SANTIAGO NUMÉRICO III

Noveno Encuentro de Análisis Numérico de Ecuaciones Diferenciales Parciales

Facultad de Matemáticas, Pontificia Universidad Católica de Chile
Santiago, Junio 28 - 30, 2017

PROGRAM and ABSTRACTS

Contents

1 Introduction

The **Noveno Encuentro de Análisis Numérico de Ecuaciones Diferenciales Parciales** has been organized in sequential talks of **45**, **30** and **15** minutes length (40, 25 and 10 minutes for the presentation, respectively, and 5 minutes for questions and comments). All the talks will be given at Auditorium Erika Himmel König of Centro **Mide UC**.

In the following pages, we describe the corresponding program. In case of a multi-authored contribution, the speaker is underlined.

The organizers acknowledge financial support by:

- Centro de Modelamiento Matemático (CMM) de la Universidad de Chile,
- Vicerrectoría de Investigación de la Pontificia Universidad Católica de Chile,
- Facultad de Matemáticas de la Pontificia Universidad Católica de Chile,
- Centro de Investigación en Ingeniería Matemática (CI^2MA) de la Universidad de Concepción.

In addition, we express our recognition and gratitude to all speakers for making **Santiago Numérico III** possible.

Organizing Committee
Jessika Camaño
Gabriel N. Gatica
Norbert Heuer
Ricardo Oyarzúa

Santiago, June 2017

2 Wednesday, June 28

8.30-9.15 REGISTRATION

9.15-9.30 WELCOME SPEECH

[*Chairman*: N. HEUER]

9.30-10.15 Ilona Ambartsumyan, Vince J. Ervin, Truong Nguyen, Ivan Yotov: *A nonlinear Biot-Stokes model for the interaction of a non-Newtonian fluid with a poroelastic medium.*

10.15-10.45 Gabriel Acosta, Juan Pablo Borthagaray, Norbert Heuer: *FE approximations of the nonhomogeneous fractional Dirichlet problem.*

10.45-11.00 Ricardo Oyarzúa, Manuel Solano, Paulo Zúñiga: *A high order mixed-FEM for the Stokes problem on curved domains.*

11.00-11.30 COFFEE BREAK

11.30-12.00 Christopher Feuillade, Carlos Jerez-Hanckes, Elwin van 't Wout: *The low-frequency resonance of acoustic scattering at bubble clouds.*

12.00-12.30 Raimund Bürger, Sudarshan K. Kenettinkara, David Zorío: *Approximate Lax-Wendroff discontinuous Galerkin methods for hyperbolic conservation laws.*

12.30-13.00 Jessika Camaño, Ricardo Oyarzúa, Ricardo Ruiz-Baier, Giordano Tierra: *Error analysis of an augmented mixed method for the Navier-Stokes problem with mixed boundary conditions.*

13.00-15.00 LUNCH

[*Chairman*: C. JEREZ-HANCKES]

15.00-15.45 Tan Bui-Thanh: *The upwind hybridized discontinuous Galerkin (HDG) framework: Theory and application to magnetohydrodynamic and atmospheric applications.*

15.45-16.15 Thomas Führer, Norbert Heuer, Ernst P. Stephan: *On the DPG method for Signorini problems.*

16.15-16.45 Alexis Jawtuschenko, Ariel Lombardi: *A mixed VEM scheme for a problem with edge and vertex singularities.*

16.45-17.15 COFFEE BREAK

17.15-17.30 Felipe Lepe, Salim Meddahi, David Mora, Rodolfo Rodríguez: *Acoustic interaction between dissipative fluids.*

17.30-18.00 Nelson O. Moraga, Roberto C. Cabrales, Marcelo A. Marambio: *Solving unsteady coupled fluid mechanics and convective heat transfer problems by a geometric multigrid finite volume method.*

18.00-18.30 Michael Karkulik: *Variational formulation of time-fractional parabolic equations.*

18.30-19.00 Fernando Henríquez, Carlos Jerez-Hanckes: *Multiple traces formulation and semi-implicit scheme for modeling packed cells under electrical stimulation.*

19.30 WELCOME COCKTAIL

3 Thursday, June 29

[*Chairman*: G. GATICA]

9.30-10.15 Erik Burman: *Stabilized finite element methods for ill-posed problems with conditional stability.*

10.15-10.45 Enrique Otárola: *Optimization with respect to order in a fractional diffusion model: analysis, approximation and algorithmic aspects.*

10.45-11.00 Raimund Bürger, Enrique Fernández-Nieto, Víctor Osores: *Polydisperse sedimentation in inclined channels.*

11.00-11.30 COFFEE BREAK

11.30-12.00 Sergio González-Andrade, Sofía López: *A multigrid approach for a class of quasilinear PDEs arising in optimization problems.*

12.00-12.30 Pedro Merino, Alexander Nenjer: *FEM approximation of sparse optimal control problems with finite–dimensional control space.*

