Kadanoff Sand Pile Model

Avalanches and Fixed Point.

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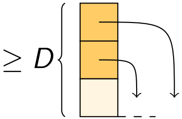
KSPM - Definition

Suggested by L. Kadanoff *et al* (1989).

- A parameter D
- An initial configuration $(N, 0, 0, \ldots)$
- One transition rule



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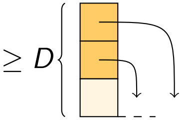

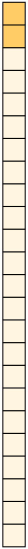

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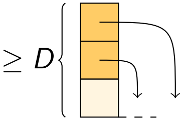

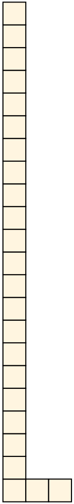

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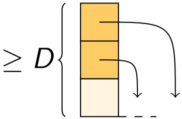

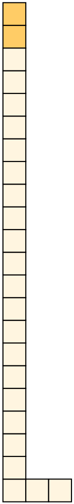

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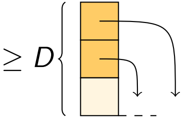

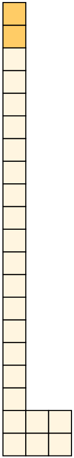

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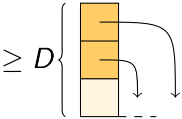

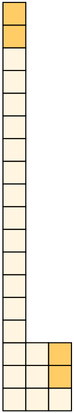

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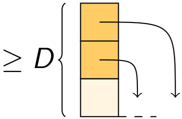

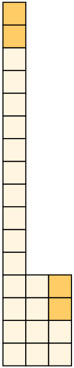

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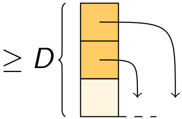

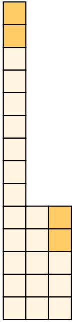

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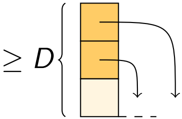

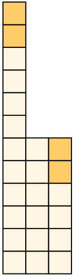

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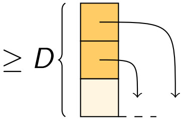

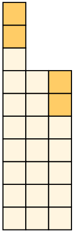

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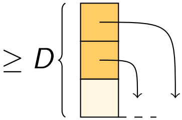

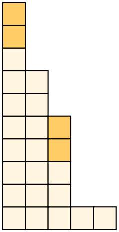

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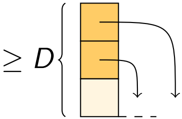

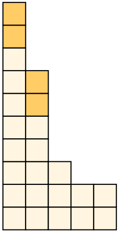

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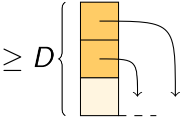

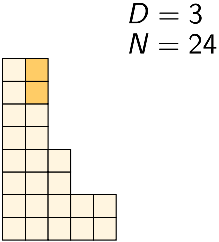

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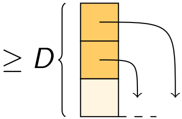

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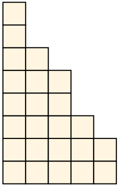

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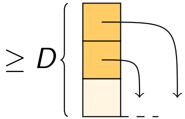

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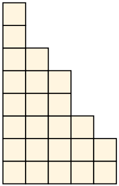

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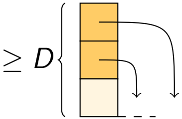
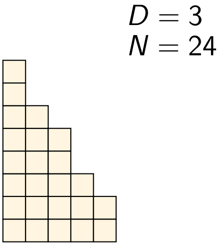


Remark : KSPM(2)=SPM



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Motivations

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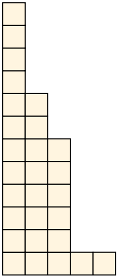
  Adding a single grain can create an unbounded avalanche !

