Fast arithmetical algorithms in Möbius number systems

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Iterative systems

X compact metric space, A finite alphabet
$(F_a : X \to X)_{a \in A}$ continuous.
$(F_u : X \to X)_{u \in A^*}$, $F_{uv} = F_u \circ F_v$, $F_\lambda = \mathrm{Id}$

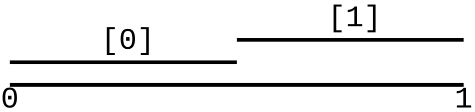
Theorem(Barnsley) If $(F_a : X \to X)_{a \in A}$ are contractions, then there exists a unique attractor $Y \subseteq X$ with $Y = \bigcup_{a \in A} F_a(Y)$, and a continuous surjective symbolic mapping $\Phi : A^{\mathbb{N}} \to Y$

$$\{\Phi(u)\} = \bigcap_{n>0} F_{u_{[0,n)}}(X), \;\; u \in A^{\mathbb{N}}$$

Binary system $A = \{0, 1\}$, $\Phi_2 : A^{\mathbb{N}} \to [0, 1]$

$$F_0(x) = \frac{x}{2}, \; F_1(x) = \frac{x+1}{2}$$

$$\Phi_2(u) = \sum_{i \geq 0} u_i \cdot 2^{-i-1}, \; u \in A^{\mathbb{N}}$$



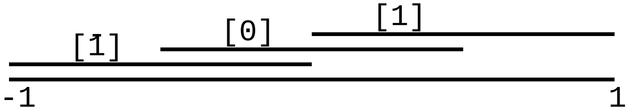
Expansion graph: $x \xrightarrow{a} F_a^{-1}(x)$ if $x \in W_a = F_a[0, 1]$

Binary signed system is redundant

$$A = \{\bar{1}, 0, 1\}, \;\; \Phi_3 : A^{\mathbb{N}} \to [-1, 1]$$

$$F_{\bar{1}}(x) = \frac{x-1}{2}, \;\; F_0(x) = \frac{x}{2}, \;\; F_1(x) = \frac{x+1}{2}$$

$$\Phi_3(u) = \sum_{i \geq 0} u_i \cdot 2^{-i-1}, \;\; u \in A^{\mathbb{N}}$$



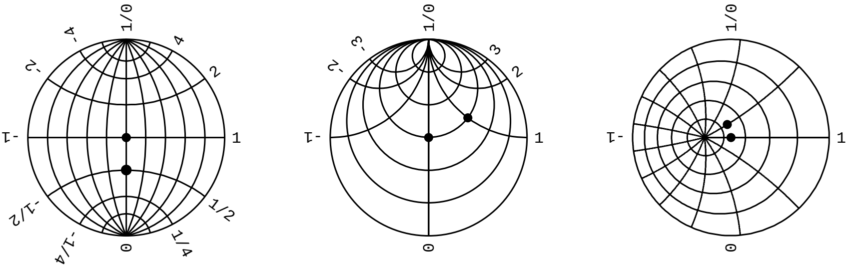
Real orientation-preserving Möbius transformations

$$M_{a,b,c,d}(x) = \frac{ax+b}{cx+d}, \; ad - bc > 0$$

act on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$
and on $\mathbb{U} = \{z \in \mathbb{C} : \; \Im(z) > 0\}$

Stereographic projection $\mathfrak{d}(z) = \frac{iz+1}{z+i}$
$\mathfrak{d} : \overline{\mathbb{R}} \to \partial\mathbb{D} = \{z \in \mathbb{C} : \; |z| = 1\}$ unit circle
$\mathfrak{d} : \mathbb{U} \to \mathbb{D} = \{z \in \mathbb{C} : \; |z| < 1\}$ unit disc

Disc Möbius transformations $\widehat{M} = \mathbf{d} \circ M \circ \mathbf{d}^{-1}$



$\widehat{F}_0(z) = \frac{3z-i}{iz+3}$	$\widehat{F}_1(z) = \frac{(2+i)z+1}{z+2-i}$	$\widehat{F}_2(z) = \frac{(7+2i)z+i}{-iz+(7-2i)}$
$F_0(x) = x/2$	$F_1(x) = x+1$	$F_2(x) = \frac{4x+1}{3-x}$
hyperbolic	parabolic	elliptic

Mean value $\mathcal{E}(\widehat{M}\ell) = \int_{\partial\mathbb{D}} z\, d(\widehat{M}\ell) = \widehat{M}(0)$

Circle metric and derivation

the length of arc between $\mathbf{d}(x)$ and $\mathbf{d}(y)$:

