Counting perfect matchings of cubic graphs in the geometric dual

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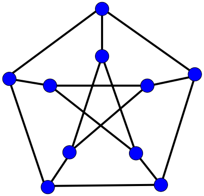
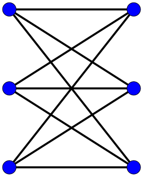
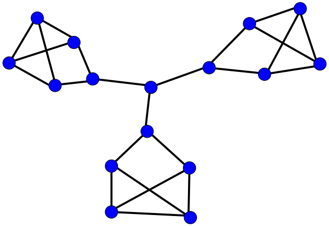
Basic Definitions







Cubic graphs

Motivation

Petersen (1891)

Every cubic bridgeless graph has a perfect matching.

Conjecture by Lovász and Plummer from the mid-1970's

For every cubic bridgeless graph G, the number of perfect matchings is exponential in $|V(G)|$.

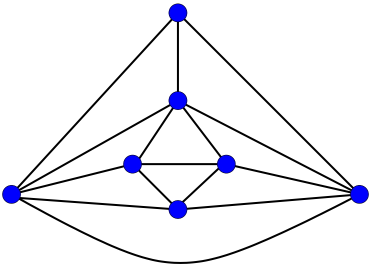
Positive resolution of the conjecture announced by Esperet, Kardos, King, Kral and Norine (Dec. 2010).

Known results for special classes

- Voorhoeve (1979): Bipartites
- Chudnovsky and Seymour (2008): Planar graphs
- Sang-il Oum (2009): Claw-free

Preliminaries

Dual graph: $G \leftrightarrow G^*$



Some simple observations

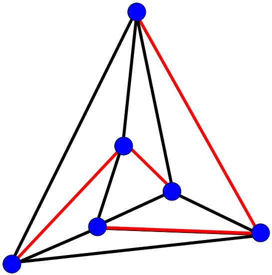
Let G be a cubic bridgeless planar graph.

Proposition

G^* is a planar triangulation.

Intersecting sets

Planar triangulation: Δ



Definition: [Intersecting set of Δ]

Set of edges of Δ with exactly one edge from each of its faces.

Intersecting sets (cont.)

Let G be a cubic bridgeless planar graph.

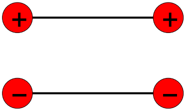
Proposition

M is a perfect matching of $G \iff M^*$ is an intersecting set of G^*.

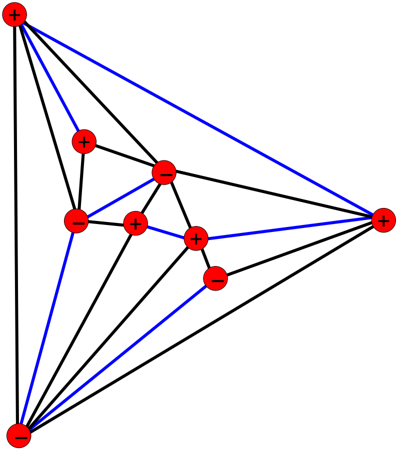
Ising Model on frustrated triangulations

Let $\Delta = (V, E)$ be a planar triangulation.

- A state of Δ is any function $\mathrm{s} : V \to \{+1, -1\}$.
- Edges frustrated by s



- A state s is a groundstate if it frustrates the minimum possible number of edges of Δ.
- The degeneracy of Δ is its number of groundstates, denoted $g(\Delta)$.



Groundstate and frustrated edges.

A new concept

Definition: [Satisfying states]

A spin assignment that frustrates exactly one edge of each face of a triangulation Δ.

Every satisfying state is a groundstate!

Converse is true if the triangulation Δ is planar. Not true in general (more on this later!).

Reformulation of Lovász and Plummer's conjecture

Let G be a cubic bridgeless planar graph and Δ_G its dual graph.

Theorem

The number of perfect matchings of G is $\frac{1}{2}g(\Delta_G)$.

Main results

Let $\varphi = (1+\sqrt{5})/2 \approx 1.6180$ be the golden ratio.

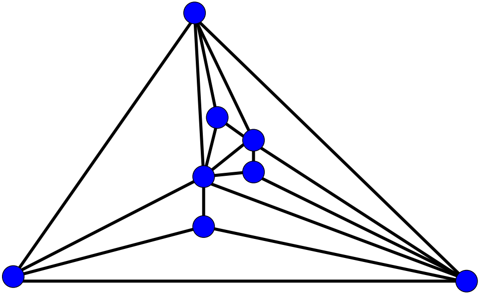
Theorem

The degeneracy of a stack triangulation Δ with $|\Delta|$ vertices is at least $6\varphi^{(|\Delta|+3)/36}$.

Corollary

The number of perfect matchings of a cubic graph G, whose dual graph is a stack triangulation is at least $3\varphi^{|V(G)|/72}$.