KSPM - Representation

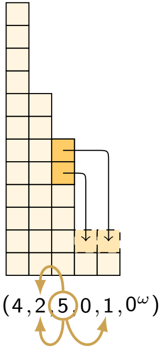
- sequence of height differences



$(4, 2, 5, 0, 1, 0^\omega)$

KSPM - Representation

- sequence of height differences
- rule application $\Rightarrow$ a column gives some height difference



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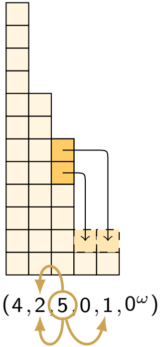
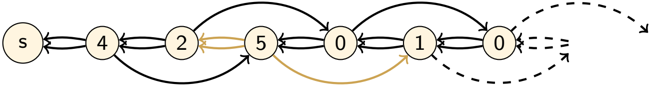


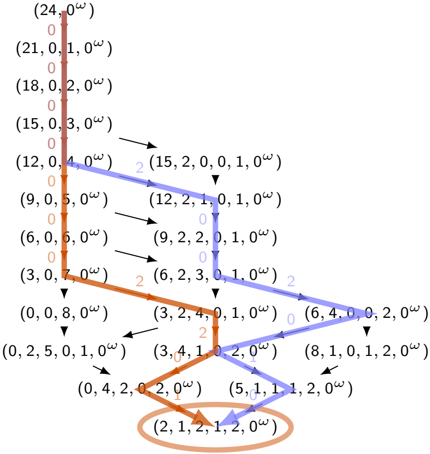
- KSPM is a *Linear Chip Firing Game*.



KSPM - Known Results

Goles, Morvan, Phan (2002)

$D = 3$ and $N = 24$



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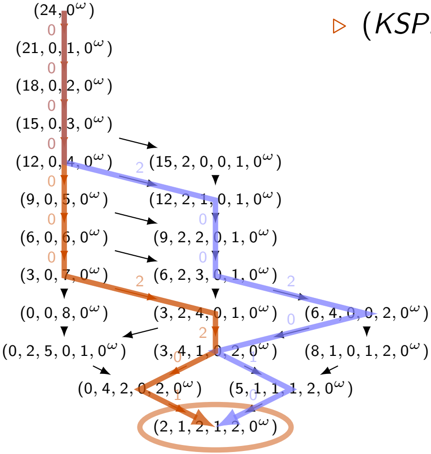

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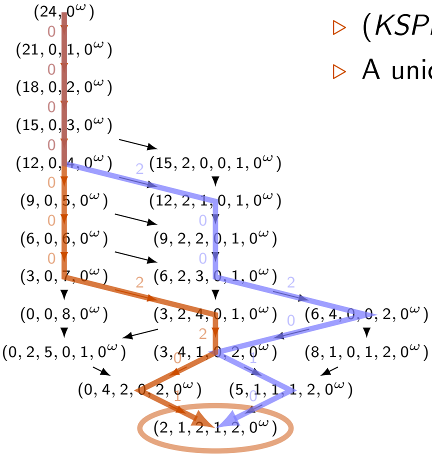

- ▷ $(KSPM(D, N), \to)$ is a lattice.
- ▷ A unique fixed point $\pi(n)$.

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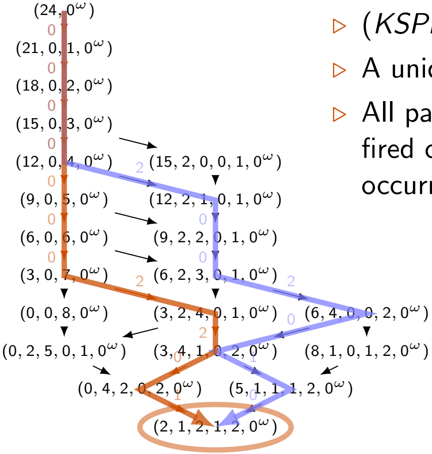

