Synchronization and Limit Behaviors in Cellular Automata

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Overview of the talk

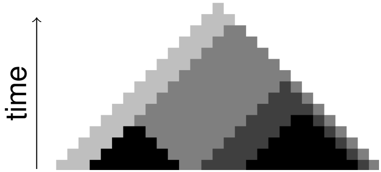
1. Cellular Automata & Limit Behaviors
2. Possible Limit
3. Typical Limit

Overview of the talk

1. **Cellular Automata & Limit Behaviors**
2. Possible Limit
3. Typical Limit

One-dimensional CA

- Q set of **states**
- r **radius** of neighborhood
- $f : Q^{2r+1} \to Q$ **local** transition function
- $Q^{\mathbb{Z}}$ set of **configurations**
- $F : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ **global** transition function



- $Q = \{0, 1, 2, 3, 4\}$
- $r = 1$
- $f(x, y, z) = \mathtt{max}(x, y, z)$

Limit behaviors

Possible

u limit word
$\Leftrightarrow$
$F^{-t}(u)$ **never empty**

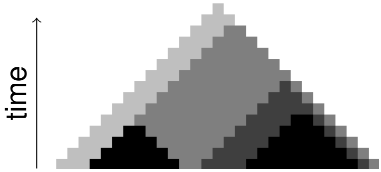
Ω
configurations made exclusively of limit words

Typical

u μ-limit word
$\Leftrightarrow$
$F^{-t}(u)$ **don't get negligible**

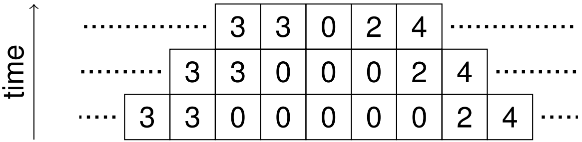
Ω_μ
configurations made exclusively of μ-limit words

Limit set (Ω)



- $Q = \{0, 1, 2, 3, 4\}$
- $r = 1$
- $f(x, y, z) = \mathtt{max}(x, y, z)$

- $\Omega =$ "*decreasing then increasing*" configurations



μ-limit set (Ω_μ)

- $[u]$: configurations where word u occurs in the center
- μ a translation invariant measure (**in this talk: Bernouilli**)

Definition

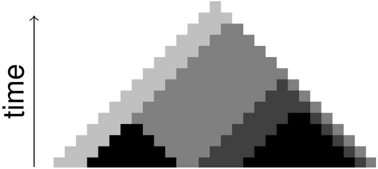
- u is a μ-limit word if

$$\lim_{t\to\infty} \mu\left(F^{-t}([u])\right) \not\to 0$$

- Ω_μ is the set of configurations made only of μ-limit words

Limit Sets of Cellular Automata Associated to Probability Measures
P. Kůrka, A. Maass, 2000

μ-limit set (Ω_μ)



- $Q = \{0, 1, 2, 3, 4\}$
- $r = 1$
- $f(x, y, z) = \mathtt{max}(x, y, z)$

- $\Omega_\mu = \{{}^\omega 4^\omega\}$

1. $u \in (Q \setminus \{4\})^* \Rightarrow$ pre-images of u in $(Q \setminus \{4\})^*$
2. $\mu\big((Q \setminus \{4\})^n\big) \to 0$ when $n \to \infty$

Ω_μ and density

- density of word u in configuration c

$$d_c(u) = \limsup_{n\to\infty} \frac{\left|c_{-n,n}\right|_u}{2n+1}$$

- configuration c is μ-generic if $d_c(u) = \mu([u])$ for all u

Property

The following are equivalent:

1. u is a μ-limit word for F
2. for any μ-generic configuration c

$$d_{F^t(c)}(u) \not\to 0$$

when $t \to \infty$

Overview of the talk

1. Cellular Automata & Limit Behaviors

2. **Possible Limit**

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Synchronization task #1

Find some F such that...