Stack triangulations or 3-trees



Δ

Degeneracy of stack triangulations

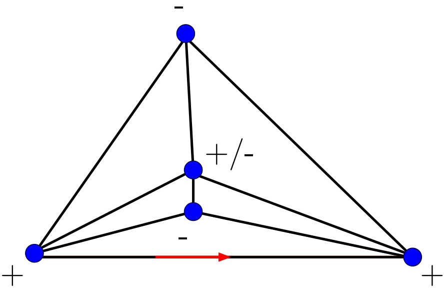
Goal

Given a stack triangulation Δ, find a *degeneracy vector* $\mathbf{v}_{\vec{\Delta}} \in \mathrm{R}^4$ such that $||\mathbf{v}_{\vec{\Delta}}||_1 = \frac{1}{2}g(\Delta)$.

Description of $\mathbf{v}_{\vec{\Delta}}$:

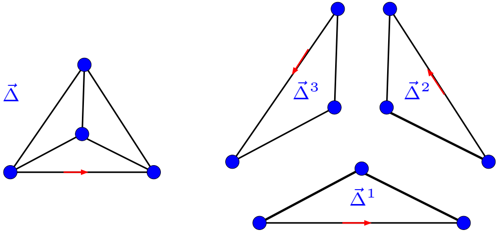
- Coordinates indexed by $I = \{+++, ++-, +-+, +--\}$.
- For $\phi \in I$, $\mathbf{v}_{\vec{\Delta}}[\phi]$ is the number of satisfying states of Δ when the spin assignment of its outer-face is ϕ.

Example: $|\Delta| = 5$



$$\mathbf{v}_{\vec{\Delta}} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \quad \begin{matrix} +++ \\ ++- \\ +-+ \\ -++ \end{matrix}$$

Recursive construction of $\mathbf{v}_{\vec{\Delta}}$ for stack triangulations

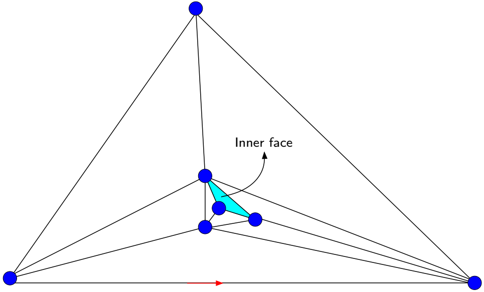


Proposition

For $j \in \{1,2,3\}$, let $\mathbf{v}_{\vec{\Delta}^j} = (v_j^k)_{k\in\{0,1,2,3\}}$. Then,

$$\mathbf{v}_{\vec{\Delta}} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix} v_1^0 v_2^0 v_3^0 + v_1^1 v_2^1 v_3^1 \\ v_1^0 v_2^2 v_3^3 + v_1^1 v_2^3 v_3^2 \\ v_1^2 v_2^3 v_3^0 + v_1^3 v_2^2 v_3^1 \\ v_1^2 v_2^1 v_3^3 + v_1^3 v_2^0 v_3^2 \end{pmatrix}.$$

Particular case: Strip stacks



But $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ is linear in each $\mathbf{v}_j$, and two of the $\mathbf{v}_j$'s are $(0, 1, 1, 1)^t$.

Degeneracy vector of strips

Lemma

If $\vec{\Delta}_0$ is a strip triangulation with inner face $\vec{\Delta}_\ell$, then for some $M_1, \ldots, M_\ell \in \{A, B, C\}$

$$\mathbf{v}_{\vec{\Delta}_\ell} = M_\ell \cdot M_{\ell-1} \cdots M_1 \cdot \mathbf{v}_{\vec{\Delta}_0} ,$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Degeneracy of strip stacks

Theorem

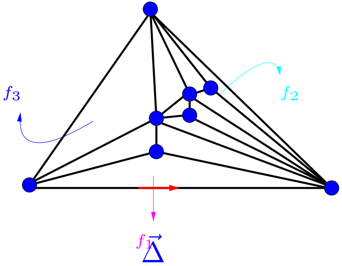
If $\vec{\Delta}$ is a strip triangulation of length ℓ with inner face $\vec{\Delta}'$, then

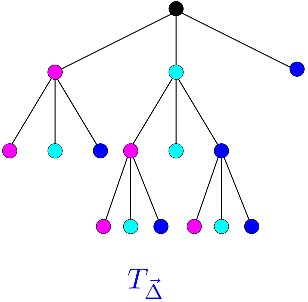
$$\mathbf{v}_{\vec{\Delta}'} \geq (\varphi^{e'_j})_{j=0,1,2,3} \Longrightarrow \mathbf{v}_{\vec{\Delta}} \geq (\varphi^{e_j})_{j=0,1,2,3} \, ,$$