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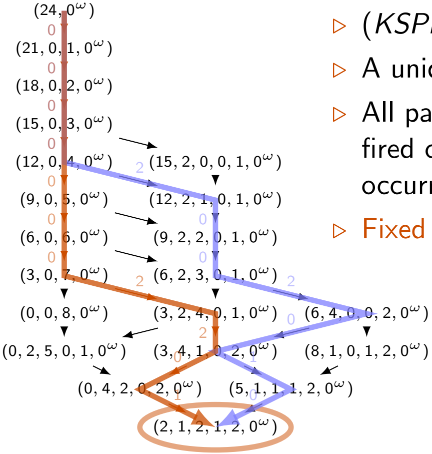

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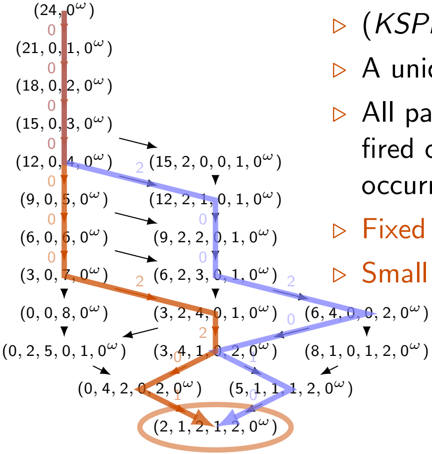

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- A unique fixed point $\pi(n)$.
- All paths are equivalent : same fired columns, with same occurrences.
- Fixed point shape ?
- Small perturbation effect ?

Inductive Approach - Avalanches

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This allows an inductive way for computing $\pi(k)$.

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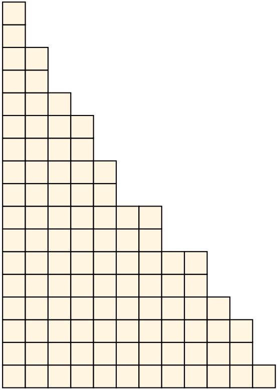
① Each column is fired at most once in an avalanche

Illustration on examples



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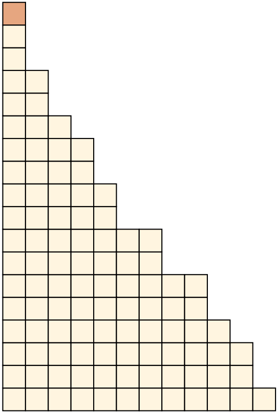

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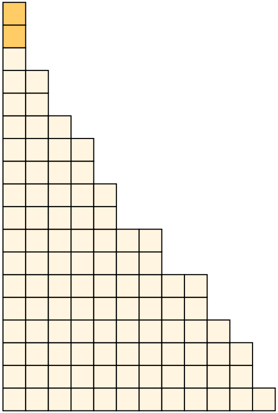

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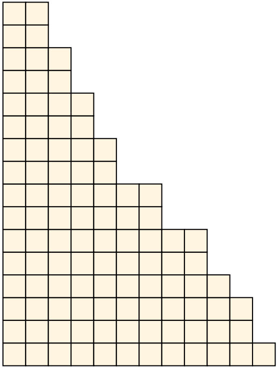

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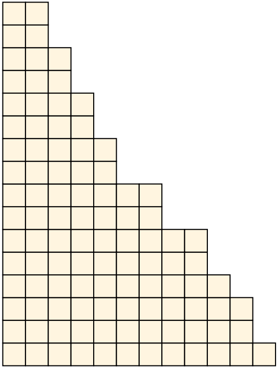

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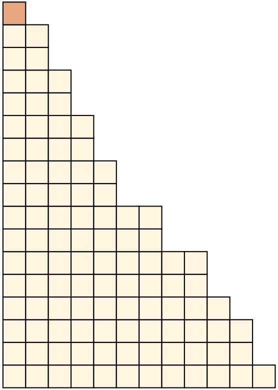

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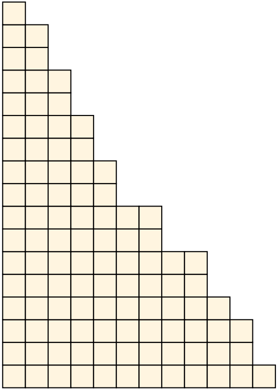

