Universal Pattern Generation by Cellular Automata

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Nicolas Ollinger, in his Cellular Automata tutorial at Unconventional Computation 2011, asked a question he attributed to Stanislaw Ulam:

- "Does there exist a cellular automaton and a finite initial configuration c such that the evolution of c contains all finite patterns over the state set of the CA ?"

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- "Does there exist a cellular automaton and a finite initial configuration c such that the evolution of c contains all finite patterns over the state set of the CA ?"

Ollinger also proposed a stronger variant:

- "Does there exist a cellular automaton and a finite initial configuration whose orbit is dense in the Cantor topology ?"

The first question is reformulated from
S. Ulam (1960) "A Collection of Mathematical Problems", p. 30:

2. A problem on matrices arising in the theory of automata

The theory of automata leads to some interesting questions which in the simplest case reduce to matrix theory formulations. Suppose one has an infinite regular system of lattice points in E^n, each capable of existing in various states $S_1, \ldots, S_k$. Each lattice point has a well defined system of m neighbors, and it is assumed that the state of each point at time $t + 1$ is uniquely determined by the states of all its neighbors at time t. Assuming that at time t only a finite set of points are active, one wants to know how the activation will spread. In particular, do there exist "universal" systems which are capable of generating arbitrary systems of states? Do there exist subsystems which are able to "reproduce," i.e., to produce other subsystems like the

In this talk, a positive answer is provided to the first question:

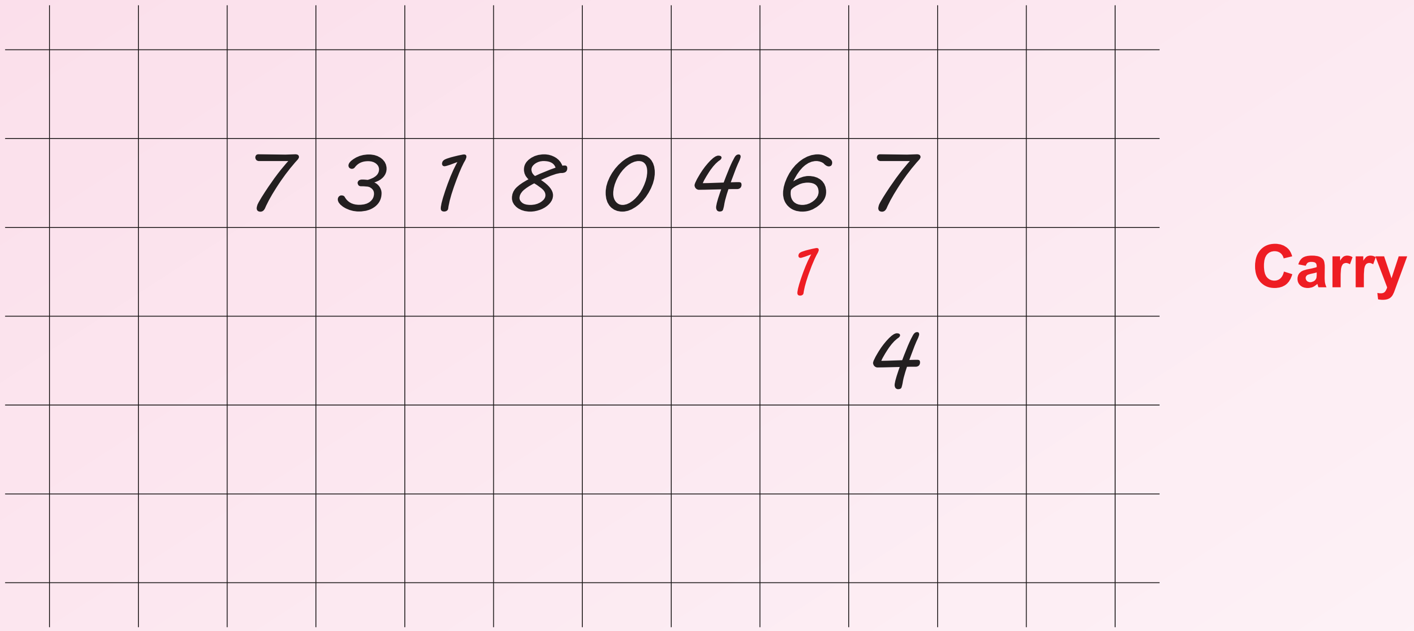
A one-dimensional, **reversible** cellular automaton with six states exists, with the property that the orbit of **every** non-trivial finite initial configuration contains all finite patterns over the state set.

The automaton multiplies numbers in base 6 by constant 3.

Multiplication by 2 in base 10:

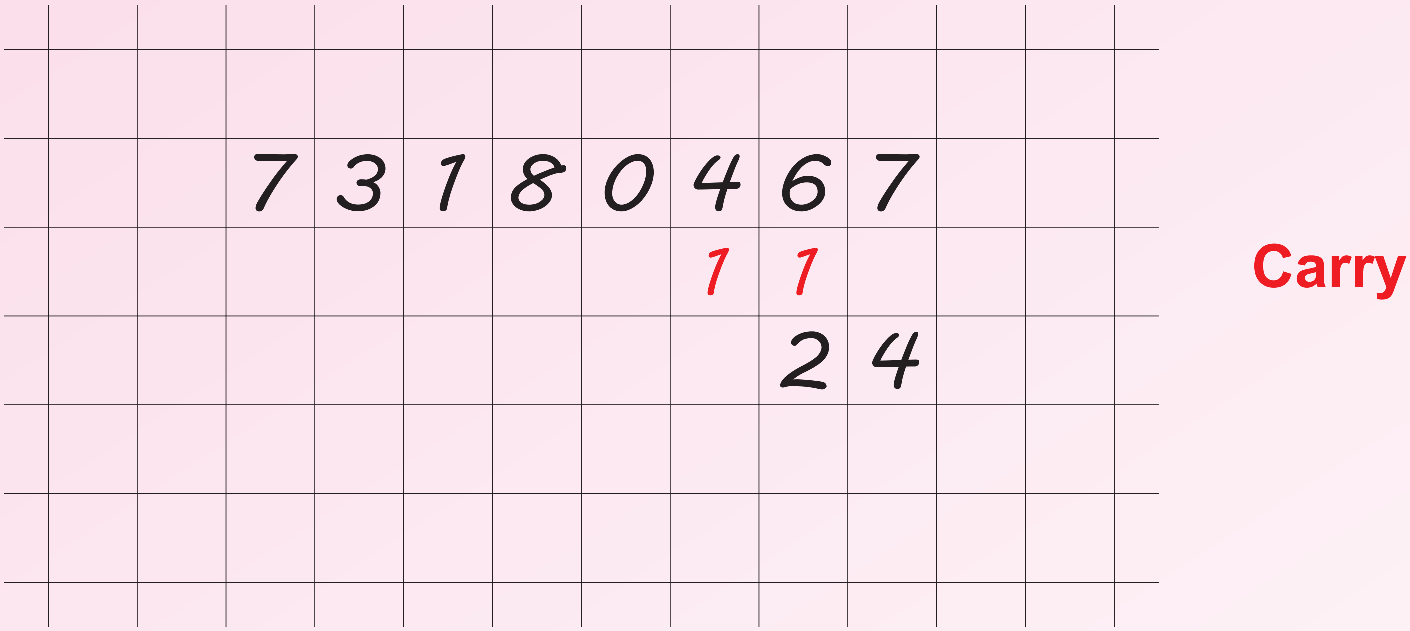
		7	3	1	8	0	4	6	7		

Multiplication by 2 in base 10:



$7 \times 2 = 14$. Number 1 is carried to the next digit place. The carries will be added to the digits at the end of the computation.