12.30-13.00 Antti Niemi: *Simple triangular shell finite elements based on shell theory.*

13.00-15.00 OFFICIAL PICTURE/LUNCH

[*Chairman*: R. BÜRGER]

15.00-15.45 Martin Costabel, Monique Dauge, Serge Nicaise, Jérôme Tomezyk: *The time-harmonic Maxwell equations with impedance boundary conditions.*

15.45-16.15 Ana Alonso Rodríguez, Francesca Rapetti: *The discrete relations between fields and potentials with high order Whitney forms.*

16.15-16.45 Jérôme Bonelle, Pierre Cantin, Erik Burman, Alexandre Ern: *A compact-stencil scheme on polyhedral meshes for steady transport equations.*

16.45-17.15 COFFEE BREAK

17.15-17.30 Carlos Garcia Vera, Gabriel N. Gatica, Antonio Márquez, Salim Meddahi: *A fully discrete scheme for the pressure-stress formulation of a time-domain fluid-structure interaction problem.*

17.30-18.00 Jessika Camaño, Gabriel N. Gatica, Ricardo Oyarzúa, Ricardo Ruiz-Baier: *An augmented stress-based mixed finite element method for the Navier-Stokes equations with nonlinear viscosity.*

18.00-18.30 Ernesto Cáceres, Gabriel N. Gatica, Filánder A. Sequeira: *A mixed virtual element method for a pseudostress-based formulation of linear elasticity.*

20.30 CONFERENCE DINNER: **Restaurant El Mesón Nerudiano**

4 Friday, June 30

[*Chairman*: R. Oyarzúa]

9.30-10.15 Maxim Olshanskii, Arnold Reusken, Xianmin Xu: *Unfitted finite element methods for PDEs on evolving surfaces.*

10.15-10.45 Gabriel Acosta, Francisco Bersetche, Juan Pablo Borthagaray: *A finite element method for fractional evolution problems.*

10.45-11.00 Sergio Caucao, Gabriel N. Gatica, Ricardo Oyarzúa: *Analysis of an augmented fully-mixed formulation for the non-isothermal Oldroyd–Stokes problem.*

11.00-11.30 COFFEE BREAK

11.30-12.00 Raimund Bürger, Stefan Diehl, M. Carmen Martí, Pep Mulet, Ingmar Nopens, Elena Torfs, Peter A. Vanrolleghem: *A multi-class model for batch settling in WRRFs.*

12.00-12.30 Andrés I. Ávila, Andreas Meister, Martin Steigemann: *An adaptive Galerkin method for the time-dependent complex Schrödinger equation.*

12.30-13.00 Jaime E. Muñoz-Rivera, Reinhard Racke, Mauricio Sepúlveda: *On exponential stability for thermoelastic plates – a comparison of different models.*

13.00-15.00 LUNCH

[*Chairman*: N. Heuer]

15.00-15.45 Steffen Börm, Jens M. Melenk: *Directional $\mathcal{H}^2$-matrices for Helmholtz integral operators.*

15.45-16.15 Carlos Pérez Arancibia, Catalin Turc: *A high-order singularity subtraction method for the Nyström discretization of boundary integral equations.*

16.15-16.45 Mario Álvarez, Gabriel N. Gatica, Ricardo Ruiz-Baier: *A posteriori error analysis of a fully-mixed formulation for the Brinkman-Darcy problem.*

16.45-17.15 COFFEE BREAK

17.15-17.30 Gabriel N. Gatica, Mauricio Munar, Filánder Sequeira: *A mixed virtual element method for the Navier-Stokes equations.*

17.30-18.00 Roberto C. Cabrales, Francisco Guillén-González, Juan Jaime, Nelson O. Moraga: *A finite volume method for 3D convective solidification.*

18.00-18.30 Paul Escapil-Inchauspé, Carlos Jerez-Hanckes: *Wave diffraction by random surfaces: Uncertainty quantification via sparse tensor boundary elements.*

18.30-19.00 Eligio Colmenares, Gabriel N. Gatica, Ricardo Oyarzúa: *A posteriori error analyses for augmented mixed formulations of the Boussinesq model.*

19.00 CLOSING WORDS.

5 Abstracts

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SANTIAGO, CHILE, JUNIO 28 - 30, 2017

A finite element method for fractional evolution problems

Gabriel Acosta* Francisco Bersetche† Juan Pablo Borthagaray‡

Abstract

In this work we introduce and analyze a finite element scheme for fractional-in-time and in-space evolution problems. The left-sided fractional order derivative in time we consider is employed to represent memory effects, while a non-local differentiation operator in space accounts for long-range dispersion processes. We discuss well-posedness and obtain regularity estimates for the evolution problems under consideration. The discrete scheme we develop is based on piecewise linear elements for the space variable and a convolution quadrature for the time component. The numerical experiments that we have carried out show a good agreement with our theoretical estimates.