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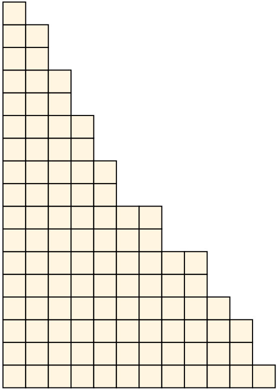

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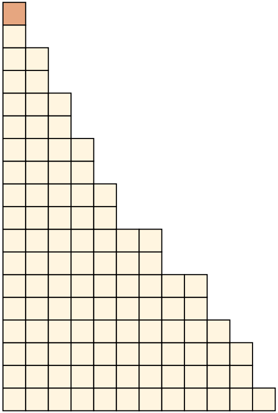

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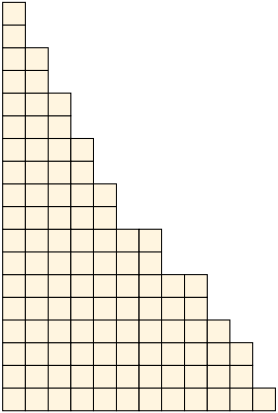

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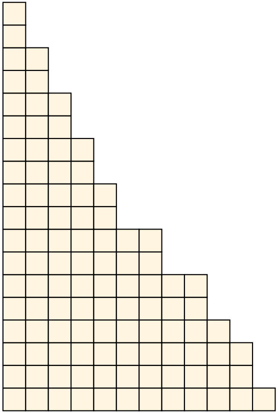

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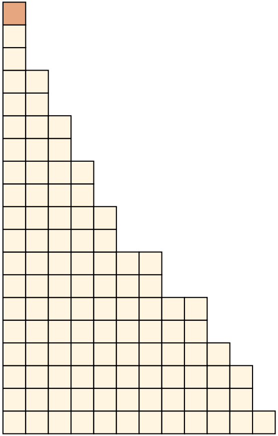

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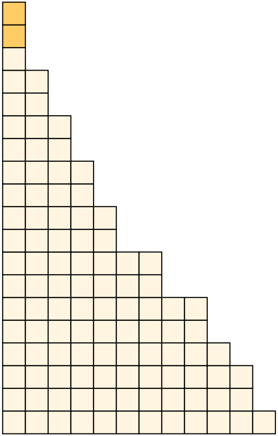

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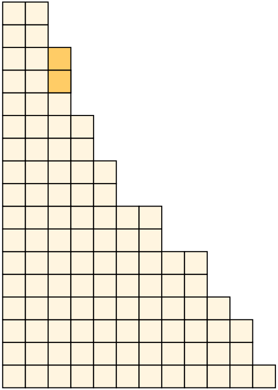

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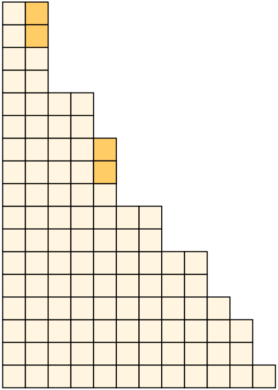

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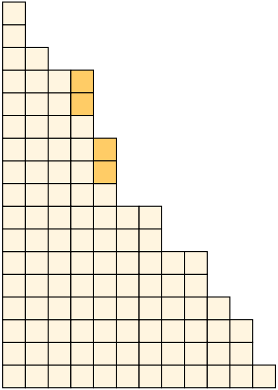

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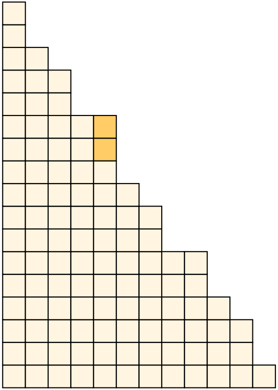