Multiplication by 2 in base 10:



$6 \times 2 = 12$, produces again carry 1.

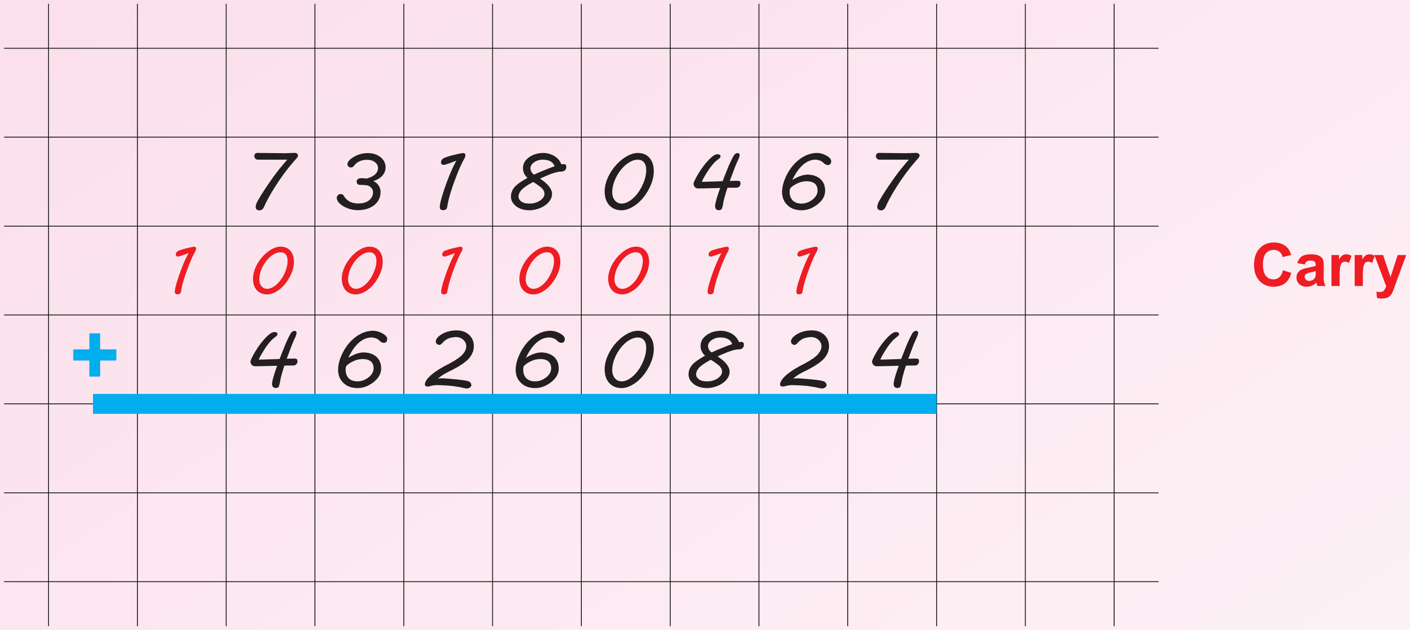
Multiplication by 2 in base 10:

7	3	1	8	0	4	6	7	
				0	1	1		Carry
					8	2	4	

Multiplication by 2 in base 10:

	7	3	1	8	0	4	6	7	
1	0	0	1	0	0	1	1		Carry
	4	6	2	6	0	8	2	4	

Multiplication by 2 in base 10:



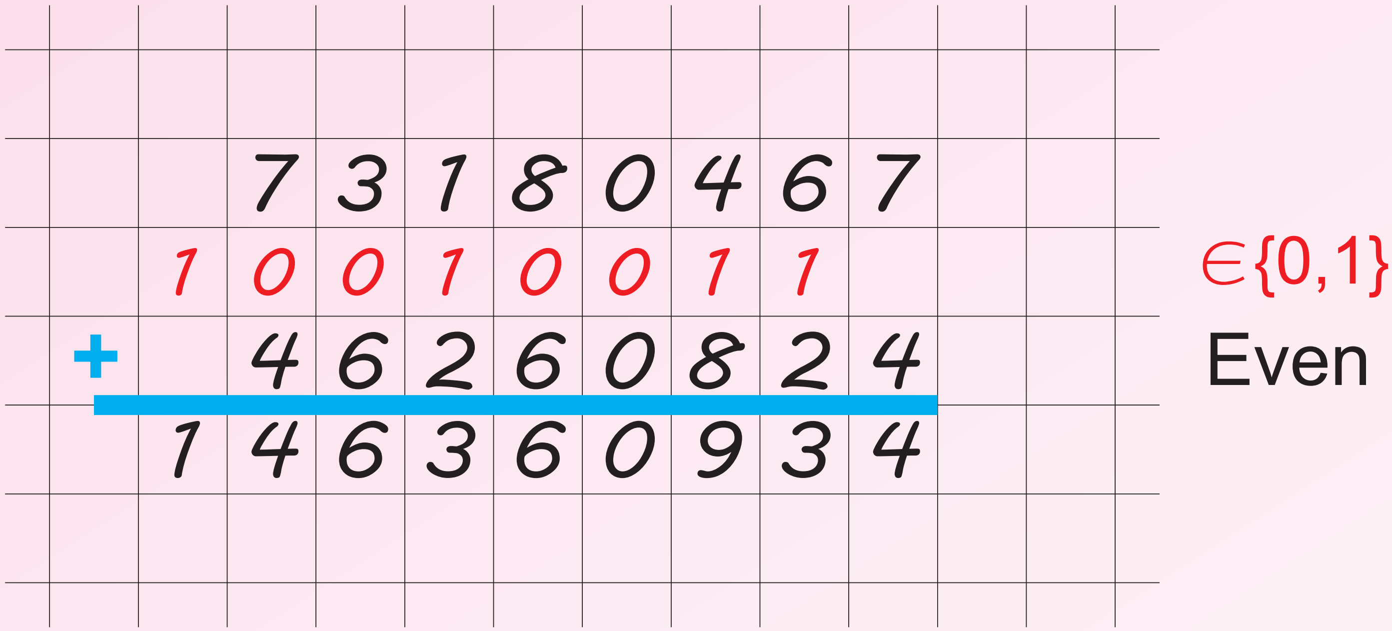
In the end, carries are added to the digits.

