, EXAMPLE 2)

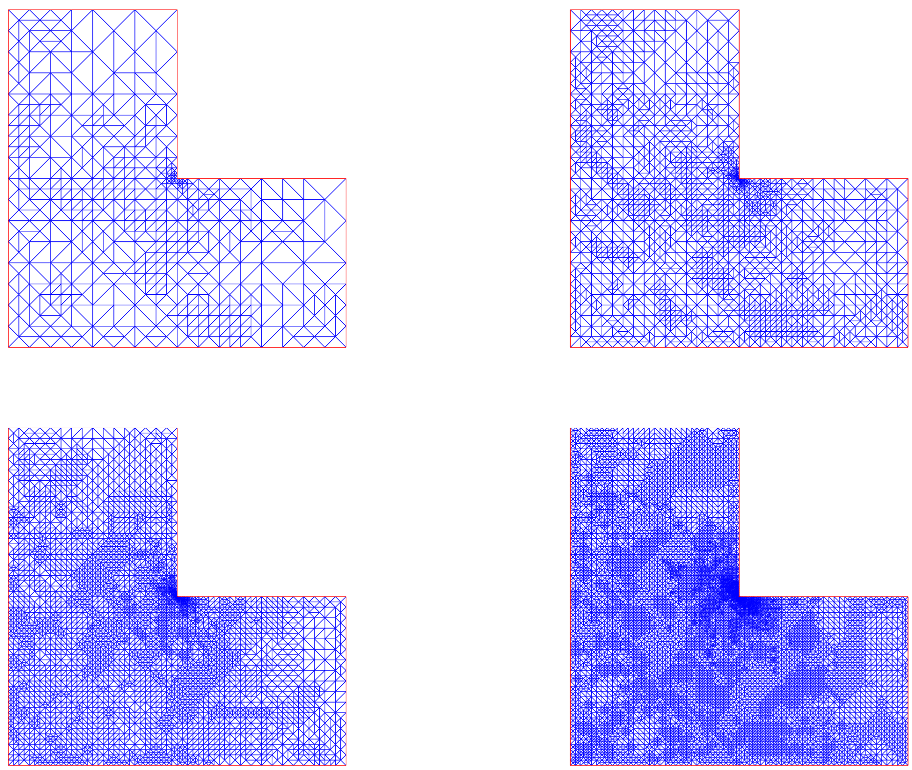

Figure 3.2: EXAMPLE 2: adapted meshes for $N \in \{6636, 24695, 60717, 136714\}$

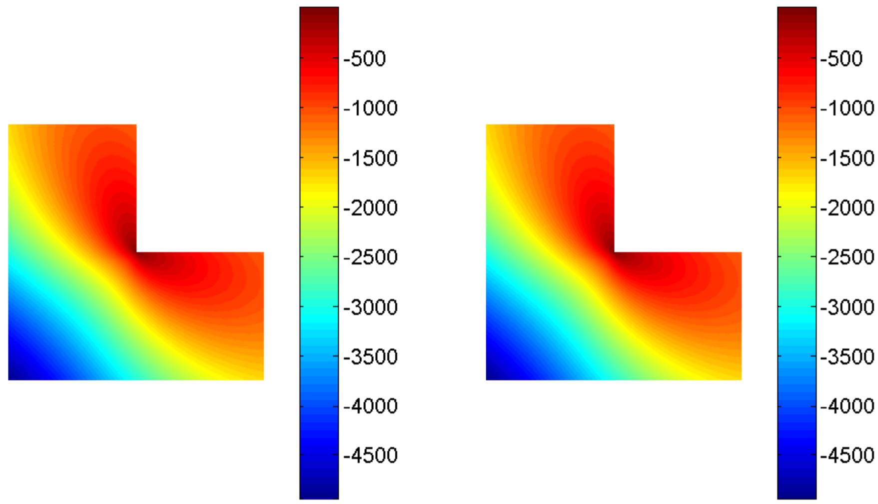

Figure 3.3: Approximate and exact σ_{11} ($N = 1453383$, EXAMPLE 2)

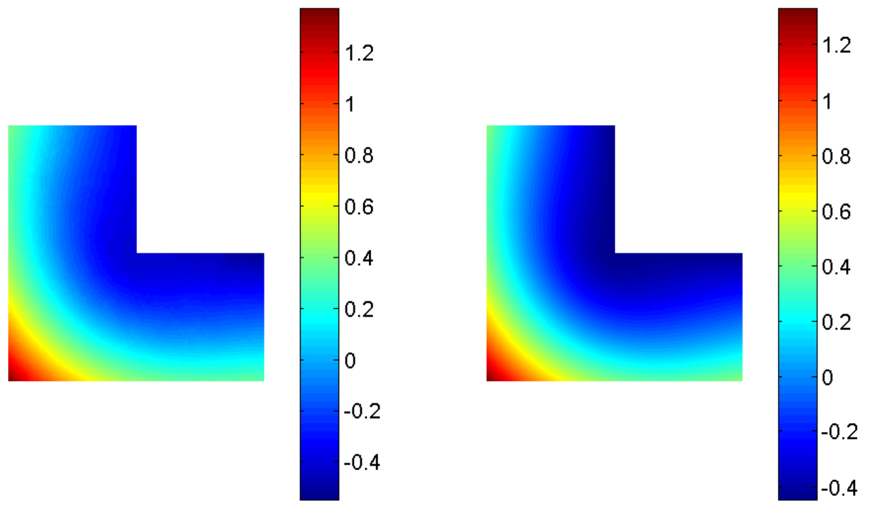

Figure 3.4: Approximate and exact u_2 ($N = 1453383$, EXAMPLE 2)

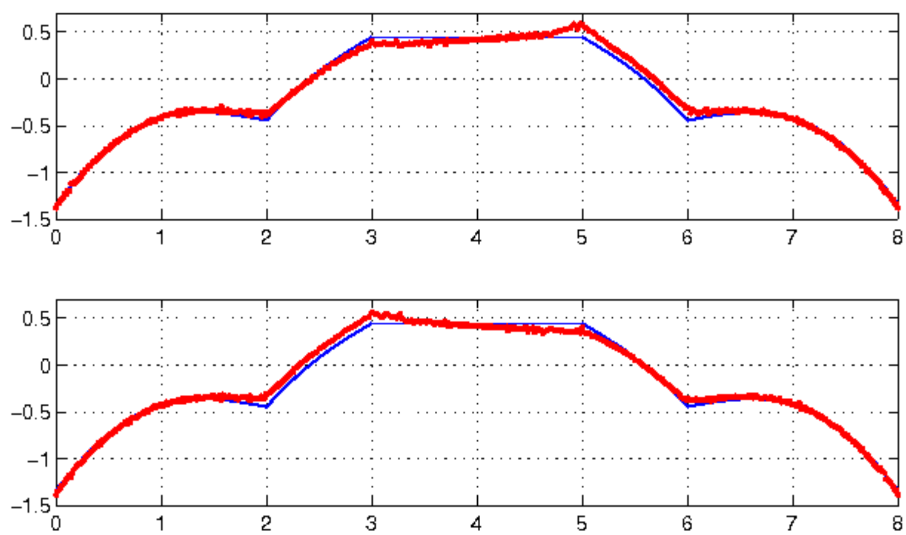

Figure 3.5: Approximate (red) and exact (blue) $\boldsymbol{\varphi}$ ($N = 1453383$, Example 2)

Chapter 4

A posteriori error analysis of a fully-mixed finite element method for a two-dimensional fluid-solid interaction problem

4.1 Introduction

In the recent paper [28] we introduced and analyzed a fully-mixed finite element method for the two-dimensional fluid-solid interaction problem studied originally in [37] (see also [39]). The respective model consists of an elastic body which is subject to a given incident wave that travels in the fluid surrounding it. Actually, the fluid is supposed to occupy an annular region, and hence a Robin boundary condition imitating the behavior of the scattered field at infinity is imposed on its exterior boundary, which is located far from the obstacle. The media are governed by the elastodynamic and acoustic equations in time-harmonic regime, respectively, and the transmission conditions are given by the equilibrium of forces and the equality of the corresponding normal displacements. Differently from the analysis in [37] where dual and primal methods are utilized in the solid and fluid, respectively, dual-mixed approaches are applied in both domains in [28], and the governing equations are employed to eliminate the displacement $\mathbf{u}$ of the solid and the pressure p of the fluid. In addition, since both transmission conditions become essential, they are enforced weakly by means of two suitable Lagrange multipliers. In this way, the Cauchy stress tensor and the rotation of the solid, together with the gradient of p and the traces of $\mathbf{u}$ and p on the boundary of the fluid, constitute the unknowns of the coupled problem. The solvability of the resulting continuous formulation is analyzed in [28] by incorporating first suitable decompositions of the spaces to which the stress and the gradient of p belong, and then by applying the Babuška-Brezzi theory and the Fredholm alternative. The unknowns of the solid and the fluid are approximated by a conforming Galerkin scheme defined in terms of PEERS elements in the solid, Raviart-Thomas of lowest order in the fluid, and continuous piecewise linear functions on the boundary. The analysis of the discrete method relies on a stable decomposition of the corresponding finite element spaces and also on the classical result on projection methods for Fredholm operators of index zero.

On the other hand, it is well known that in order to guarantee a good convergence behaviour of the finite element solutions, specially under the presence of complex geometries leading eventually to singularities, one needs to apply an adaptive strategy based on a posteriori error estimates. These are usually represented by global quantities $\boldsymbol{\theta}$ that are expressed in terms of local estimators θ_T defined on each element T of a given triangulation of the domain. The estimator $\boldsymbol{\theta}$ is said to be reliable (resp. efficient) if there exists $C_{\mathtt{rel}} > 0$ (resp. $C_{\mathtt{eff}} > 0$), independent of the meshsizes, such that

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} + \text{h.o.t.} \;\leq\; \|error\| \;\leq\; C_{\mathtt{rel}}\,\boldsymbol{\theta} + \text{h.o.t.}\,,$$

where h.o.t. is a generic expression denoting one or several terms of higher order. Concerning the Helmholtz and elasticity equations, several approaches have already been developed independently in the literature. In particular, a posteriori error analyses for interior Helmholtz problems, which are based on local computations or explicit residuals, can be found in [15] and [56], respectively. In addition, a reliable residual-based a posteriori error estimator, which follows the nowadays standard approach from [72], is proposed in [57]. In turn, a posteriori error estimators for the mixed finite element formulation of the linear elasticity problem, which are based on residuals and on the solution of local problems, are provided in [2]. The main novelty of the approach there has to do with the utilization of a Helmholtz decomposition of the stress-type unknown to derive the corresponding reliability and efficiency estimates. For related approaches employing the Helmholtz decomposition technique as well we refer to [22] and [61].

Furtermore, to the best of our knowledge, [35] is the only work available in the literature dealing with the a posteriori error analysis of fluid-solid interaction problems involving the acoustic and elastodynamic equations in time-harmonic regime. In fact, a reliable and efficient residual-based a posteriori error estimator for the dual-mixed/primal formulation of the model problem analyzed in [37] was derived in [35]. More precisely, suitable auxiliary problems, the continuous inf-sup conditions satisfied by the bilinear forms involved, a discrete Helmholtz decomposition, and the local approximation properties of the Clément interpolant and Raviart-Thomas operator are the main tools for proving the reliability of the estimator in [35]. Then, Helmholtz decomposition, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions are employed to show the efficiency. According to the preceding remarks, and in order to additionaly contribute in this direction, the main purpose of the present paper is to derive a reliable and efficient residual-based a posteriori error estimator for the fully-mixed formulation introduced and analyzed in [28]. The rest of this work is organized as follows. In Section 4.2 we recall from [28] the fluid-solid interaction problem and its continuous and discrete fully-mixed variational formulations. The kernel of the present work is given by Section 4.3, where we develop the a posteriori error analysis. Our tools for showing reliability and efficiency are basically the same ones utilized in [35]. More precisely, in Section 4.3.1 we employ the global inf-sup condition for the continuous variational formulation, discrete Helmholtz decompositions in both domains, and the above mentioned properties of the Clément interpolant and Raviart-Thomas operator, to derive a reliable residual-based a posteriori error estimator. Even, at some point of this analysis we are able to identify independent terms related to the fluid and solid, respectively, which allows us to apply, separately, some of the arguments employed for the a posteriori error analyses of each equation. Next, in Section 4.3.2 we apply discrete trace and inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions to show the efficiency of the estimator. In this part we take advantage of the fact that either the efficiency estimates for some terms or the way to derive them, are already available in the literature (see, e.g. [22], [35], and [72]). However, and for sake of completeness, we sketch at least most of the corresponding proofs. For the remaining terms defining the a posteriori error estimator we certainly provide full proofs. Finally, some numerical examples confirming the reliability and efficiency of the a posteriori error estimator, and showing the good performance of the associated adaptive algorithm are provided in Section 4.4.

We end this section with further notations to be used below. Since in the sequel we deal with complex valued functions, we let $\mathbb{C}$ be the set of complex numbers, use the symbol $\imath$ for $\sqrt{-1}$, denote by $\overline{z}$ and $|z|$ the conjugate and modulus, respectively, of each $z \in \mathbb{C}$, and let $\mathbf{I}$ be the identity matrix of $\mathbb{C}^{2\times 2}$. On the other hand, in what follows tr denotes the matrix trace and $^{\mathtt{t}}$ stands for the transpose of a matrix. Also, given $\boldsymbol{\tau}_s := (\tau_{ij})$, $\boldsymbol{\zeta}_s := (\zeta_{ij}) \in \mathbb{C}^{2\times 2}$, we define the deviator tensor $\boldsymbol{\tau}_s^{\mathtt{d}} := \boldsymbol{\tau}_s - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau}_s)\,\mathbf{I}$, the tensor product $\boldsymbol{\tau}_s : \boldsymbol{\zeta}_s := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$, and the conjugate tensor $\overline{\boldsymbol{\tau}_s} := (\overline{\tau}_{ij})$. In turn, in what follows we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, $\mathcal{S}$ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}, \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^2 .$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote H^2 and $H^{2\times 2}$, respectively. In addition, we use $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ to denote the usual duality pairings between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, and between $\mathbf{H}^{-1/2}(\mathcal{S})$ and $\mathbf{H}^{1/2}(\mathcal{S})$. Furthermore, the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \left\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\,\mathbf{w} \in L^2(\mathcal{O})\right\},$$

is standard in the realm of mixed problems (see [19], [47]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O})$, where $\mathbf{div}$ stands for the usual divergence operator div acting on each row of the tensor, The Hilbert norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div};\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. Finally, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

4.2 The fluid-solid interaction problem

For the benefit of reader, we repeat here some of the main features in Chapter 2.

4.2.1 The model problem

We consider the two-dimensional fluid-solid interaction problem whose a priori error analysis was provided recently in [28] (see also [37] for a previous analysis of this problem). In other words, given an incident acoustic wave upon a bounded elastic body (obstacle) fully surrounded by a fluid, we are interested in determining both the response of the body and the scattered wave. The obstacle is supposed to be a long cylinder parallel to the x_3-axis whose cross-section is Ω_s. The boundary of Ω_s is denoted by Σ. The incident wave and the volume force acting on the body are assumed to exhibit a time-harmonic behaviour with $e^{-\imath\,\omega\,t}$ ansatz and phasors p_i and $\mathbf{f}$, respectively, so that p_i satisfies the Helmholtz equation in $\mathbb{R}^2\backslash\Omega_s$. Hence, since the phenomenon is supposed to be invariant under a translation in the x_3-direction, we may consider a bidimensional interaction problem posed in the frequency domain. In this way, and since we employ mixed formulations in both domains (solid and fluid), the main unknowns of our interaction problem are given by $\boldsymbol{\sigma}_s : \Omega_s \to \mathbb{C}^{2\times 2}$, $\mathbf{u} : \Omega_s \to \mathbb{C}^2$, $p : \mathbb{R}^2\backslash\Omega_s \to \mathbb{C}$, and $\boldsymbol{\sigma}_f : \mathbb{R}^2\backslash\Omega_s \to \mathbb{C}^2$, corresponding to the amplitudes of the Cauchy stress tensor, the displacement field, the total (incident + scattered) pressure, and the gradient of p, respectively.

The fluid is assumed to be perfect, compressible, and homogeneous, with density ρ_f and wave number $\kappa_f := \dfrac{\omega}{v_0}$, where v_0 is the speed of sound in the linearized fluid, whereas the solid is supposed to be isotropic and linearly elastic with density ρ_s and Lamé constants μ and λ. The latter means, in particular, that the corresponding constitutive equation is given by Hooke's law, that is

$$\boldsymbol{\sigma}_s \,=\, \lambda \operatorname{tr} \boldsymbol{\varepsilon}(\mathbf{u})\, \mathbf{I} \,+\, 2\,\mu\, \boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in} \quad \Omega_s\,,$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathtt{t}}\right)$ is the strain tensor of small deformations and ∇ is the gradient tensor. Consequently, under the hypotheses of small oscillations, both in the solid and the fluid, the unknowns $\boldsymbol{\sigma}_s$, $\mathbf{u}$, $\boldsymbol{\sigma}_f$, and p satisfy the elastodynamic and acoustic equations in time-harmonic regime, that is:

$$\begin{aligned} \mathbf{div}\, \boldsymbol{\sigma}_s \,+\, \kappa_s^2\, \mathbf{u} \;&=\; -\,\mathbf{f} \qquad \text{in} \quad \Omega_s\,, \\ \operatorname{div} \boldsymbol{\sigma}_f \,+\, \kappa_f^2\, p \;&=\; 0 \qquad \text{in} \quad \mathbb{R}^2 \backslash \Omega_s\,, \end{aligned}$$

where κ_s is defined by $\sqrt{\rho_s}\,\omega$, together with the transmission conditions:

$$\begin{aligned} \boldsymbol{\sigma}_s\, \boldsymbol{\nu} \;&=\; -\,p\, \boldsymbol{\nu} \qquad \text{on} \quad \Sigma\,, \\ \boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} \;&=\; \rho_f\, \omega^2\, \mathbf{u} \cdot \boldsymbol{\nu} \qquad \text{on} \quad \Sigma\,. \end{aligned} \tag{4.1}$$

and the Sommerfeld radiation condition

$$\frac{\partial (p - p_i)}{\partial \mathtt{r}} \,-\, \imath\, \kappa_f\, (p - p_i) \,=\, o(\mathtt{r}^{-1})\,, \tag{4.2}$$

as $\mathtt{r} := \|\mathbf{x}\| \to +\infty$, uniformly for all directions $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$. Hereafter, $\|\mathbf{x}\|$ is the euclidean norm of a vector $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \mathbb{R}^2$, and $\boldsymbol{\nu}$ denotes the unit outward normal on Σ, that is pointing toward $\mathbb{R}^2 \backslash \Omega_s$.

Next, according to the condition at infinity given by (4.2), which basically says that the outgoing waves are absorbed by the far field, and in order to obtain a convenient simplification of our model, we now proceed as in [28] and [37] and introduce a sufficiently large polyhedral surface Γ approximating a sphere centered at the origin, whose interior contains Ω_s. Then, we define Ω_f as the annular region bounded by Σ and Γ, and consider the Robin boundary condition:

$$\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} \,-\, \imath\, \kappa_f\, p \,=\, g \,:=\, \nabla p_i \cdot \boldsymbol{\nu} \,-\, \imath\, \kappa_f\, p_i \quad \text{on} \quad \Gamma\,, \tag{4.3}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal on Γ as well. Therefore, given $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ and $g \in H^{-1/2}(\Gamma)$, we are now interested in the following fluid-solid interaction problem: Find $\boldsymbol{\sigma}_s \in \mathbb{H}(\mathbf{div}; \Omega_s)$, $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$, $\boldsymbol{\sigma}_f \in \mathbf{H}(\operatorname{div}; \Omega_f)$, and $p \in H^1(\Omega_f)$, such that there hold in the distributional sense:

$$
\begin{aligned}
\boldsymbol{\sigma}_s &= \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in} \quad \Omega_s\,, \\
\mathbf{div}\,\boldsymbol{\sigma}_s + \kappa_s^2\,\mathbf{u} &= -\,\mathbf{f} && \text{in} \quad \Omega_s\,, \\
\boldsymbol{\sigma}_f &= \nabla p && \text{in} \quad \Omega_f\,, \\
\operatorname{div}\boldsymbol{\sigma}_f + \kappa_f^2\,p &= 0 && \text{in} \quad \Omega_f\,, \\
\boldsymbol{\sigma}_s\,\boldsymbol{\nu} &= -\,p\,\boldsymbol{\nu} && \text{on} \quad \Sigma\,, \\
\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu} &= \rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu} && \text{on} \quad \Sigma\,, \\
\boldsymbol{\sigma}_f\cdot\boldsymbol{\nu} - \imath\,\kappa_f\,p &= g && \text{on} \quad \Gamma\,.
\end{aligned}
\tag{4.4}
$$

4.2.2 The fully-mixed variational formulation

In order to recall from [28] the fully-mixed variational formulation of (4.4), we need to introduce the auxiliary unknowns given by the trace of the displacement

$$\boldsymbol{\varphi}_s \;:=\; \mathbf{u}|_\Sigma \,\in\, \mathbf{H}^{1/2}(\Sigma)\,,$$

the traces of the pressure

$$\boldsymbol{\varphi}_f \;=\; (\varphi_\Sigma, \varphi_\Gamma) \;:=\; (p|_\Sigma, p|_\Gamma) \,\in\, H^{1/2}(\Sigma)\times H^{1/2}(\Gamma)\,,$$

and the rotation

$$\boldsymbol{\gamma} \;:=\; \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}) \,\in\, \mathbb{L}^2_{\mathtt{skew}}(\Omega_s)$$

where $\mathbb{L}^2_{\mathtt{skew}}(\Omega_s)$ denotes the space of skew-symmetric tensors with entries in $L^2(\Omega_s)$. In addition, we let

$$\mathbf{H} \;:=\; \mathbb{H}(\mathbf{div};\Omega_s)\times\mathbf{H}(\operatorname{div};\Omega_f) \quad \text{and} \quad \mathbf{Q} \;:=\; \mathbb{L}^2_{\mathtt{asym}}(\Omega_s)\times\mathbf{H}^{1/2}(\Sigma)\times H^{1/2}(\partial\Omega_f)$$

endowed with the usual product norms. Hereafter, given $t\in\mathbb{R}$, we make the identification $H^t(\partial\Omega_f)\equiv H^t(\Sigma)\times H^t(\Gamma)$ with the norm $\|\boldsymbol{\psi}_f\|_{t,\partial\Omega_f} := \|\psi_\Sigma\|_{t,\Sigma} + \|\psi_\Gamma\|_{t,\Gamma}$ for each $\boldsymbol{\psi}_f := (\psi_\Sigma,\psi_\Gamma)\in H^t(\partial\Omega_f)$.