Key words: fractional Laplacian, Caputo derivative, evolution problems

Mathematics subject classifications (2010): 65R20, 65M60, 35R11

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FE approximations of the nonhomogeneous fractional Dirichlet problem *

Gabriel Acosta† Juan Pablo Borthagaray‡ Norbert Heuer§

Abstract

We study finite element approximations of the following non-homogeneous Dirichlet problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \Omega^c, \end{cases} \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$. The operator $(-\Delta)^s$ stands for the Fractional Laplacian and the functions f and g belong to suitable spaces. Our approach is based on weak imposition of the Dirichlet condition and incorporating a nonlocal analogous of the normal derivative as a Lagrange multiplier in the formulation of the problem. In order to obtain convergence orders for our scheme, regularity estimates are developed, both for the solution and its nonlocal derivative. The method we propose requires that, as meshes are refined, the discrete problems be solved in a family of domains of growing diameter.

Key words: Fractional Laplacian, Mixed Methods, a priori error analysis

Mathematics subject classifications (1991): 65N30, 65N12, 35S15

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The discrete relations between fields and potentials with high order Whitney forms

Ana Alonso Rodríguez* Francesca Rapetti†

Abstract

Besides the list of nodes and of their positions, the mesh data structure also contains incidence matrices, saying which node belongs to which oriented edge, which oriented edge bounds which oriented face and so on. These matrices contain all the information about the topology of the domain. Moreover, when using Whitney elements on simplices [2], they connect the dofs describing potentials to dofs describing fields. As an example, the relation $\mathbf{E} = -\mathrm{grad}\, V$ between the electric field $\mathbf{E}$ and the scalar electric potential V become at the discrete level $\mathbf{e} = -G\mathbf{v}$ where G is the transpost of the node-to-edge incidence matrix and $\mathbf{e}$ and $\mathbf{v}$ are the vectors of edge circulations and values at nodes of $\mathbf{E}$ and V respectively. When fields and potentials are approximated by polynomial differential forms of higher degree, the discrete equivalent of the field/potential relation is more structured. The involved matrices present a structure by blocks, each block taking into account of the transmission of dofs associated to a geometrical dimension. We wish to investigate the block-structure of these matrices, when fields and potentials are approximated by high order Whitney forms [5], with dofs given either by the wellknown moments [4, 1] or by the more recent weights on the small simplices [3].

Key words: Discrete potentials, Whitney forms, inicidence matrices, high order approximations.

Mathematics subject classifications (1991): 78M10, 65N30, 68U20

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A posteriori error analysis of a fully-mixed formulation for the Brinkman-Darcy problem*

MARIO ÁLVAREZ[†] GABRIEL N. GATICA[‡] RICARDO RUIZ-BAIER[§]

Abstract

We develop the *a posteriori* error analysis for a mixed finite element method applied to the coupling of Brinkman and Darcy equations in 3D, modelling the interaction of viscous and non-viscous flow effects across a given interface. The system is formulated in terms of velocity and pressure within the Darcy subdomain, together with vorticity, velocity and pressure of the fluid in the Brinkman region, and a Lagrange multiplier enforcing pressure continuity across the interface. The solvability of the fully-mixed formulation along with *a priori* error estimates for a finite element method have been recently established in [M. Alvarez et al., Comput. Methods Appl. Mech. Engrg. 307 (2016) 68–95]. Here we derive a residual-based *a posteriori* error estimator for such a scheme, and we prove its reliability exploiting a global inf-sup condition in combination with suitable Helmholtz decompositions, and properties of Clément and Raviart-Thomas operators. The estimator is also shown to be efficient, following a localisation strategy and appropriate inverse inequalities. We present some numerical tests to confirm the features of the estimator and to illustrate the performance of the method in a number of application-oriented problems.

Key words: Brinkman-Darcy equations, vorticity-based formulation, mixed finite element methods, *a posteriori* error analysis.

Mathematics subject classifications (1991): 65N30, 65N12, 76D07, 65N15

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A nonlinear Biot-Stokes model for the interaction of a non-Newtonian fluid with a poroelastic medium

ILONA AMBARTSUMYAN* VINCE J. ERVIN† TRUONG NGUYEN‡ IVAN YOTOV§

Abstract

We develop and analyze a nonlinear model for the interaction of a quasi-Newtonian free fluid with a poroelastic medium. The flow in the fluid region is described by the Stokes equations and in the poroelastic medium by the quasi-static Biot model. We estabilish existence and uniqueness of a weak solution. A mixed finite element method is developed and analyzed for the approximation of the model, using a Lagrange multiplier to enforce weakly the continuity of flux on the interface. We establish stability and optimal order a priori error estimates. Computational experiments confirming the theoretical convergence rates, as well as applications to flows in filters and hydraulic fracturing are presented.