Multiplication by 2 in base 10:

		7	3	1	8	0	4	6	7	
	1	0	0	1	0	0	1	1		Carry
+		4	6	2	6	0	8	2	4	
	1	4	6	3	6	0	9	3	4	

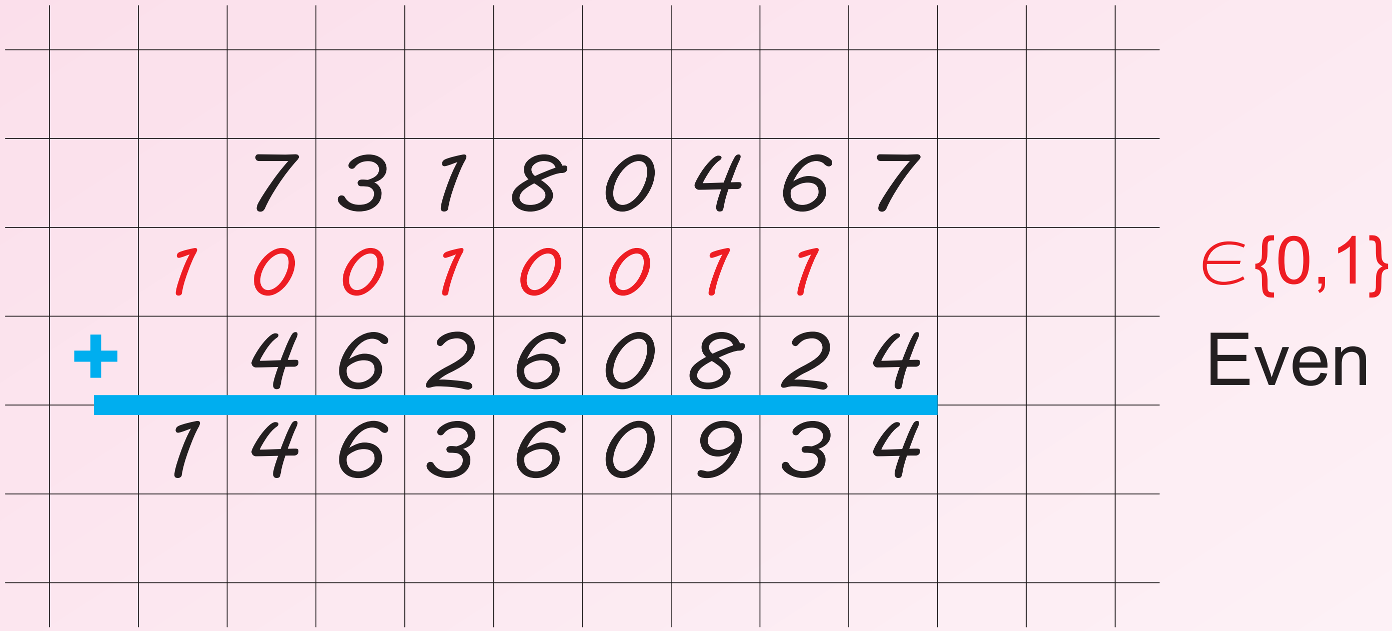
In the end, carries are added to the digits.

Multiplication by 2 in base 10:



The carries do not propagate because, at each position, numbers in $\{0, 1\}$ and $\{0, 2, 4, 6, 8\}$ are added together. Such sum does not exceed 9.

Multiplication by 2 in base 10:



Because the carries do not propagate, the multiplication is local. Digit i of the result only depends on the digits i and $i-1$ of the input. A cellular automaton can execute the multiplication.

In the same way, any number expressed in base 6 can be multiplied by 3 using a CA. The carries do not propagate because 3 divides 6. The local rule of the CA is

i \ $i+1$	**0**	**1**	**2**	**3**	**4**	**5**
0	**0**	**0**	**1**	**1**	**2**	**2**
1	**3**	**3**	**4**	**4**	**5**	**5**
2	**0**	**0**	**1**	**1**	**2**	**2**
3	**3**	**3**	**4**	**4**	**5**	**5**
4	**0**	**0**	**1**	**1**	**2**	**2**
5	**3**	**3**	**4**	**4**	**5**	**5**