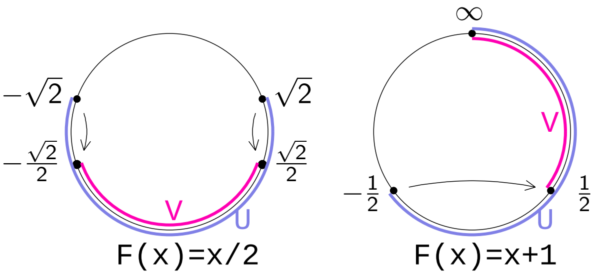
$$\varrho(x, y) = 2 \arcsin \frac{|x - y|}{\sqrt{(x^2 + 1)(y^2 + 1)}}$$

circle derivation of $M(x) = (ax + b)/(cx + d)$:

$$\begin{aligned} M^{\bullet}(x) &= \lim_{y \to x} \frac{\varrho(M(x), M(y))}{\varrho(x, y)} \\ &= \frac{(ad - bc)(x^2 + 1)}{(ax + b)^2 + (cx + d)^2} \end{aligned}$$

Contracting and expanding intervals

$$\mathbf{U}_u = \{x \in \overline{\mathbb{R}} : F_u^\bullet(x) < 1\}, \quad F_u(\mathbf{U}_u) = \mathbf{V}_u$$
$$\mathbf{V}_u = \{x \in \overline{\mathbb{R}} : (F_u^{-1})^\bullet(x) > 1\}$$



Möbius number system(MNS) $(F, \mathcal{W})$

$(F_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}})_{a \in A}$ Möbius transformations
$W_a \subseteq \mathbf{V}_a$ expansion intervals $\bigcup_{a \in A} \overline{W_a} = \overline{\mathbb{R}}$

Expansion graph: $x \xrightarrow{a} F_a^{-1}(x)$ if $x \in W_a$

$$x \xrightarrow{u_0} F_{u_0}^{-1}(x) \xrightarrow{u_1} F_{u_0 u_1}^{-1}(x) \xrightarrow{u_2} \cdots$$

$$x \in W_{u_0},\ F_{u_0}^{-1}(x) \in W_{u_1},\ F_{u_0 u_1}^{-1}(x) \in W_{u_2}$$

$$W_u := W_{u_0} \cap F_{u_0}(W_{u_1}) \cap \cdots \cap F_{u_{[0,n)}}(W_{u_n})$$

$x \in W_u$ iff u is the label of a path with source x:

Expansion subshift:

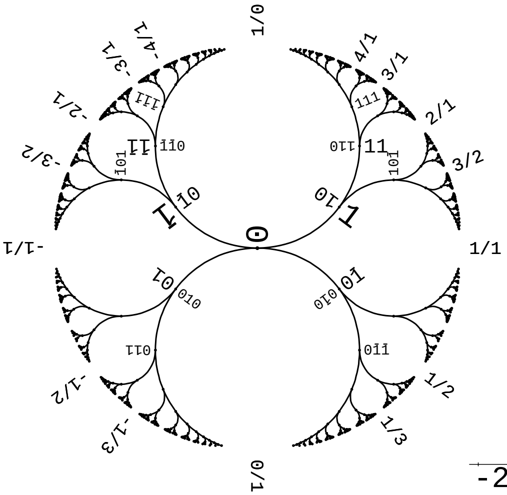
$$\mathcal{S}_{\mathcal{W}} := \{u \in A^{\mathbb{N}} : \ \forall n, W_{u_{[0,n)}} \neq \emptyset\}$$

Symbolic extension:

$$\Phi(u) = \lim_{n\to\infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}, \ u \in \mathcal{S}_{\mathcal{W}}$$

$\Phi : \mathcal{S}_{\mathcal{W}} \to \overline{\mathbb{R}}$ is continuous and surjective.

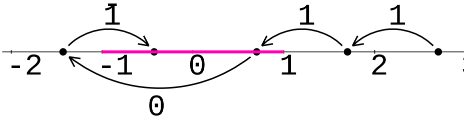
Continued fractions $a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cdots}} = F_1^{a_0} F_0 F_1^{a_1} F_0 \cdots$



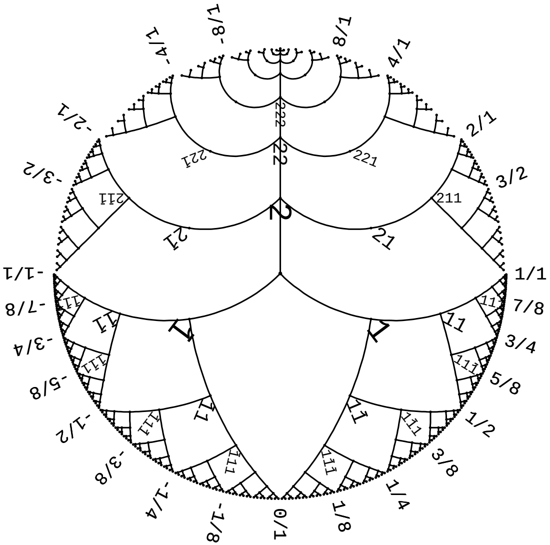
$F_{\bar{1}}(x) = x - 1, \quad W_{\bar{1}} = (\infty, -1)$
$F_0(x) = -1/x, \quad W_0 = (-1, 1)$
$F_1(x) = x + 1, \quad W_1 = (1, \infty)$