Key words: nonlinear Biot-Stokes equations, non-Newtonial fluid, fluid-structure interaction

Mathematics subject classifications (2010): 35M13, 65M12, 65M60, 76D07, 76S05

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An adaptive Galerkin method for the time-dependent complex Schrödinger equation *

Andrés I. Ávila† Andreas Meister‡ Martin Steigemann§

Abstract

Nonlinear time-dependent Schrödinger equations (NLSE) model several important problems in quantum physics and morphogenesis. Recently, vortex lattice formation were experimentally found in Bose-Einstein condensate and Fermi superfluids, which are modeled by adding a rotational term in the NLSE equation. Numerical solutions have been computed by using separate approaches for time and space variables. If we see the complex equation as a system, wave methods can be used. In this article, we consider finite element approximations using continuous Galerkin schemes in time and space by adaptive mesh balancing both errors. To get this balance, we adapt the dual weighted residual method used for wave equations and estimates of error indicators for adaptive space-time finite element discretization. The results show how important is dynamic refinement to control the degrees of freedom in space.

Key words: nonlinear time-dependent Schrödinger equation, dual weighted residual method, adaptive Galerkin method

Mathematics subject classifications (2000): 35B40, 35P30, 35Q55, 65N25, 81Q05.

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The low-frequency resonance of acoustic scattering at bubble clouds*

Christopher Feuillade[†] Carlos Jerez-Hanckes[‡] Elwin van 't Wout[§]

Abstract

When air bubbles in water are excited by a low-frequency acoustic signal, they exhibit resonant behaviour. This has a strong impact on the accuracy of underwater sonar surveillance systems, which typically operate at frequencies close to the resonance mode of fish with swim bladders. Even though the resonance of a single air bubble can be calculated analytically, computational methods have to be used when considering a cloud of bubbles. In the case of bubbles situated close to each other, the standard techniques based on low-frequency approximations fail to predict the pronounced frequency shift accurately. In this study, a boundary integral equation of the transmission problem is being discretized with the multi-trace formulation. The numerical results show an accurate simulation of the low-frequency behaviour of different bubble cloud configurations.

Key words: acoustics, resonance, boundary integral equation

Mathematics subject classifications (1991): 65R20, 78A45

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SANTIAGO, CHILE, JUNIO 28 - 30, 2017

A compact-stencil scheme on polyhedral meshes for steady transport equations*

Jérôme Bonelle† Pierre Cantin‡ Erik Burman§ Alexandre Ern¶

Abstract

In this work [1], we present a new vertex-based scheme for the steady transport problem on polyhedral meshes. This scheme extends the stabilized Lagrange finite element on general meshes while containing the total number of degrees of freedom, *i.e.* considering only those attached to mesh vertices. The key idea is to consider scalar degrees of freedom attached to both mesh vertices and mesh cells (as for VAG schemes [2]). Taking inspiration from the recent analysis of composite finite element schemes in [3], the scheme is partially stabilized using the Continuous Interior Penalty approach (see [4]) so as to not hamper the possibility to eliminate locally cell-based unknowns. Well-posedness is obtained from an inf-sup condition and a priori error estimates are inferred for smooth and rough solutions. Numerical results are finally presented on three-dimensional polyhedral meshes, and the benefit of our approach is illustrated in terms of computational cost.

Key words: polyhedral meshes, transport equations, a priori error analysis

Mathematics subject classifications (1991): 65N12, 65N30, 65N08

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Directional $\mathcal{H}^2$-matrices for Helmholtz integral operators

Steffen Börm* Jens M. Melenk†

Abstract

Boundary Element Methods (BEM) are an important tool for the numerical solution of acoustic and electromagnetic scattering problems. These BEM matrices are fully populated so that data-sparse approximations are required to reduce the complexity from quadratic to log-linear. For the high-frequency case of large wavenumber, standard blockwise low-rank approaches are insufficient. One possible data-sparse matrix format for this problem class that can lead to log-linear complexity are *directional* $\mathcal{H}^2$-matrices. [1, 2, 4, 9]. We present a full analysis of a specific incarnation of this approach, [4]. Directional $\mathcal{H}^2$-matrices are blockwise low rank matrices, where the block structure is determined by the so-called parabolic admissibility condition, [6]. In order to achieve log-linear complexity with this admissibility condition, a nested multilevel structure such as $\mathcal{H}^2$-matrices [7] is essential, which provides a data-sparse connection between clusters of source and target points on different levels. We present a particular variant of directional $\mathcal{H}^2$ matrices in which all pertinent objects are obtained by polynomial interpolation. This allows us to rigorously establish exponential convergence in the block rank in conjunction with log-linear complexity. We will also discuss the relation of the directional $\mathcal{H}^2$-matrices to Butterfly Algorithms, [8, 5, 3].