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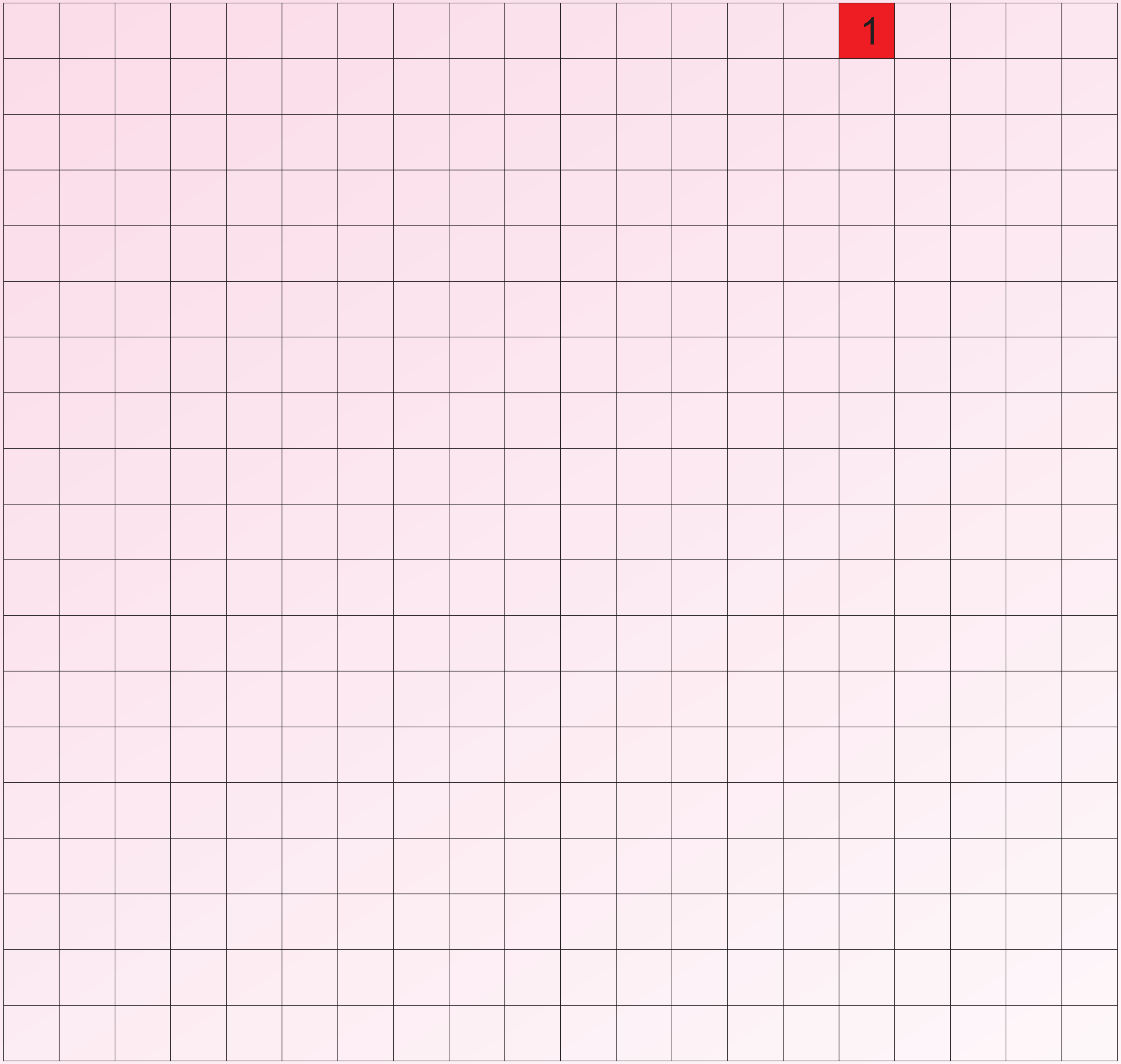
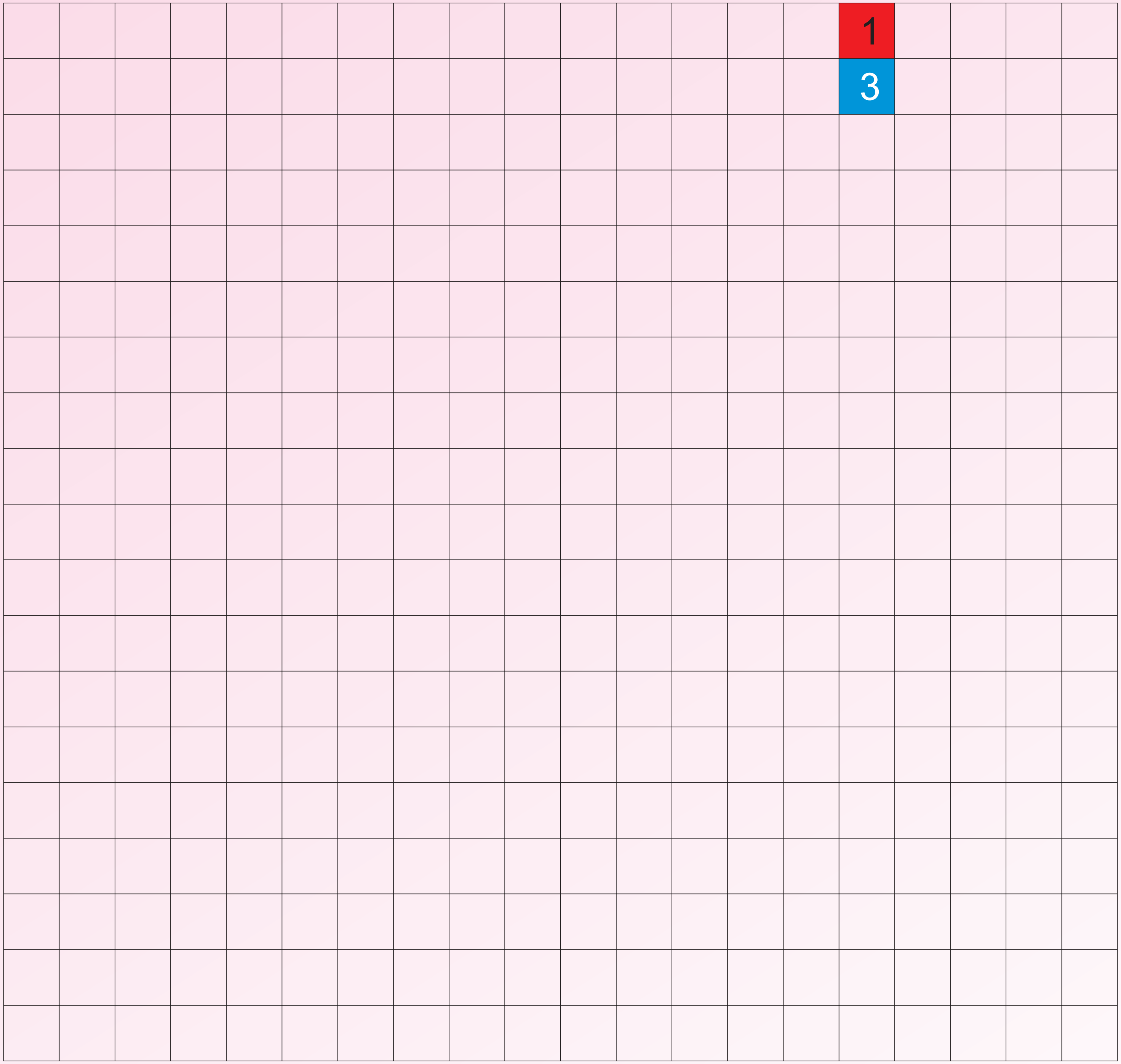
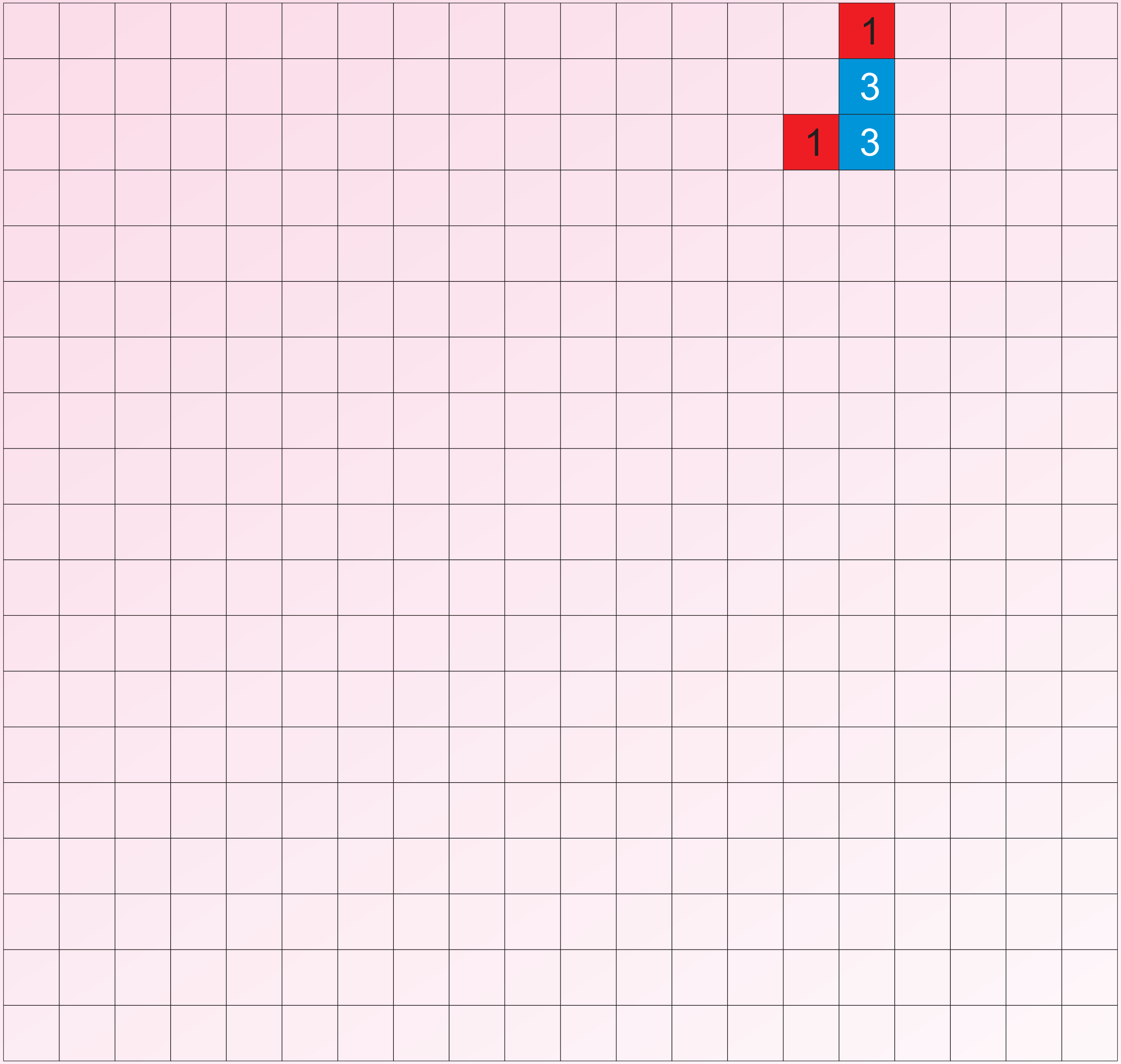
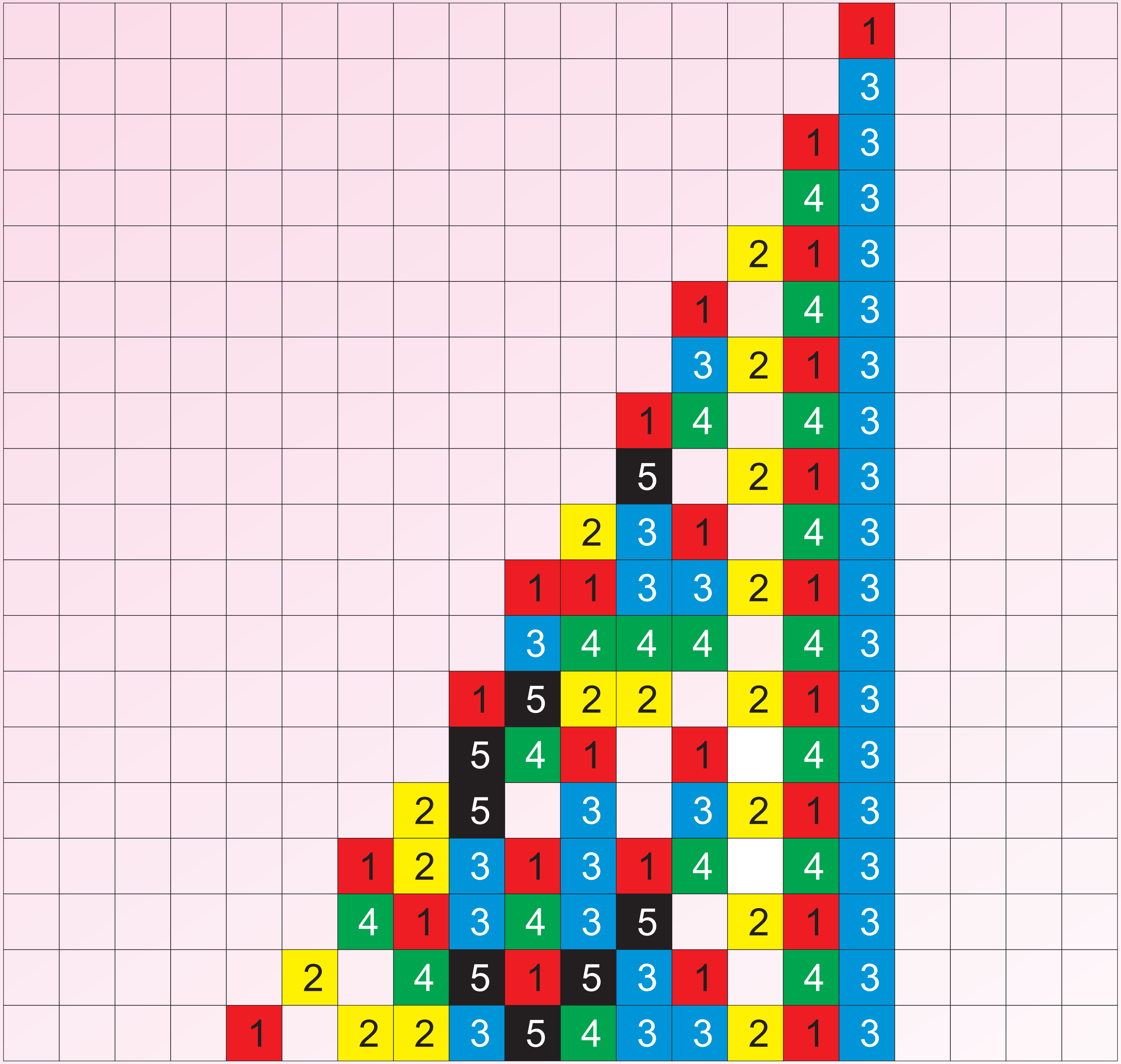
0	**0**	**0**	**1**	**1**	**2**	**2**
1	**3**	**3**	**4**	**4**	**5**	**5**
2	**0**	**0**	**1**	**1**	**2**	**2**
3	**3**	**3**	**4**	**4**	**5**	**5**
4	**0**	**0**	**1**	**1**	**2**	**2**
5	**3**	**3**	**4**	**4**	**5**	**5**