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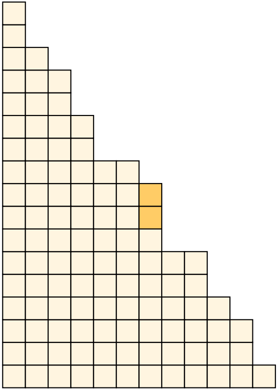

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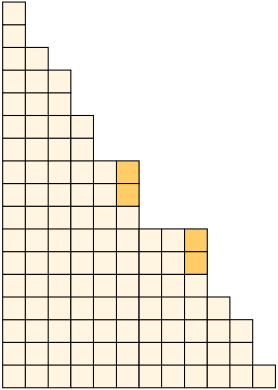

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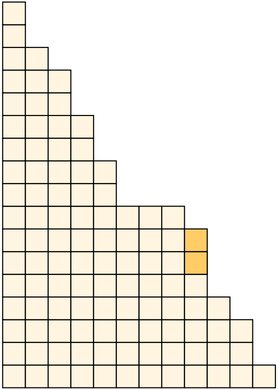

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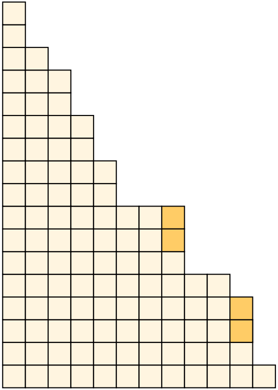

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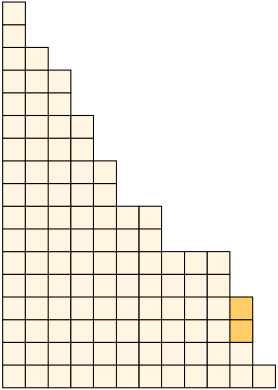

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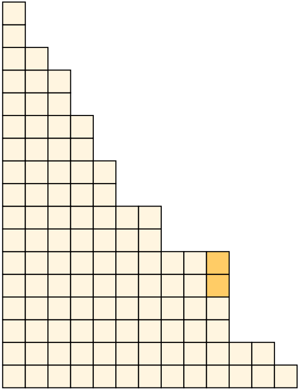

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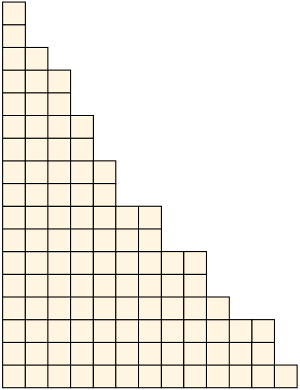

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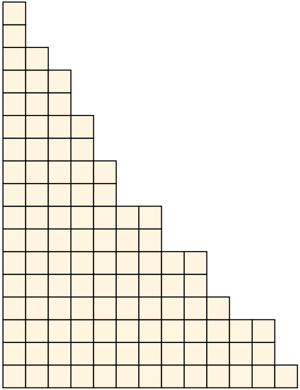

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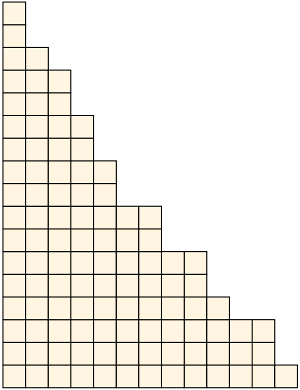

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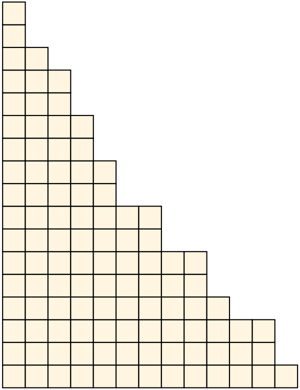