for all t there is an initial configuration c_t with

1. all cells are in state 0 at time t
2. no 0 appears before time t

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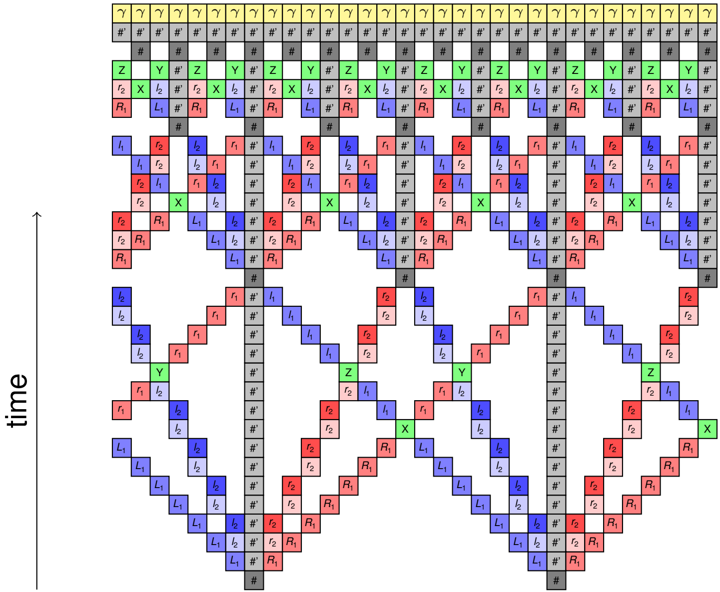
Well known solutions: firing squad CA

J. Kari's firing squad



Applications to Ω

Firing Squad Elevator
Raises any configuration to the limit set

- fix some F over states Q
- by adding a firing squad component to F, we can
  1. make any word in Q^* a limit word
  2. without changing the dynamics of F over $Q^{\mathbb{Z}}$

Formally: G over $(Q' \times Q) \cup Q$ such that

1. the whole set $Q^{\mathbb{Z}}$ is in $\Omega(G)$
2. G restricted to $Q^{\mathbb{Z}}$ is exactly F

Applications to Ω

Theorem (J. Kari, 1994)

Any non-trivial property of limit sets is undecidable

Theorem (P. Guillon, P.E. Meunier, GT, 2010)

There is an intrinsically universal CA with a simple limit set

(simple = logspace computable)

Rice Theorem for Ω

Firing Squad Elevator
+ Switch


Definition

F **nilpotent** if $\Omega(F)$ is a singleton

Construction: $F, H \to G$

Is H nilpotent?

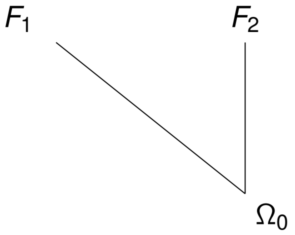
- **YES**: $\Omega(G) = \Omega(F)$
- **NO**: $\Omega(G) = \Omega_0$ independent of F

Rice Theorem for Ω

J. Kari, 1992

Nilpotency is an undecidable property

- fix some property $\mathcal{P}$ of limit sets
- choose F_1 and F_2 with
  - $\Omega(F_1) \in \mathcal{P}$
  - $\Omega(F_2) \notin \mathcal{P}$
- aplly construction twice with the same H



H not nilpotent

Rice Theorem for Ω

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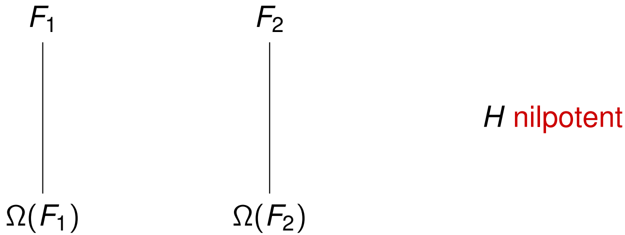

Overview of the talk

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2. Possible Limit
3. **Typical Limit**

Synchronization task #2

- fix some $n \geq 2$

Find some F such that...

for **almost all** initial configuration c

- any cell, after some time, is in state $t \bmod n$ at time t

Synchronization task #2

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Find some F such that...

for **almost all** initial configuration c

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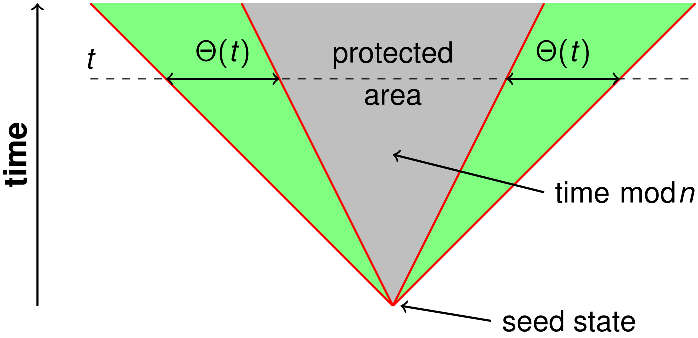
A solution exists!