Key words: Helmholtz equation, boundary element method, matrix compression, multipole method

Mathematics subject classifications (1991): 35J05, 65D05, 65N38, 41A10, 65N12

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The upwind hybridized discontinuous Galerkin (HDG) framework: Theory and application to magnetohydrodynamic and atmospheric applications

Tan Bui-Thanh*

Abstract

By revisiting the classical Godunov approach for linear system of hyperbolic Partial Differential Equations (PDEs) we show that it is hybridizable. As such, it is a natural recipe for us to constructively and systematically establish a unified HDG framework for a large class of PDEs including those of Friedrichs type. The unification is fourfold. First, it provides a single constructive procedure to devise HDG schemes for elliptic, parabolic, hyperbolic, and mixed-type PDEs. Second, it reveals the nature of the trace unknowns as the Riemann solutions. Third, it provides a parameter free HDG framework, and hence eliminating the usual complaint that HDG is a parameter-dependent method. Fourth, it allows us to construct the existing HDG methods in a systematic manner. In particular, using the unified framework we can rediscover most of the existing HDG methods and furthermore discover new ones. We present a rigorous wellposedness of the upwind HDG framework for abstract PDEs of Friedriechs' type. Convergent analysis will be established for PDEs arising from Magnetohydrodynamic and atmospheric applications. For nonlinear PDEs, we present an IMEX scheme that exploits the HDG method to solve a single small linear system in each time step: a tremendous advantage over traditional approaches. Part of the talk are the multilevel HDG and iterative HDG approaches that we have developed to solve the HDG systems efficiently on parallel supercomputers. Serial and parallel numerical results for various PDEs will be presented to verify and demonstrate our upwind HDG framework.

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A multi-class model for batch settling in WRRFs *

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Abstract

In order to achieve a unified description of the settling processes in water resource recovery facilities (WRRFs) taking place in both primary settling tanks (PSTs) and secondary settling tanks (SSTs) in conventional wastewater treatment, a new framework, based on the state of the art Bürger-Diehl settling model for SSTs [2], was introduced in [4]. This new unified framework is built on the idea that the distributed properties of the sludge can be captured by dividing the total sludge concentration into a number of classes, depending on the settling velocity distribution. From the mathematical point of view, the extension to a multi-class scenario leads us to a system of nonlinear convection-diffusion equations of the type

$$\frac{\partial \boldsymbol{X}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{X})}{\partial z} = \frac{\partial}{\partial z}\left(\boldsymbol{B}(\boldsymbol{X})\frac{\partial \boldsymbol{X}}{\partial z}\right), \tag{1}$$

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where $\boldsymbol{X} = (X_1, \dots, X_N)^{\mathrm{T}}$ is the sought solution depending on the spatial position z and time t, $X_i = X_i(z,t)$ the mass concentration of class i, $i = 1, \dots, N$, where N is the number of classes considered, $\boldsymbol{f}(\boldsymbol{X}) = (f_1(\boldsymbol{X}), \dots, f_N(\boldsymbol{X}))$ is a vector of convective flux density functions modelling the settling of the sludge and $\boldsymbol{B}(\boldsymbol{X})$ is a given $N \times N$ matrix expressing the diffusive correction, in this case, due to the solids compressibility. This system has to be supplied with initial and boundary conditions. It is well known that under the typical assumptions of sedimentation with compression, (1) is a strongly degenerate parabolic system, while when settling effects are dominant, and $\boldsymbol{B}(\boldsymbol{X}) = \mathbf{0}$, it is a first-order, nonlinear hyperbolic system of conservation laws. Due to the nonlinearity of $\boldsymbol{f}$ as a function of $\boldsymbol{X}$ in combination with the degenerate behaviour, discontinuities or sharp gradients are expected to develop. This property calls for specific techniques for the numerical simulations. The use of implicit-explicit Runge-Kutta (IMEX-RK) schemes [1], along with the weighted essentially non-oscillatory (WENO) shock-capturing technology for the discretization of the set of equations (1), is advocated in [3]. These schemes combine an explicit treatment for the time discretization of the convective terms with an implicit treatment of the diffusive ones, with the result that the resulting IMEX scheme enjoys a less restrictive stability condition than a fully explicit scheme. The use of high resolution shock-capturing finite difference WENO schemes for the discretization of the convective term ensure obtaining precise numerical approximations, accurately resolving the shocks arising and avoiding the spurious oscillations that otherwise often appear.

Key words: multi-class kinematic flow model, wastewater treatment, convection-diffusion equation, primary settling tank, secondary settling tank, settling velocity distribution

Mathematics subject classifications (1991): 35L65, 65M06, 76T05

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Polydisperse sedimentation in inclined channels*

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Abstract

In this talk we consider the flow of a fluid in a channel inclined by an angle θ. The fluid carries particulate matter consisting of small particles that belong to N different species differing in size and density. Here, the polydisperse transport and sedimentation process is modelled by combining a multilayer shallow water system with a polydisperse sedimentation model. The resulting model can be written as a hyperbolic system with nonconservative products of the form

$$\partial_t \vec{w} + \mathcal{A}(\vec{w}) \partial_x \vec{w} + \mathcal{B}(\vec{w}) \partial_y \vec{w} = G(\vec{w}) \quad (1)$$

(plus initial and boundary conditions), which we solve through finite volume techniques. The main difficulty is the definition of the nonconservative products that appear in the model. The unknowns of interest are the height h and the velocity field of the fluid $\vec{v}$ along with the concentrations by layer of the different solid species $\phi_{j,\alpha}$ for $j = 1, \dots, N$ and $\alpha = 1, \dots, M$. We show how to construct a high-order method to approximate the present hyperbolic system with nonconservative products, and report several numerical tests.