The CA is partitioned, and hence trivially reversible. (To go back in time, each cell needs to look only at its left neighbor.)


1


1
3


1
3
1 3



1
3
1 3
4 3
2 1 3
1 4 3
3 2 1 3
1 4 4 3
5 2 1 3
2 3 1 4 3
1 1 3 3 2 1 3
3 4 4 4 4 3
1 5 2 2 2 1 3
5 4 1 1 4 3
2 5 3 3 2 1 3
1 2 3 1 3 1 4 4 3
4 1 3 4 3 5 2 1 3
2 4 5 1 5 3 1 4 3
1 2 2 3 5 4 3 3 2 1 3

This CA has been studied before:

- F. Blanchard and A. Maass (1997) “Dynamical properties of expansive one-sided cellular automata”
  The same update rule but one-sided configurations.

- D. Rudolph (1990) “$\times 2$ and $\times 3$ invariant measures and entropy”
  Relates one-sided variant to the Furstenberg conjecture in ergodic theory.

- S. Wolfram (2002) “A New Kind of Science”
  Displays the two-sided variant.

State set is $S = \{0, 1, 2, 3, 4, 5\}$.

State 0 is quiescent. Configurations with only finite number of non-0 states are **finite**.

Let $F_{\times 3} : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ be our CA map.

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Let $F_{\times 3} : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ be our CA map.

For each finite configuration $x \in S^{\mathbb{Z}}$ we define

$$\alpha(x) = \sum_{i=-\infty}^{\infty} x_i \cdot 6^{-i}$$

to be the number it represents in base 6.

Lemma. For every finite configuration $x \in S^{\mathbb{Z}}$,

$$\alpha(F_{\times 3}(x)) = 3\alpha(x).$$

We can analogously define $F_{\times 2} : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ that multiplies by 2 in base 6.

The left-shift σ multiplies by 6, and the right shift σ^{-1} divides by 6.

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We have

$$\alpha(\sigma^{-1} \circ F_{\times 2} \circ F_{\times 3}(x)) = \alpha(x)$$

for all finite x. Therefore

$$\sigma^{-1} \circ F_{\times 2} \circ F_{\times 3}(x) = x$$

for all finite x, and consequently for all $x \in S^{\mathbb{Z}}$. The **inverse** of $F_{\times 3}$ is

$$\sigma^{-1} \circ F_{\times 2}.$$

Theorem. Let $x \neq \ldots 000 \ldots$ be a finite initial configuration. For every word $w \in S^*$ there is time t such that

$$F_{\times 3}^{t}(x) = \ldots 00\ w \ldots$$

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Proof. Let

$$\xi = \alpha(x) > 0.$$

Let $I \subseteq (0,1)$ be an open interval such that the base 6 representation of every element of I begins

$$0.w \ldots$$

Let

$$J = \log_6(I/\xi) = \{\log_6(a/\xi) \mid a \in I\}.$$

Number $\log_6(3)$ is **irrational,** and therefore the set

$$\{ \operatorname{Frac}[t \log_6(3)] \mid t \in \mathbb{N} \}$$

is **dense** in $[0,1)$. [$\operatorname{Frac}(a)$ is the fractional part of $a \in \mathbb{R}$.]