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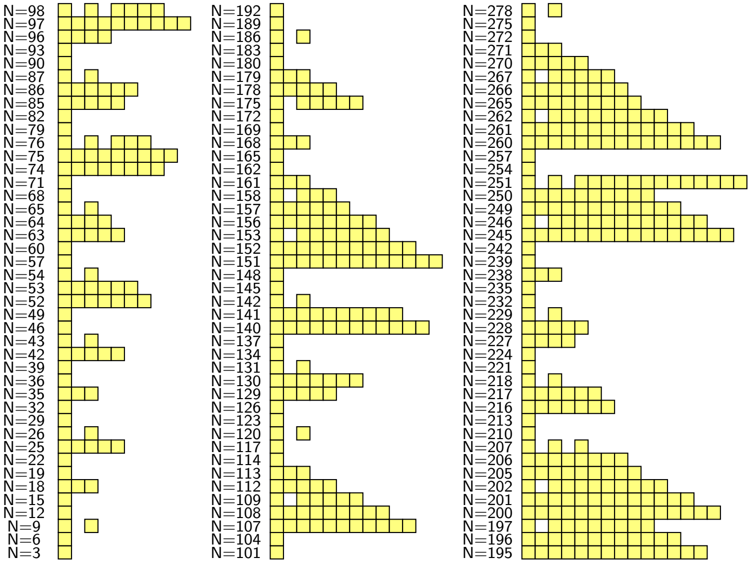
Avalanches $\longleftrightarrow$ SOC

First Avalanches



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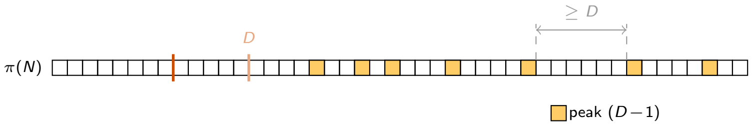
- **Consequence :** except for leftmost columns,

  we do not care about holes

  (since the support of the sand pile $\pi(N)$ is in $\Theta(\sqrt{N})$).

Beyond the Holes : Long Avalanches

- *Peak* : column whose height difference is $D-1$.
- Peaks are masters of long avalanches.



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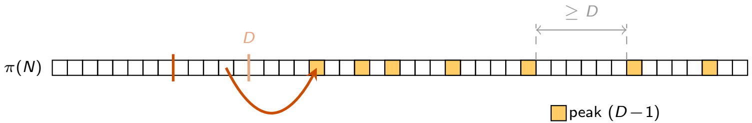

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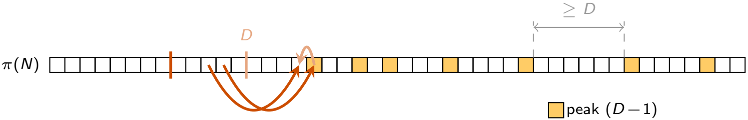

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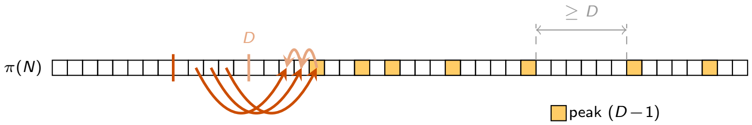

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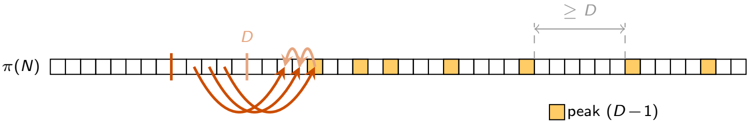

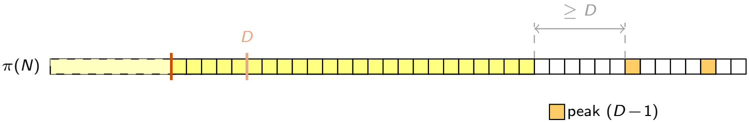

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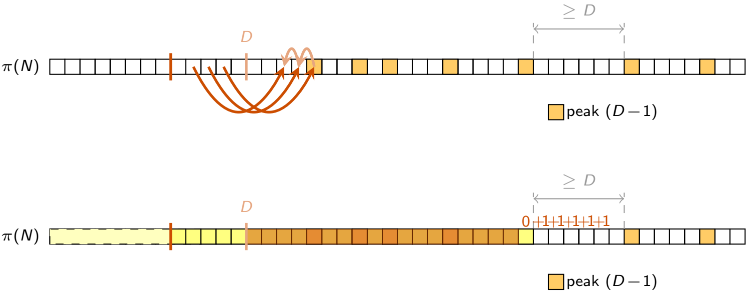


