The Dynamics Group of Asynchronous Systems

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Dedicated to Eric Goles for his 60th birthday!

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VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY

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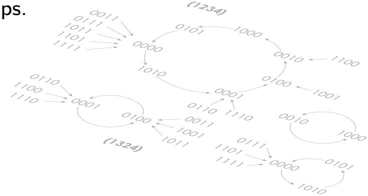
Talk Overview – Dynamics Groups

▶ Using tools from group theory to assess long-term dynamics of asynchronous discrete dynamical systems.

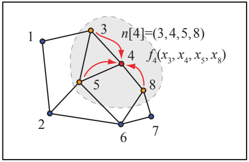
- The notion of update sequence independence.
- The dynamics group of an update sequence independent system.
- Relations to Coxeter theory and Coxeter groups.
- Outlook and open questions.



Sequential Dynamical Systems (SDS)

- A subclass of graph dynamical systems (GDS). Constructed from:
  - A (dependency) graph X with vertex set $\mathrm{v}[X] = \{1, 2, \ldots, n\}$.
  - For each vertex v a state $x_v \in K$ (e.g. $K = \mathbb{F}_2 = \{0, 1\}$) and an *$X$-local function* $F_v \colon K^n \longrightarrow K^n$

$$F_v(x_1, x_2, \ldots, x_n) = (x_1, \ldots, \underbrace{f_v(x[v])}_{\text{vertex function } v}, \ldots, x_n) \ .$$



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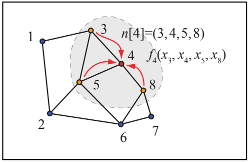


  - A word $w = w_1 w_2 \cdots w_k$ over the vertex set of X.

- The SDS map $\mathbf{F}_w \colon K^n \longrightarrow K^n$ is:

$$\boxed{\mathbf{F}_w = F_{w(k)} \circ F_{\pi(k-1)} \circ \cdots \circ F_{w(1)}}$$

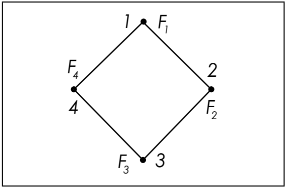
SDS – An example

- System components:
  - Circle graph on 4 vertices: $X = \mathrm{Circle}_4$
  - Update sequence: $\pi = (1, 2, 3, 4)$
  - Vertex functions:

$$\mathrm{nor}_3(x_1, x_2, x_3) = (1 + x_1)(1 + x_2)(1 + x_3)$$

  - The X-local map for vertex 1:

$$F_1(x_1, x_2, x_3, x_4) = (\mathrm{nor}_3(x_1, x_2, x_4), x_2, x_3, x_4)$$



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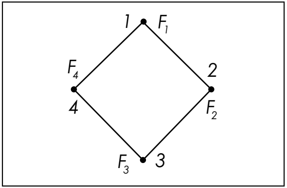

Dependency graph

- System update:

$$(x_1, x_2, x_3, x_4) = (0, 0, 0, 0) \overset{F_1}{\mapsto} (1, 0, 0, 0) \text{ and}$$

$$(1, 0, 0, 0) \overset{F_2}{\mapsto} (1, 0, 0, 0) \overset{F_3}{\mapsto} (1, 0, 1, 0) \overset{F_4}{\mapsto} (1, 0, 1, 0)$$

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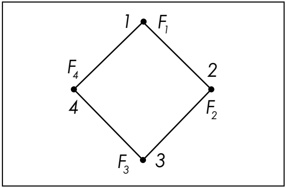
- SDS map:

$$\boxed{\mathbf{F}_\pi(0, 0, 0, 0) = (1, 0, 1, 0)}$$



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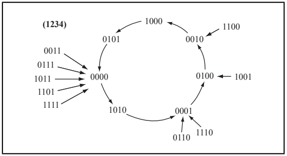

Phase space

Definition (Update sequence independence)

A sequence $\mathbf{F} = (F_i)_{i=1}^n$ of X-local maps over a finite state space K^n are word (resp. permutation) update sequence independent, if there exists $P \subset K^n$ such that for all fair words $w \in W'_X$ (resp. $w \in S_X$) we have

$$\mathrm{Per}(\mathbf{F}_w) = P \ .$$

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- **Questions:**
  - Are there word independent SDSs, and is this a common property?
  - Why should we care about this in the first place?

Properties of π-independent SDS

Proposition

Let X be a graph and $\mathbf{F} = (F_i)_i$ a π-independent sequence of X-local functions with periodic points P. Then each restricted function

$$F_i|_P \colon P \longrightarrow P$$

is a well-defined bijection.

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- Let π be a permutation with $\pi(1) = i$, let $P = \mathrm{Per}(\mathbf{F}_\pi)$ and $P' = \mathrm{Per}(\mathbf{F}_{\sigma_1(\pi)})$ [cyclic 1-shift].