$\mathcal{S}_{\mathcal{W}}$ is a SFT.
forbidden words:
$00, 1\bar{1}, \bar{1}1, 101, \bar{1}0\bar{1}$

$x \xrightarrow{a} F_a^{-1}(x)$ if $x \in W_a$



Binary signed system, $A = \{\bar{1}, 1, 2\}$



$$F_{\bar{1}}(x) = (x-1)/2$$
$$F_1(x) = (x+1)/2$$
$$F_2(x) = 2x$$

$$W_{\bar{1}} = (-1, 0) \xrightarrow{\bar{1}} (-1, 1)$$
$$W_1 = (0, 1) \xrightarrow{1} (-1, 1)$$
$$W_2 = (1, -1) \xrightarrow{2} (\tfrac{1}{2}, -\tfrac{1}{2})$$

$\mathcal{S}_{\mathcal{W}}$ is a SFT.
forbidden words:
$\bar{1}2, 12, 2\bar{1}1, 21\bar{1}$

$$\mathcal{S}_{\mathcal{W}} = \{2^n u : \ u \in \{\bar{1}, 1\}^{\mathbb{N}}\}$$
$$\Phi(2^n u) = \sum_{i=0}^{\infty} u_i \cdot 2^{n-i}$$

Fractional bilinear functions

$$P(x, y) = \frac{axy + bx + cy + d}{exy + fx + gy + h}, \ M(x) = \frac{ax + b}{cx + d}.$$

$$M^x = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}, \ M^y = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

$P(Mx, y) = PM^x(x, y)$, $P(x, My) = PM^y(x, y)$, $MP(x, y)$ are fractional bilinear functions.

Bilinear graph

vertices: (P, u, v), $u, v \in \mathcal{S}_{\mathcal{W}}$.

$$(P, u, v) \stackrel{a}{\to} (F_a^{-1} P, u, v) \quad \text{if} \quad P(\overline{W_{u_0}}, \overline{W_{v_0}}) \subseteq \overline{W_a}$$
$$(P, u, v) \stackrel{\lambda}{\to} (P F_{u_0}^x, \sigma(u), v)$$
$$(P, u, v) \stackrel{\lambda}{\to} (P F_{v_0}^y, u, \sigma(v))$$

Proposition If $u, v \in \mathcal{S}_{\mathcal{W}}$ and $w \in A^{\mathbb{N}}$ is a label of a path with source (P, u, v), then $w \in \mathcal{S}_{\mathcal{W}}$ and $\Phi(w) = P(\Phi(u), \Phi(v))$.

Linear graph

vertices: $(M, u) \in \mathcal{M}_1 \times \mathcal{S}_{\mathcal{W}}$,

$$\text{emission: } (M,u) \overset{a}{\to} (F_a^{-1}M, u) \quad \text{if} \quad M(\overline{W_{u_0}}) \subseteq \overline{W_a}$$
$$\text{absorption: } (M,u) \overset{\lambda}{\to} (MF_{u_0}, \sigma(u))$$

Proposition There exists a path with source (M, u) whose label $w = f(u) \in \mathcal{S}_{\mathcal{W}}$ and $\Phi(w) = M(\Phi(u))$.

If M remain bounded, then the algorithm has linear time complexity.

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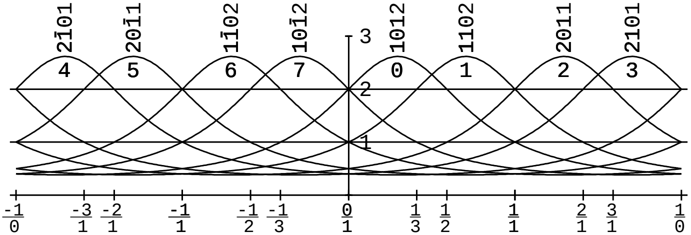
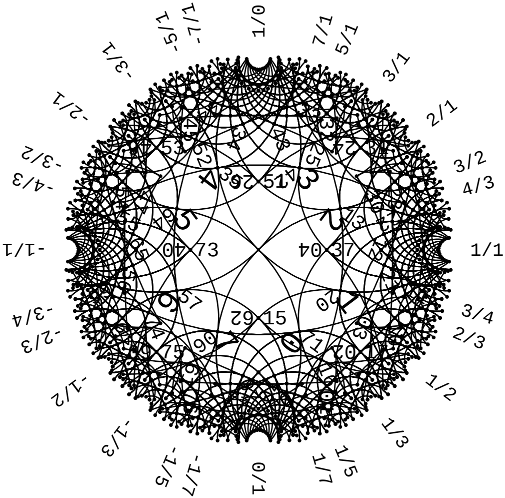
Bimodular group: $\det(M) = 2^p$



$$\det(M) = 2,\ ||M|| = 6,\ \mathrm{tr}(M) = 3$$

a	0	1	2	3
F_a	$\frac{x}{x+2}$	$\frac{x+1}{2}$	$\frac{2x}{x+1}$	$2x+1$
$W_a = \mathbf{V}_a$	$(-\frac{1}{3}, 1)$	$(0, 2)$	$(\frac{1}{2}, \infty)$	$(1, -3)$