Because $J \subseteq \mathbb{R}$ is open, there exist $t \in \mathbb{N}$ and $n \in \mathbb{Z}$ such that

$$t \log_6(3) + n \in J = \log_6(I/\xi).$$

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Raising 6 to this power and multiplying by ξ gives

$$\xi 3^t 6^n \in I.$$

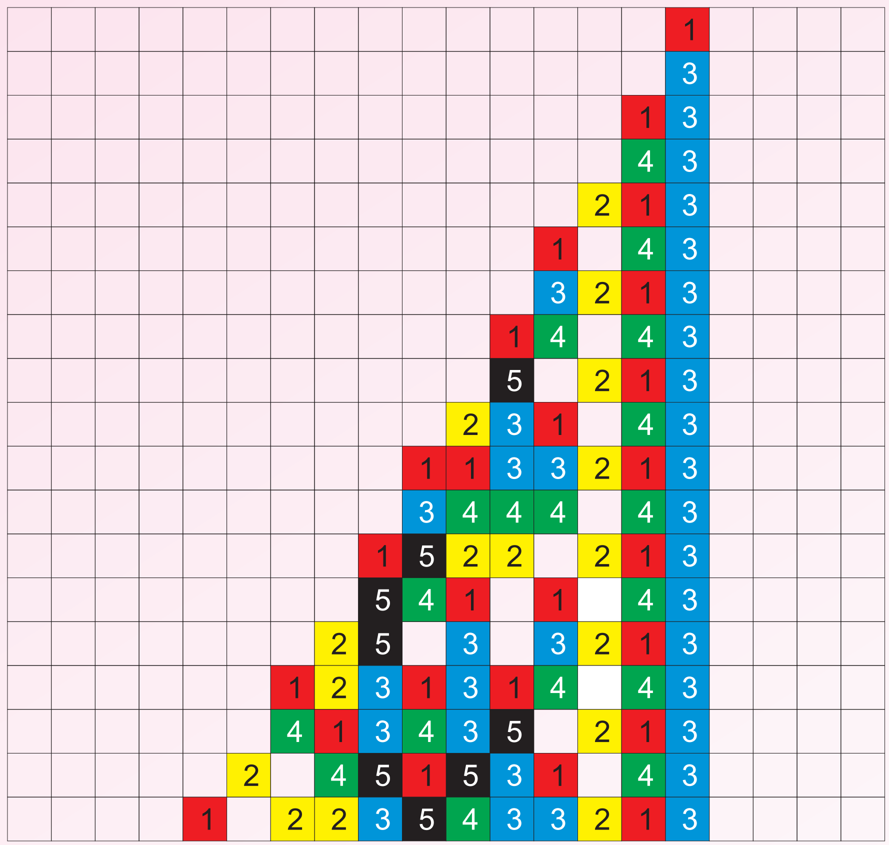
This means that

$$F_{\times 3}^t(x) = \ldots 00\ w \ldots$$

as claimed. $\square$

Ollinger's second questions asked whether it is possible to generate every finite pattern in **every position** (or, equivalently, **centered at the origin.**)

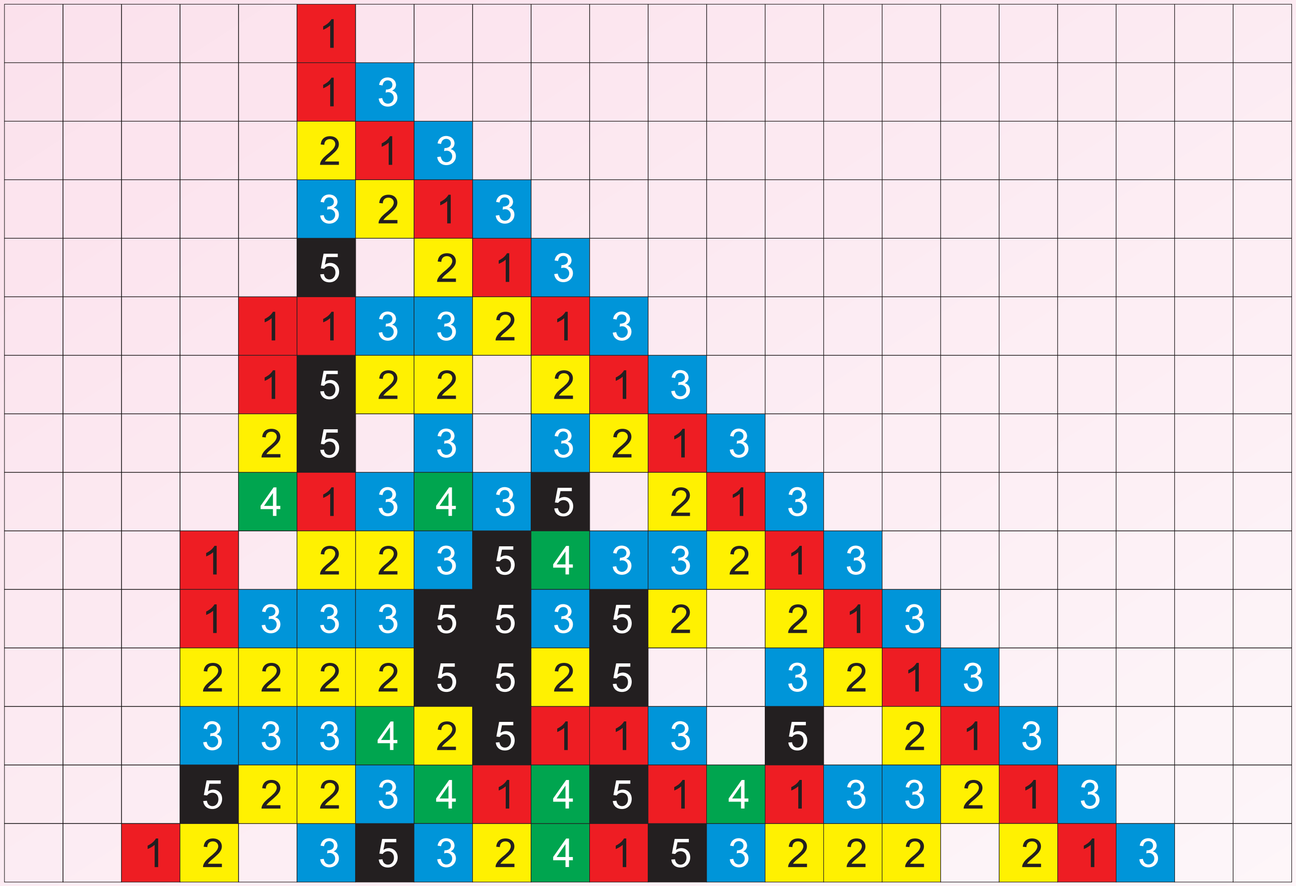
$F_{\times 3}$ clearly does not have this property: it has one-sided neighborhood.



Let us make the cone expanding to both sides. Consider

$$F_{\times \frac{3}{2}} = \sigma^{-1} \circ F_{\times 3} \circ F_{\times 3}.$$

It multiplies by 3/2 in base 6.



The orbit of finite $x \in S^{\mathbb{Z}}$ contains every word $w \in S^*$ positioned **starting at cell 0** if and only if the set

$$A_\xi = \{ \ \mathrm{Frac}[\xi(3/2)^t] \mid t \in \mathbb{N} \ \}$$

is dense for $\xi = \alpha(x)$.

Every word appears positioned **everywhere** iff the set A_ξ is dense for $\xi = 6^n \alpha(x)$, for all $n \in \mathbb{Z}$.

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Unfortunately, denseness of A_ξ is very difficult to establish for specific ξ – it has been studied for over 100 years.

H. Weyl H (1916) "Über die Gleichverteilung von Zahlen modulo Eins"
Shows that A_ξ is dense for almost all ξ. The set of exceptions has measure zero.