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- From $F_{\pi(1)} \circ \mathbf{F}_\pi = \mathbf{F}_{\sigma_1(\pi)} \circ F_{\pi(1)}$ it follows that $F_i(P) \subset P'$.

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- From $F_{\pi(1)} \circ \mathbf{F}_\pi = \mathbf{F}_{\sigma_1(\pi)} \circ F_{\pi(1)}$ it follows that $F_i(P) \subset P'$.
- Repeated application of this n times yields $|P| = |P'|$.
- Upshot: $F_i(P) = P'$ and by π-independence we have $P = P'$.

Dynamics Group – A first look

- For π-independent SDS each $F_i|_P$ is a permutation P.
- We set

$$F_i^* := F_i|_P$$

- If $|P| = m$ and we label the periodic points $1, 2, \ldots, m$, then each $F_i^* \leftrightarrow n_i \in S_m$.

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Let K be a finite set and $\mathbf{F} = (F_i)_i$ be a π-independent sequence of X-local functions. The *dynamics group* of $\mathbf{F}$ is

$$G(\mathbf{F}) = \langle F_1^*, \ldots, F_n^* \rangle \ .$$

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- Clearly, $G(\mathbf{F})$ is isomorphic to a subgroup of S_m.
- Sometimes more convenient to consider the group generated by the permutations n_i – it is denoted by $\widetilde{G}(\mathbf{F})$.

An Example of *w*-Independence

Proposition

SDS induced by Nor*-functions are* w*-independent for any graph* X.

An Example of w-Independence

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SDS induced by Nor*-functions are* w*-independent for any graph* X*.*

Proof idea.

Establish a 1-1 correspondence between $\mathrm{Per}(\mathbf{Nor}_w)$ and the set of independent sets of X. Of course, the latter quantity does not depend on w.

Example (brute force)

Take $X = K_3$ (complete graph on 3 vertices) and $\mathbf{F} = \mathbf{Nor} = (\mathrm{Nor}_i)_i$.

Periodic point	Label	Nor_1	Nor_2	Nor_3
$(0,0,0)$	0	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
$(1,0,0)$	1	$(0,0,0)$	$(1,0,0)$	$(1,0,0)$
$(0,1,0)$	2	$(0,1,0)$	$(0,0,0)$	$(0,1,0)$
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- Clearly, $\widetilde{G}(\mathbf{Nor}) < S_4$. From $n_3 n_2 n_1 = (0,1,2,3)$, and the fact that $S_4 = \langle\{(0,1),(0,1,2,3)\}\rangle$ it follows that $S_4 < G(\mathbf{Nor})$.
- Hence $G(\mathbf{Nor}) \cong S_4$.

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▶ Hence $G(\mathbf{Nor}) \cong S_4$.

▶ **Significance:** We can organize the periodic points in any cycle configuration we like by a suitable choice of update sequence w.

How common is π-independence?

Theorem (Theorem [1])

For SDS over $X = \mathrm{Circle}_n$, precisely 104 of the 256 elementary cellular automaton rules induce sequences $(F_i)_i$ that are π-independent for any $n \geq 3$. Of these, 86 are w-independent for any $n \geq 3$.

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- Thus, roughly 40% of ECA SDS over Circle_n are π-independent.
- Currently, it is unclear how this generalizes to other graph classes.
- The following classes have been analyzed more generally:
  - Invertible SDS are (of course) w-independent.
  - Nor-SDS, Nand-SDS, $(\mathrm{Nor} + \mathrm{Nand})$-SDS, threshold SDS, and trivial SDS are all w-independent.
  - SDS with monotone functions are not necessarily w-independent (example, ECA rule 240).

π-independence does not imply w-independence

- Example due to Kevin Ahrendt and Collin Bleak.
- Take $X = \mathrm{Circle}_n$ and let **F** be induced by ECA 32 which has function table

(x_{i-1}, x_i, x_{i+1})	111	110	101	100	011	010	001	000
f	0	0	1	0	0	0	0	0

- Claim: $\mathrm{Per}(\mathbf{F}_\pi^{32}) = \{0\}$ for any permutation $\pi \in S_X$.
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- The state $x = 0$ is the only fixed point (use local fixed point graph).
- Non-isolated 0-blocks will persist and grow by each application of $\mathbf{F}^{32}$ since

$$\ldots \underbrace{100}_{\to 0}\, 00\, \overbrace{001}^{\to 0} \ldots\ ,$$

and

$$\ldots 10000\, \overbrace{011}^{\to 0} \ldots \quad \text{and} \quad \ldots 10000\, \overbrace{010}^{\to 0} \ldots$$

π-independence does not imply w-independence - cont.