where

$$\sum_{j=0}^{4} e_j \;\geq\; \frac{1}{2}(\ell - 3) + \sum_{j=0}^{4} e'_j \, .$$

General Case

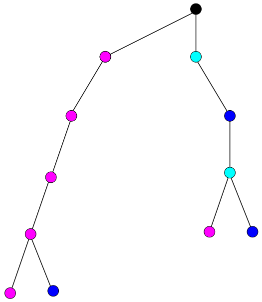




Root of $T_{\vec{\Delta}}$ $\leftrightarrow$ Outer-face of $\vec{\Delta}$

Leaves of $T_{\vec{\Delta}}$ $\leftrightarrow$ Inner faces of $\vec{\Delta}$

Proof at a glance



1. Build $T_{\vec{\Delta}}$.
2. Prune leafs of $T_{\vec{\Delta}}$.
3. Prune and obtain $\tilde{T}_{\vec{\Delta}}$ s.t. $|\tilde{T}_{\vec{\Delta}}| \geq \frac{1}{3}|\Delta| - 1$.
4. Note that $\mathbf{v}_{\Delta_u} \geq (1,1,1,1)^t$ for every leaf of u of $\tilde{T}_{\vec{\Delta}}$.
5. Lower bound $\mathbf{v}_{\Delta}$ working bottom up
   - Show that progress is made at vertices v of $\tilde{T}_{\vec{\Delta}}$ with more than one children.
   - Show that progress is made if the subtree of $\tilde{T}_{\vec{\Delta}}$ rooted at v is a path $P_{v,w}$ of length at least 5 plus a tree $\tilde{T}_w$ rooted at a node w with at least two descendants.
   - Observe that either (a) or (b) must happen before too long.

Natural question

Can the proof argument be extended to the general (non-planar) case?

Seems so! By Polyhedral Embedding Conjecture: *Every cubic bridgeless graph may be embedded to an orientable surface so that every two faces that intersect do so in a single edge.*

But! Every triangulation has groundstates (by definition), but not necesarily has satisfying states (although, planar triangulations do).

Hearsay

Physicists expect that geometrically frustrated systems (like surface triangulations) are such that if they admit satisfying states, then they have an exponential number of them.

What aboud the complexity of deciding existence and enumerating satisfying states?

- It is NP-complete to decide, given a surface triangulation, whether or not it admits a satisfying state.
- It is $\#P$-complete (under parsimonious reductions) to enumerate, given a surface triangulation, the number of satisfying states it admits.
- (Maybe already known) It is $\#P$-complete to enumerate, given a surface triangulation, the number of its groundstates. Same holds for the number of Max-Cuts.

Reduction sketch

Reduction from `Positive-Not-All-Equal-3SAT`. Follows the usual gadget type construction. But, gadgets are rather atypical.

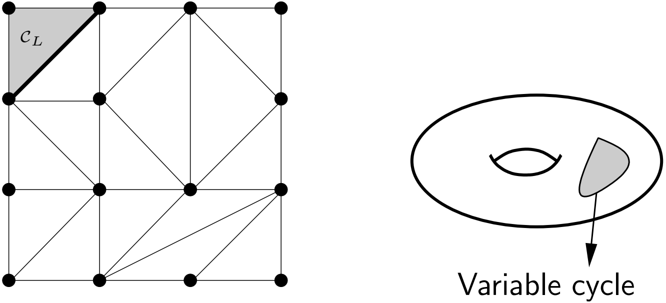


Figure: Choice gadget.

Characteristics: 8 nodes and genus 1.

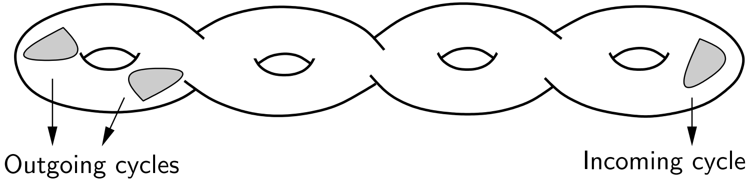


Figure: Block replicator gadget sketch

Characteristics: 25 nodes and genus 4.

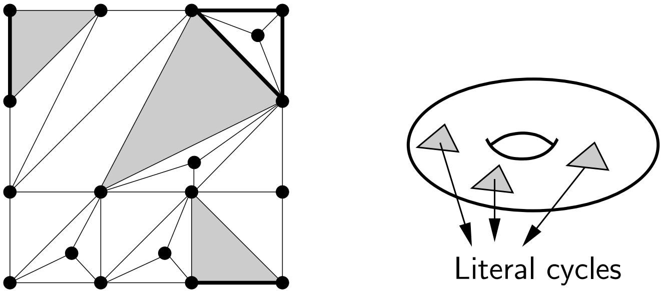


Figure: Choice gadget.

Characteristics: 11 nodes and genus 1.

Wrapp up

Given an instance φ of `Positive-Not-All-Equal-3SAT` with n variables and m clauses, the reduction computes a rotation system for a triangulation Δ_φ of a surface of genus

$$m + 2(n+1) + 4\sum_{i=1}^{n} 2^{k_i - 1},$$

where $k_i = 2\max\{1, \lceil 0.5 \log_2 t_i \rceil\}$ and t_i denotes the number of clauses in which the i-th variable of φ appears. Moreover, the number of satisfying states of Δ_φ is 4 times the number of satisfying assignments of φ.

Is there an infinite family of triangulations that admit satisfying states, but no more than a given constant?

YES! Contrary to physicist's intuition.

THE END!