Next, as explained in [28], we employ a dual-mixed approach in the solid Ω_s as well as in the fluid Ω_f, and observe that both transmission conditions (cf. (4.1)) and the Robin boundary condition (4.3) become now essential. In addition, we use the elastodynamic and the Helmholtz equations (cf. second and fourth equation of (4.4)), respectively, to eliminate $\mathbf{u}$ and p according to the formulae

$$\mathbf{u} = -\frac{1}{\kappa_s^2}(\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_s)\,, \tag{4.5}$$

and

$$p \,=\, -\,\frac{1}{\kappa_f^2}\,\mathrm{div}\,\boldsymbol{\sigma}_f \quad \text{in} \quad \Omega_f\,. \tag{4.6}$$

In this way, we arrive at the following fully-mixed variational formulation of (4.4): Find $\widehat{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_f) \in \mathbf{H}$ and $\widehat{\boldsymbol{\gamma}} := (\boldsymbol{\gamma}, \boldsymbol{\varphi}_s, \boldsymbol{\varphi}_f) \in \mathbf{Q}$ such that

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) \,+\, B(\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\gamma}}) &\;=\; F(\widehat{\boldsymbol{\tau}}) \qquad \forall\, \widehat{\boldsymbol{\tau}} \,:=\, (\boldsymbol{\tau}_s, \boldsymbol{\tau}_f) \,\in\, \mathbf{H}\,, \\ B(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\eta}}) \,+\, K(\widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\eta}}) &\;=\; G(\widehat{\boldsymbol{\eta}}) \qquad \forall\, \widehat{\boldsymbol{\eta}} \,:=\, (\boldsymbol{\eta}, \boldsymbol{\psi}_s, \boldsymbol{\psi}_f) \,\in\, \mathbf{Q}\,, \end{aligned} \tag{4.7}$$

where $F : \mathbf{H} \to \mathbb{C}$ and $G : \mathbf{Q} \to \mathbb{C}$ are the linear functionals

$$\begin{gathered} F(\widehat{\boldsymbol{\tau}}) \,:=\, \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{f}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \quad \forall\, \widehat{\boldsymbol{\tau}} \,:=\, (\boldsymbol{\tau}_s, \boldsymbol{\tau}_f) \,\in\, \mathbf{H}\,, \\ G(\widehat{\boldsymbol{\eta}}) \,:=\, -\,\langle\, g\,,\, \psi_\Gamma \,\rangle_\Gamma \quad \forall\, \widehat{\boldsymbol{\eta}} \,:=\, (\boldsymbol{\eta}, \boldsymbol{\psi}_s, \boldsymbol{\psi}_f) \,:=\, (\boldsymbol{\eta}, \boldsymbol{\psi}_s, (\psi_\Sigma, \psi_\Gamma)) \,\in\, \mathbf{Q}\,, \end{gathered} \tag{4.8}$$

and $A : \mathbf{H}\times\mathbf{H} \to \mathbb{C}$, $\;B : \mathbf{H}\times\mathbf{Q} \to \mathbb{C}$, and $K : \mathbf{Q}\times\mathbf{Q} \to \mathbb{C}$ are the bilinear forms defined by

$$A(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) \,:=\, \int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\zeta}_s : \boldsymbol{\tau}_s \;-\; \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{div}\,\boldsymbol{\zeta}_s\cdot\mathbf{div}\,\boldsymbol{\tau}_s \;+\; \int_{\Omega_f} \boldsymbol{\zeta}_f\cdot\boldsymbol{\tau}_f \,-\, \frac{1}{\kappa_f^2}\int_{\Omega_f} \mathrm{div}\,\boldsymbol{\zeta}_f\,\mathrm{div}\,\boldsymbol{\tau}_f$$

$$\forall\, (\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) \,:=\, ((\boldsymbol{\zeta}_s, \boldsymbol{\zeta}_f), (\boldsymbol{\tau}_s, \boldsymbol{\tau}_f)) \,\in\, \mathbf{H}\times\mathbf{H}\,,$$

$$B(\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\eta}}) \,:=\, B_s(\boldsymbol{\tau}_s, (\boldsymbol{\eta}, \boldsymbol{\psi}_s)) \,+\, B_f(\boldsymbol{\tau}_f, \boldsymbol{\psi}_f) \qquad \forall\, (\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\eta}}) \,:=\, ((\boldsymbol{\tau}_s, \boldsymbol{\tau}_f), (\boldsymbol{\eta}, \boldsymbol{\psi}_s, \boldsymbol{\psi}_f)) \,\in\, \mathbf{H}\times\mathbf{Q}\,,$$

with

$$\begin{aligned} B_s(\boldsymbol{\tau}_s, (\boldsymbol{\eta}, \boldsymbol{\psi}_s)) &\,:=\, \int_{\Omega_s} \boldsymbol{\tau}_s : \boldsymbol{\eta} \,-\, \langle \boldsymbol{\tau}_s\,\boldsymbol{\nu}, \boldsymbol{\psi}_s\rangle_\Sigma\,, \\ B_f(\boldsymbol{\tau}_f, \boldsymbol{\psi}_f) &\,:=\, \langle \boldsymbol{\tau}_f\cdot\boldsymbol{\nu}, \psi_\Sigma\rangle_\Sigma \,-\, \langle \boldsymbol{\tau}_f\cdot\boldsymbol{\nu}, \psi_\Gamma\rangle_\Gamma\,, \end{aligned}$$

and

$$K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\eta}}) \,:=\, -\,\langle \xi_\Sigma\,\boldsymbol{\nu}, \boldsymbol{\psi}_s\rangle_\Sigma \,-\, \rho_f\,\omega^2\,\langle \boldsymbol{\xi}_s\cdot\boldsymbol{\nu}, \psi_\Sigma\rangle_\Sigma \,+\, \imath\,\kappa_f\,\langle \xi_\Gamma, \psi_\Gamma\rangle_\Gamma$$

$$\forall\, \widehat{\boldsymbol{\chi}} \,:=\, (\boldsymbol{\chi}, \boldsymbol{\xi}_s, \boldsymbol{\xi}_f) \,:=\, (\boldsymbol{\chi}, \boldsymbol{\xi}_s, (\xi_\Sigma, \xi_\Gamma)) \,\in\, \mathbf{Q}\,,$$

$$\forall\, \widehat{\boldsymbol{\eta}} \,:=\, (\boldsymbol{\eta}, \boldsymbol{\psi}_s, \boldsymbol{\psi}_f) \,:=\, (\boldsymbol{\eta}, \boldsymbol{\psi}_s, (\psi_\Sigma, \psi_\Gamma)) \,\in\, \mathbf{Q}\,.$$

The main result concerning the solvability analysis of (4.7) is stated as follows. To this respect, notice that irrespective of the particular functionals defined in (4.8), the following result is actually valid for any pair $(F, G) \in \mathbf{H}'\times\mathbf{Q}'$.

Theorem 4.1 *Assume that the homogeneous problem associated to* (4.7) *has only the trivial solution. Then, given* $F \in \mathbf{H}'$ *and* $G \in \mathbf{Q}'$*, there exists a unique* $(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\gamma}}) \in \mathbf{H}\times\mathbf{Q}$ *solution to* (4.7)*. In addition, there exists* $C_{\mathtt{cd}} > 0$ *such that*

$$\|(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\gamma}})\|_{\mathbf{H}\times\mathbf{Q}} \;\leq\; C_{\mathtt{cd}}\left\{\, \|F\|_{\mathbf{H}'} \,+\, \|G\|_{\mathbf{Q}'} \,\right\}. \tag{4.9}$$

Proof. The proof basically consists of showing that the left hand side of (4.7) constitutes a Fredholm operator of index zero. We omit further details and refer to the whole analysis developed in [28, Section 3]. □

We end this section with the converse of the derivation of (4.7). Indeed, the following theorem establishes that the unique solution of (4.7) together with $\mathbf{u}$ and p given by (4.5) and (4.6), respectively, solves the original fluid-solid interaction problem (4.4). This result will be used later on in Section 4.3.2 to prove the efficiency of the a posteriori error estimator. Note that no extra regularity assumptions on the data, but only $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ and $g \in H^{-1/2}(\Gamma)$, are needed here.

Theorem 4.2 *Let* $((\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_f), (\gamma, \boldsymbol{\varphi}_s, \boldsymbol{\varphi}_f)) \in \mathbf{H} \times \mathbf{Q}$ *be the unique solution of* (4.7)*, where* $\boldsymbol{\varphi}_f := (\varphi_\Sigma, \varphi_\Gamma) \in H^{1/2}(\Sigma) \times H^{1/2}(\Gamma)$*, and let* $\mathbf{u} \in \mathbf{L}^2(\Omega_s)$ *and* $p \in L^2(\Omega_f)$ *be defined according to* (4.5) *and* (4.6)*. Then* $\nabla \mathbf{u} = \mathcal{C}^{-1}\boldsymbol{\sigma}_s + \boldsymbol{\gamma}$ *in* Ω_s *(which yields* $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$*),* $\mathbf{u} = \boldsymbol{\varphi}_s$ *on the interface* Σ*,* $\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s^{\mathtt{t}}$ *in* Ω_s*, and* $\boldsymbol{\gamma} = \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}\right)$ *in* Ω_s *(which yields* $\boldsymbol{\sigma}_s = \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u})$*). In addition, there hold* $\boldsymbol{\sigma}_f = \nabla p$ *in* Ω_f *(which yields* $p \in H^1(\Omega_f)$*),* $\operatorname{div} \boldsymbol{\sigma}_f + \kappa_f^2\, p = 0$ *in* Ω_f*,* $\varphi_\Sigma = p|_\Sigma$ *on* Σ*,* $\varphi_\Gamma = p|_\Gamma$ *on* Γ*, and hence* $\boldsymbol{\sigma}_s \boldsymbol{\nu} = -\varphi_\Sigma\, \boldsymbol{\nu} = -p\, \boldsymbol{\nu}$ *on* Σ*,* $\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} = \rho_f\, \omega^2\, \boldsymbol{\varphi}_s \cdot \boldsymbol{\nu} = \rho_f\, \omega^2\, \mathbf{u} \cdot \boldsymbol{\nu}$ *on* Σ*, and* $\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} - \imath\, \kappa_f\, \varphi_\Gamma = \boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} - \imath\, \kappa_f\, p = g$ *on* Γ*.*

Proof. It basically follows by applying integration by parts backwardly in (4.7) and using suitable test functions. We omit further details. □

4.2.3 The Galerkin scheme

In this section we recall from [28] the definition of the Galerkin approximation of (4.7). To this end, we first let $\left\{\mathcal{T}_h^s\right\}_{h>0}$ and $\left\{\mathcal{T}_h^f\right\}_{h>0}$ be regular families of triangulations of the polygonal regions $\bar{\Omega}_s$ and $\bar{\Omega}_f$, respectively, by triangles T of diameter h_T, with global mesh sizes

$$h_s := \max\left\{\, h_T : T \in \mathcal{T}_h^s \,\right\}, \quad h_f := \max\left\{\, h_T : T \in \mathcal{T}_h^f \,\right\}, \quad \text{and} \quad h := \max\left\{ h_s,\, h_f \right\},$$

such that they are quasi-uniform around Σ and Γ, and so that their vertices coincide on Σ. In what follows, given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^2$, $P_\ell(S)$ denotes the space of polynomials defined in S of total degree $\leq \ell$. According to the notation convention given in the introduction, we denote $\mathbf{P}_\ell(S) := [P_\ell(S)]^2$. Furthermore, given $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ and $\mathbf{x} := (x_1, x_2)^{\mathtt{t}}$ a generic vector of $\mathbb{R}^2$, we let $\mathrm{RT}_0(T) := \operatorname{span}\left\{(1,0), (0,1), (x_1, x_2)\right\}$ be the local Raviart-Thomas space of order 0 (cf. [19], [69]), and let $\mathbf{curl}^{\mathtt{t}}\, b_T := \left(\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1}\right)$, where b_T is the usual cubic bubble function on T. Then we define

$$\mathbf{H}_h^s := \left\{ \mathbf{v}_{s,h} \in \mathbf{H}(\operatorname{div}; \Omega_s) : \quad \mathbf{v}_{s,h}|_T \in \mathrm{RT}_0(T) \oplus P_0(T)\, \mathbf{curl}^{\mathtt{t}}\, b_T \quad \forall\, T \in \mathcal{T}_h^s \right\},$$

$$\mathbb{H}_h^s := \left\{ \boldsymbol{\tau}_{s,h} \in \mathbb{H}(\mathbf{div};\Omega_s) : \quad \mathbf{c}^{\mathrm{t}}\, \boldsymbol{\tau}_{s,h} \in \mathbf{H}_h^s \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \right\}, \tag{4.10}$$

$$\mathbf{H}_h^f := \left\{ \boldsymbol{\tau}_{f,h} \in \mathbf{H}(\mathrm{div};\Omega_f) : \quad \boldsymbol{\tau}_{f,h}|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^f \right\}, \tag{4.11}$$

$$\mathbb{Q}_h^s := \left\{ \boldsymbol{\eta}_h := \begin{pmatrix} 0 & \eta_h \\ -\eta_h & 0 \end{pmatrix} : \quad \eta_h \in C(\bar{\Omega}_s), \quad \eta_h|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h^s \right\}. \tag{4.12}$$

Next, in order to set the finite dimensional subspaces on the boundaries of the domains, we let Σ_h and Γ_h be the partitions of Σ and Γ, respectively, inherited from the triangulations, and suppose, without loss generality, that the numbers of edges of Σ_h and Γ_h are both even. The case of an odd number of edges is easily reduced to the even case (see [45]). Then, we let Σ_{2h} (resp. Γ_{2h}) be the partition of Σ (resp. Γ) arising by joining pairs of adjacent edges of Σ_h (resp. Γ_h). Because of the assumptions on the triangulations, Σ_h and Γ_h are automatically of bounded variation, and, therefore, so are Σ_{2h} and Γ_{2h}. Hence, we now define

$$\Lambda_h(\Sigma) := \left\{ \psi_h \in C(\Sigma) : \quad \psi_h|_e \in P_1(e) \quad \forall e \text{ edge of } \Sigma_{2h} \right\}, \tag{4.13}$$

$$\Lambda_h(\Gamma) := \left\{ \psi_h \in C(\Gamma) : \quad \psi_h|_e \in P_1(e) \quad \forall e \text{ edge of } \Gamma_{2h} \right\}, \tag{4.14}$$

$$\mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma), \tag{4.15}$$

$$\mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma), \tag{4.16}$$

and introduce the global finite element spaces

$$\mathbf{H}_h := \mathbb{H}_h^s \times \mathbf{H}_h^f \qquad \text{and} \qquad \mathbf{Q}_h := \mathbb{Q}_h^s \times \mathbf{Q}_h^s \times \mathbf{Q}_h^f. \tag{4.17}$$

In addition, our analysis below will also require the subspaces

$$\mathbf{U}_h^s := \left\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega_s) : \quad \mathbf{v}_h|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^s \right\} \tag{4.18}$$

and

$$U_h^f := \left\{ v_h \in L^2(\Omega_f) : \quad v_h|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^f \right\}. \tag{4.19}$$

Notice here that $\mathbb{H}_h^s \times \mathbf{U}_h^s \times \mathbb{Q}_h^s$ constitutes the well known PEERS space introduced in [4] for a mixed finite element aproximation of the linear elasticity problem in the plane. In turn, $\mathbf{H}_h^f \times U_h^f$ is the lowest order Raviart-Thomas mixed finite element approximation of the Poisson problem for the Laplace equation (see [19], [69]).

According to the above, the Galerkin scheme associated with our continuous problem (4.7) reduces to: Find $\widehat{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{s,h}, \boldsymbol{\sigma}_{f,h}) \in \mathbf{H}_h$ and $\widehat{\boldsymbol{\gamma}}_h := (\boldsymbol{\gamma}_h, \boldsymbol{\varphi}_{s,h}, \boldsymbol{\varphi}_{f,h}) \in \mathbf{Q}_h$ such that

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\tau}}_h) + B(\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\gamma}}_h) &= F(\widehat{\boldsymbol{\tau}}_h) \qquad \forall\, \widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h, \\ B(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\eta}}_h) + K(\widehat{\boldsymbol{\gamma}}_h, \widehat{\boldsymbol{\eta}}_h) &= G(\widehat{\boldsymbol{\eta}}_h) \qquad \forall\, \widehat{\boldsymbol{\eta}}_h := (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}, \boldsymbol{\psi}_{f,h}) \in \mathbf{Q}_h. \end{aligned} \tag{4.20}$$

The following theorem establishes the well-posedness and convergence of the discrete scheme (4.20).

Theorem 4.3 *Assume that the homogeneous problem associated to* (4.7) *has only the trivial solution, and let $h_0 > 0$ be the constant provided by* [28, Lemma 4.10]. *Then there exists $h_1 \in (0, h_0]$ such that for each $h \in (0, h_1]$, the fully-mixed finite element scheme* (4.20) *has a unique solution $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h) := ((\boldsymbol{\sigma}_{s,h}, \boldsymbol{\sigma}_{f,h}), (\gamma_h, \boldsymbol{\varphi}_{s,h}, \boldsymbol{\varphi}_{f,h})) \in \mathbf{H}_h \times \mathbf{Q}_h$, with $\boldsymbol{\varphi}_{f,h} := (\varphi_{\Sigma,h}, \varphi_{\Gamma,h}) \in \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)$. In addition, there exist $C_1, C_2 > 0$, independent of h, such that for each $h \in (0, h_1]$ there hold*

$$\|(\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C_1 \left\{ \sup_{\widehat{\boldsymbol{\tau}}_h \in \mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{|F(\widehat{\boldsymbol{\tau}}_h)|}{\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} + \sup_{\widehat{\boldsymbol{\eta}}_h \in \mathbf{Q}_h\setminus\{\mathbf{0}\}} \frac{|G(\widehat{\boldsymbol{\eta}}_h)|}{\|\widehat{\boldsymbol{\eta}}_h\|_{\mathbf{Q}}} \right\} \le C_1 \left\{ \|\mathbf{f}\|_{0,\Omega_s} + \|g\|_{-1/2,\Gamma} \right\}$$

and

$$\|(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) - (\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C_2 \inf_{(\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\eta}}_h) \in \mathbf{H}_h\times\mathbf{Q}_h} \|(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) - (\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\eta}}_h)\|_{\mathbf{H}\times\mathbf{Q}},$$

where $(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) := ((\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_f), (\boldsymbol{\gamma}, \boldsymbol{\varphi}_s, \boldsymbol{\varphi}_f)) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution of (4.7). *Furthermore, if there exists $\delta \in (0, 1]$ such that $\boldsymbol{\sigma}_s \in \mathbb{H}^\delta(\Omega_s)$, $\mathbf{div}\, \boldsymbol{\sigma}_s \in \mathbf{H}^\delta(\Omega_s)$, $\boldsymbol{\sigma}_f \in \mathbf{H}^\delta(\Omega_f)$, $\mathrm{div}\, \boldsymbol{\sigma}_f \in H^\delta(\Omega_f)$, $\boldsymbol{\gamma} \in \mathbb{H}^\delta(\Omega_s)$, $\boldsymbol{\varphi}_s \in \mathbf{H}^{1/2+\delta}(\Sigma)$, and $\boldsymbol{\varphi}_f \in H^{1/2+\delta}(\partial\Omega_f)$, then there exists $C_3 > 0$, independent of h, such that for each $h \in (0, h_1]$ there holds*

$$\begin{aligned} \|(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) - (\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C_3\, h^\delta \Big\{ & \|\boldsymbol{\sigma}_s\|_{\delta,\Omega_s} + \|\mathbf{div}\, \boldsymbol{\sigma}_s\|_{\delta,\Omega_s} + \|\boldsymbol{\sigma}_f\|_{\delta,\Omega_f} \\ & + \|\mathrm{div}\, \boldsymbol{\sigma}_f\|_{\delta,\Omega_f} + \|\boldsymbol{\gamma}\|_{\delta,\Omega_s} + \|\boldsymbol{\varphi}_s\|_{1/2+\delta,\Sigma} + \|\boldsymbol{\varphi}_f\|_{1/2+\delta,\partial\Omega_f} \Big\}. \end{aligned}$$

Proof. See [28, Theorem 4.1] and the whole analysis in [28, Section 4] for full details. □