Key words: nonconservative products, finite volume method, hyperbolic systems

Mathematics subject classifications (1991): 65M06, 65M12, 76M25

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Approximate Lax-Wendroff discontinuous Galerkin methods for hyperbolic conservation laws *

Raimund Bürger† Sudarshan K. Kenettinkara‡ David Zorío§

Abstract

The Lax-Wendroff time discretization [4] is an alternative method to the popular total variation diminishing Runge-Kutta time discretization of discontinuous Galerkin schemes for the numerical solution of hyperbolic conservation laws. The resulting fully discrete schemes are known as LWDG and RKDG methods, respectively. Although LWDG methods are in general more compact and efficient than RKDG methods (cf., e.g., [2]) of comparable order of accuracy, the formulation of LWDG methods [3, 5, 6] involves the successive computation of exact flux derivatives. This procedure allows to construct schemes of arbitrary formal order of accuracy in space and time. A new approximation procedure, implemented in [7] for finite difference schemes, avoids the computation of exact flux derivatives. The resulting approximate LWDG schemes, addressed as ALDWG schemes, are easier to implement than their original LWDG versions. Numerical results for the scalar and system cases in one and two space dimensions indicate that ALWDG methods are more efficient in terms of error reduction per CPU time than LWDG methods of the same order of accuracy. Moreover, increasing the order of accuracy leads to substantial reductions of numerical error and gains in efficiency for solutions that vary smoothly. This contribution summarizes results of [1].

Key words: Discontinuous Galerkin scheme, Lax-Wendroff time discretization, systems of conservation laws

Mathematics subject classifications (1991): 76S05, 65M08, 65M60, 65M12

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Stabilized finite element methods for ill-posed problems with conditional stability *

Erik Burman†

Abstract

The design and analysis of computational methods for partial differential equations typically relies heavily on the well-posedness of the system under study, both to prove that the finite dimensional system resulting from discretization is invertible and for error estimates on the discrete solution. In practice a large class of problems are indefinite, with stability properties that are not so easily exploited for the design of numerical methods, or even ill-posed, with only some conditional stability. In such cases typically standard finite element methods may fail: the discrete system may not be invertible and when it is, the solution may be inaccurate, even if the ill-posed problem admits a unique solution in a neighbour hood of the available data. The standard way to approach ill-posed problems is through regularisation of the continuous problem and then discretization of the resulting, well-posed, perturbed problem, this however leads to the need of balancing errors due to regularization and discretization.Here we will discuss a different approach based on a stabilized primal-dual finite element formulation, introduced in [1] for the approximation of indefinite elliptic problems. An analysis in the ill-posed case was proposed in [2, 3]. This method is based on the discretization of the ill-posed problem *without any regularization on the continuous level*, instead the discrete system is regularized using ideas from stabilized finite element methods. Sometimes these stabilizing terms coincide with typical Tikhonov regularizations, but in many cases the regularizations do not have an interpretation on the continuous level. We show that the discrete system is always invertible and that the approximate solution satisfies (conditional) a priori and a posteriori error estimates that match the approximation order of the finite element space and the conditional stability properties of the discrete solution. Data perturbations also enter the analysis in a natural way. Three different problems will be discussed to illustrate the theory: The elliptic Cauchy problem, where Dirichlet and Neumann data are set only on a subset of the boundary; an elliptic data assimilation problem where the solution is known in a subset of the bulk, but no boundary data is available; finally a parabolic problem with Dirichlet boundary conditions, where the solution is known in some space time cylinder, but the initial data is unknown [4].

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Key words: ill-posed problems, data assimilation, stabilized finite element methods

Mathematics subject classifications (1991): 65N30, 35R25, 65N20

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A finite volume method for 3D convective solidification *

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Juan Jaime[§] Nelson O. Moraga[§]

Abstract

Solidification is a phase change problem appearing in several engineering applications such as liquid to solid phase transformations of metals and alloys, food freezing and freeze-drying for pharmaceutical drugs production. The mathematical model (see [1]) for this problem is formed by a system of nonlinear partial differential equations jointly with auxiliary expressions relating the variables of the problem: the velocity of the fluid $\boldsymbol{u}$, the pressure p, the enthalpy H, the temperature T and the phase change function f_{pc}. The last variable depends on T and describes the state of every point on the calculation domain Ω: if T is lower than the solid temperature T_s, $f_{pc}(T) = 0$ and we are in solid phase; if T is greater than the liquid temperature T_l, $f_{pc}(T) = 1$ and we are in liquid phase, if $T_s < T < T_l$, $f_{pc}(T) = (T_l - T_s)^{-1}(T - T_s)$, and we have coexistence of both phases. Additionally, it is assumed that the only external force acting on the system is the gravity $\boldsymbol{g}$. The numerical method is based on the finite volume method. Each variable is calculated in a sequential form and the velocity-pressure coupling is managed by using a SIMPLE-like algorithm developed in [2]. To verify the computational implementation of the algorithm, we consider 3D natural convection problem for several Rayleigh numbers from 10^3 to 10^6 and compare with those presented in [3]. The investigated case is the solidification of a Al-Si alloy in a 3D cavity with a graphite mold, extending the 2D results of [4].