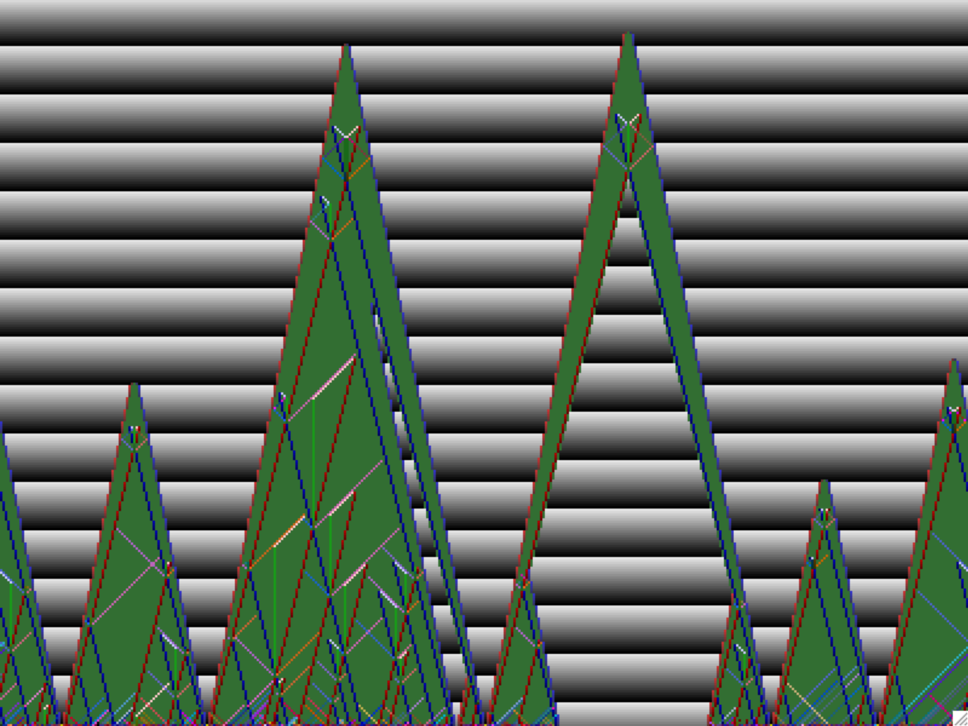
Directional Dynamics along Arbitrary Curves in Cellular Automata
M. Delacourt, V. Poupet, M. Sablik, GT, 2010