- A state $x \in \mathbb{F}_2^n$ where each 0-block is isolated will eventually map to a state containing a non-isolated zero-block. Consider the the configuration $\ldots 101 \ldots$ around vertex i.

Case 1: if $i - 1 <_\pi i$ or $i + 1 <_\pi i$ then a non-isolated 0-block is created immediately.

Case 2: if $i <_\pi i - 1$ then a 0-block of length ≥ 2 appears after two iterations. Here it is crucial that π is a permutation and not a fair word.

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▶ However, $\mathbf{F}^{32}$ is not w-independent: take $x = (1, 1, \ldots, 1)$ and update sequence $w = (1, 1, 2, 2, \ldots, n, n)$.

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▶ **Observation:**

- If $(F_i)_i$ is π-independent with periodic points P then there may be states in $K^n \setminus P$ that are "locally periodic": F_i applied to x two or more times in succession gives x.
- Can still form the dynamics group, but in this case it only gives information about the points in P (the "permutation periodic" points).

Relations to Coxeter Theory

▶ A (finitely generated) Coxeter group with generating set $S = \{s_1, \ldots, s_n\}$ and symmetric Coxeter matrix $M = [m_{ij}]_{ij}$ where $m_{ij} \in \mathbb{N} \cup \{\infty\}$ and $m_{ij} = 1$ iff $i = j$ is the group with presentation

$$W(S) = \langle s_1, \ldots, s_n | (s_i s_j)^{m_{ij}} \rangle \ .$$

▶ Every group G generated by a finite set of involutions can therefore be viewed as a quotient of a Coxeter group. One defines m_{ij} to be the order of the product of the corresponding generators.

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Theorem

If $\mathbf{F} = (F_i)_i$ is π-independent and $K = \{0, 1\}$, then each F_i^* is either trivial or an involution. As a result, $G(\mathbf{F})$ is either trivial or a quotient of a Coxeter group.

Orders of $F_i^* \circ F_j^*$ for $X = \mathrm{Circle}_n$

- Let $X = \mathrm{Circle}_n$ and consider induced sequences $(F_i)_i$. What are the possible values for m_{ij}, the order of $F_i^* \circ F_j^*$?

- Clearly, i and j must differ by 1 for this to be interesting. Since $F_{i+1}^* \circ F_i^*$ may only change the states x_i and x_{i+1}, and since there are only four sub-configuration for these, we see that any $x \in P$ under $F_{i+1}^* \circ F_i^*$ must have period $1 \le p \le 4$.

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Example ($m_{i,i+1}$ in the case of the parity function)

	$i-1$	i	$i+1$	$i+2$
0	x_{i-1}	x_i	x_{i+1}	x_{i+2}
1	x_{i-1}	$x_{i-1}+x_i+x_{i+1}$	$x_{i-1}+x_i+x_{i+2}$	x_{i+2}
2	x_{i-1}	$x_{i-1}+x_{i+1}+x_{i+2}$	$x_i+x_{i+1}+x_{i+2}$	x_{i+2}
3	x_{i-1}	x_i	x_{i+1}	x_{i+2}

and conclude that $m_{i,i+1} = 3$. (Actually, we computed the order of $F_{i+1} \circ F_i$. Why is that okay?)

Theorem

Let $(F_i)_i$ be π-independent with periodic points P. Then: (i) $G(\mathbf{F}) = 1$ if and only if all $x \in P$ are fixed points. (ii) If $G(\mathbf{F})$ acts transitively on P and p is a prime dividing $|P|$, then there exists a word $w \in W$ such that (a) $|\mathrm{Fix}(\mathbf{F}_w)|$ is divisible by p, and (b) all periodic orbits of length ≥ 2 of $\mathbf{F}_w$ have length p.

Theorem

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Proof.

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- Let $x \in P$. For a finite group acting on a set X we always have $|Gx| = [G : G_x] = |G|/|G_x|$ where $G_x = \{\phi \in G | \phi(x) = x\}$. Since the action is assumed to be transitive, we conclude that $Gx = P$ and derive

$$|G| = |P||G_x| \ ,$$

and thus that p divides $|G|$. By Cauchy's Theorem, it follows that G has a subgroup of order p, and this subgroup is cyclic with generator $\phi = \prod_i F^*_{w(i)}$, say. Let $n \in \widetilde{G}$ be the corresponding permutation representation of ϕ. It is clear that n is a product of cycles of length either 1 or p, and also that at least one cycle of length p must exists.

Proposition ([3])

The group $G(\textbf{Nor})$ *acts transitively on* $\mathrm{Per}(\textbf{Nor})$.