Conjecture. The orbit of every non-zero finite initial configuration under $F_{\times \frac{3}{2}}$ is dense.

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CA $F_{\times\frac{3}{2}}$ is related also to other difficult open problems.

K. Mahler (1968) “An unsolved problem on the powers of 3/2”

Mahler's Problem. Does there exist $\xi > 0$ such that

$$\forall t \in \mathbb{N} \ : \ \mathrm{Frac}[\xi(3/2)^t] < \frac{1}{2} \qquad ?$$

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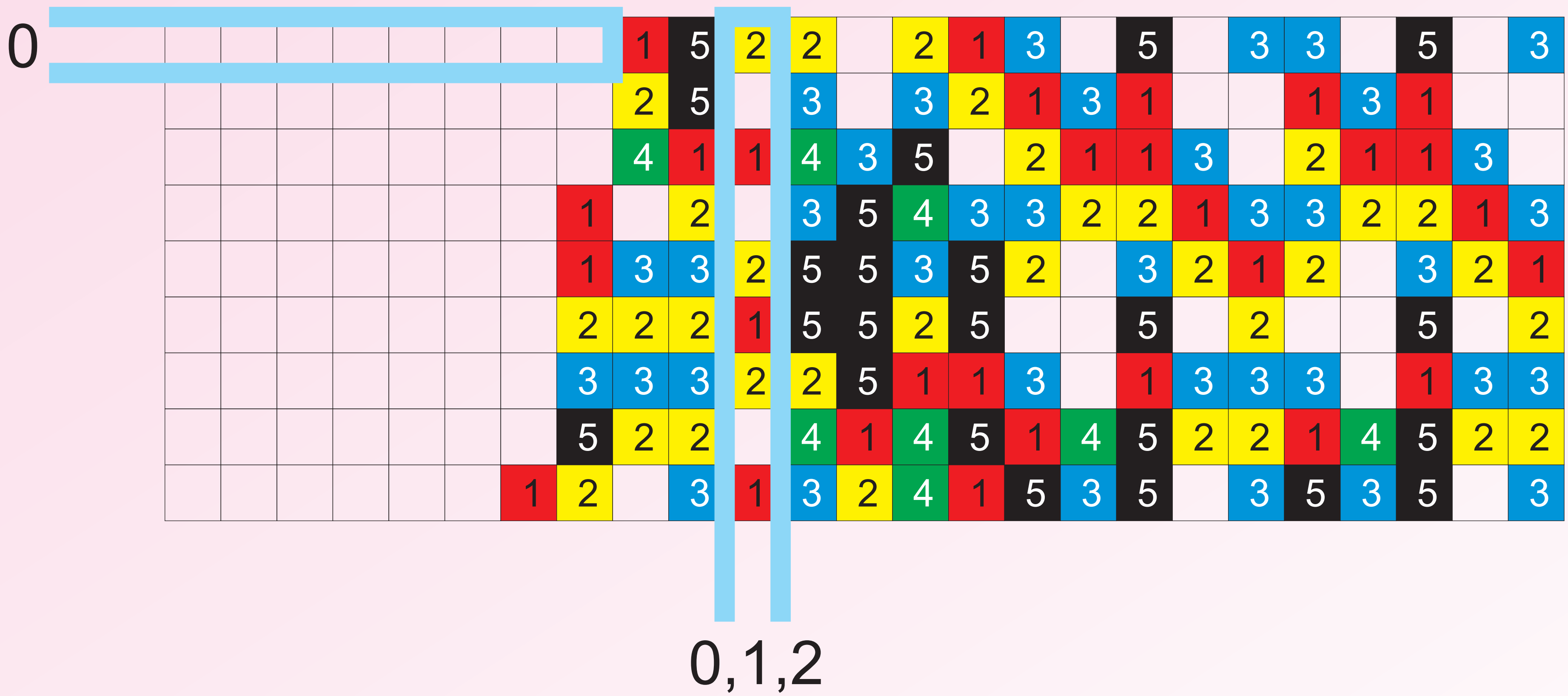
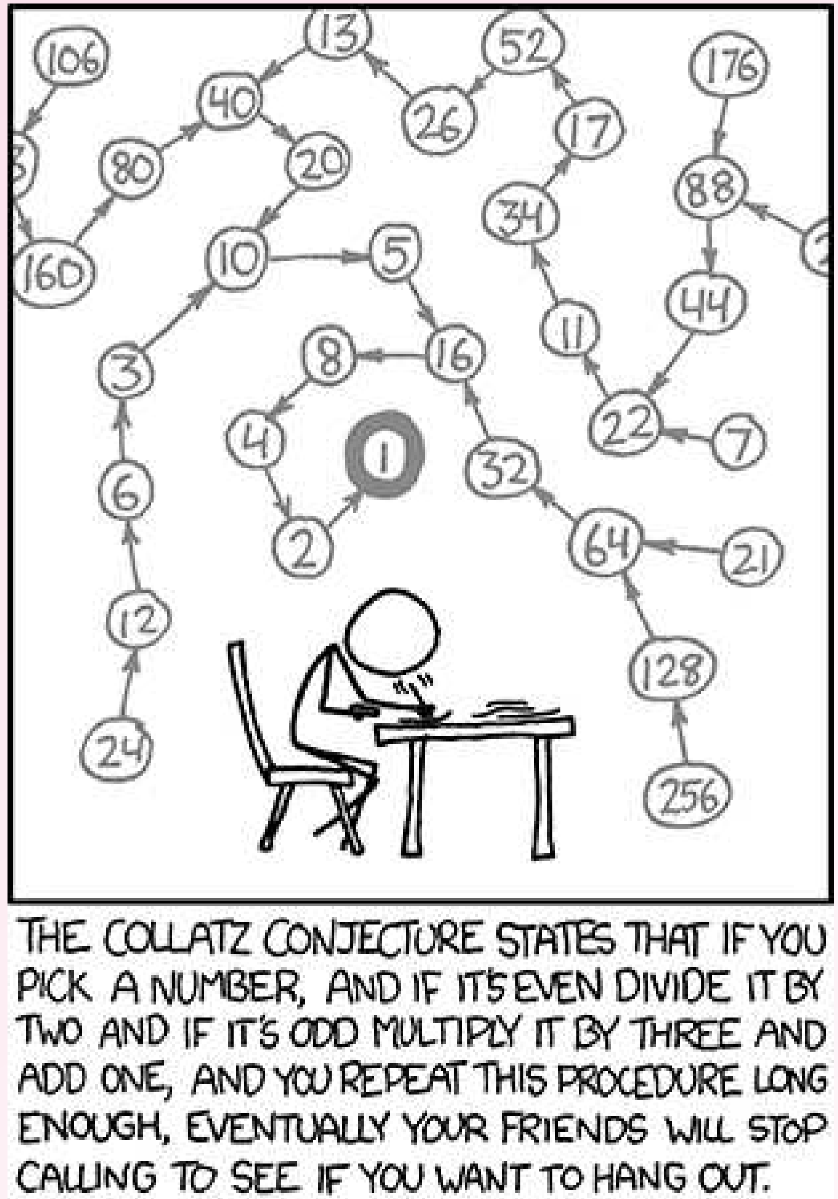


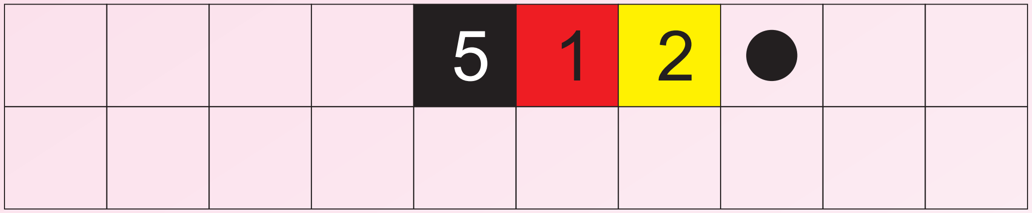
A small modification to $F_{\times 3}$ makes it compute the Collatz-function

$$n \mapsto \begin{cases} n/2, & \text{for } n \text{ even,} \\ 3n+1, & \text{for } n \text{ odd.} \end{cases}$$



We add a new state ● that indicates the position of the decimal point.