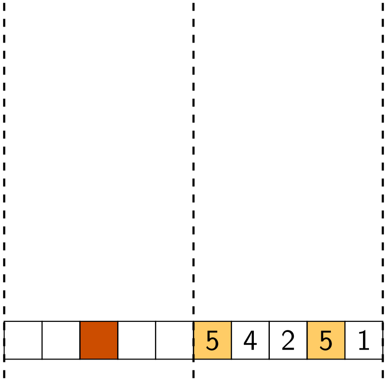
Avalanche effect :

- the last reached peak $:= 0$,
- the $D-1$ columns just after the last reached peak : $+\ 1$,
- other columns : unchanged.

③ The avalanche effect is easily computable beyond the holes

Transduction

Governing peak : rightmost peak before the interval



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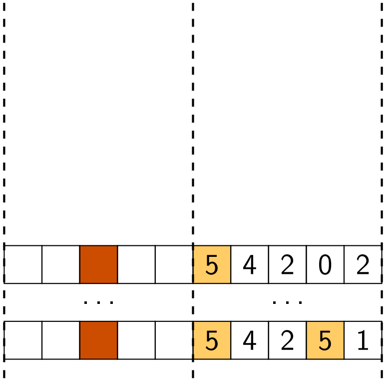

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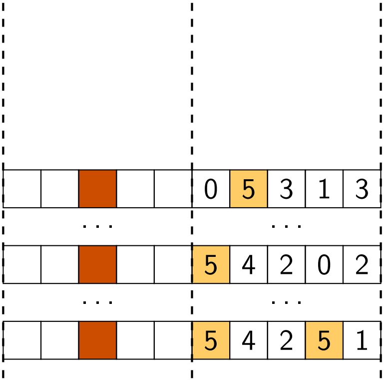

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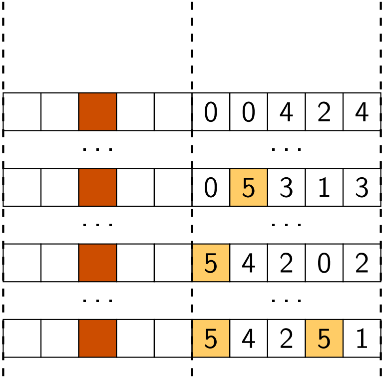

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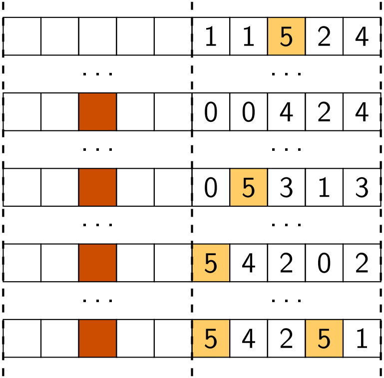

The governing peak really governs the evolution of the interval

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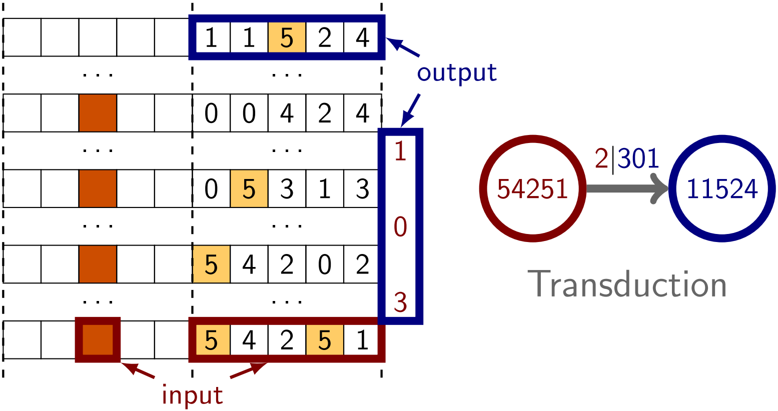