4.3 A residual-based a posteriori error estimator

In this section we derive reliable and efficient residual based *a posteriori* error estimators for (4.20). We begin by introducing further notations. Given $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$, we let $\mathcal{E}(T)$ be the set of edges of T, and denote by $\mathcal{E}_h$ be the set of all edges of $\mathcal{T}_h^s \cup \mathcal{T}_h^f$. Then we can write

$$\mathcal{E}_h = \mathcal{E}_h(\Omega_s) \cup \mathcal{E}_h(\Sigma) \cup \mathcal{E}_h(\Omega_f) \cup \mathcal{E}_h(\Gamma), \tag{4.21}$$

where $\mathcal{E}_h(\Omega_s) := \{e \in \mathcal{E}_h : \ e \subseteq \Omega_s\}$, $\mathcal{E}_h(\Sigma) := \{e \in \mathcal{E}_h : \ e \subseteq \Sigma\}$, and similarly for $\mathcal{E}_h(\Omega_f)$ and $\mathcal{E}_h(\Gamma)$. In what follows, h_e stands for the length of the edge $e \in \mathcal{E}_h$. Also, for each edge $e \in \mathcal{E}_h$ we fix a unit normal vector $\boldsymbol{\nu} := (\nu_1, \nu_2)^{\mathtt{t}}$, and let $\mathbf{s} := (-\nu_2, \nu_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Now, let $\mathbf{w}_s \in \mathbf{L}^2(\Omega_s)$ such that $\mathbf{w}_s|_T \in \mathbf{C}(T)$ for each $T \in \mathcal{T}_h^s$. Then, given $T \in \mathcal{T}_h^s$ and $e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_s)$, we denote by $[\mathbf{w}_s]$ the jump of $\mathbf{w}_s$ across e, that is $[\mathbf{w}_s] := (\mathbf{w}_s|_T)|_e - (\mathbf{w}_s|_{T'})|_e$, where T and T' are the triangles of $\mathcal{T}_h^s$ having e as a common edge. Also, given $e \in \mathcal{E}_h(\Omega_s)$ and $\boldsymbol{\tau}_s \in \mathbb{L}(\Omega_s)$ such that $\boldsymbol{\tau}_s|_T \in \mathbb{C}(T)$ on each $T \in \mathcal{T}_h^s$, we let $[\boldsymbol{\tau}_s\,\mathbf{s}] := (\boldsymbol{\tau}_s|_T - \boldsymbol{\tau}_s|_{T'})|_e\,\mathbf{s}$. Similar definitions hold for $\mathbf{v}_f \in \mathbf{L}^2(\Omega_f)$ such that $\mathbf{v}_f|_T \in \mathbf{C}(T)$ for each $T \in \mathcal{T}_h^f$. In fact, given $e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_f)$, we define $[\mathbf{v}_f \cdot \boldsymbol{\nu}] := \big((\mathbf{v}_f|_T)|_e - (\mathbf{v}_f|_{T'})|_e\big)|_e \cdot \boldsymbol{\nu}$. Finally, given a scalar function q, a vector $\boldsymbol{\chi} := (\boldsymbol{\chi}_1, \boldsymbol{\chi}_2)$ and a tensor $\boldsymbol{\tau} := (\tau_{ij})$, we let

$$\mathbf{curl}(q) := \begin{pmatrix} \frac{\partial q}{\partial x_2} \\ \\ -\frac{\partial q}{\partial x_1} \end{pmatrix}, \quad \underline{\mathbf{curl}}(\boldsymbol{\chi}) := \begin{pmatrix} \frac{\partial \boldsymbol{\chi}_1}{\partial x_2} & -\frac{\partial \boldsymbol{\chi}_1}{\partial x_1} \\ \\ \frac{\partial \boldsymbol{\chi}_2}{\partial x_2} & -\frac{\partial \boldsymbol{\chi}_2}{\partial x_1} \end{pmatrix},$$

$$\operatorname{rot} \boldsymbol{\chi} := \frac{\partial \boldsymbol{\chi}_2}{\partial x_1} - \frac{\partial \boldsymbol{\chi}_1}{\partial x_2} \quad \text{and} \quad \operatorname{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \\ \\ \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

Next, letting $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h) := ((\boldsymbol{\sigma}_{s,h}, \boldsymbol{\sigma}_{f,h}), (\boldsymbol{\gamma}_h, \boldsymbol{\varphi}_{s,h}, \boldsymbol{\varphi}_{f,h})) \in \mathbf{H}_h \times \mathbf{Q}_h$ be the unique solution of (4.20), with $\boldsymbol{\varphi}_{f,h} := (\varphi_{\Sigma,h}, \varphi_{\Gamma,h}) \in \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)$, and denoting by $\mathcal{P}_h^s$ the $\mathbf{L}^2(\Omega_s)$-orthogonal projector onto $\mathbf{U}_h^s$ (cf. (4.18)), we define for each $T \in \mathcal{T}_h^s$, and for each $T \in \mathcal{T}_h^f$, respectively, the *a posteriori* error indicators:

$$\begin{aligned} \theta_{T,s}^2 := &\|\boldsymbol{\sigma}_{s,h} - \boldsymbol{\sigma}_{s,h}^{\mathtt{t}}\|_{0,T}^2 + \|(\mathbf{I} - \mathcal{P}_h^s)\,\mathbf{f}\,\|_{0,T}^2 + h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\|_{0,T}^2 \\ &+ h_T^2\,\|\operatorname{curl}(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_s)} h_e\,\|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2\,, \end{aligned} \tag{4.22}$$

$$\theta_{T,f}^2 := h_T^2\,\|\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 + h_T^2\,\|\operatorname{rot}(\boldsymbol{\sigma}_{f,h})\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_f)} h_e\,\|[\boldsymbol{\sigma}_{f,h} \cdot \mathbf{s}]\|_{0,e}^2\,. \tag{4.23}$$

Similarly, for each $e \in \mathcal{E}_h(\Sigma)$ we define

$$\begin{aligned} \theta_{e,\Sigma}^2 := &h_e\,\|\boldsymbol{\varphi}_{s,h} - \mathbf{u}_h\|_{0,e}^2 + h_e\,\|\boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f \omega^2 \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}\|_{0,e}^2 + h_e\,\|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}\|_{0,e}^2 \\ &+ h_e \left\|(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\,\mathbf{s} - \frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}}\right\|_{0,e}^2 + h_e \left\|\boldsymbol{\sigma}_{f,h} \cdot \mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}}\right\|_{0,e}^2 + h_e\,\|\varphi_{\Sigma,h} - p_h\|_{0,e}^2\,, \end{aligned} \tag{4.24}$$

where, resembling (4.5) and (4.6) (see also [28]), we set

$$\mathbf{u}_h := -\frac{1}{\kappa_s^2}\Big(\mathcal{P}_h^s(\mathbf{f}) + \mathbf{div}\,\boldsymbol{\sigma}_{s,h}\Big) \quad \text{in} \quad \Omega_s \tag{4.25}$$

and

$$p_h := -\frac{1}{\kappa_f^2}\operatorname{div}\boldsymbol{\sigma}_{f,h} \quad \text{in} \quad \Omega_f\,. \tag{4.26}$$

In addition, assuming that the Robin datum $g \in L^2(\Gamma)$, we set for each $e \in \mathcal{E}_h(\Gamma)$

$$\theta_{e,\Gamma}^2 := h_e \left\| \boldsymbol{\sigma}_{f,h} \cdot \mathbf{s} - \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}} \right\|_{0,e}^2 + h_e \, \|\varphi_{\Gamma,h} - p_h\|_{0,e}^2 + h_e \, \|\boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \imath\, \kappa_f \, \varphi_{\Gamma,h} - g\|_{0,e}^2 \,. \tag{4.27}$$

Therefore, we introduce the global *a posteriori* error estimator

$$\boldsymbol{\theta} := \left\{ \sum_{T \in \mathcal{T}_h^s} \theta_{T,s}^2 + \sum_{T \in \mathcal{T}_h^f} \theta_{T,f}^2 + \sum_{e \in \mathcal{E}_h(\Sigma)} \theta_{e,\Sigma}^2 + \sum_{e \in \mathcal{E}_h(\Gamma)} \theta_{e,\Gamma}^2 \right\}^{1/2}, \tag{4.28}$$

and state the main result of this section as follows.

Theorem 4.4 *Assume that the homogeneous problem associated to* (4.7) *has only the trivial solution, and let* $(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) := \big((\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_f), (\gamma, \boldsymbol{\varphi}_s, \boldsymbol{\varphi}_f)\big) \in \mathbf{H} \times \mathbf{Q}$ *and* $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h) := \big((\boldsymbol{\sigma}_{s,h}, \boldsymbol{\sigma}_{f,h}), (\gamma_h, \boldsymbol{\varphi}_{s,h}, \boldsymbol{\varphi}_{f,h})\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the unique solutions of* (4.7) *and* (4.20)*, respectively. In addition, let* $\mathbf{u} \in \mathbf{L}^2(\Omega_s)$ *and* $p \in L^2(\Omega_f)$ *be defined according to* (4.5) *and* (4.6)*, respectively, that is* $\mathbf{u} := -\frac{1}{\kappa_s^2}\big(\mathbf{f} + \mathbf{div}\, \boldsymbol{\sigma}_s\big)$ *and* $p = -\frac{1}{\kappa_f^2} \mathrm{div}\, \boldsymbol{\sigma}_f$*, and assume that the Robin datum* g *belongs to* $L^2(\Gamma)$*. Then, there exist* $C_{\mathrm{eff}}, C_{\mathrm{rel}} > 0$ *independent of* h*, such that*

$$C_{\mathrm{eff}}\ \boldsymbol{\theta} \ \leq\ \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_s} + \|p - p_h\|_{0,\Omega_f} + \|\widehat{\boldsymbol{\sigma}} - \widehat{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} + \|\widehat{\gamma} - \widehat{\gamma}_h\|_{\mathbf{Q}} \ \leq\ C_{\mathrm{rel}}\ \boldsymbol{\theta} \,. \tag{4.29}$$

The lower and upper estimates given by (4.29) constitute what we call the efficiency and reliability of $\boldsymbol{\theta}$, respectively.

4.3.1 Reliability of the a posteriori error estimator

We begin with the upper bounds for $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_s}$ and $\|p - p_h\|_{0,\Omega_f}$. In fact, according to the definitions of $\mathbf{u}$ (cf. (4.5)), p (cf. (4.6)), $\mathbf{u}_h$ (cf. (4.25)), and p_h (cf. (4.26)), we easily find that

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_s} \leq \frac{1}{\kappa_s^2} \left\{ \|(I - \mathcal{P}_h^s)\mathbf{f}\|_{0,\Omega_s} + \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{\mathbf{div};\Omega_s} \right\} \tag{4.30}$$

and

$$\|p - p_h\|_{0,\Omega_f} \leq \frac{1}{\kappa_f^2} \, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{\mathrm{div};\Omega_f} \,. \tag{4.31}$$

We continue our analysis by recalling that the continuous dependence result given by (4.9) (cf. Theorem 4.1) is equivalent to the global inf-sup condition for the continuous formulation (4.7) with the constant $\alpha = \frac{1}{C_{\mathrm{cd}}} > 0$. Then, by applying this estimate to the error $(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) - (\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h) \in \mathbf{H} \times \mathbf{Q}$, we obtain

$$\alpha \, \|(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) - (\widehat{\boldsymbol{\sigma}}_h, \widehat{\gamma}_h)\|_{\mathbf{H} \times \mathbf{Q}} \ \leq \sup_{(\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\eta}}) \in \mathbf{H} \times \mathbf{Q} \setminus \{\mathbf{0}\}} \frac{|E(\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\eta}})|}{\|(\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\eta}})\|_{\mathbf{H} \times \mathbf{Q}}} \,,$$

where

$$E(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}}) \,:=\, A(\widehat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\sigma}}_h,\widehat{\boldsymbol{\tau}}) \,+\, B(\widehat{\boldsymbol{\tau}},\widehat{\gamma}-\widehat{\gamma}_h) \,+\, B(\widehat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\sigma}}_h,\widehat{\boldsymbol{\eta}}) \,+\, K(\widehat{\gamma}-\widehat{\gamma}_h,\widehat{\boldsymbol{\eta}})\,,$$

for all $(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}}) \,:=\, ((\boldsymbol{\tau}_s,\boldsymbol{\tau}_f),(\boldsymbol{\eta},\boldsymbol{\psi}_s,\boldsymbol{\psi}_f)) \,\in\, \mathbf{H}\times\mathbf{Q}$, with $\boldsymbol{\psi}_f \,=\, (\psi_\Sigma,\psi_\Gamma)\in H^{1/2}(\Sigma)\times H^{1/2}(\Gamma)$. More precisely, thanks to the equations of the continuous variational formulation (4.7), we deduce that

$$E(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}}) \,=\, E_1(\boldsymbol{\tau}_s) \,+\, E_2(\boldsymbol{\tau}_f) \,+\, E_3(\boldsymbol{\eta}) \,+\, E_4(\boldsymbol{\psi}_s) \,+\, E_5(\psi_\Sigma) \,+\, E_6(\psi_\Gamma)\,, \tag{4.32}$$

where E_1 up to E_6 are the linear functionals defined by

$$E_1(\boldsymbol{\tau}_s) \,:=\, \frac{1}{\kappa_s^2}\int_{\Omega_s}\big\{\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_{s,h}\big\}\cdot\mathbf{div}\,\boldsymbol{\tau}_s - \int_{\Omega_s}\big\{\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\big\}:\boldsymbol{\tau}_s + \quad \langle\boldsymbol{\tau}_s\,\boldsymbol{\nu},\boldsymbol{\varphi}_{s,h}\rangle_\Sigma\,, \tag{4.33}$$

$$E_2(\boldsymbol{\tau}_f) \,:=\, \frac{1}{\kappa_f^2}\int_{\Omega_f}\operatorname{div}\boldsymbol{\sigma}_{f,h}\operatorname{div}\boldsymbol{\tau}_f \,-\, \int_{\Omega_f}\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\tau}_f \,-\, \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\varphi_{\Sigma,h}\rangle_\Sigma \,+\, \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\varphi_{\Gamma,h}\rangle_\Gamma\,, \tag{4.34}$$

$$E_3(\boldsymbol{\eta}) \,:=\, -\int_{\Omega_s}\boldsymbol{\sigma}_{s,h}:\boldsymbol{\eta}\,,$$

$$E_4(\boldsymbol{\psi}_s) \,:=\, \langle\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu}+\varphi_{\Sigma,h}\,\boldsymbol{\nu},\boldsymbol{\psi}_s\rangle_\Sigma\,,$$

$$E_5(\psi_\Sigma) \,:=\, -\langle\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, \rho_f\,\omega^2\,\boldsymbol{\varphi}_{s,h}\cdot\boldsymbol{\nu},\psi_\Sigma\rangle_\Sigma\,,$$

and

$$E_6(\psi_\Gamma) \,:=\, \langle\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, i\,\kappa_f\,\varphi_{\Gamma,h} - g,\psi_\Gamma\rangle_\Gamma\,.$$

In addition, it is not difficult to see that

$$\begin{aligned}\sup_{(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})\in\mathbf{H}\times\mathbf{Q}\backslash\{\mathbf{0}\}}\frac{|E(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})|}{\|(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})\|_{\mathbf{H}\times\mathbf{Q}}} \,\le\, & \sup_{\boldsymbol{\tau}_s\in\mathbb{H}(\mathbf{div};\Omega_s)\backslash\{\mathbf{0}\}}\frac{|E_1(\boldsymbol{\tau}_s)|}{\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}} + \sup_{\boldsymbol{\tau}_f\in\mathbf{H}(\operatorname{div};\Omega_f)\backslash\{\mathbf{0}\}}\frac{|E_2(\boldsymbol{\tau}_f)|}{\|\boldsymbol{\tau}_f\|_{\operatorname{div};\Omega_f}}\\ & + \sup_{\boldsymbol{\eta}\in\mathbb{L}^2_{\mathtt{skew}}(\Omega_s)\backslash\{\mathbf{0}\}}\frac{|E_3(\boldsymbol{\eta})|}{\|\boldsymbol{\eta}\|_{0,\Omega_s}} + \sup_{\boldsymbol{\psi}_s\in\mathbf{H}^{1/2}(\Sigma)\backslash\{\mathbf{0}\}}\frac{|E_4(\boldsymbol{\psi}_s)|}{\|\boldsymbol{\psi}_s\|_{1/2,\Sigma}}\\ & + \sup_{\psi_\Sigma\in H^{1/2}(\Sigma)\backslash\{0\}}\frac{|E_5(\psi_\Sigma)|}{\|\psi_\Sigma\|_{1/2,\Sigma}} + \sup_{\psi_\Gamma\in H^{1/2}(\Gamma)\backslash\{0\}}\frac{|E_6(\psi_\Gamma)|}{\|\psi_\Gamma\|_{1/2,\Gamma}}\,.\end{aligned} \tag{4.35}$$

Furthermore, the "*Galerkin orthogonality condition*" arising from (4.7) and (4.20) establishes that

$$E(\widehat{\boldsymbol{\tau}}_h,\widehat{\boldsymbol{\eta}}_h) \,=\, 0 \qquad \forall\,(\widehat{\boldsymbol{\tau}}_h,\widehat{\boldsymbol{\eta}}_h) \,\in\, \mathbf{H}_h\times\mathbf{Q}_h\,,$$

and hence, in order to estimate the above norms of the six functionals defining $E(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})$, we could replace $(\boldsymbol{\tau}_s,\boldsymbol{\tau}_f,\boldsymbol{\eta},\boldsymbol{\psi}_s,\psi_\Sigma,\psi_\Gamma)$ by $(\boldsymbol{\tau}_s-\boldsymbol{\tau}_{s,h},\boldsymbol{\tau}_f-\boldsymbol{\tau}_{f,h},\boldsymbol{\eta}-\boldsymbol{\eta}_h,\boldsymbol{\psi}_s-\boldsymbol{\psi}_{s,h},\psi_\Sigma-\psi_{\Sigma,h},\psi_\Gamma-\psi_{\Gamma,h})$ with any suitable choice of $\widehat{\boldsymbol{\tau}}_h \,:=\, (\boldsymbol{\tau}_{s,h},\boldsymbol{\tau}_{f,h}) \,\in\, \mathbf{H}_h$ and $\widehat{\boldsymbol{\eta}}_h \,:=\, (\boldsymbol{\eta}_h,\boldsymbol{\psi}_{s,h},(\psi_{\Sigma,h},\psi_{\Gamma,h})) \,\in\, \mathbf{Q}_h$, whenever it is necessary. However, this procedure is applied in what follows only to estimate the first two suprema on the right hand side of (4.35).

We begin the estimates of all these suprema with the last four of them.

Lemma 4.1 *There holds*

$$\|E_3\| := \sup_{\boldsymbol{\eta}\in\mathbb{L}^2_{\mathtt{skew}}(\Omega_s)\backslash\{\mathbf{0}\}} \frac{|E_3(\boldsymbol{\eta})|}{\|\boldsymbol{\eta}\|_{0,\Omega_s}} \le \frac{1}{2}\,\|\boldsymbol{\sigma}_{s,h} - \boldsymbol{\sigma}^{\mathtt{t}}_{s,h}\|^2_{0,\Omega_s}\,. \tag{4.36}$$

Proof. It suffices to see that $\boldsymbol{\sigma}_{s,h} = \frac{1}{2}(\boldsymbol{\sigma}_{s,h} + \boldsymbol{\sigma}^{\mathtt{t}}_{s,h}) + \frac{1}{2}(\boldsymbol{\sigma}_{s,h} - \boldsymbol{\sigma}^{\mathtt{t}}_{s,h})$, which yields

$$\int_{\Omega_s} \boldsymbol{\sigma}_{s,h} : \boldsymbol{\eta} = \frac{1}{2}\int_{\Omega_s}\big(\boldsymbol{\sigma}_{s,h} - \boldsymbol{\sigma}^{\mathtt{t}}_{s,h}\big) : \boldsymbol{\eta} \qquad \forall\,\boldsymbol{\eta}\in\mathbb{L}^2_{\mathtt{skew}}(\Omega_s)\,,$$

and hence the Cauchy-Schwarz inequality completes the proof. □

The upper bounds for the norms of E_4, E_5, and E_6, being all consequence of the same arguments, are collected in the following lemma.