A CA to calculate the Collatz-function:



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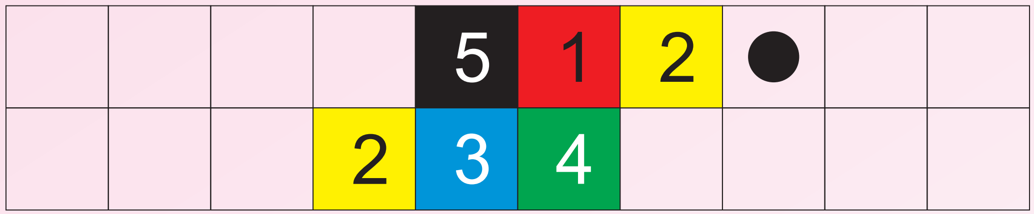

1) Apply $F_{\times 3}$ to multiply the current number by 3.

We add a new state ● that indicates the position of the decimal point.

A CA to calculate the Collatz-function:

				5	1	2	●		
			2	3	4	●			

2) If the last digit before ● becomes zero (i.e. the number was even), the point ● is moved one position to the left.

This amount to dividing the number by 6, so the combined operation is

$$n \mapsto n/2.$$

We add a new state ● that indicates the position of the decimal point.

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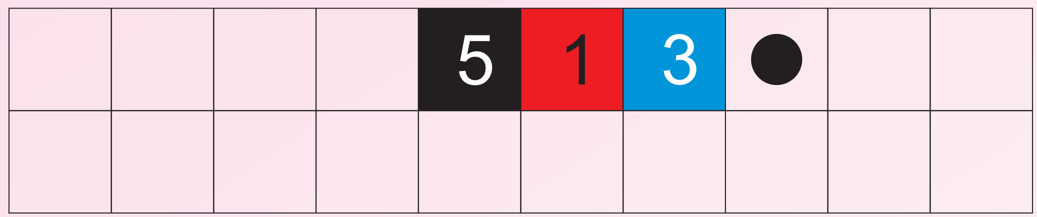
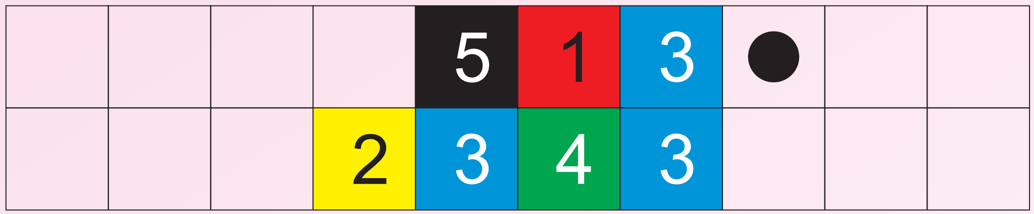


On odd numbers. . .

We add a new state ● that indicates the position of the decimal point.

A CA to calculate the Collatz-function:



2) If the last digit before ● becomes three (i.e. the number was odd), it is incremented to become four. The point ● stays put.

This amount to operation

$$n \mapsto 3n + 1.$$

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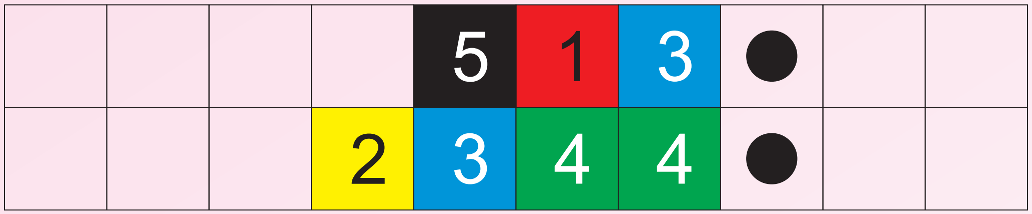

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This amount to operation

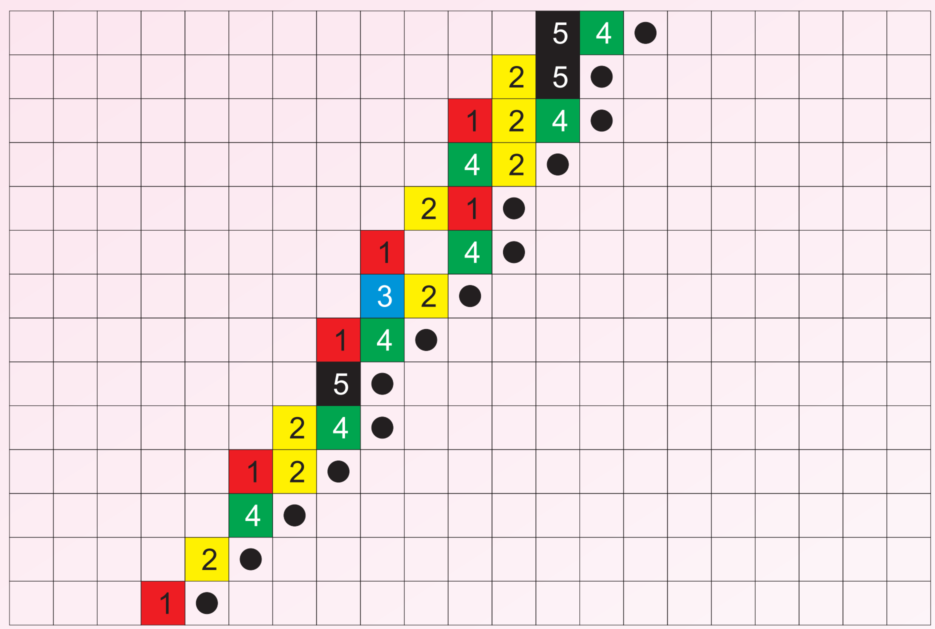
$$n \mapsto 3n + 1.$$

The **Collatz conjecture** then states that for any $w \in S^* \setminus 0^*$, the finite initial configuration

$$\ldots 00\ w\ \boxed{\bullet}\ 00 \ldots$$

becomes eventually

$$\ldots 00\ 1\ \boxed{\bullet}\ 00 \ldots$$



T. Cloney, E. Goles, G.Y. Vichniac (1987) "The 3x+1 Problem: A Quasi Cellular Automaton"

The first paper where a cellular automaton like model was used to study the Collatz-function!

Open problems

Ollinger's second question:

Problem 1. Does there exist a CA with a finite initial configuration whose orbit is dense ? Does $F_{\times \frac{3}{2}}$ have this property ?

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Problem 2. Do there exist higher dimensional cellular automata with the property that the orbit of some finite configuration contains all finite patterns ?

$F_{\times 3}$ has six states. This method does not provide examples with fewer states.

Problem 3. Does there exist a binary state CA with the property that the orbit of some finite configuration contains all finite patterns ?