Redundant bimodular system $W_a = \mathbf{V}(F_a)$



Top five algorithm:
Keeps five matrices with the smallest norm.

Ergodic theory of singular transformations

$M(x) = \frac{ax+b}{cx+d}$, $ad - bc = 0$

$$\begin{aligned} \widetilde{M} &= \{(x,y) \in \overline{\mathbb{R}}^2 : \ (ax_0 + bx_1)y_1 = (cx_0 + dx_1)y_0\} \\ &= (\overline{\mathbb{R}} \times \{s_M\}) \cup (\{u_m\} \times \overline{\mathbb{R}}) \end{aligned}$$

$s_M = M(i) \in \{\frac{a}{c}, \frac{b}{d}\} \cap \overline{\mathbb{R}}$: stable point
$u_M = M^{-1}(i) \in \{-\frac{b}{a}, -\frac{d}{c}\} \cap \overline{\mathbb{R}}$: unstable point

If M is singular, then $s_{MF} = s_M$, $u_{FM} = u_M$.
Emission acts on columns, absorption acts on rows of singular matrices.

Growth of norm $||x|| = \sqrt{x_0^2 + x_1^2}$

$$M^{\bullet}\left(\tfrac{x_0}{x_1}\right) = \frac{(ad - bc)(x_0^2 + x_1^2)}{(ax_0 + bx_1)^2 + (cx_0 + dx_1)^2}$$

$$= \frac{\det(M) \cdot ||x||^2}{||M(x))||^2}$$

$$\frac{||M(x)||}{||x||} = \sqrt{\frac{\det(M)}{M^{\bullet}(x)}}$$

Invariant emission measure

Partition of unity $w_a : \overline{\mathbb{R}} \to [0,1], \quad a \in A$

$$\mathrm{supp}(\mathrm{w_a}) \subseteq \overline{\mathrm{W_a}}, \; \sum_{\mathrm{a} \in \mathrm{A}} \mathrm{w_a}(\mathrm{x}) = 1$$

Emission process $x \overset{w_a}{\to} F_a^{-1}(x)$ has a unique Lebesgue-continuous invariant measure μ.

$$\mathbf{e}_n = \sum_{u \in \mathcal{L}^n(\mathcal{S}_\mathcal{W})} \frac{1}{2} \int \ln \frac{\det(F_u)}{(F_u^{-1})^\bullet} \, d\mu$$

$\mathbf{e}_{n+m} \leq \mathbf{e}_n + \mathbf{e}_m$: emission quotients

Invariant absorption measure

$$P_a = \int w_a \, d\mu, \;\; P_{ab} = \int w_a \cdot w_b \circ F_a^{-1} \, d\mu$$

$\mathcal{S}_{\mathcal{W}}$ is a SFT of order 2: $R_{ab} = P_{ab}/P_a$.

Absorption process $(x, a) \xrightarrow{R_{ab}} (F_b^t(x), b)$ in $\overline{\mathbb{R}} \times A$ has a unique invariant measure $\nu(U, a) = P_a \nu_a(U)$

$$\mathbf{a}_n \;\; = \;\; \sum_{au \in \mathcal{L}^n(\mathcal{S}_{\mathcal{W}})} \frac{1}{2} P_{au} \int \ln \frac{\det(F_u)}{(F_u^t)^\bullet} \, d\nu_a$$

$\mathbf{a}_{n+m} \leq \mathbf{a}_n + \mathbf{a}_m$: absorption quotients

Transaction quotient

$$\begin{aligned}
\mathbf{E} &= \lim_{n\to\infty} \exp(\mathbf{e}_n/n) : \quad \text{Emission quotient} \\
\mathbf{A} &= \lim_{n\to\infty} \exp(\mathbf{a}_n/n) : \quad \text{Absorption quotient} \\
\mathbf{T} &= \mathbf{E} \cdot \mathbf{A} : \quad \text{Transaction quotient}
\end{aligned}$$

$\mathbf{T} > \sqrt{r}$: positional r-ary system (Heckmann).

$\mathbf{T} < 1.1$: redundant bimodular system.

Conjecture There exists a multiplication algorithm with average linear time complexity.