Lemma 4.2 *There exist C_4, C_5, $C_6 \ge 0$, independent of h, such that*

$$\|E_4\| := \sup_{\boldsymbol{\psi}_s\in\mathbf{H}^{1/2}(\Sigma)\backslash\{\mathbf{0}\}} \frac{|E_4(\boldsymbol{\psi}_s)|}{\|\boldsymbol{\psi}_s\|_{1/2,\Sigma}} \le C_4 \left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}\|^2_{0,e}\right\}^{1/2}, \tag{4.37}$$

$$\|E_5\| := \sup_{\psi_\Sigma\in H^{1/2}(\Sigma)\backslash\{0\}} \frac{|E_5(\psi_\Sigma)|}{\|\psi_\Sigma\|_{1/2,\Sigma}} \le C_5 \left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\boldsymbol{\varphi}_{s,h}\cdot\boldsymbol{\nu}\|^2_{0,e}\right\}^{1/2}, \tag{4.38}$$

and

$$\|E_6\| := \sup_{\psi_\Gamma\in H^{1/2}(\Gamma)\backslash\{0\}} \frac{|E_6(\psi_\Gamma)|}{\|\psi_\Gamma\|_{1/2,\Gamma}} \le C_6 \left\{\sum_{e\in\mathcal{E}_h(\Gamma)} h_e\,\|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} - i\,\kappa_f\,\varphi_{\Gamma,h} - g\|^2_{0,e}\right\}^{1/2}. \tag{4.39}$$

Proof. It follows easily from the definitions of the functionals involved that

$$\|E_4\| = \|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}\|_{-1/2,\Sigma}\,,$$

$$\|E_5\| = \|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} - \rho_f\,\omega^2\,\boldsymbol{\varphi}_{s,h}\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma}\,,$$

and

$$\|E_6\| = \|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} - i\,\kappa_f\,\varphi_{\Gamma,h} - g\|_{-1/2,\Gamma}\,.$$

Next, we observe from the equations forming the Galerkin scheme (4.20), that the discrete versions of the transmission and Robin boundary conditions become, respectively,

$$\langle\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}, \boldsymbol{\psi}_{s,h}\rangle_\Sigma = 0 \qquad \forall\,\boldsymbol{\psi}_{s,h}\in\Lambda_h(\Sigma)\times\Lambda_h(\Sigma)\,,$$

$$\langle \boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, \rho_f\,\omega^2\,\boldsymbol{\varphi}_{s,h}\cdot\boldsymbol{\nu}, \psi_{\Sigma,h}\rangle_\Sigma \qquad \forall\,\psi_{\Sigma,h}\in\Lambda_h(\Sigma)\,,$$

and

$$\langle \boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, i\,\kappa_f\,\varphi_{\Gamma,h} - g, \psi_{\Gamma,h}\rangle_\Gamma \qquad \forall\,\psi_{\Gamma,h}\in\Lambda_h(\Gamma)\,,$$

which say, equivalently, that each expression on the left hand side of the above dualities is orthogonal to the corresponding finite element subspace indicated at the end of each equation. In particular, $\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu}+\varphi_{\Sigma,h}\,\boldsymbol{\nu}$ is $\mathbf{L}^2(\Sigma)$-orthogonal to $\Lambda_h(\Sigma)\times\Lambda_h(\Sigma)$, and therefore, a straightforward application of [20, Theorem 2] and the fact that Σ_h and Σ_{2h} are of bounded variation, yield the existence of a constant $C_4>0$, independent of h, such that, denoting by $\mathcal{E}_{2h}(\Sigma)$ the set of edges of Σ_{2h}, there holds

$$\|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu}+\varphi_{\Sigma,h}\,\boldsymbol{\nu}\|_{-1/2,\Sigma} \,\le\, C \sum_{e\in\mathcal{E}_{2h}(\Sigma)} h_e\|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu}+\varphi_{\Sigma,h}\,\boldsymbol{\nu}\|_{0,e}^2 \,\le\, C_4 \sum_{e\in\mathcal{E}_{h}(\Sigma)} h_e\|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu}+\varphi_{\Sigma,h}\,\boldsymbol{\nu}\|_{0,e}^2\,,$$

which shows (4.37). The proofs of (4.38) and (4.39), being also based on [20, Theorem 2] and the above mentioned properties of Σ_h and Σ_{2h}, are derived similarly. We omit further details. $\square$

We now aim to establish the upper bounds of $\|E_1\|$ and $\|E_2\|$, for which, as announced before, we plan to use that

$$E_1(\boldsymbol{\tau}_s) \,=\, E_1(\boldsymbol{\tau}_s-\boldsymbol{\tau}_{s,h}) \quad\text{and}\quad E_2(\boldsymbol{\tau}_f) \,=\, E_2(\boldsymbol{\tau}_f-\boldsymbol{\tau}_{f,h}) \qquad \forall\,\widehat{\boldsymbol{\tau}}_h \,:=\, (\boldsymbol{\tau}_{s,h},\boldsymbol{\tau}_{f,h}) \,\in\, \mathbf{H}_h\,. \tag{4.40}$$

To this end, we also need to consider the space of pure Raviart-Thomas tensors of order 0, that is

$$\mathbb{RT}_h^s \,:=\, \Big\{\boldsymbol{\tau}_{s,h}\in\mathbb{H}(\mathbf{div};\Omega_s): \quad \mathbf{c}^{\mathrm{t}}\,\boldsymbol{\tau}_{s,h}|_T \,\in\, \mathrm{RT}_0(T) \quad \forall\,T\in\mathcal{T}_h^s\,, \quad \forall\,\mathbf{c}\in\mathbb{R}^2\Big\}\,,$$

which is clearly contained in $\mathbb{H}_h^s$ (cf. (4.10)). Then, we let $\Pi_h^s:\mathbb{H}^1(\Omega_s)\to\mathbb{RT}_h^s$ and $\Pi_h^f:\mathbf{H}^1(\Omega_f)\to\mathbf{H}_h^f$ be the usual Raviart–Thomas interpolation operators, which are characterized by the identities

$$\int_e \Pi_h^s(\boldsymbol{\zeta}_s)\,\boldsymbol{\nu} = \int_e \boldsymbol{\zeta}_s\,\boldsymbol{\nu} \qquad \forall\,e\in\mathcal{T}_h^s\,, \quad \forall\,\boldsymbol{\zeta}_s\in\mathbb{H}^1(\Omega_s)\,, \tag{4.41}$$

and

$$\int_e \Pi_h^f(\boldsymbol{\zeta}_f)\cdot\boldsymbol{\nu} = \int_e \boldsymbol{\zeta}_f\cdot\boldsymbol{\nu} \qquad \forall\,e\in\mathcal{T}_h^f\,, \quad \forall\,\boldsymbol{\zeta}_f\in\mathbf{H}^1(\Omega_f)\,. \tag{4.42}$$

It is easy to show, using (4.41) and (4.42), that

$$\mathbf{div}(\Pi_h^s(\boldsymbol{\zeta}_s)) \,=\, \mathcal{P}_h^s(\mathbf{div}\,\boldsymbol{\zeta}_s) \quad\text{and}\quad \mathrm{div}(\Pi_h^f(\boldsymbol{\zeta}_f)) \,=\, \mathcal{P}_h^f(\mathrm{div}\,\boldsymbol{\zeta}_f)\,, \tag{4.43}$$

where, as said before, $\mathcal{P}_h^s$ is the $\mathbf{L}^2(\Omega_s)$-orthogonal projector onto $\mathbf{U}_h^s$ (cf. (4.18)), and $\mathcal{P}_h^f$ is the $L^2(\Omega_f)$-orthogonal projector onto U_h^f (cf. (4.19)). In addition, it is well known (see, e.g. [19], [69], and [44, Theorem 4.5]) that Π_h^s and Π_h^f satisfy the following approximation properties:

$$\|\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s)\|_{0,T} \leq C\, h_T \|\boldsymbol{\zeta}_s\|_{1,T} \quad \forall\, T \in \mathcal{T}_h^s, \quad \forall\, \boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega_s)\,, \tag{4.44}$$

$$\|(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}\|_{0,e} \leq C\, h_e^{1/2} \|\boldsymbol{\zeta}_s\|_{1,T_e} \quad \forall\, e \in \mathcal{T}_h^s, \quad \forall\, \boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega_s)\,, \tag{4.45}$$

$$\|\boldsymbol{\zeta}_f - \Pi_h^f(\boldsymbol{\zeta}_f)\|_{0,T} \leq C\, h_T \|\boldsymbol{\zeta}_f\|_{1,T} \quad \forall\, T \in \mathcal{T}_h^f, \quad \forall\, \boldsymbol{\zeta}_f \in \mathbf{H}^1(\Omega_f)\,, \tag{4.46}$$

$$\|(\boldsymbol{\zeta}_f - \Pi_h^f(\boldsymbol{\zeta}_f)) \cdot \boldsymbol{\nu}\|_{0,e} \leq C\, h_e^{1/2} \|\boldsymbol{\zeta}_f\|_{1,T_e} \quad \forall\, e \in \mathcal{T}_h^f, \quad \forall\, \boldsymbol{\zeta}_f \in \mathbf{H}^1(\Omega_f)\,, \tag{4.47}$$

where T_e in (4.45) (resp. in (4.47)) is a triangle of $\mathcal{T}_h^s$ (resp. $\mathcal{T}_h^f$) containing e on its boundary.

We now let $I_{s,h} : H^1(\Omega_s) \to X_{s,h}$ and $I_{f,h} : H^1(\Omega_f) \to X_{f,h}$ be the usual Clément interpolation operators (cf. [23]), where

$$X_{s,h} := \left\{ v \in C(\bar{\Omega}_s) : \quad v|_T \in P_1(T), \quad \forall\, T \in \mathcal{T}_h^s \right\},$$

$$X_{f,h} := \left\{ v \in C(\bar{\Omega}_f) : \quad v|_T \in P_1(T), \quad \forall\, T \in \mathcal{T}_h^f \right\}.$$

A vectorial version of $I_{s,h}$, say $\mathbf{I}_{s,h} : \mathbf{H}^1(\Omega_s) \to \mathbf{X}_{s,h} := X_{s,h} \times X_{s,h}$, which is defined componentwise by $I_{s,h}$, is also required. The following lemma provides the local approximation properties of $I_{s,h}$. Analogue estimates hold for the operator $I_{f,h}$.

Lemma 4.3 *There exist constants c_1, $c_2 > 0$, independent of h_s, such that for all $v \in H^1(\Omega_s)$ there holds*

$$\|v - I_{s,h}(v)\|_{0,T} \;\leq\; c_1\; h_T\, \|v\|_{1,\Delta(T)} \qquad \forall\, T \in \mathcal{T}_h^s$$

and

$$\|v - I_{s,h}(v)\|_{0,e} \;\leq\; c_2\; h_e^{1/2}\, \|v\|_{1,\Delta(e)} \qquad \forall\, e \in \mathcal{E}_h(\Omega_s) \cup \mathcal{E}_h(\Sigma)\,,$$

where $\Delta(T) := \cup\{T' \in \mathcal{T}_h^s : \quad T' \cap T \neq \emptyset\}$ and $\Delta(e) := \cup\{T' \in \mathcal{T}_h^s : \quad T' \cap e \neq \emptyset\}$.

Proof. See [23]. □

Next, in order to define a suitable $\widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h$ to be employed in (4.40), we first demonstrate the existence of continuous Helmholtz decompositions of the spaces $\mathbb{H}(\mathbf{div};\Omega_s)$ and $\mathbf{H}(\mathrm{div};\Omega_f)$. More precisely, we adapt the analysis from [35, Section 3.2.2] to establish the following result.

Lemma 4.4 *For each $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$ there exist $\boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega_s)$ and $\boldsymbol{\chi}_s := (\chi_1,\chi_2)^{\mathtt{t}} \in \mathbf{H}^1(\Omega_s)$, with $\int_{\Omega_s} \chi_1 = \int_{\Omega_s} \chi_2 = 0$, such that $\boldsymbol{\tau}_s = \boldsymbol{\zeta}_s + \underline{\mathbf{curl}}\,\boldsymbol{\chi}_s$ in Ω_s and*

$$\|\boldsymbol{\zeta}_s\|_{1,\Omega_s} + \|\boldsymbol{\chi}_s\|_{1,\Omega_s} \,\le\, C_s\,\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}\,, \tag{4.48}$$

where C_s is a positive constant independent of $\boldsymbol{\tau}_s$. In turn, for each $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$ there exist $\mathbf{w}_f \in \mathbf{H}^1(\Omega_f)$ and $\phi_f \in H^1(\Omega_f)$, such that $\boldsymbol{\tau}_f = \mathbf{w}_f + \mathbf{curl}\,\phi_f$ in Ω_f and

$$\|\mathbf{w}_f\|_{1,\Omega_f} + \|\phi_f\|_{1,\Omega_f} \,\le\, C_f\,\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}\,, \tag{4.49}$$

where C_f is a positive constant independent of $\boldsymbol{\tau}_f$.

Proof. We proceed as in [35, Section 3.2.2] by considering first a convex domain $\widetilde{\Omega}$ containing Ω_s. Then, given $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, we define the auxiliary function $\mathbf{q} \in \mathbf{L}^2(\widetilde{\Omega})$ by

$$\mathbf{q} := \begin{cases} \mathbf{div}\,\boldsymbol{\tau}_s & \text{in} \quad \Omega_s \\ \mathbf{0} & \text{in} \quad \widetilde{\Omega}\backslash\bar{\Omega}_s \end{cases} .$$

and let $\mathbf{z} \in \mathbf{H}^1_0(\widetilde{\Omega})$ be the unique weak solution of the boundary value problem:

$$\Delta\mathbf{z} \,=\, \mathbf{q} \quad \text{in} \quad \widetilde{\Omega}\,, \qquad \mathbf{z} = 0 \quad \text{on} \quad \partial\widetilde{\Omega}\,.$$

The elliptic regularity result for the above problem guarantees that $\mathbf{z} \in \mathbf{H}^2(\widetilde{\Omega})$ and

$$\|\mathbf{z}\|_{2,\widetilde{\Omega}} \,\le\, C\,\|\mathbf{q}\|_{0,\widetilde{\Omega}} \,=\, \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}\,.$$

It follows that $\boldsymbol{\zeta}_s := \nabla\mathbf{z}|_{\Omega_s}$ belongs to $\mathbb{H}^1(\Omega_s)$,

$$\mathbf{div}\,\boldsymbol{\zeta}_s \,=\, \mathbf{div}\,\boldsymbol{\tau}_s \quad \text{in} \quad \Omega_s\,, \tag{4.50}$$

and

$$\|\boldsymbol{\zeta}_s\|_{1,\Omega_s} \,\le\, C\,\|\mathbf{z}\|_{2,\Omega_s} \,\le\, C\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}\,. \tag{4.51}$$

In this way, since $\mathbf{div}(\boldsymbol{\tau}_s - \boldsymbol{\zeta}_s) = 0$ in Ω_s, and Ω_s is connected, there exist $\boldsymbol{\chi}_s := (\chi_1,\chi_2)^{\mathtt{t}} \in \mathbf{H}^1(\Omega_s)$, with $\int_{\Omega_s}\chi_1 = \int_{\Omega_s}\chi_2 = 0$, such that $\boldsymbol{\tau}_s - \boldsymbol{\zeta}_s = \underline{\mathbf{curl}}\,\boldsymbol{\chi}_s$. Note that this identity, the generalized Poincaré inequality, and (4.51) imply that

$$\|\boldsymbol{\chi}_s\|_{1,\Omega_s} \,\le\, C\,|\boldsymbol{\chi}_s|_{1,\Omega_s} \,=\, C\,\|\boldsymbol{\tau}_s - \boldsymbol{\zeta}_s\|_{0,\Omega_s} \,\le\, C\left\{\|\boldsymbol{\tau}_s\|_{0,\Omega_s} + \|\boldsymbol{\zeta}_s\|_{0,\Omega_s}\right\} \,\le\, C\,\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}\,,$$

which, together with (4.51) again, yields (4.48).

In turn, given $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, and since Ω_f is not connected, we first need to perform a suitable extension of $\boldsymbol{\tau}_f$ to the domain $\Omega := \Omega_s \cup \Sigma \cap \Omega_f$. To this end, we now let $v \in H^1(\Omega_s)$ be the unique solution of the Neumann problem:

$$\Delta v = -\frac{\langle \boldsymbol{\tau}_f \cdot \boldsymbol{\nu}, 1\rangle_\Sigma}{|\Omega_s|} \quad \text{in} \quad \Omega_s\,, \quad \frac{\partial v}{\partial \boldsymbol{\nu}} = \boldsymbol{\tau}_f \cdot \boldsymbol{\nu} \quad \text{on} \quad \Sigma\,, \quad \int_{\Omega_s} v = 0\,.$$

The unique solvability of the above problem is guaranteed by the Lax-Milgram Lemma, whose corresponding continuous dependence result establishes that

$$\|v\|_{1,\Omega_s} \leq c\,\|\boldsymbol{\tau}_f \cdot \boldsymbol{\nu}\|_{-1/2,\Sigma}\,. \tag{4.52}$$

Then we define

$$\widetilde{\boldsymbol{\tau}} := \begin{cases} \boldsymbol{\tau}_f & \text{in} \quad \Omega_f\,, \\ \nabla v & \text{in} \quad \Omega_s\,, \end{cases}$$

which clearly belongs to $\mathbf{H}(\mathrm{div};\Omega)$, and observe, using (4.52), that

$$\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div};\Omega} \leq \|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f} + \|\nabla v\|_{\mathrm{div};\Omega_s} \leq \|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f} + \widetilde{c}\,\|\boldsymbol{\tau}_f \cdot \boldsymbol{\nu}\|_{-1/2,\Sigma} \leq C\,\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}\,.$$

In this way, proceeding as in the first part of the present proof, but now applied to $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div};\Omega)$, we deduce the existence of $\widetilde{\mathbf{w}} \in \mathbf{H}^1(\Omega)$ and $\widetilde{\phi} \in H^1(\Omega)$, with $\int_\Omega \widetilde{\phi} = 0$, such that $\widetilde{\boldsymbol{\tau}} = \widetilde{\mathbf{w}} + \mathbf{curl}(\widetilde{\phi})$ in Ω and

$$\|\widetilde{\mathbf{w}}\|_{1,\Omega} + \|\widetilde{\phi}\|_{1,\Omega} \leq C\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div};\Omega} \leq C\,\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}\,.$$

Finally, the proof is completed by defining $\mathbf{w}_f := \widetilde{\mathbf{w}}|_{\Omega_f}$ and $\phi_f := \widetilde{\phi}|_{\Omega_f}$. □

Estimating $\|E_1\|$

Given $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, we use (4.40) to estimate $E_1(\boldsymbol{\tau}_s) = E_1\big(\boldsymbol{\tau}_s - \boldsymbol{\tau}_{s,h}\big)$ with a suitable chosen $\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s$. More precisely, as suggested by the Helmholtz decomposition for $\boldsymbol{\tau}_s$ provided by Lemma 4.4, that is $\boldsymbol{\tau}_s = \boldsymbol{\zeta}_s + \underline{\mathbf{curl}}(\boldsymbol{\chi}_s)$, with $\boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega_s)$ and $\boldsymbol{\chi}_s \in \mathbf{H}^1(\Omega_s)$, we consider in what follows

$$\boldsymbol{\chi}_{s,h} := \mathbf{I}_{s,h}(\boldsymbol{\chi}_s) \in \mathbf{X}_{s,h} \quad \text{and} \quad \boldsymbol{\tau}_{s,h} := \Pi_h^s(\boldsymbol{\zeta}_s) + \underline{\mathbf{curl}}(\boldsymbol{\chi}_{s,h}) \in \mathbb{RT}_h^s \subseteq \mathbb{H}_h^s\,,$$

which yields

$$\boldsymbol{\tau}_s - \boldsymbol{\tau}_{s,h} = \boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s) + \underline{\mathbf{curl}}\big(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}\big)\,.$$

In particular, using (4.43) and (4.50) we find from the above identity that

$$\mathbf{div}\big(\boldsymbol{\tau}_s - \boldsymbol{\tau}_{s,h}\big) = \big(\mathbf{I} - \mathcal{P}_h^s\big)(\mathbf{div}\,\boldsymbol{\zeta}_s) = \big(\mathbf{I} - \mathcal{P}_h^s\big)(\mathbf{div}\,\boldsymbol{\tau}_s)\,,$$

and hence, according to the definition of E_1 (cf. (4.33)), we find that

$$E_1(\boldsymbol{\tau}_s - \boldsymbol{\tau}_{s,h}) = E_{11}(\boldsymbol{\tau}_s) + E_{12}(\boldsymbol{\zeta}_s) + E_{13}(\boldsymbol{\chi}_s)\,,$$

where

$$E_{11}(\boldsymbol{\tau}_s) = \frac{1}{\kappa_s^2}\int_{\Omega_s}\big\{\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{s,h}\big\}(\mathbf{I}-\mathcal{P}_h^s)(\mathbf{div}\,\boldsymbol{\tau}_s) = \frac{1}{\kappa_s^2}\int_{\Omega_s}(\mathbf{I}-\mathcal{P}_h^s)(\mathbf{f})\cdot(\mathbf{div}\,\boldsymbol{\tau}_s)\,,$$

$$E_{12}(\boldsymbol{\zeta}_s) = -\int_{\Omega_s}\big\{\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big\} : (\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s)) + \langle(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}, \boldsymbol{\varphi}_{s,h}\rangle_\Sigma\,,$$

and

$$E_{13}(\boldsymbol{\chi}_s) = -\int_{\Omega_s}\big\{\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big\} : \underline{\mathbf{curl}}(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}) + \langle\underline{\mathbf{curl}}(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h})\,\boldsymbol{\nu}, \boldsymbol{\varphi}_{s,h}\rangle_\Sigma\,.$$

Note that the second expression defining $E_{11}(\boldsymbol{\tau}_s)$ follows from the fact that $\mathcal{P}_h^s$ is self-adjoint and that, according to the definitions of $\mathbb{H}_h^s$ (cf. (4.10)) and $\mathbf{U}_h^s$ (cf. (4.18)), there holds $\mathbf{div}\big(\mathbb{H}_h^s\big) \subseteq \mathbf{U}_h^s$, whence $\big(\mathbf{I}-\mathcal{P}_h^s\big)(\mathbf{div}\,\boldsymbol{\sigma}_{s,h}) = \mathbf{0}$.

The following three lemmata provide the upper bounds for $E_{11}(\boldsymbol{\tau}_s)$, $E_{12}(\boldsymbol{\zeta}_s)$, and $E_{13}(\boldsymbol{\chi}_s)$.