Example ($X = \mathrm{Circle}_4$ and $(\mathrm{Nor}_i)_i$)

- Periodic points $0 \leftrightarrow (0,0,0,0)$, $1 \leftrightarrow (1,0,0,0)$, $2 \leftrightarrow (0,1,0,0)$, $3 \leftrightarrow (0,0,1,0)$, $4 \leftrightarrow (1,0,1,0)$, $5 \leftrightarrow (0,0,0,1)$ and $6 \leftrightarrow (0,1,0,1)$.
- Permutation representations n_i of Nor_i for $0 \leq i \leq 3$ (cycle form): $n_0 = (0,1)(3,4)$, $n_1 = (0,2)(5,6)$, $n_2 = (0,3)(1,4)$ and $n_3 = (0,5)(2,6)$.
- A_7 has a presentation $\langle x, y \,|\, x^3 = y^5 = (xy)^7 = (xy^{-1}xy)^2 = (xy^{-2}xy^2) = 1\rangle$, and $a = (0,1,2)$ and $b = (2,3,4,5,6)$ are two elements of S_7 that will generate A_7.
- Now, $a' = n_2(n_0n_3n_1)^2 = (0,4,1,6,3)$ and $b' = (n_3n_2)^2(n_2n_1)^2 = (2,5,3)$, and after relabeling of the periodic points using the permutation $(0,3,2)(1,5)$ we transform a' into a and b' into b.
- Since every generator n_i is even we conclude that $G(\mathbf{Nor}) \cong A_7$.

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- Since every generator n_i is even we conclude that $G(\mathbf{Nor}) \cong A_7$.

- Is there a sequence w such that the map $\mathbf{Nor}_w$ above has (a) two 3-cycles and a fixed point, (b) five fixed points and a 2-cycle, (c) a 3-cycle, a 2-cycle and two fixed points?

Example (Function $\mathbf{F}^{232}$)

This function has table

(x_{i-1}, x_i, x_{i+1})	111	110	101	100	011	010	001	000
f	1	1	1	0	1	0	0	0

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▶ Isolated zeroes are removed but never introduced, and non-isolated 0-blocks may never shrink.

▶ The function assigning to x the number of non-isolated zeros minus the number of isolated zeroes is a non-decreasing potential function.

▶ All periodic points are fixed points for any $w \in W'_X$ and thus the dynamics group is trivial.

▶ The same argument allows us to conclude that functions 160, 164, 168 and 172 are w-independent as well.

Example ($G(\mathbf{F}^{51})$)

Since $\mathbf{F}^{51}$ is invertible we have $P = \mathbb{F}_2^n$. The function table is

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- Every generator is an involution and $m_{i,i+1} = 2$.
- It follows directly that $G(\mathbf{F}^{51})$ is a quotient of $\mathbb{Z}_2^n$. Since every composition of distinct sets of generators toggles a different subset of vertex states, it follows that $G(\mathbf{F}^{51})$ contains at least 2^n elements, and we conclude that this dynamics group is isomorphic to $\mathbb{Z}_2^n$.

Example ($G(\mathbf{F}^{60})$)

ECA rule 60 has table

(x_{i-1}, x_i, x_{i+1})	111	110	101	100	011	010	001	000
f	0	0	1	1	1	1	0	0

It is the linear function given by $(x_{i-1}, x_i, x_{i+1}) \mapsto x_{i-1} + x_i$.

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It is the linear function given by $(x_{i-1}, x_i, x_{i+1}) \mapsto x_{i-1} + x_i$.

▶ Since the vertex functions are linear so are the X-local functions – may represent each of them as a matrix. That is, F_i has matrix representation $A_i := I + E_{i,i-1}$ (standard basis.

▶ Each matrix A_i has determinant 1, so the matrix group generated by $A = \{A_1, \ldots, A_n\}$ is a subgroup of $\mathrm{SL}_n(\mathbb{F}_2)$.

▶ It is a known fact that A generates the entire $\mathrm{SL}_n(\mathbb{F}_2)$, so $G(\mathbf{F}^{60})$ is isomorphic to $\mathrm{SL}_n(\mathbb{F}_2)$.

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▶ $G(\mathbf{F}^{150})$ isomorphic to group with GAP index (96,227). G. Miller: 230/231.

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What choices of Ω are useful?

▶ How do we compute dynamics groups efficiently?

- if X is a graph union of X_1 and X_2, can we derive the dynamics group for X from those over X_1 and X_2 when functions are suitably defined?
- Is there a result analogous to the Seifert/van Kampen Theorem from algebraic topology?

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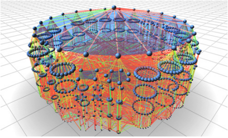

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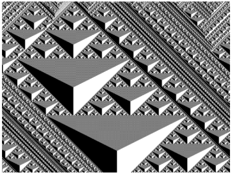

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Cheers & Happy Birthday!