Key words: finite volume method, SIMPLE algorithm, 3D convective solidification

Mathematics subject classifications (2010): 65M08, 80A22, 76M12

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S A N T I A G O   N U M É R I C O   III
Noveno Encuentro de Análisis Numérico de Ecuaciones Diferenciales Parciales
Departamento de Matemática, Pontificia Universidad Católica de Chile
SANTIAGO, CHILE, JUNIO 28 - 30, 2017

A mixed virtual element method for quasi-Newtonian Stokes flows*

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Abstract

In this paper we introduce and analyze a virtual element method (VEM) for an augmented mixed variational formulation of a class of nonlinear Stokes models arising in quasi-Newtonian fluids. While the original unknowns are given by the pseudostress, the velocity, and the pressure, the latter is eliminated by using the incompressibility condition, and in order to handle the nonlinearity involved, the velocity gradient is set as an auxiliary one. In this way, and adding a redundant term arising from the constitutive equation relating the psdeudostress and the velocity, an augmented formulation showing a saddle point structure is obtained, whose well-posedness has been established previously by using known results from nonlinear functional analysis. Then, following the basic principles and ideas of the mixed-VEM approach, we introduce a Galerkin scheme employing generic virtual element subspaces and projectors satisfying suitable abstract conditions, and derive the corresponding solvability analysis, along with the associated *a priori* error estimates for the virtual element solution as well as for the fully computable projection of it. Next, we provide two specific choices of subspaces and local projectors verifying the required hypotheses, one of them yielding an optimally convergent mixed-VEM for the fully nonlinear problem studied here, and the other one providing a new approach for the linear version of it, that is for the Stokes problem. In addition, we are able to apply a second element-by-element postprocessing formula for the pseudostress, which yields an optimally convergent approximation of it with respect to the broken $\mathbb{H}(\mathbf{div})$-norm. Finally, several numerical results illustrating the good performance of the method and confirming the theoretical rates of convergence are reported.

Key words: nonlinear Stokes equations, virtual element method, a priori error analysis

Mathematics subject classifications (1991): 65N30, 65N12, 65N15, 76D07

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SANTIAGO NUMÉRICO III
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A mixed virtual element method for a pseudostress-based formulation of linear elasticity *

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Abstract

In this talk we introduce and analyze a mixed virtual element method (mixed-VEM) for a pseudostress-displacement formulation of the linear elasticity problem with non-homogeneous Dirichlet boundary conditions. More precisely, we employ a mixed formulation which does not require symmetric tensor spaces in the finite element discretization. The main unknowns here are given by the pseudostress and the velocity, whereas physical quantities such as the stress, the strain tensor of small deformations, and the rotation, are computed through a simple postprocessing in terms of the pseudostress variable. We first recall the corresponding variational formulation, and then summarize the main mixed-VEM ingredients that are required for our discrete analysis. In particular, we utilize a well-known local projector onto a suitable polynomial subspace, in order to define a calculable version of our discrete bilinear form, whose continuous version requires information of the variables on the interior of each element. Next, we show that the global discrete bilinear form satisfies the hypotheses required by the Babuška-Brezzi theory. In this way, we conclude the well-posedness of our mixed-VEM scheme and derive the associated *a priori* error estimates for the virtual solution as well as for the fully computable projection of it. Furthermore, we also introduce a second element-by-element postprocessing formula for the pseudostress, which yields an optimally convergent approximation of this unknown with respect to the broken $\mathbb{H}(\mathbf{div})$-norm. In addition, this postprocessing formula can also be applied to the stress variable. Finally, several numerical results illustrating the good performance of the method and confirming the theoretical rates of convergence are presented.