Lemma 4.5 *There holds*

$$|E_{11}(\boldsymbol{\tau}_s)| \le \frac{1}{\kappa_s^2}\left\{\sum_{T\in\mathcal{T}_h^s}\|(\mathbf{I}-\mathcal{P}_h^s)\,\mathbf{f}\,\|_{0,T}^2\right\}^{1/2}\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}\,.$$

Proof. It follows from a straightforward application of the Cauchy-Schwarz inequality. □

Lemma 4.6 *There exists $C > 0$, independent of μ, λ, and κ_s, such that*

$$|E_{12}(\boldsymbol{\zeta}_s)| \le C\left\{\sum_{T\in\mathcal{T}_h^s} h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\varphi}_{s,h} - \mathbf{u}_h\|_{0,e}^2\right\}^{1/2}\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}\,.$$

Proof. The present estimate was actually proved in [35, Lemma 5]. For sake of completeness we provide here the main aspects of the corresponding proof. We first observe, thanks to the fact that $\boldsymbol{\zeta}_s$ belongs to $\mathbb{H}^1(\Omega_s)$, that $(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}|_\Sigma \in \mathbf{L}^2(\Sigma)$, and hence

$$\langle(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}, \boldsymbol{\varphi}_{s,h}\rangle_\Sigma = \sum_{e\in\mathcal{E}_h(\Sigma)}\int_e \boldsymbol{\varphi}_{s,h}\cdot(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}\,. \tag{4.53}$$

Next, it is clear from (4.25) that $\mathbf{u}_h \in \mathbf{U}_h^s$, which means, in particular, that for each $e \in \mathcal{E}_h(\Sigma)$ there holds $\mathbf{u}_h|_e \in \mathbf{P}_0(e)$, and therefore the identity (4.41) yields

$$\sum_{e\in\mathcal{E}_h(\Sigma)}\int_e \mathbf{u}_h\cdot(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu} = 0\,.$$

Thus, by introducing the above null expression in the right hand side of (4.53), and then re-incorporating the resulting equation in the definition of E_{12}, we find that

$$E_{12}(\boldsymbol{\zeta}_s) \,=\, -\sum_{T\in\mathcal{T}_h^s}\int_T \left\{\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\right\} : (\boldsymbol{\zeta}_s-\Pi_h^s(\boldsymbol{\zeta}_s)) + \sum_{e\in\mathcal{E}_h(\Sigma)}\int_e \left(\boldsymbol{\varphi}_{s,h}-\mathbf{u}_h\right)\cdot(\boldsymbol{\zeta}_s-\Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}\,,$$

where we have replaced the original integration $\displaystyle\int_{\Omega_s}$ by $\displaystyle\sum_{T\in\mathcal{T}_h^s}\int_T$. In this way, the rest of the proof reduces to apply the Cauchy-Schwarz inequality, the approximation properties (4.44) and (4.45), and finally the upper bound given by (4.51). We omit further details. □

Lemma 4.7 *There exists $C>0$, independent of μ, λ and κ_s, such that*

$$|E_{13}(\boldsymbol{\chi}_s)| \,\le\, C\left\{\sum_{T\in\mathcal{T}_h^s} h_T^2\,\|\mathrm{curl}(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h)\|_{0,T}^2 \,+\, \sum_{e\in\mathcal{E}_h(\Omega_s)} h_e\,\|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2 \right.$$
$$\left. +\, \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\left\|(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h)\,\mathbf{s}-\frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}}\right\|_{0,e}^2\right\}^{1/2} \|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}\,.$$

Proof. While this result is also available in several places (see, e.g. [35, Lemma 6]), here we proceed similarly as for the previous lemma and provide an sketch of its proof. Indeed, since

$$\underline{\mathbf{curl}}(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h})\,\boldsymbol{\nu} \,=\, -\frac{d}{d\mathbf{s}}(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h}) \quad\text{and}\quad \frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}}\in\mathbf{L}^2(\Sigma)\,,$$

we deduce, using integration by parts on Σ, that

$$\langle\underline{\mathbf{curl}}(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h})\,\boldsymbol{\nu},\boldsymbol{\varphi}_{s,h}\rangle_\Sigma \,=\, -\left\langle\frac{d}{d\mathbf{s}}(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h}),\boldsymbol{\varphi}_{s,h}\right\rangle_\Sigma \,=\, \int_\Sigma \frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}}\cdot(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h})\,. \tag{4.54}$$

In turn, integrating by parts on each $T\in\mathcal{T}_h^s$, we obtain that

$$-\int_{\Omega_s}\left\{\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\right\} : \underline{\mathbf{curl}}(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h}) \,=\, -\sum_{T\in\mathcal{T}_h^s}\int_T\left\{\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\right\} : \underline{\mathbf{curl}}(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h})$$
$$=\sum_{T\in\mathcal{T}_h^s}\left\{\int_T \mathrm{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\big)\cdot(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h}) - \int_{\partial T}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\big)\,\mathbf{s}\cdot(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h})\right\}$$
$$=\sum_{T\in\mathcal{T}_h^s}\int_T \mathrm{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\big)\cdot(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h}) - \sum_{e\in\mathcal{E}_h(\Omega_s)}\int_e\big[\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\big)\,\mathbf{s}\big]\cdot(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h})$$
$$-\sum_{e\in\mathcal{E}_h(\Sigma)}\int_e\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\big)\,\mathbf{s}\cdot(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h})\,,$$

which, together with (4.54), yields

$$E_{13}(\boldsymbol{\chi}_s) \,=\, \sum_{T\in\mathcal{T}_h^s}\int_T \mathrm{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\big)\cdot(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h}) - \sum_{e\in\mathcal{E}_h(\Omega_s)}\int_e\big[\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\big)\,\mathbf{s}\big]\cdot(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h})$$
$$-\sum_{e\in\mathcal{E}_h(\Sigma)}\int_e\left\{\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\big)\,\mathbf{s}-\frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}}\right\}\cdot(\boldsymbol{\chi}_s-\boldsymbol{\chi}_{s,h})\,.$$

In this way, and recalling that $\boldsymbol{\chi}_{s,h} = \mathbf{I}_{s,h}(\boldsymbol{\chi}_s)$, the rest of the proof follows from obvious applications of the Cauchy-Schwarz inequality and the approximation properties of the Clément interpolation operator $\mathbf{I}_{s,h}$ (cf. Lemma 4.3), taking into account as well that the number of elements in $\Delta(T)$ and $\Delta(e)$ are bounded and that $\|\boldsymbol{\chi}_s\|_{1,\Omega_s} \le C_s \|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}$ (cf. (4.48)). Further details are omitted. $\square$

As a direct consequence of Lemmata 4.5, 4.6, and 4.7, the norm of the functional E_1 (cf. (4.33)) is estimated as follows.

Lemma 4.8 *There exists $C > 0$, independent of μ, λ and κ_s, such that*

$$\begin{aligned}\|E_1\| \,\le\, C \Bigg\{ & \frac{1}{\kappa_s^4} \sum_{T\in\mathcal{T}_h^s} \|(\mathbf{I}-\mathcal{P}_h^s)\,\mathbf{f}\,\|_{0,T}^2 \,+\, \sum_{T\in\mathcal{T}_h^s} h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\|_{0,T}^2 \\ & + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\varphi}_{s,h} - \mathbf{u}_h\|_{0,e}^2 \,+\, \sum_{T\in\mathcal{T}_h^s} h_T^2\,\|\mathrm{curl}(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\|_{0,T}^2 \\ & + \sum_{e\in\mathcal{E}_h(\Omega_s)} h_e\,\|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2 \,+\, \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\|(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\,\mathbf{s} - \frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}}\right\|_{0,e}^2 \Bigg\}^{1/2}.\end{aligned}$$

Estimating $\|E_2\|$

We proceed analogously to the case of $\|E_1\|$. This means that, given $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, we consider from Lemma 4.4 its Helmholtz decomposition $\boldsymbol{\tau}_f = \mathbf{w}_f + \mathbf{curl}\,\phi_f$ in Ω_f, with $\mathbf{w}_f \in \mathbf{H}^1(\Omega_f)$ and $\phi_f \in H^1(\Omega_f)$, and define

$$\phi_{f,h} := I_{f,h}(\phi_f) \quad \text{and} \quad \boldsymbol{\tau}_{f,h} := \Pi_h^f(\mathbf{w}_f) + \mathbf{curl}(\phi_{f,h})\,,$$

so that, using the second equality in (4.40), we can write $E_2(\boldsymbol{\tau}_f) = E_2\big(\boldsymbol{\tau}_f - \boldsymbol{\tau}_{f,h}\big)$. It follows that

$$\boldsymbol{\tau}_f \,-\, \boldsymbol{\tau}_{f,h} \,=\, \mathbf{w}_f - \Pi_h^f(\mathbf{w}_f) \,+\, \mathbf{curl}(\phi_f - \phi_{f,h})\,,$$

from which, employing the second identity in (4.43), and noting from the definitions (4.11) and (4.19) that $\mathrm{div}\,\boldsymbol{\sigma}_{f,h} \in U_h^f$, we find that

$$\int_{\Omega_f} \mathrm{div}\,\boldsymbol{\sigma}_{f,h}\,\mathrm{div}\left(\boldsymbol{\tau}_f - \boldsymbol{\tau}_{f,h}\right) \,=\, \int_{\Omega_f} \mathrm{div}\,\boldsymbol{\sigma}_{f,h}\left(\mathbf{I} - \mathcal{P}_h^f\right)(\mathrm{div}\,\mathbf{w}_f) \,=\, 0\,.$$

Hence, according to (4.34) and the above computation, we get

$$E_2(\boldsymbol{\tau}_f - \boldsymbol{\tau}_{f,h}) \,=\, E_{21}(\mathbf{w}_f) \,+\, E_{22}(\phi_f)\,,$$

where

$$E_{21}(\mathbf{w}_f) := -\int_{\Omega_f} \boldsymbol{\sigma}_{f,h}\cdot\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) - \langle\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}, \varphi_{\Sigma,h}\rangle_\Sigma + \langle\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}, \varphi_{\Gamma,h}\rangle_\Gamma$$

and

$$E_{22}(\phi_f) \;:=\; -\int_{\Omega_f} \boldsymbol{\sigma}_{f,h} \cdot \mathbf{curl}(\phi_f - \phi_{f,h}) - \langle \mathbf{curl}(\phi_f - \phi_{f,h}) \cdot \boldsymbol{\nu}, \varphi_{\Sigma,h} \rangle_\Sigma$$
$$+ \quad \langle \mathbf{curl}(\phi_f - \phi_{f,h}) \cdot \boldsymbol{\nu}, \varphi_{\Gamma,h} \rangle_\Gamma \,.$$

The following two lemmata establish the upper bounds for $|E_{21}(\mathbf{w}_f)|$ and $|E_{22}(\phi_f)|$.

Lemma 4.9 *There exists $C > 0$, independent of κ_f and h, such that*

$$|E_{21}(\mathbf{w}_f)| \;\le\; C \left\{ \sum_{T\in\mathcal{T}_h^f} h_T^2 \, \|\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+\, \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \, \|\varphi_{\Sigma,h} - p_h\|_{0,e}^2 \right.$$
$$\left. + \quad \sum_{e\in\mathcal{E}_h(\Gamma)} h_e \, \|\varphi_{\Gamma,h} - p_h\|_{0,e}^2 \right\} \|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f} \,.$$

Proof. We proceed as in the proof of Lemma 4.6. Indeed, since $\mathbf{w}_f \in \mathbf{H}^1(\Omega_f)$ it is clear that

$$\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) \cdot \boldsymbol{\nu}|_\Sigma \in L^2(\Sigma) \quad \text{and} \quad \big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) \cdot \boldsymbol{\nu}|_\Gamma \in L^2(\Gamma)\,,$$

which, together with the fact that $p_h|_e \in P_0(e) \;\; \forall\, e \in \mathcal{E}_h(\Sigma) \cup \mathcal{E}_h(\Gamma)$ (cf. (4.26) and (4.11)), and thanks to the characterization property (4.42), allow to show that

$$\langle \big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) \cdot \boldsymbol{\nu}, \varphi_{\Sigma,h} \rangle_\Sigma \;=\; \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e (\varphi_{\Sigma,h} - p_h) \, \big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) \cdot \boldsymbol{\nu}$$

and

$$\langle \big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) \cdot \boldsymbol{\nu}, \varphi_{\Gamma,h} \rangle_\Gamma \;=\; \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e (\varphi_{\Gamma,h} - p_h) \, \big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) \cdot \boldsymbol{\nu} \,.$$

In this way, we find that

$$E_{21}(\mathbf{w}_f) := -\sum_{T\in\mathcal{T}_h^f} \int_T \boldsymbol{\sigma}_{f,h} \cdot \big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) - \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e (\varphi_{\Sigma,h} - p_h) \, \big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) \cdot \boldsymbol{\nu}$$
$$+ \quad \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e (\varphi_{\Gamma,h} - p_h) \, \big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) \cdot \boldsymbol{\nu}\,,$$

and hence, the proof is completed by applying the Cauchy-Schwarz inequality, the approximation properties (4.46) and (4.47), and the fact that $\|\mathbf{w}_f\|_{1,\Omega_f} \;\le\; C_f \, \|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}$ (cf. (4.49)). We omit further details. □

Lemma 4.10 *There exists $C > 0$, independent of κ_f and h, such that*

$$|E_{22}(\phi_f)| \;\le\; C \left\{ \sum_{T\in\mathcal{T}_h^f} h_T^2 \, \|\mathrm{rot}(\boldsymbol{\sigma}_{f,h})\|_{0,T}^2 \,+\, \sum_{e\in\mathcal{E}_h(\Omega_f)} h_e \, \|[\boldsymbol{\sigma}_{f,h} \cdot \mathbf{s}]\|_{0,e}^2 \right.$$
$$\left. + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\| \boldsymbol{\sigma}_{f,h} \cdot \mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}} \right\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(\Gamma)} h_e \left\| \boldsymbol{\sigma}_{f,h} \cdot \mathbf{s} - \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}} \right\|_{0,e}^2 \right\}^{1/2} \|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}.$$

Proof. The analysis here is analogous to the proof of Lemma 4.7. In fact, we begin by noticing that

$$\mathbf{curl}(\phi_f - \phi_{f,h}) \cdot \boldsymbol{\nu} \,=\, -\frac{d}{d\mathbf{s}}(\phi_f - \phi_{f,h})\,, \quad \frac{d\varphi_{\Sigma,h}}{d\mathbf{s}} \in L^2(\Sigma)\,, \quad \text{and} \quad \frac{d\varphi_{\Gamma,h}}{d\mathbf{s}} \in L^2(\Gamma)\,,$$

which, together with integration by parts procedures on Σ, Γ, and on each $T \in \mathcal{T}_h^f$, yield

$$\begin{aligned} E_{22}(\phi_f) \,=\, & -\sum_{T\in\mathcal{T}_h^f} \int_T \operatorname{rot}(\boldsymbol{\sigma}_{f,h})\,(\phi_f - \phi_{f,h}) \,+ \sum_{e\in\mathcal{E}_h(\Omega_f)} \int_e [\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s}](\phi_f - \phi_{f,h}) \\ - & \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \Big(\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}}\Big)\,(\phi_f - \phi_{f,h}) \,+ \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e \Big(\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}}\Big)(\phi_f - \phi_{f,h})\,. \end{aligned}$$

Consequently, and similarly as for Lemma 4.7, the rest of the proof follows from straightforward applications of the Cauchy-Schwarz inequality, the approximation properties of the Clément interpolator $\phi_{f,h} := I_{f,h}(\phi_f)$ (cf. Lemma 4.3), the fact that the cardinalities of $\Delta(T)$ and $\Delta(e)$ are bounded, and the upper bound $\|\phi_f\|_{1,\Omega_f} \,\leq\, C_f\,\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}$ (cf. (4.49)). We omit further details. □

The norm of E_2 (cf. (4.34) is bounded now as a consequence of Lemmata 4.9 and 4.10.

Lemma 4.11 *There exists $C > 0$, independent of κ_f and h, such that*

$$\begin{aligned} \|E_2\| \,\leq\, C \Bigg\{ & \sum_{T\in\mathcal{T}_h^f} h_T^2\,\|\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\varphi_{\Sigma,h} - p_h\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}_h(\Gamma)} h_e\,\|\varphi_{\Gamma,h} - p_h\|_{0,e}^2 \,+ \sum_{T\in\mathcal{T}_h^f} h_T^2\,\|\operatorname{rot}(\boldsymbol{\sigma}_{f,h})\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}_h(\Omega_f)} h_e\,\|[\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s}]\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\|\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}}\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(\Gamma)} h_e \left\|\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}}\right\|_{0,e}^2 \Bigg\}^{1/2}\,. \end{aligned}$$

We end this section by observing that the reliability estimate (cf. Theorem 4.4) is a direct consequence of (4.30) and (4.31), together with Lemmata 4.1, 4.2, 4.8, and 4.11.

4.3.2 Efficiency of the a posteriori error estimator

In this section we prove the efficiency of our a posteriori error estimator $\boldsymbol{\theta}$ (lower bound in (4.29)). We begin with the first two terms defining $\theta_{T,s}^2$ (cf. (4.22)). In fact, since $\boldsymbol{\sigma}_s$ is symmetric in Ω_s, we easily notice, adding and substracting $\boldsymbol{\sigma}_s$, that there holds

$$\|\boldsymbol{\sigma}_{s,h} - \boldsymbol{\sigma}_{s,h}^{\mathtt{t}}\|_{0,T}^2 \,\leq\, 4\,\|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T}^2\,. \tag{4.55}$$

Next, according to the definitions of $\mathbf{u}$ (cf. (4.5)) and $\mathbf{u}_h$ (cf. (4.25)), we find that

$$\|(\mathbf{I} - \mathcal{P}_h^s)\mathbf{f}\|_{0,T}^2 \,\leq\, 2\,\kappa_s^4\,\|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \,+\, 2\,\|\mathbf{div}(\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h})\|_{0,T}^2\,. \tag{4.56}$$

Throughout the rest of the section we provide the corresponding upper bounds for the terms in (4.22), (4.23), (4.24), and (4.27) that involve the mesh parameters h_T and h_e. Actually, most of these estimates are already available in the literature (see, e.g. [21], [22], [35], and [43]), but for sake of completeness we sketch here some of their proofs, which employ the localization technique based on triangle-bubble and edge-bubble functions, together with extension operators, discrete trace and inverse inequalities, and certainly the original identities recovered by Theorem 4.2. To this end, we now introduce further notations and preliminary results. Given $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ and $e \in \mathcal{E}(T)$, we let ψ_T and ψ_e be the usual triangle-bubble and edge-bubble functions, respectively (see [72, eqs. (1.5) and (1.6)]), which satisfy:

ii) $\psi_T \in P_3(T)$, $\psi_T = 0$ on ∂T, $\mathrm{supp}(\psi_T) \subseteq T$, and $0 \leq \psi_T \leq 1$ in T.

ii) $\psi_e|_T \in P_2(T)$, $\psi_e = 0$ on $\partial T \setminus e$, $\mathrm{supp}(\psi_e) \subseteq w_e := \cup\{T' \in \mathcal{T}_h^s \cup \mathcal{T}_h^f : \quad e \in \mathcal{E}(T')\}$, and $0 \leq \psi_e \leq 1$ in w_e.

We also recall from [71] that, given $k \in \mathbb{N} \cup \{0\}$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p) \in P_k(T)$ and $L(p)|_e = p$ for all $p \in P_k(e)$. Additional properties of ψ_T, ψ_e and L are collected in the following lemma.

Lemma 4.12 *Given $k \in \mathbb{N} \cup \{0\}$, there exist positive constants c_1, c_2 and c_3, depending only on k and the shape regularity of the triangulations (minimun angle condition), such that for each $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ and $e \in \mathcal{E}(T)$, there hold*

$$\|q\|_{0,T}^2 \leq c_1 \|\psi_T^{1/2} q\|_{0,T}^2 \qquad \forall q \in P_k(T) \tag{4.57}$$

$$\|p\|_{0,e}^2 \leq c_2 \|\psi_e^{1/2} p\|_{0,e}^2 \qquad \forall p \in P_k(e) \tag{4.58}$$

and

$$\|\psi_e^{1/2} L(p)\|_{0,T}^2 \leq c_3\, h_e \|p\|_{0,e}^2 \qquad \forall p \in P_k(e) \tag{4.59}$$

Proof. See [71, Lemma 1.3]. □

The following inverse and discrete trace inequalities will also be used.

Lemma 4.13 *Let k, l, $m \in \mathbb{N} \cup \{0\}$ such that $l \leq m$. Then there exists $c > 0$, depending only on k, l, m and the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ there holds*

$$|q|_{m,T} \,\leq\, c\, h_T^{l-m}\, |q|_{l,T} \qquad \forall\, q \in P_k(T)\,. \tag{4.60}$$

Proof. See [24, Theorem 3.2.6]. □

Lemma 4.14 *There exists $C > 0$, depending only on the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ and $e \in \mathcal{E}(T)$, there holds*

$$\|v\|_{0,e}^2 \,\le\, C \left\{ h_e^{-1} \, \|v\|_{0,T}^2 \,+\, h_e \, |v|_{1,T}^2 \right\} \quad \forall\, v \in H^1(T)\,. \tag{4.61}$$

Proof. See [1, Theorem 3.10] or [3, eq. (2.4)]. □

The following three lemmas, whose proofs make use of the techniques and results described above, provide the upper bounds for the remaining terms defining $\theta_{T,s}^2$ (cf. (4.22)).

Lemma 4.15 *There exists $C > 0$, independent of h and λ, such that for each $T \in \mathcal{T}_h^s$ there holds*

$$h_T^2 \, \|\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\|_{0,T}^2 \,\le\, C \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \,+\, h_T^2 \, \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T}^2 \,+\, h_T^2 \, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\}.$$

Proof. See [22, Lemma 6.6]. □

Lemma 4.16 *There exists $C > 0$, independent of h and λ, such that for each $T \in \mathcal{T}_h^s$ there holds*

$$h_T^2 \, \|\mathrm{curl}\,(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\|_{0,T}^2 \,\le\, C \left\{ \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T}^2 \,+\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\}.$$

Proof. See [22, Lemma 6.3] or [13, Lemma 4.7]. □

Lemma 4.17 *There exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Omega_s)$ there holds*

$$h_e \, \|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\mathbf{s}]\|_{0,e}^2 \,\le\, C \sum_{T \subseteq \omega_e} \left\{ \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T}^2 \,+\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 \right\},$$

where $\omega_e := \cup\big\{ T' \in \mathcal{T}_h^s : \quad e \in \mathcal{E}(T') \big\}$.

Proof. See [22, Lemma 6.4]. □

The analogue of the above three lemmas for the terms defining $\theta_{T,f}^2$ (cf. (4.23)) are stated next.

Lemma 4.18 *There exists $C > 0$, independent of h, such that for each $T \in \mathcal{T}_h^f$ there holds*

$$h_T^2 \, \|\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,\le\, C \left\{ h_T^2 \, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+\, \|p - p_h\|_{0,T}^2 \right\}.$$

Proof. It is a slight modification of [21, Lemma 6.3] (see also [43, Lemma 4.13]). In fact, given $T \in \mathcal{T}_h^f$, we apply (4.57), use that $\boldsymbol{\sigma}_f = \nabla p$ in Ω_f and $\nabla p_h = 0$ in T (which follows from the fact that p_h is piecewise constant in virtue of (4.11) and (4.26)), and then integrate by parts. In this way, we find that

$$\begin{aligned}\|\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,\le\, C\,\|\psi_T^{1/2}\,\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,=\, C\int_T \psi_T\,\overline{\boldsymbol{\sigma}_{f,h}}\cdot\Big\{(\boldsymbol{\sigma}_{f,h}-\boldsymbol{\sigma}_f) - \nabla(p_h - p)\Big\}\\ = C\Big\{\int_T \psi_T\,\overline{\boldsymbol{\sigma}_{f,h}}\cdot(\boldsymbol{\sigma}_{f,h}-\boldsymbol{\sigma}_f) \,+\, \int_T \operatorname{div}(\psi_T\,\overline{\boldsymbol{\sigma}_{f,h}})\,(p-p_h)\Big\}\,.\end{aligned}$$

Then, employing the Cauchy- Schwarz inequality, the inverse estimate (4.60) (cf. Lemma 4.13), and the fact that $0 \le \psi_T \le 1$, we get

$$\|\boldsymbol{\sigma}_{f,h}\|_{0,T} \,\le\, C\Big\{\|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+\, h_T^{-1}\,\|p-p_h\|_{0,T}^2\Big\}\,,$$

which implies the required bound and completes the proof. □

Lemma 4.19 *There exists $C > 0$, independent of h, such that for each $T \in \mathcal{T}_h^f$ there holds*

$$h_T^2\,\|\operatorname{rot}\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,\le\, C\,\|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2\,.$$

Proof. It basically follows from the general estimate provided by [13, Lemma 4.3]. Indeed, a row-wise interpretation of this result allows to show that, given a piecewise polynomial $\rho_h \in \mathbf{L}^2(\Omega_f)$ of degree $k \ge 0$ on each $T \in \mathcal{T}_h^f$, and $\rho \in \mathbf{L}^2(\Omega_f)$ such that $\operatorname{rot}\rho = 0$ in Ω_f, there exists $c > 0$, independent of h, such that

$$h_T\,\|\operatorname{rot}\rho_h\|_{0,T} \,\le\, c\,\|\rho - \rho_h\|_{0,T} \qquad \forall\, T \in \mathcal{T}_h^f\,. \tag{4.62}$$

Hence, since $\operatorname{rot}\boldsymbol{\sigma}_f = \operatorname{rot}(\nabla p) = 0$, it suffices to apply (4.62) to $\rho_h = \boldsymbol{\sigma}_{f,h}$ and $\rho = \boldsymbol{\sigma}_f$. □

Lemma 4.20 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Omega_f)$ there holds*

$$h_e\,\|[\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s}]\|_{0,e}^2 \,\le\, C\,\|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,\omega_e}^2\,,$$

where $\omega_e := \cup\big\{T' \in \mathcal{T}_h^f : \quad e \in \mathcal{E}(T')\big\}$.