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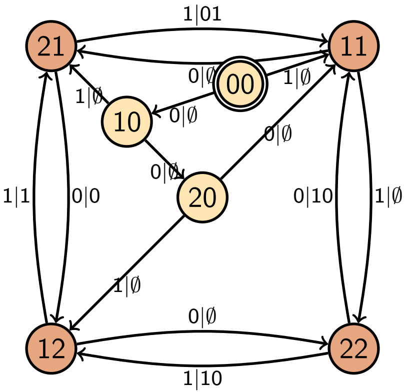
Wave Pattern - Example for $D = 3$



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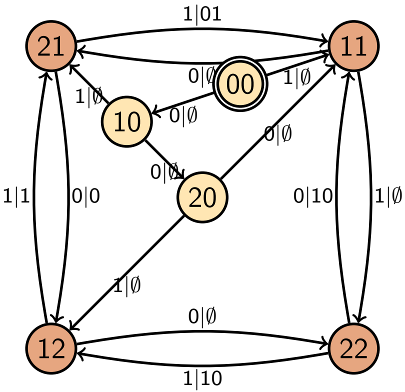

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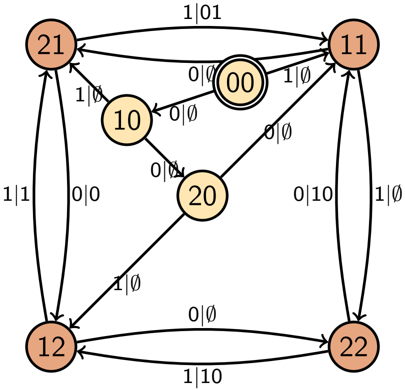

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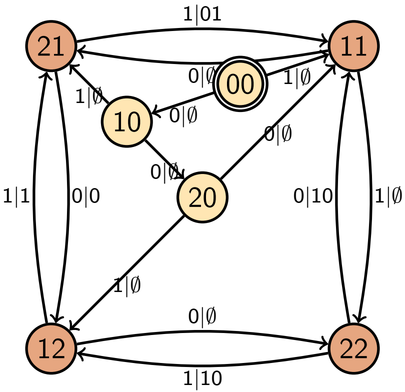

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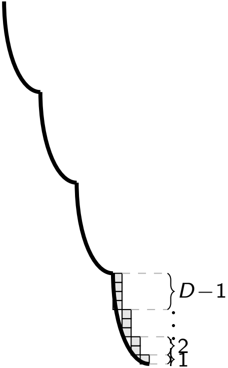

Wave $(D-1, \ldots, 1)^k$

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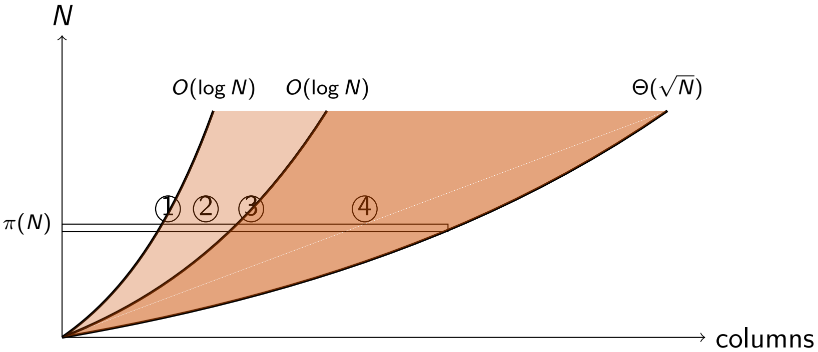
Conclusion



	D=3	general D
① emergent density on avalanches	■	
② density used to built a transducer	■	■
③ transducer iteration involves a balanced evolution	■	
④ balanced evolution involves the wave pattern	■	■

Thank you for Sandpiling, Mister Goles