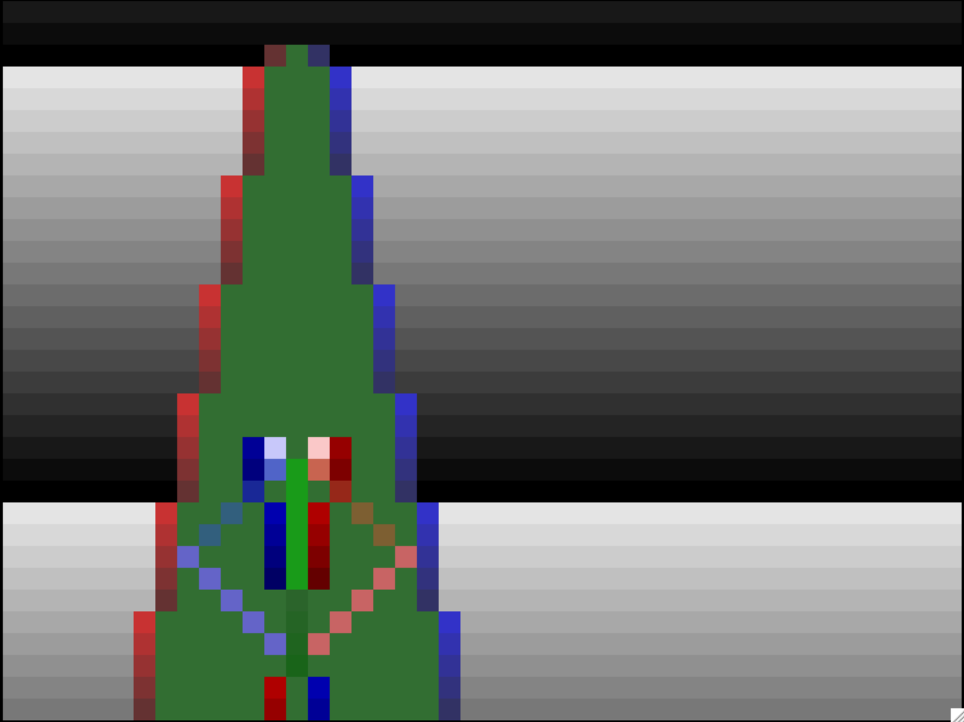
Time Counters Construction

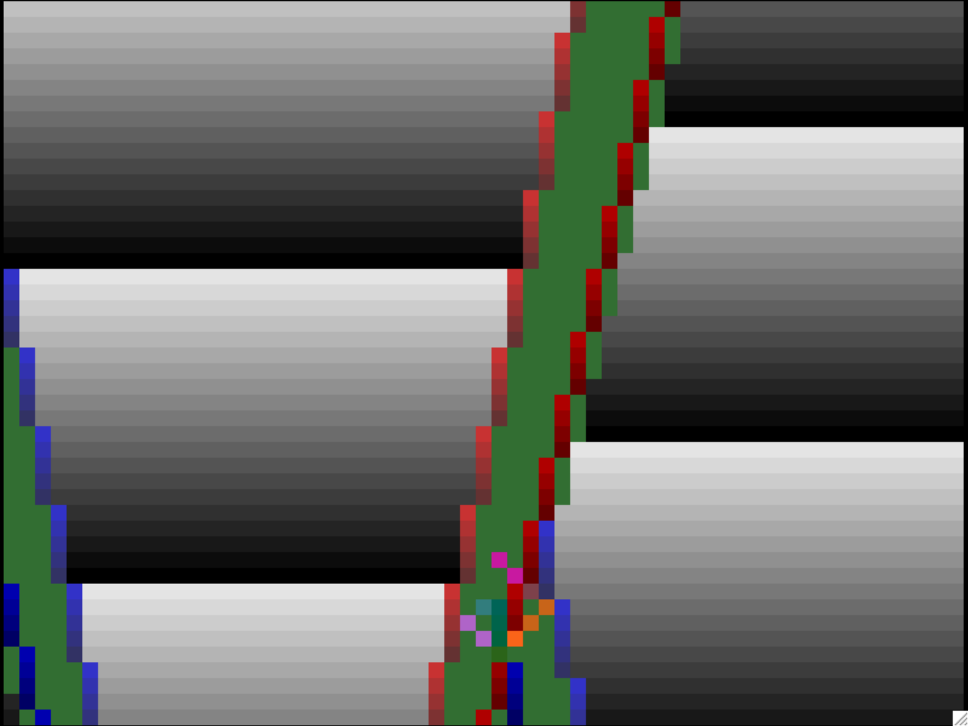
Outline

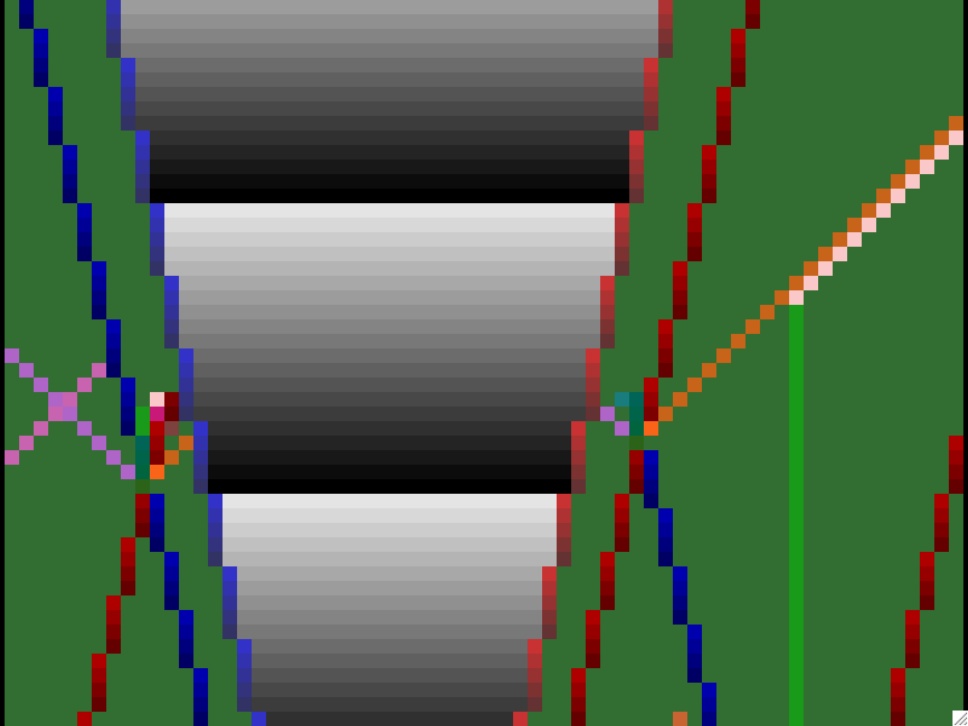


1. only a valid zone can stop a valid zone
2. when two valid zones meet, the older is destroyed
3. two valid zones of equal age merge when they meet









Time Counters Construction

Implementation details

Construction for n=20:

- 2733 states
- radius 4

Question

Is there a significantly smaller solution?

- Kari’s firing squad: 16 states, radius 1
- Mazoyer’s firing squad: 6 states, radius 1

Time Counters Construction

Implementation details

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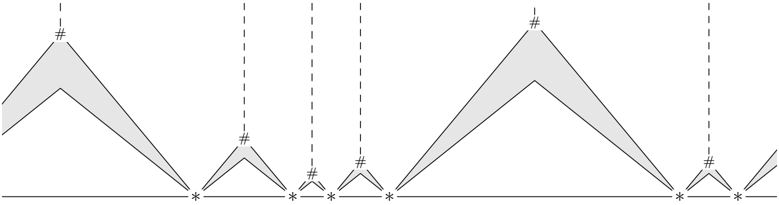
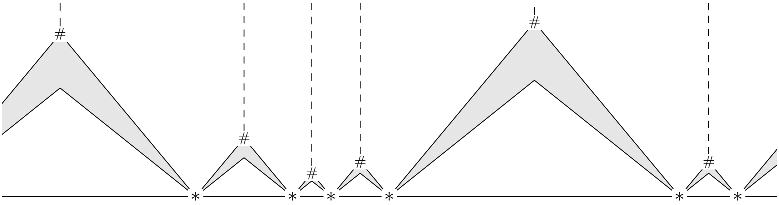
Other property

CA with equicontinuous points but none in the image set $F(Q^{\mathbb{Z}})$

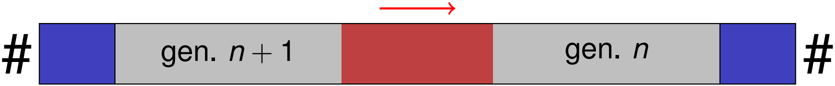
Applications to Ω_μ



Applications to Ω_μ



Computation segment:



= **computation area** (Turing head + working space)

= **merging process info** (time, length, random bits,...)

= write once **output**

Applications to Ω_μ