Proof. We first observe that a slight modification of the proof of [13, Lemma 4.4] allows to show that, under the same hypotheses leading to (4.62), that is given a piecewise polynomial $\rho_h \in \mathbf{L}^2(\Omega_f)$ of degree $k \ge 0$ on each $T \in \mathcal{T}_h^f$, and $\rho \in \mathbf{L}^2(\Omega_f)$ such that $\operatorname{rot}\rho = 0$ in Ω_f, there exists $c > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Omega_f)$ there holds

$$h_e\,\|[\rho_h\cdot\mathbf{s}]\|_{0,e}^2 \,\le\, c\,\|\rho - \rho_h\|_{0,\omega_e}^2\,. \tag{4.63}$$

Hence, the present proof is a straightforward application of (4.63) to $\rho_h = \boldsymbol{\sigma}_{f,h}$ and $\rho = \boldsymbol{\sigma}_f = \nabla p$. □

We now aim to bound the first three terms defining $\theta_{e,\Sigma}^2$ (cf. (4.24)).

Lemma 4.21 *There exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e \, \|\boldsymbol{\varphi}_{s,h} - \mathbf{u}_h\|_{0,e}^2 \,\le\, C \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 + h_T^2 \, \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T}^2 + h_T^2 \, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T}^2 + h_e \, \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h^s$ having e as an edge.

Proof. It is based mainly on the discrete trace inequality (4.61), the fact that $\nabla \mathbf{u} = \mathcal{C}^{-1} \boldsymbol{\sigma}_s + \boldsymbol{\gamma}$ in Ω_s, and the upper bound for $h_T^2 \, \|\mathcal{C}^{-1} \boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\|_{0,T}^2$ provided by Lemma 4.15. We omit further details and refer to [35, Lemma 22]. □

Lemma 4.22 *There exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e \, \|\boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f \, \omega^2 \, \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}\|_{0,e}^2 \,\le\, C \left\{ \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 + h_T^2 \, \|\mathrm{div}(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h})\|_{0,T}^2 + h_e \, \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. We proceed similarly as in [12, Lemma 4.7] (see also [46, Lemma 3.15]). Indeed, given $e \in \mathcal{E}_h(\Sigma)$, we let T be the triangle of $\mathcal{T}_h^f$ having e as an edge, define $v_e := \boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f \, \omega^2 \, \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}$ on e, and consider the extension operator $L : C(e) \to C(T)$. Then, applying (4.58), recalling that $\psi_e = 0$ on $\partial T \backslash e$, extending $\psi_e \, \overline{L(v_e)}$ by zero in $\Omega_f \backslash T$ so that the resulting function belongs to $H^1(\Omega_f)$, and adding and substracting $\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} = \rho_f \, \omega^2 \, \boldsymbol{\varphi}_s \cdot \boldsymbol{\nu}$ on Σ, we get

$$\begin{aligned} \|v_e\|_{0,e}^2 \,&\le\, c_2 \, \|\psi_e^{1/2} \, v_e\|_{0,e}^2 \,=\, c_2 \int_e \psi_e \, \overline{v_e} \, (\boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f \, \omega^2 \, \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}) \\ &= \, c_2 \, \langle \boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f \, \omega^2 \, \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}, \psi_e \, \overline{L(v_e)} \rangle_\Sigma \\ &= c_2 \left\{ - \, \langle (\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}) \cdot \boldsymbol{\nu}, \psi_e \, \overline{L(v_e)} \rangle_\Sigma \,+\, \rho_f \, \omega^2 \, \langle (\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \cdot \boldsymbol{\nu}, \psi_e \, \overline{L(v_e)} \rangle_\Sigma \right\}, \end{aligned} \tag{4.64}$$

where, as indicated in Section 4.1, $\langle \cdot, \cdot \rangle_\Sigma$ stands here for the duality pairing between $H^{-1/2}(\Sigma)$ and $H^{1/2}(\Sigma)$. Next, integrating by parts in Ω_f, and then employing the Cauchy-Schwarz inequality, the inverse estimate (4.60) (cf. Lemma 4.13), and (4.59), we find that

$$\begin{aligned} \langle (\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}) \cdot \boldsymbol{\nu}, \psi_e \overline{L(v_e)} \rangle_\Sigma \,&=\, \int_T \nabla (\psi_e \, \overline{L(v_e)}) \cdot (\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}) \,+\, \int_T \psi_e \, \overline{L(v_e)} \, \mathrm{div}(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}) \\ &\le\, |\psi_e \, L(v_e))|_{1,T} \, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T} \,+\, \|\psi_e \, L(v_e)\|_{0,T} \, \|\mathrm{div}(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h})\|_{0,T} \\ &\le\, C \left\{ h_T^{-1} \, h_e^{1/2} \, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T} \,+\, h_e^{1/2} \, \|\mathrm{div}(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h})\|_{0,T} \right\} \|v_e\|_{0,e} \,. \end{aligned} \tag{4.65}$$

In turn, noting that $(\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \cdot \boldsymbol{\nu} \in L^2(\Sigma)$, recalling that $0 \le \psi_e \le 1$ in w_e, and applying again the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \langle (\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \cdot \boldsymbol{\nu}, \psi_e \overline{L(v_e)} \rangle_\Sigma &= \int_e (\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \cdot \boldsymbol{\nu}\, \psi_e\, \overline{v_e} \\ &\le \|(\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \cdot \boldsymbol{\nu}\|_{0,e}\, \|\psi_e\, v_e\|_{0,e} \;\le\; \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,e}\, \|v_e\|_{0,e}\,. \end{aligned} \tag{4.66}$$

Finally, inserting the estimates (4.65) and (4.66) into (4.64), and using that $h_e \le h_T$, we get after minor simplifications the required upper bound for $h_e\, \|\boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f\, \omega^2\, \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}\|_{0,e}^2$. □

Lemma 4.23 *There exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e\, \|\boldsymbol{\sigma}_{s,h} \cdot \boldsymbol{\nu} \,+\, \varphi_{\Sigma,h}\, \boldsymbol{\nu}\|_{0,e}^2 \;\le\; C \left\{ \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T}^2 \,+\, h_T^2\, \|\mathbf{div}(\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h})\|_{0,T}^2 \,+\, h_e\, \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h^s$ having e as an edge.

Proof. It proceeds similarly as for Lemma 4.22. This means that given $e \in \mathcal{E}_h(\Sigma)$, we now let T be the triangle of $\mathcal{T}_h^s$ having e as an edge, consider the extension operator $L : C(e) \to C(T)$, define $v_e := \boldsymbol{\sigma}_{s,h} \cdot \boldsymbol{\nu} \,+\, \varphi_{\Sigma,h}\, \boldsymbol{\nu}$ on e, and extend $\psi_e\, L(v_e)$ by zero in $\Omega_s \backslash T$ so that the resulting function belongs to $H^1(\Omega_s)$. The rest of the proof follows basically by applying (4.58), using that $\boldsymbol{\sigma}_s \cdot \boldsymbol{\nu} = \varphi_\Sigma\, \boldsymbol{\nu}$ on Σ, integrating by parts and applying Cauchy-Schwarz and inverse inequalities. We omit further details.

□

The upper bounds for the terms of $\theta_{e,\Sigma}^2$ and $\theta_{e,\Gamma}^2$ involving tangential derivatives are given now.

Lemma 4.24 *There exists $C > 0$, independent of h and λ, such that*

$$\begin{aligned} &\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \left\| (\mathcal{C}^{-1}\, \boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\, \mathbf{s} \,-\, \frac{d\, \boldsymbol{\varphi}_{s,h}}{d\mathbf{s}} \right\|_{0,e}^2 \\ &\qquad \le C \left\{ \sum_{e \in \mathcal{E}_h(\Sigma)} \left\{ \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T_e}^2 \,+\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T_e}^2 \right\} \,+\, \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{1/2,\Sigma}^2 \right\}, \end{aligned}$$

where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^s$ having e as an edge.

Proof. It makes use of the extension operator $\mathbf{L} : \mathbf{C}(e) \to \mathbf{C}(T)$ (vector version of $L : C(e) \to C(T)$), the fact that $\nabla \mathbf{u} = \mathcal{C}^{-1}\, \boldsymbol{\sigma}_s + \boldsymbol{\gamma}$ in Ω_s, the boundedness of the tangential derivative $\frac{d}{d\mathbf{s}} : \mathbf{H}^{1/2}(\Sigma) \to \mathbf{H}^{-1/2}(\Sigma)$, the inverse and the Cauchy-Schwarz inequalities, and the upper bound for $h_{T_e}^2 \, \|\mathrm{curl}\,(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\|_{0,T_e}^2$ (cf. Lemma 4.16). We omit further details and refer to [35, Lemma 20] where this result was established and proved. □

We remark that the upper bound provided by Lemma 4.24 is one of the three non-local estimates of the present efficiency analysis (see Lemma 4.26 below for the other two). However, the following lemma establishes that, under an additional regularity assumption on $\boldsymbol{\varphi}_s$, a corresponding local estimate can also be obtained.

Lemma 4.25 *Assume that $\boldsymbol{\varphi}_s|_e \in \mathbf{H}^1(e)$ for each $e \in \mathcal{E}_h(\Sigma)$. Then there exists $C > 0$, independent of h and λ, such that*

$$h_e \left\| (\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\,\mathbf{s} - \frac{d\,\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}} \right\|_{0,e}^2 \leq C \left\{ \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T_e}^2 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T_e}^2 + h_e \left\| \frac{d}{d\mathbf{s}}(\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \right\|_{0,e}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^s$ having e as an edge.

Proof. See [35, Lemma 21]. □

Lemma 4.26 *There exist $C_1, C_2 > 0$, independent of h, such that*

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\| \boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}} \right\|_{0,e}^2 \leq C_1 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T_e}^2 + \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{1/2,\Sigma}^2 \right\}$$

and

$$\sum_{e\in\mathcal{E}_h(\Gamma)} h_e \left\| \boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}} \right\|_{0,e}^2 \leq C_2 \left\{ \sum_{e\in\mathcal{E}_h(\Gamma)} \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T_e}^2 + \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{1/2,\Gamma}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Sigma) \cup \mathcal{E}_h(\Gamma)$, T_e is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. Having the same structure of the estimate provided by Lemma 4.24, the present bounds follow from slight modifications of the proof of [35, Lemma 20]. □

Similarly as for Lemma 4.25, the following result establishes that, under additional regularity assumptions on φ_Σ and φ_Γ, corresponding local estimates can also be obtained.

Lemma 4.27 *Assume that $\varphi_\Sigma|_e \in H^1(e)$ for each $e \in \mathcal{E}_h(\Sigma)$ and $\varphi_\Gamma|_e \in H^1(e)$ for each $e \in \mathcal{E}_h(\Gamma)$. Then there exist $C_1, C_2 > 0$, independent of h, such that*

$$h_e \left\| \boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}} \right\|_{0,e}^2 \leq C_1 \left\{ \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T_e}^2 + h_e \left\| \frac{d}{d\mathbf{s}}(\varphi_\Sigma - \varphi_{\Sigma,h}) \right\|_{0,e}^2 \right\}$$

and

$$h_e \left\| \boldsymbol{\sigma}_{f,h} \cdot \mathbf{s} - \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}} \right\|_{0,e}^2 \,\leq\, C_2 \left\{ \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T_e}^2 \,+\, h_e \left\| \frac{d}{d\mathbf{s}} \left(\varphi_\Gamma - \varphi_{\Gamma,h}\right) \right\|_{0,e}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Sigma) \cup \mathcal{E}_h(\Gamma)$, T_e is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. These bounds follow from slight modifications of the proof of [35, Lemma 21]. □

The remaining three terms defining $\theta_{e,\Sigma}^2$ and $\theta_{e,\Gamma}^2$ are bounded in what follows.

Lemma 4.28 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e \, \|\varphi_{\Sigma,h} - p_h\|_{0,e}^2 \,\leq\, C \left\{ h_T^2 \, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+\, \|p - p_h\|_{0,T}^2 \,+\, h_e \, \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. Adding and substracting $\varphi_\Sigma = p$ on Σ, and then employing the discrete trace inequality (4.61) (cf. Lemma 4.14), we obtain for each $e \in \mathcal{E}_h(\Sigma)$

$$\begin{aligned} h_e \, \|\varphi_{\Sigma,h} - p_h\|_{0,e}^2 \,&\leq\, 2\, h_e \left\{ \|\varphi_{\Sigma,h} - \varphi_\Sigma\|_{0,e}^2 \,+\, \|p - p_h\|_{0,e}^2 \right\} \\ &\leq\, C \left\{ h_e \, \|\varphi_{\Sigma,h} - \varphi_\Sigma\|_{0,e}^2 \,+\, \|p - p_h\|_{0,T}^2 \,+\, h_T^2 \, |p - p_h|_{1,T}^2 \right\}, \end{aligned} \tag{4.67}$$

where the last term uses that $h_e \leq h_T$. Then, recalling that p_h is piecewise constant (cf. (4.26)), using that $\boldsymbol{\sigma}_f = \nabla p$ in Ω_f, adding and substracting $\boldsymbol{\sigma}_{f,h}$, and employing the upper bound from Lemma 4.18, we find that

$$\begin{aligned} h_T^2 \, |p - p_h|_{1,T}^2 \,=\, h_T^2 \, \|\nabla p\|_{0,T}^2 \,=\, h_T^2 \, \|\boldsymbol{\sigma}_f\|_{0,T}^2 \,&\leq\, 2\, h_T^2 \left\{ \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+\, \|\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \right\} \\ &\leq\, C \left\{ h_T^2 \, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+\, \|p - p_h\|_{0,T}^2 \right\}. \end{aligned} \tag{4.68}$$

Finally, (4.67) and (4.68) yield the required estimate and finish the proof. □

Lemma 4.29 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Gamma)$ there holds*

$$h_e \, \|\varphi_{\Gamma,h} - p_h\|_{0,e}^2 \,\leq\, C \left\{ h_T^2 \, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+\, \|p - p_h\|_{0,T}^2 \,+\, h_e \, \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. It follows exactly as in the proof of Lemma 4.28 replacing Σ by Γ everywhere. □

We complete the efficiency analysis of the a posteriori error estimator $\boldsymbol{\theta}$ with the upper bound for the term concerning the Robin boundary condition on Γ. To this end, and for simplicity, we assume that g is piecewise polynomial on Γ. Otherwise, one would proceed as in the proof of [35, Lemma 23] by adding and substracting a suitable projection of g onto a polynomial space.

Lemma 4.30 *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Gamma)$ there holds*

$$h_e \, \|\boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \imath \, \kappa_f \, \varphi_{\Gamma,h} - g\|_{0,e}^2 \, \leq \, C \left\{ \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 + h_T^2 \, \|\mathrm{div}(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h})\|_{0,T}^2 + h_e \, \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. We proceed analogously to the proofs of Lemmas 4.22 and 4.23. In fact, given $e \in \mathcal{E}_h(\Gamma)$, we let T be the triangle of $\mathcal{T}_h^f$ having e as an edge, define $v_e \, := \, \boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} \, - \, \imath \, \kappa_f \, \varphi_{\Gamma,h} \, - \, g$ on e, and consider the extension operator $L : C(e) \to C(T)$. Then, applying (4.58), recalling that $\psi_e = 0$ on $\partial T \backslash e$, extending $\psi_e \, \overline{L(v_e)}$ by zero in $\Omega_f \backslash T$ so that the resulting function belongs to $H^1(\Omega_f)$, and replacing the datum g by $\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} \, - \, \imath \, \kappa_f \, \varphi_\Gamma$ on Γ, we get

$$\begin{aligned} \|v_e\|_{0,e}^2 \, &\leq \, c_2 \int_e \psi_e \, \overline{v_e} \left(\boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} \, - \, \imath \, \kappa_f \, \varphi_{\Gamma,h} - g \right) \, = \, c_2 \langle \boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} \, - \, \imath \, \kappa_f \, \varphi_{\Gamma,h} - g, \psi_e \, \overline{L(v_e)} \rangle_\Gamma \\ &= \, c_2 \left\{ \, - \, \langle (\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}) \cdot \boldsymbol{\nu}, \psi_e \, \overline{L(v_e)} \rangle_\Gamma \, + \, \imath \, \kappa_f \, \langle \varphi_\Gamma - \varphi_{\Gamma,h}, \psi_e \, \overline{L(v_e)} \rangle_\Gamma \right\}. \end{aligned}$$

The rest of the proof proceeds exactly as in Lemma 4.22, that is integrating by parts in Ω_f, and then employing the Cauchy-Schwarz and inverse inequalities, the estimate (4.59), and the obvious fact that $\varphi_\Gamma - \varphi_{\Gamma,h} \, \in \, L^2(\Gamma)$. We omit further details here and refer to that lemma. $\square$

We end this section by remarking that the efficiency of $\boldsymbol{\theta}$ follows straightforwardly from estimates (4.55) and (4.56), together with Lemmas 4.15 - 4.24, 4.26, 4.28 - 4.30, after summing up over triangles $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^s$ and edges $e \in \mathcal{E}_h$ (cf. (4.21)), and using that the number of triangles on each domain ω_e is bounded by two. In particular, note that the global efficiency estimates induced by the terms of the form $h_e \, \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,e}^2$, $h_e \, \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{0,e}^2$, and $h_e \, \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{0,e}^2$ (cf. Lemmas 4.21, 4.22, 4.23, 4.28, and 4.29), follow easily from the fact that

$$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \, \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,e}^2 \, \leq \, h \, \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,\Sigma}^2 \, \leq \, C \, h \, \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{1/2,\Sigma}^2 \,,$$

$$\sum_{e \in \mathcal{E}_h(\Sigma)} h_e \, \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{0,e}^2 \, \leq \, h \, \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{0,\Sigma}^2 \, \leq \, C \, h \, \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{1/2,\Sigma}^2 \,,$$

and

$$\sum_{e \in \mathcal{E}_h(\Gamma)} h_e \, \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{0,e}^2 \, \leq \, h \, \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{0,\Gamma}^2 \, \leq \, C \, h \, \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{1/2,\Sigma}^2 \,.$$

4.4 Numerical results

In this section we present some numerical results confirming the reliability and efficiency of the a posteriori error estimator $\boldsymbol{\theta}$ analyzed in Section 4.3. We begin by introducing additional notations. The variable N stands for the number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$ (equivalently, the number of unknowns of (4.20)), and the individual and global errors are denoted by:

$$\mathsf{e}(\boldsymbol{\sigma}_s) := \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{\mathbf{div};\Omega_s}, \quad \mathsf{e}(\boldsymbol{\sigma}_f) := \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{\mathrm{div};\Omega_f}, \quad \mathsf{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega_s},$$

$$\mathsf{e}(\boldsymbol{\varphi}_s) := \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{1/2,\Sigma}, \quad \mathsf{e}(\varphi_\Sigma) := \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{1/2,\Sigma}, \quad \mathsf{e}(\varphi_\Gamma) := \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{1/2,\Gamma},$$

$$\mathsf{e}(\widehat{\boldsymbol{\sigma}}) := \Big\{[\mathsf{e}(\boldsymbol{\sigma}_s)]^2 + [\mathsf{e}(\boldsymbol{\sigma}_f)]^2\Big\}^{1/2}, \quad \mathsf{e}(\widehat{\boldsymbol{\gamma}}) := \Big\{[\mathsf{e}(\boldsymbol{\gamma})]^2 + [\mathsf{e}(\boldsymbol{\varphi}_s)]^2 + [\mathsf{e}(\varphi_\Sigma)]^2 + [\mathsf{e}(\varphi_\Gamma)]^2\Big\}^{1/2},$$

$$\mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_s}, \quad \mathsf{e}(p) := \|p - p_h\|_{0,\Omega_f}, \quad \text{and}$$

$$\mathsf{e} := \Big\{[\mathsf{e}(\widehat{\boldsymbol{\sigma}})]^2 + [\mathsf{e}(\widehat{\boldsymbol{\gamma}})]^2 + [\mathsf{e}(\mathbf{u})]^2 + [\mathsf{e}(p)]^2\Big\}^{1/2},$$

where $\boldsymbol{\varphi}_f := (\varphi_\Sigma, \varphi_\Gamma) \in H^{1/2}(\Sigma) \times H^{1/2}(\Gamma)$ and $\boldsymbol{\varphi}_{f,h} := (\varphi_{\Sigma,h}, \varphi_{\Gamma,h}) \in \mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)$. Bear in mind here that $\mathbf{u}_h$ and p_h are the postprocessed variables computed according to (4.25) and (4.26). Also, we define the effectivity index

$$\texttt{eff}(\boldsymbol{\theta}) := \mathsf{e}/\boldsymbol{\theta}.$$

In turn, we let $r(\boldsymbol{\sigma}_s)$, $r(\boldsymbol{\sigma}_f)$, $r(\boldsymbol{\gamma})$, $r(\boldsymbol{\varphi}_s)$, $r(\varphi_\Sigma)$, $r(\varphi_\Gamma)$, $r(\mathbf{u})$, $r(p)$, and r be the experimental rates of convergence given by

$$r(\%) := \frac{\log\big(\mathsf{e}(\%)/\mathsf{e}'(\%)\big)}{\log(h/h')} \quad \forall\, \% \in \{\boldsymbol{\sigma}_s,\, \boldsymbol{\sigma}_f,\, \boldsymbol{\gamma},\, \boldsymbol{\varphi}_s,\, \varphi_\Sigma,\, \varphi_\Gamma,\, \mathbf{u},\, p\}, \quad \text{and} \quad r := \frac{\log\big(\mathsf{e}/\mathsf{e}'\big)}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with corresponding individual errors $\mathsf{e}(\%)$ and $\mathsf{e}'(\%)$, and global errors e and e', respectively. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

In what follows we describe the examples to be considered. We first consider $\Omega_s := (-0.2, 0.2) \times (-0.4, 0.4)$ and let the artificial boundary Γ be the ellipse centered at the origin with minor and major semiaxis given by 0.4 and 0.6, respectively, that is

$$\Omega_f := \Big\{(x_1, x_2)^{\mathtt{t}} \in \mathbb{R}^2 : \quad \frac{x_1^2}{0.4^2} + \frac{x_2^2}{0.6^2} < 1\Big\} \setminus \overline{\Omega}_s.$$

We take $\rho_s = \rho_f = \lambda = \mu = 1$, and the rest of parameters are given by the sets

$$\Big\{v_0 = 1;\, \omega = 5;\, \kappa_s = 5;\, \kappa_f = 5\Big\} \quad \text{and} \quad \Big\{v_0 = 0.7;\, \omega = 7;\, \kappa_s = 7;\, \kappa_f = 10\Big\},$$

which define Examples 1 and 2, respectively. Furthermore, let K_0, K_1 and K_2 be the modified Bessel functions of the second kind and order 0, 1, and 2, respectively, and let $H_0^{(1)}$ be the Hankel function of the first kind and order zero. Then, we choose the data in such a way that the exact solution of (4.4) (or (4.7)) is determined by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \dfrac{1}{2\pi}\,\psi(\mathbf{x}) \;-\; \dfrac{(x_1-1)^2}{r_1^2}\,\chi(\mathbf{x}) \\[2ex] -\,\dfrac{(x_1-1)\,x_2}{r_1^2}\,\chi(\mathbf{x}) \end{pmatrix} \quad \forall\,\mathbf{x} := (x_1,x_2)^{\mathbf{t}} \in \Omega_s\,, \text{ and } \; p(\mathbf{x}) = H_0^{(1)}(\kappa_f\,|\mathbf{x}|) \;\; \forall\,\mathbf{x}\in\Omega_f\,,$$

where $\quad r_1 := \sqrt{(x_1-1)^2 + x_2^2}\,, \quad \psi(\mathbf{x}) := K_0(\imath\,\omega\,r_1) + \frac{1}{\imath\,\omega\,r_1}\Big\{K_1(\imath\,\omega\,r_1) - \frac{1}{\sqrt{3}}\,K_1\Big(\frac{\imath\,\omega\,r_1}{\sqrt{3}}\Big)\Big\}$, and $\quad \chi(\mathbf{x}) := K_2(\imath\,\omega\,r_1) - \frac{1}{3}\,K_2\Big(\frac{\imath\,\omega\,r_1}{\sqrt{3}}\Big)$. Actually, $\mathbf{u}$ is the fundamental solution, centered at $(1,0)^{\mathbf{t}}$, of the elastodynamic equation, which yields $\mathbf{f} = \mathbf{0}$ in Ω_s, and p is the fundamental solution, centered at the origin, of the Helmholtz equation in Ω_f.

Then, for Example 3 we let Ω_s be the L-shaped domain $(-0.3,0.3)^2\setminus(0,0.3)^2$ and consider Γ as the boundary of the unit circle $B(\mathbf{0},1)$. In addition, we take $\rho_s = \rho_f = \lambda = \mu = 1$, $v_0 = 10$, and $\omega = 10$, so that $\kappa_s = 10$ and $\kappa_f = 1$. Then, we choose the data in such a way that the exact solution of (4.4) (or (4.7)) is given by

$$\mathbf{u}(\mathbf{r},\theta) \;:=\; \mathbf{r}^{5/3}\,\sin\big((2\,\theta-\pi)/3\big)\begin{pmatrix} 1+\imath \\[1ex] 1+\imath \end{pmatrix} \qquad \forall\,(\mathbf{r},\theta)\,\in\,\Omega_s\,,$$

in polar coordinates, and

$$p(\mathbf{x}) = H_0^{(1)}(\kappa_f\,|\mathbf{x}+(0.15,0)|) \qquad \forall\,\mathbf{x}\in\Omega_f\,,$$

Note that $\mathbf{u}$ becomes singular at the origin, the corner of the L. More precisely, it is not difficult to see that around this singularity $\mathbf{div}\,\boldsymbol{\sigma}_s$ behaves of order $\mathbf{r}^{-1/3}$. It follows that $\mathbf{div}\,\boldsymbol{\sigma}_s$ belongs to $\mathbf{H}^{2/3-\epsilon}(\Omega_s)$ for each $\epsilon > 0$, and hence, according to Theorem 4.3, we expect experimental rates of convergence, particularly $r(\boldsymbol{\sigma}_s)$, close to $2/3$. According to the preceding remarks, this example is utilized to illustrate the behavior of the adaptive algorithm associated with $\boldsymbol{\theta}$, which applies the following procedure from [72]:

1) Start with coarse meshes $\mathcal{T}_h^s$ and $\mathcal{T}_h^f$.

2) Solve the discrete problem (4.20) for the actual meshes $\mathcal{T}_h^s$ and $\mathcal{T}_h^f$.

3) Compute the error indicators θ_T on each triangle $T\in\mathcal{T}_h^s\cup\mathcal{T}_h^f$ as follows:

$$\theta_T^2 \;:=\; \begin{cases} \theta_{T,s}^2 \,+\, \dfrac{1}{2}\displaystyle\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \theta_{e,\Sigma}^2 & \text{if} \quad T\,\in\,\mathcal{T}_h^s\,, \\[3ex] \theta_{T,f}^2 \,+\, \dfrac{1}{2}\displaystyle\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \theta_{e,\Sigma}^2 \,+\, \displaystyle\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} \theta_{e,\Gamma}^2 & \text{if} \quad T\,\in\,\mathcal{T}_h^f\,. \end{cases}$$

4) Evaluate stopping criterion and decide to finish or go to next step.

5) Use *blue-green* procedure to refine each $T' \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ whose local error indicator $\theta_{T'}$ satisfies

$$\theta_{T'} \geq \frac{1}{2} \max\left\{\theta_T : \quad T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f\right\}.$$

6) Define resulting meshes as actual meshes $\mathcal{T}_h^s$ and $\mathcal{T}_h^f$, and go to step 2.

The numerical results shown below were obtained using a MATLAB code. In Tables 4.1 up to 4.6 we summarize the convergence history of our fully-mixed finite element scheme (4.20) as applied to Examples 1 and 2, for finite sequences of quasi-uniform triangulations of the computational domain $\overline{\Omega}_s \cup \overline{\Omega}_f$. While these examples coincide with the ones presented in [28, Section 5], the novelty now is certainly the computation of the effectivity indexes. We observe in those tables, looking at the corresponding experimental rates of convergence, that the $O(h)$ predicted by Theorem 4.3 when $\delta = 1$ (see [28, Theorem 4.1]) is attained in all the unknowns for both examples. In addition, we notice from the last columns of Tables 4.3 and 4.6 that the effectivity indexes $\texttt{eff}(\boldsymbol{\theta})$ remain always in neighborhoods of 0.74 and 1.75 for Examples 1 and 2, respectively, which illustrates the reliability and efficiency of $\boldsymbol{\theta}$ in the case of regular solutions.

Then, in Tables 4.7 up to 4.12 we provide the convergence history of the quasi-uniform and adaptive refinements, as applied to Example 3. As already announced, we notice in the quasi-uniform case that $r(\boldsymbol{\sigma}_s)$ oscillates in fact around 2/3, whereas the rates of convergence of the other unknowns are not affected by the lack of regularity of $\boldsymbol{\sigma}_s$. However, since $\texttt{e}(\boldsymbol{\sigma}_s)$ is the dominant component of the total error $\texttt{e}$, the above feature is also reflected in the global rate of convergence r (see Table 4.9). Furthermore, it is clear from these tables that the total errors of the adaptive scheme decrease faster than those obtained by the quasi-uniform one, which is confirmed by the global experimental rates of convergence provided in Table 4.12. This fact is also illustrated by Figure 4.1 where we display the total errors $\texttt{e}$ vs. the number of degrees of freedom N for both refinements. Moreover, as shown by these values of r, the adaptive method is able to recover the quasi-optimal rate of convergence $O(h)$ for $\texttt{e}$. On the other hand, the effectivity indexes remain bounded from above and below for both the quasi-uniform and adaptive schemes, which confirms the reliability and efficiency of $\boldsymbol{\theta}$ in the present case of a non-smooth solution. Intermediate meshes obtained with the adaptive refinement are displayed in Figure 4.2. We remark from there that the method is able to recognize the origin as a singularity of the solution of this example. Finally, some components of the approximate (left) and exact (right) solutions for Example 3 are displayed in Figures 4.3 up to 4.8. Note in the case of the unknowns on the boundaries, that they are depicted along straight lines beginning at the points $(0.3, 0)$ and $(0, 1)$, and then continuing clockwise and counterclockwise, for Σ and Γ, respectively. The fact that the approximate and exact solutions do not distinguish from each other in all the components shown illustrates the accurateness of the proposed fully-mixed method and the corresponding adaptive scheme.

h	N	$\mathtt{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathtt{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathtt{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	1117	6.150E−02	–	8.865E−01	–	6.642E−03	–
$2\pi/96$	2090	4.264E−02	0.903	5.996E−01	0.964	3.975E−03	1.266
$2\pi/128$	3686	3.112E−02	1.095	4.414E−01	1.065	2.570E−03	1.516
$2\pi/192$	7869	2.107E−02	0.962	3.044E−01	0.917	1.530E−03	1.279
$2\pi/256$	13666	1.586E−02	0.987	2.249E−01	1.053	1.018E−03	1.415
$2\pi/384$	31282	1.038E−02	1.046	1.489E−01	1.017	6.623E−04	1.061
$2\pi/512$	55438	7.784E−03	1.000	1.106E−01	1.035	4.324E−04	1.482
$2\pi/768$	125069	5.152E−03	1.017	7.397E−02	0.991	2.745E−04	1.121
$2\pi/1024$	221848	3.871E−03	0.994	5.540E−02	1.005	2.034E−04	1.041
$2\pi/1536$	498545	2.579E−03	1.001	3.670E−02	1.016	1.298E−04	1.109
$2\pi/2048$	887629	1.927E−03	1.014	2.770E−02	0.978	9.678E−05	1.019

Table 4.1: Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (Example 1)

N	$\mathtt{e}(\boldsymbol{\varphi}_s)$	$r(\boldsymbol{\varphi}_s)$	$\mathtt{e}(\varphi_\Sigma)$	$r(\varphi_\Sigma)$	$\mathtt{e}(\varphi_\Gamma)$	$r(\varphi_\Gamma)$
1117	9.684E−03	–	1.689E−01	–	4.819E−02	–
2090	4.899E−03	1.681	7.439E−02	2.022	2.030E−02	2.133
3686	2.727E−03	2.037	4.415E−02	1.813	1.226E−02	1.752
7869	1.427E−03	1.598	2.362E−02	1.542	5.610E−03	1.928
13666	8.446E−04	1.822	1.348E−02	1.951	3.850E−03	1.308
31282	4.023E−04	1.829	6.741E−03	1.708	1.834E−03	1.830
55438	2.521E−04	1.625	3.849E−03	1.948	1.187E−03	1.511
125069	1.266E−04	1.699	1.896E−03	1.746	6.280E−04	1.571
221848	8.236E−05	1.494	1.290E−03	1.339	4.437E−04	1.208
498545	4.112E−05	1.713	6.765E−04	1.592	2.231E−04	1.695
887629	2.633E−05	1.550	4.455E−04	1.452	1.533E−04	1.305

Table 4.2: Convergence history for $\boldsymbol{\varphi}_s$, φ_Σ, and φ_Γ (Example 1)

N	$\mathtt{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
1117	2.207E−03	–	3.419E−02	–	9.065E−01	–	0.7495
2090	1.547E−03	0.877	2.317E−02	0.960	6.065E−01	0.991	0.7315
3686	1.131E−03	1.087	1.706E−02	1.064	4.452E−01	1.075	0.7424
7869	7.671E−04	0.958	1.177E−02	0.916	3.063E−01	0.922	0.7328
13666	5.781E−04	0.983	8.700E−03	1.050	2.260E−01	1.057	0.7437
31282	3.781E−04	1.044	5.760E−03	1.017	1.495E−01	1.019	0.7417
55438	2.840E−04	0.999	4.277E−03	1.035	1.110E−01	1.036	0.7377
125069	1.881E−04	1.018	2.863E−03	0.991	7.423E−02	0.992	0.7445
221848	1.413E−04	0.993	2.144E−03	1.005	5.559E−02	1.005	0.7413
498545	9.417E−05	1.001	1.420E−03	1.016	3.682E−02	1.016	0.7366
887629	7.036E−05	1.013	1.072E−03	0.978	2.779E−02	0.978	0.7360

Table 4.3: Convergence history for $\mathbf{u}$, p, $\mathtt{e}$, and effectivity index (Example 1)

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	1117	1.592E−01	–	4.981E−00	–	1.422E−02	–
$2\pi/96$	2090	8.706E−02	1.489	3.252E−00	1.052	6.901E−03	1.783
$2\pi/128$	3686	6.061E−02	1.259	2.371E−00	1.098	4.045E−03	1.857
$2\pi/192$	7869	3.967E−02	1.045	1.626E−00	0.931	2.231E−03	1.468
$2\pi/256$	13666	2.927E−02	1.057	1.199E−00	1.057	1.458E−03	1.480
$2\pi/384$	31282	1.893E−02	1.074	7.931E−01	1.020	9.090E−04	1.164
$2\pi/512$	55438	1.416E−02	1.010	5.886E−01	1.036	5.821E−04	1.549
$2\pi/768$	125069	9.337E−03	1.027	3.937E−01	0.992	3.642E−04	1.157
$2\pi/1024$	221848	7.007E−03	0.998	2.949E−01	1.004	2.673E−04	1.076
$2\pi/1536$	498545	4.664E−03	1.004	1.954E−01	1.016	1.685E−04	1.138
$2\pi/2048$	887629	3.486E−03	1.012	1.474E−01	0.979	1.248E−04	1.043

Table 4.4: Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (Example 2)

N	$\mathsf{e}(\boldsymbol{\varphi}_s)$	$r(\boldsymbol{\varphi}_s)$	$\mathsf{e}(\varphi_\Sigma)$	$r(\varphi_\Sigma)$	$\mathsf{e}(\varphi_\Gamma)$	$r(\varphi_\Gamma)$
1117	2.843E−02	–	4.104E−01	–	1.388E−01	–
2090	1.217E−02	2.092	1.898E−01	1.901	5.923E−02	2.100
3686	6.413E−03	2.228	1.085E−01	1.945	3.262E−02	2.073
7869	3.053E−03	1.831	5.654E−02	1.607	1.517E−02	1.888
13666	1.722E−03	1.990	3.159E−02	2.023	9.679E−03	1.562
31282	8.131E−04	1.851	1.560E−02	1.740	4.454E−03	1.914
55438	5.253E−04	1.518	8.850E−03	1.970	2.832E−03	1.575
125069	2.394E−04	1.938	4.339E−03	1.758	1.512E−03	1.547
221848	1.605E−04	1.391	2.868E−03	1.440	9.982E−04	1.444
498545	8.008E−05	1.714	1.438E−03	1.703	5.174E−04	1.621
887629	5.005E−05	1.633	9.361E−04	1.492	3.530E−04	1.329

Table 4.5: Convergence history for $\boldsymbol{\varphi}_s$, φ_Σ, and φ_Γ (Example 2)

N	$\mathtt{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
1117	3.080E−03	–	4.950E−02	–	5.003E−00	–	1.8347
2090	1.686E−03	1.486	3.232E−02	1.051	3.259E−00	1.057	1.7396
3686	1.178E−03	1.247	2.356E−02	1.098	2.374E−00	1.101	1.7641
7869	7.713E−04	1.044	1.616E−02	0.930	1.627E−00	0.932	1.7431
13666	5.694E−04	1.055	1.192E−02	1.056	1.200E−00	1.058	1.7676
31282	3.686E−04	1.072	7.885E−03	1.020	7.935E−01	1.021	1.7623
55438	2.758E−04	1.009	5.852E−03	1.036	5.889E−01	1.037	1.7586
125069	1.819E−04	1.027	3.915E−03	0.992	3.939E−01	0.992	1.7698
221848	1.365E−04	0.997	2.932E−03	1.004	2.951E−01	1.005	1.7655
498545	9.086E−05	1.004	1.943E−03	1.016	1.955E−01	1.016	1.7601
887629	6.790E−05	1.012	1.466E−03	0.978	1.475E−01	0.979	1.7611

Table 4.6: Convergence history for $\mathbf{u}$, p, $\mathtt{e}$, and effectivity index (EXAMPLE 2)

h	N	$\mathtt{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathtt{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathtt{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	2215	9.127E−01	–	4.267E−01	–	3.210E−02	–
$2\pi/96$	4767	6.802E−01	0.725	1.896E−01	2.000	1.371E−02	2.098
$2\pi/128$	8495	5.408E−01	0.797	1.185E−01	1.634	9.156E−03	1.403
$2\pi/192$	19067	4.465E−01	0.472	6.492E−02	1.484	4.033E−03	2.022
$2\pi/256$	33331	3.898E−01	0.472	4.851E−02	1.013	2.828E−03	1.234
$2\pi/384$	75077	2.800E−01	0.816	3.053E−02	1.142	1.630E−03	1.359
$2\pi/512$	133497	2.351E−01	0.607	2.317E−02	0.960	1.049E−03	1.532
$2\pi/768$	299000	1.883E−01	0.547	1.528E−02	1.026	6.357E−04	1.235
$2\pi/1024$	534105	1.493E−01	0.807	1.139E−02	1.023	4.391E−04	1.286
$2\pi/1536$	1199275	1.109E−01	0.735	7.601E−03	0.997	2.663E−04	1.233

Table 4.7: Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (quasi-uniform scheme, EXAMPLE 3)

N	$\mathtt{e}(\boldsymbol{\varphi}_s)$	$r(\boldsymbol{\varphi}_s)$	$\mathtt{e}(\varphi_\Sigma)$	$r(\varphi_\Sigma)$	$\mathtt{e}(\varphi_\Gamma)$	$r(\varphi_\Gamma)$
2215	6.895E−02	–	5.538E−01	–	5.233E−02	–
4767	2.300E−02	2.708	2.027E−01	2.479	1.786E−02	2.652
8495	1.417E−02	1.683	1.066E−01	2.232	8.300E−03	2.663
19067	4.631E−03	2.759	3.555E−02	2.710	2.920E−03	2.576
33331	3.500E−03	0.974	2.082E−02	1.859	1.396E−03	2.565
75077	1.520E−03	2.056	1.028E−02	1.741	6.814E−04	1.769
133497	1.019E−03	1.390	6.675E−03	1.501	3.776E−04	2.052
299000	4.515E−04	2.008	3.018E−03	1.958	2.102E−04	1.444
534105	3.266E−04	1.126	1.975E−03	1.473	1.564E−04	1.029
1199275	1.523E−04	1.882	9.444E−04	1.820	6.877E−05	2.026

Table 4.8: Convergence history for $\boldsymbol{\varphi}_s$, φ_Σ, and φ_Γ (quasi-uniform scheme, EXAMPLE 3)

N	$\mathtt{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
2215	9.444E−03	–	5.476E−02	–	1.155E−00	–	0.6179
4767	5.899E−03	1.161	2.980E−02	1.501	7.360E−01	1.111	0.6313
8495	4.430E−03	0.996	2.024E−02	1.345	5.645E−01	0.922	0.6546
19067	2.942E−03	1.010	1.292E−02	1.107	4.529E−01	0.543	0.7241
33331	2.189E−03	1.028	9.722E−03	0.988	3.935E−01	0.488	0.7679
75077	1.459E−03	1.000	6.359E−03	1.047	2.819E−01	0.823	0.7943
133497	1.091E−03	1.009	4.801E−03	0.977	2.364E−01	0.612	0.8232
299000	7.360E−04	0.971	3.191E−03	1.008	1.890E−01	0.552	0.8679
534105	5.567E−04	0.971	2.388E−03	1.008	1.498E−01	0.809	0.8806
1199275	3.685E−04	1.018	1.594E−03	0.996	1.111E−01	0.736	0.9004

Table 4.9: Convergence history for $\mathbf{u}$, p, $\mathtt{e}$, and effectivity index (quasi-uniform scheme, EXAMPLE 3)

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
0.1169	2215	9.127E−01	–	4.267E−01	–	3.210E−02	–
0.1169	2503	7.145E−01	4.006	2.996E−01	5.786	2.589E−02	3.520
0.1169	3471	5.377E−01	1.739	2.607E−01	0.851	2.394E−02	0.478
0.1169	4459	4.417E−01	1.570	1.713E−01	3.354	1.472E−02	3.883
0.1169	6355	3.477E−01	1.351	1.401E−01	1.134	1.299E−02	0.707
0.1169	9410	2.753E−01	1.189	1.088E−01	1.287	9.272E−03	1.717
0.1169	11985	2.411E−01	1.097	9.418E−02	1.196	8.363E−03	0.853
0.1169	19655	1.882E−01	1.002	7.556E−02	0.890	5.892E−03	1.416
0.0934	38391	1.406E−01	0.870	5.126E−02	1.159	4.545E−03	0.775
0.0832	65934	1.058E−01	1.051	4.117E−02	0.810	3.321E−03	1.161
0.0832	98472	9.131E−02	0.736	3.519E−02	0.783	3.022E−03	0.470
0.0622	125924	8.021E−02	1.055	3.056E−02	1.146	2.723E−03	0.847
0.0511	151119	7.225E−02	1.146	2.681E−02	1.436	2.257E−03	2.060
0.0493	196274	6.617E−02	0.673	2.456E−02	0.670	2.161E−03	0.331
0.0471	241916	6.067E−02	0.830	2.287E−02	0.684	2.065E−03	0.436
0.0467	282385	5.684E−02	0.843	2.144E−02	0.830	1.904E−03	1.051
0.0400	343470	4.852E−02	1.617	1.836E−02	1.586	1.581E−03	1.900
0.0298	570415	3.694E−02	1.075	1.382E−02	1.120	1.177E−03	1.162
0.0244	894088	3.037E−02	0.872	1.139E−02	0.861	9.605E−04	0.905
0.0240	1269053	2.654E−02	0.769	9.882E−03	0.811	8.686E−04	0.574
0.0234	1635325	2.360E−02	0.926	8.777E−03	0.935	7.831E−04	0.817