IF

- segment size $\to \infty$
- non-output part $<<$ segment size

THEN

Characterization of Ω_μ

μ-limit word **are exactly** words which are dense in the computation output (asymptotically)

Applications to Ω_μ

Construction of μ-limit sets
L. Boyer, M. Delacourt, M. Sablik (2010)

Constructions with an ergodic point of view
M. Sablik
(Information & Randomness 2010, ALEA 2011)

Rice Theorem for μ-Limit Sets
of Cellular Automata
M. Delacourt (2011)

Rice Theorem for Ω_μ

Definition

F **μ-nilpotent** if $\Omega_\mu(F)$ is a singleton

- a state is **persistent** if it cannot desappear from a cell

L. Boyer, V. Poupet, GT, 2006

- μ-nilpotency is undecidable for CA with a persistent state
- μ-limit words are enumerable for such CA

Construction: $F, H \rightarrow G$

Is H μ-nilpotent?

- **YES**: $\Omega_\mu(G) = \Omega_\mu(F)$
- **NO**: $\Omega_\mu(G) = \{^{\mathbb{Z}}q^{\mathbb{Z}}\}$ independent of F

Work in Progress / Future Work

- complex μ-limit sets
- higher complexity lower bounds for properties of limit sets
- convergence behaviors (e.g. limit vs. ceasaro mean)
- higher dimensions