Table 4.10: Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (adaptive scheme, EXAMPLE 3)

N	$\mathtt{e}(\boldsymbol{\varphi}_s)$	$r(\boldsymbol{\varphi}_s)$	$\mathtt{e}(\varphi_\Sigma)$	$r(\varphi_\Sigma)$	$\mathtt{e}(\varphi_\Gamma)$	$r(\varphi_\Gamma)$
2215	6.895E−02	–	5.538E−01	–	5.233E−02	–
2503	5.104E−02	4.921	3.576E−01	7.157	4.086E−02	4.037
3471	3.138E−02	2.975	2.942E−01	1.195	3.051E−02	1.787
4459	1.530E−02	5.738	1.346E−01	6.243	2.099E−02	2.986
6355	1.124E−02	1.741	8.971E−02	2.290	1.954E−02	0.405
9410	5.915E−03	3.270	4.522E−02	3.491	7.613E−03	4.803
11985	4.596E−03	2.085	3.356E−02	2.465	7.385E−03	0.251
19655	3.352E−03	1.277	2.590E−02	1.048	7.867E−03	-0.255
38391	1.735E−03	1.967	1.118E−02	2.510	3.919E−03	2.082
65934	1.229E−03	1.276	8.728E−03	0.915	3.104E−03	0.863
98472	9.169E−04	1.459	6.057E−03	1.821	2.989E−03	0.188
125924	7.763E−04	1.355	4.871E−03	1.772	2.240E−03	2.344
151119	5.946E−04	2.923	3.680E−03	3.074	1.914E−03	1.726
196274	5.925E−04	0.028	3.390E−03	0.628	1.738E−03	0.738
241916	5.497E−04	0.717	3.330E−03	0.171	1.583E−03	0.896
282385	4.916E−04	1.443	3.101E−03	0.921	1.455E−03	1.088
343470	4.137E−04	1.763	2.400E−03	2.617	1.007E−03	3.763
570415	2.366E−04	2.204	1.307E−03	2.395	6.893E−04	1.493
894088	1.835E−04	1.130	9.845E−04	1.262	4.778E−04	1.630
1269053	1.606E−04	0.763	9.044E−04	0.485	4.672E−04	0.129
1635325	1.343E−04	1.411	7.551E−04	1.423	3.758E−04	1.716

Table 4.11: Convergence history for $\boldsymbol{\varphi}_s$, φ_Σ, and φ_Γ (adaptive scheme, EXAMPLE 3)

N	$\mathtt{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
2215	9.444E−03	–	5.476E−02	–	1.155E−00	–	0.6179
2503	8.923E−03	0.928	4.779E−02	2.229	8.576E−01	4.868	0.5530
3471	6.348E−03	2.083	4.289E−02	0.661	6.693E−01	1.516	0.5277
4459	5.179E−03	1.625	3.797E−02	0.974	4.949E−01	2.411	0.4727
6355	4.091E−03	1.332	3.583E−02	0.328	3.880E−01	1.374	0.4537
9410	3.008E−03	1.566	3.101E−02	0.735	3.014E−01	1.287	0.4249
11985	2.772E−03	0.678	2.814E−02	0.803	2.628E−01	1.133	0.4205
19655	2.196E−03	0.942	2.250E−02	0.904	2.059E−01	0.986	0.4089
38391	1.549E−03	1.042	1.499E−02	1.214	1.510E−01	0.927	0.4300
65934	1.215E−03	0.899	1.223E−02	0.752	1.146E−01	1.018	0.3973
98472	1.013E−03	0.908	1.045E−02	0.786	9.870E−02	0.747	0.4051
125924	9.152E−04	0.822	9.149E−03	1.077	8.653E−02	1.070	0.4050
151119	8.144E−04	1.280	7.918E−03	1.585	7.762E−02	1.192	0.4108
196274	7.452E−04	0.679	7.221E−03	0.704	7.109E−02	0.672	0.4082
241916	6.858E−04	0.795	6.727E−03	0.678	6.532E−02	0.809	0.3933
282385	6.388E−04	0.917	6.308E−03	0.832	6.121E−02	0.842	0.4030
343470	5.594E−04	1.356	5.398E−03	1.591	5.225E−02	1.616	0.4038
570415	4.196E−04	1.134	4.004E−03	1.178	3.969E−02	1.084	0.4075
894088	3.470E−04	0.846	3.315E−03	0.840	3.264E−02	0.871	0.4025
1269053	3.032E−04	0.770	2.886E−03	0.792	2.850E−02	0.773	0.3792
1635325	2.680E−04	0.972	2.565E−03	0.931	2.534E−02	0.928	0.4013

Table 4.12: Convergence history for $\mathbf{u}$, p, $\mathtt{e}$, and effectivity index (adaptive scheme, Example 3)

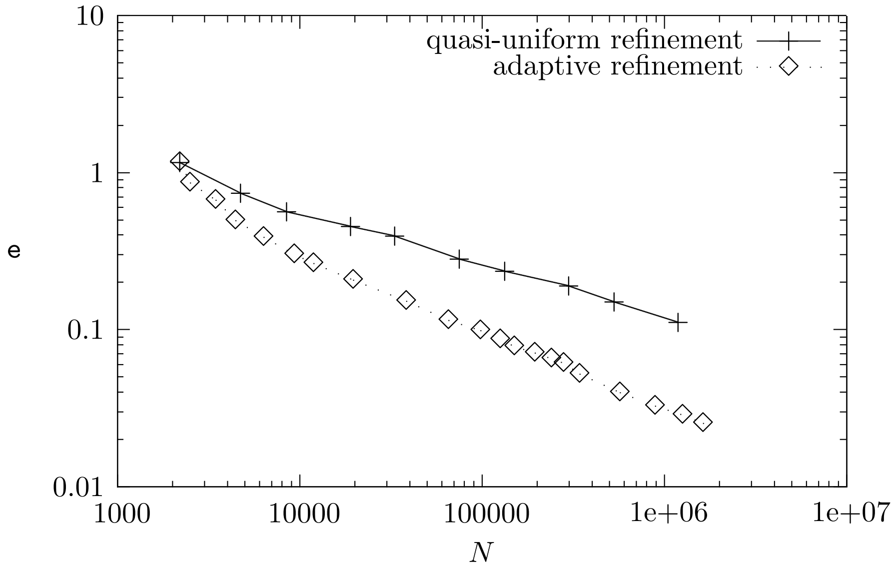

Figure 4.1: EXAMPLE 3, total error e vs. N for the quasi-uniform and adaptive schemes

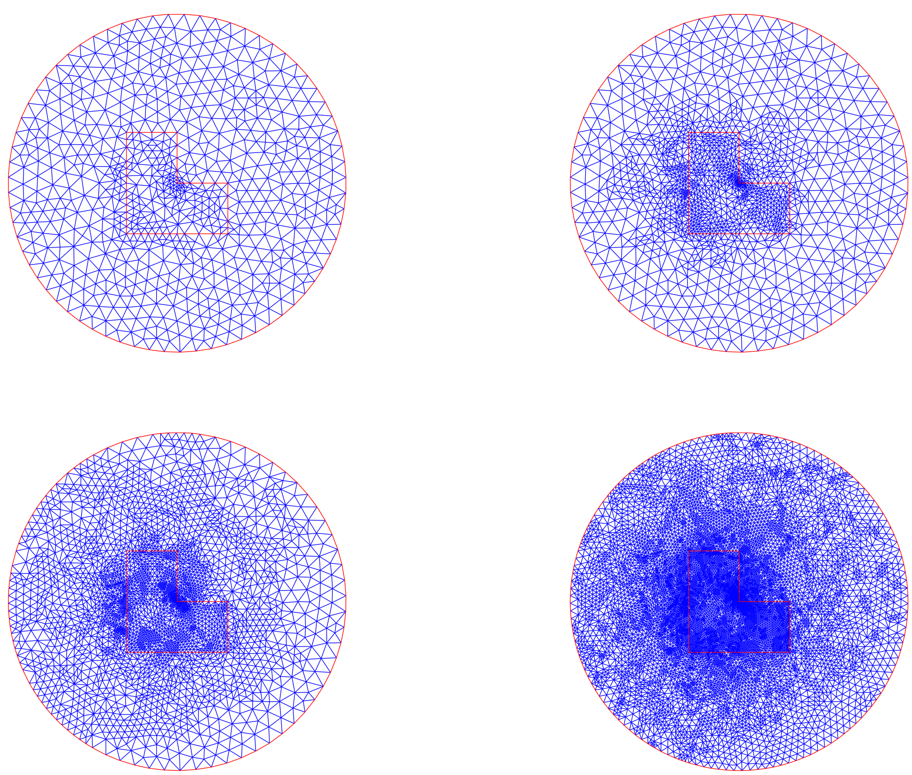

Figure 4.2: EXAMPLE 3: adapted meshes for $N \in \{3471, 9410, 19655, 65934\}$

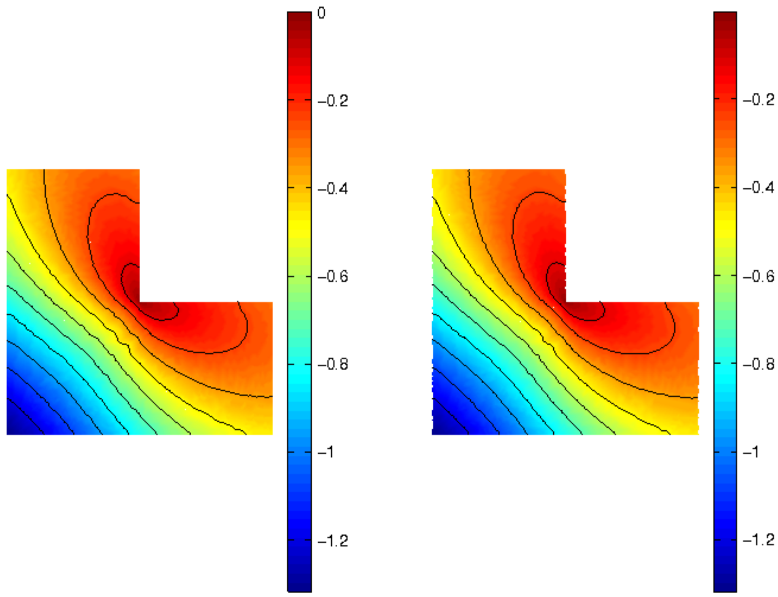

Figure 4.3: Approximate and exact real part of $\boldsymbol{\sigma}_{s,21}$ (Example 3)

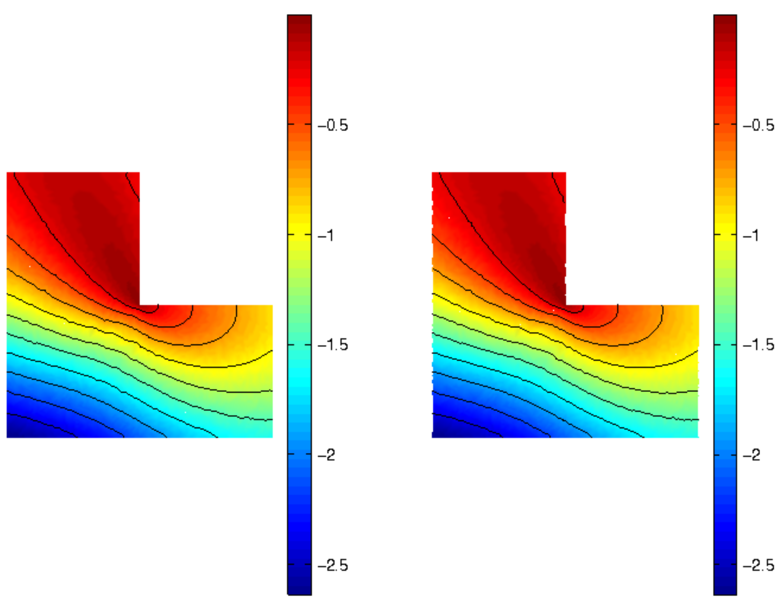

Figure 4.4: Approximate and exact imaginary part of $\boldsymbol{\sigma}_{s,22}$ (Example 3)

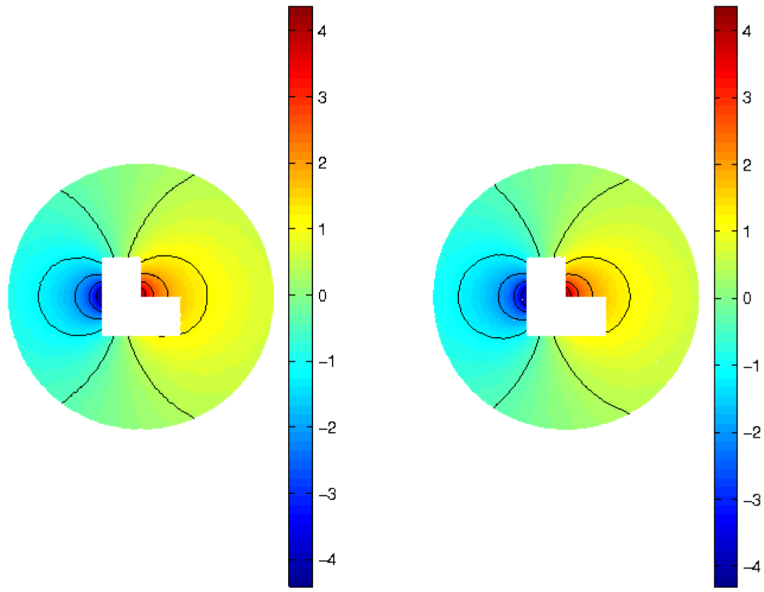

Figure 4.5: Approximate and exact imaginary part of $\boldsymbol{\sigma}_{f,1}$ (Example 3)

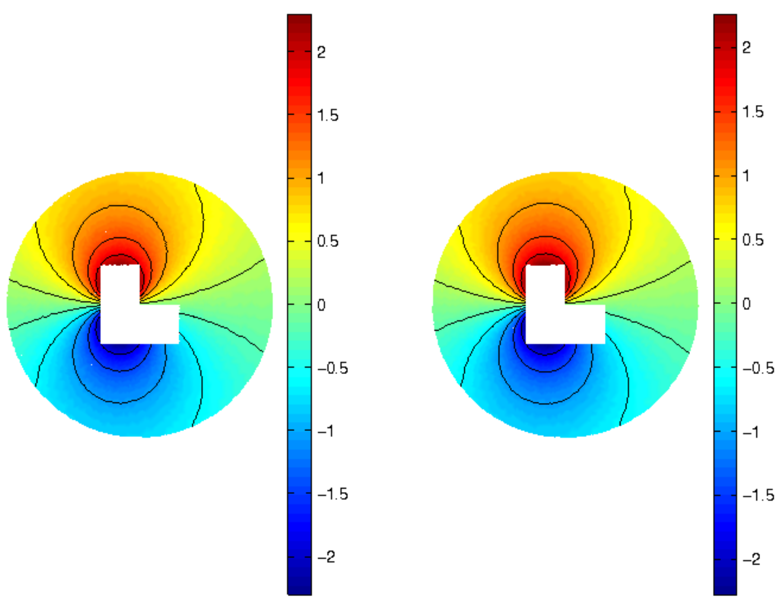

Figure 4.6: Approximate and exact imaginary part of $\boldsymbol{\sigma}_{f,2}$ (Example 3)

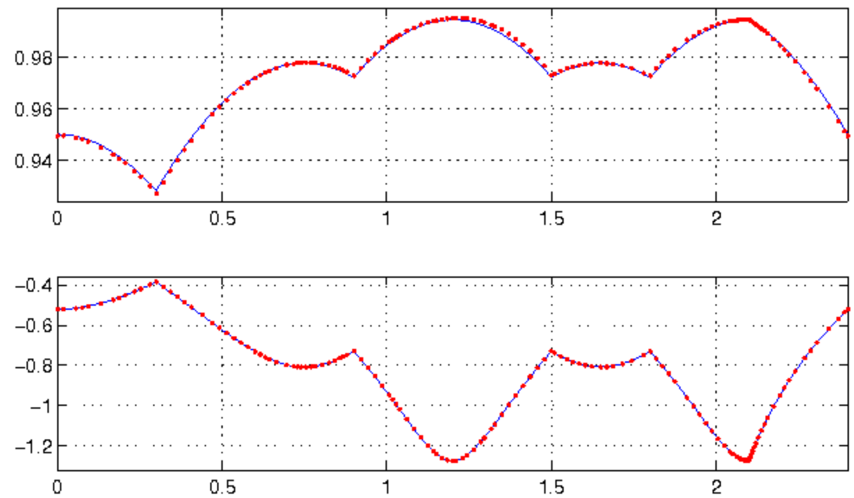

Figure 4.7: Approximate (red) and exact (blue) real and imaginary parts of φ_Σ (Example 3)

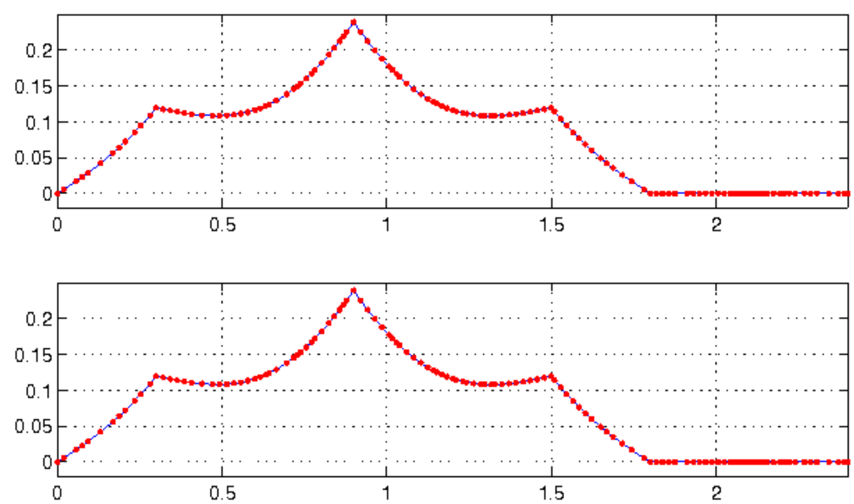

Figure 4.8: Approximate (red) and exact (blue) real and imaginary parts of $\boldsymbol{\varphi}_{s,2}$ (Example 3)

Chapter 5

Conclusiones y trabajo futuro

5.1 Conclusiones

El objetivo principal de esta tesis ha sido el desarrollo de un análisis de error a priori y a posteriori de un método de elementos finitos completamente mixto para un problema de interacción sólido-fluido bidimensional. Las conclusiones principales de esta tesis, en orden de desarrollo, son:

- Se realizó una extensión de los resultados obtenidos en [37]. En efecto, en [37] se consideró una formulación variacional mixta en el sólido y primal en el fluido, mientras que en el Capítulo 2 de nuestro trabajo se planteó una formulación variacional mixta en ambos dominios. Ello se logró introduciendo una nueva incógnita dada por el gradiente de presiones en el fluido, la cual se aproximó directamente en el esquema de Galerkin asociado.

- En el análisis del error a priori del problema acoplado se obtuvo, tanto en el esquema continuo como discreto, una estructura de punto silla por bloques en la diagonal y la presencia de una perturbación compacta. Lo anterior permitió aplicar la teoría de Babuška-Brezzi en cada uno de los bloques, y de esta manera se logró probar que el esquema de Galerkin asociado es estable y bien propuesto. Para ello, se aplicaron las técnicas de levantamientos estables para probar la condición inf-sup discreta en cada dominio. Finalmente, debido a la perturbación compacta, se aplicó un resultado clasico de métodos de proyección de operadores de Fredhoml con índice cero.

- En el problema acoplado, la formulación dual mixta en el sólido y en el fluido simplificó el código computacional correspondiente al permitir la utilización de los subespacios de elementos finitos de Raviart-Thomas en ambos dominios.

- Se obtuvo un estimador de error a posteriori residual, confiable y eficiente, para el problema de elasticidad lineal con condiciones de frontera de tracción pura. Los subespacios de elementos finitos utilizados fueron Raviart-Thomas + rotacional de las funciones burbuja en el tensor de esfuerzos, las funciones constantes a trozos en el desplazamiento, y las funciones lineales a trozos y continuas para la rotación y el multiplicador de Lagrange sobre la frontera. Finalmente, varios resultados numéricos confirmaron la confiabilidad y eficiencia del estimador, e ilustraron el buen comportamiento del esquema adaptivo asociado.

- Se obtuvo un estimador de error a posteriori residual, confiable y eficiente para el problema de interación sólido-fluido descrito por las ecuaciones de Lamé-Helmholtz. Los elementos finitos considerados fueron PEERS en el sólido, Raviart-Thomas de bajo orden en el fluido, y las funciones lineales a trozos y continuas sobre la interfase y la frontera.

5.2 Trabajo futuro

Se realizará un análisis de error a posteriori del acoplamiento entre los elementos finitos de Arnold-Falk-Winther y Lagrange para un problema de interacción sólido-fluido tridimensional. El modelo se rige por las ecuaciones de la acústica y la elastodinámica en régimen de tiempo armónico y las condiciones de transmisión están dadas por el equilibrio de fuerzas y la igualdad de los desplazamientos normales correspondientes. Se empleará una formulación variacional dual mixta en el sólido y primal en el fluido, tal cual como se planteó en [41], introduciendo la primera condición de transmisión como parte de la definición del espacio al cual pertenecen los esfuerzos en el sólido y la presión en el fluido. La principal dificultad que se vislumbra es la incorporación de dicha condición de transmisión en la descomposición de Helmholtz discreta del espacio producto correspondiente.

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