Key words: linear elasticity, mixed virtual element method, a priori error analysis

Mathematics subject classifications (1991): 65N30, 65N12, 65N15, 76D07

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S A N T I A G O   N U M É R I C O   III

Noveno Encuentro de Análisis Numérico de Ecuaciones Diferenciales Parciales

Departamento de Matemática, Pontificia Universidad Católica de Chile

SANTIAGO, CHILE, JUNIO 28 - 30, 2017

An augmented stress-based mixed finite element method for the Navier-Stokes equations with nonlinear viscosity*

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Ricardo Oyarzúa§ Ricardo Ruiz-Baier¶

Abstract

A new stress-based mixed variational formulation for the Navier-Stokes equations with constant density and variable viscosity depending on the magnitude of the strain tensor, is proposed and analyzed in this work. Our approach is a natural extension of a technique applied in a recent paper by some of the authors to the same boundary value problem but with a viscosity that depends nonlinearly on the gradient of velocity instead of the strain tensor. In the present case, and besides remarking that the strain-dependence for the viscosity yields a physically more meaningful model, we notice that in order to handle this nonlinearity we now need to incorporate not only the strain itself but also the vorticity as auxiliary unknowns. Furthermore, similarly as in that previous work, and aiming to deal with a suitable space for the velocity, the variational formulation is augmented with Galerkin type terms arising from the constitutive and equilibrium equations, the relations defining the two additional unknowns, and the Dirichlet boundary condition. In this way, and since the resulting augmented scheme can be rewritten as a fixed point operator equation, the classical Schauder and Banach theorems together with monotone operators theory are applied to derive the well-posedness of the continuous and associated discrete schemes. In particular, we show that arbitrary finite element subspaces can be utilized for the latter, and then we derive

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optimal a priori error estimates and the corresponding rates of convergence. Next, a reliable and efficient residual-based a posteriori error estimator on arbitrary polygonal and polyhedral regions is proposed. The main tools employed include Raviart-Thomas and Clément interpolation operators, inverse and discrete inequalities, and the localization technique based on triangle-bubble and edge-bubble functions. Finally, several numerical essays illustrating the good performance of the method, confirming the reliability and efficiency of the a posteriori error estimator, and showing the desired behaviour of the adaptive algorithm, are reported.

Key words: Navier-Stokes equations, nonlinear viscosity, augmented mixed formulation, fixed point theory, mixed finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

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SANTIAGO, CHILE, JUNIO 28 - 30, 2017

A finite volume method for 3D convective solidification *

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Abstract

Solidification is a phase change problem appearing in several engineering applications such as liquid to solid phase transformations of metals and alloys, food freezing and freeze-drying for pharmaceutical drugs production. The mathematical model (see [1]) for this problem is formed by a system of nonlinear partial differential equations jointly with auxiliary expressions relating the variables of the problem: the velocity of the fluid $\boldsymbol{u}$, the pressure p, the enthalpy H, the temperature T and the phase change function f_{pc}. The last variable depends on T and describes the state of every point on the calculation domain Ω: if T is lower than the solid temperature T_s, $f_{pc}(T) = 0$ and we are in solid phase; if T is greater than the liquid temperature T_l, $f_{pc}(T) = 1$ and we are in liquid phase, if $T_s < T < T_l$, $f_{pc}(T) = (T_l - T_s)^{-1}(T - T_s)$, and we have coexistence of both phases. Additionally, it is assumed that the only external force acting on the system is the gravity $\boldsymbol{g}$. The numerical method is based on the finite volume method. Each variable is calculated in a sequential form and the velocity-pressure coupling is managed by using a SIMPLE-like algorithm developed in [2]. To verify the computational implementation of the algorithm, we consider 3D natural convection problem for several Rayleigh numbers from 10^3 to 10^6 and compare with those presented in [3]. The investigated case is the solidification of a Al-Si alloy in a 3D cavity with a graphite mold, extending the 2D results of [4].

Key words: finite volume method, SIMPLE algorithm, 3D convective solidification

Mathematics subject classifications (2010): 65M08, 80A22, 76M12

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SANTIAGO, CHILE, JUNIO 28 - 30, 2017

Analysis of an augmented fully-mixed formulation for the non-isothermal Oldroyd–Stokes problem*

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Abstract

In this work we present an augmented mixed finite element method for the Oldroyd–Stokes problem describing the motion of a non-isothermal incompressible fluid subject to a heat source. The model is described by a system of equations where the Stokes and heat equations are coupled through the convective term and the viscosity of the fluid. We introduce the strain, stress and vorticity tensors, as well as the gradient of the temperature, as further unknowns, which together with the velocity, and the temperature of the fluid, constitute the main unknowns of the system. The pressure is eliminated from the system and can be recovered through a simple post-process of the solution. Since the convective term in the heat equation forces both the velocity and the temperature to live in a smaller space than usual, we augment the variational formulation by using the constitutive and equilibrium equations, the relation defining the strain and vorticity tensors, and the temperature boundary condition. Next, we combine the well-known Schauder and Banach fixed-point theorems with the Lax–Milgram lemma and prove existence and uniqueness of solution of the resulting augmented fully-mixed formulation. The associated Galerkin scheme is defined by Raviart–Thomas elements of order k for the stress tensor and the heat flux vector, continuous piecewise polynomials of order $< k+1$ for velocity and temperature, and piecewise polynomials of order $< k$ for the strain tensor and the vorticity, and its solvability is established similarly to its continuous counterpart, using in this case Brouwer fixed-point theorem for the existence of solution. Finally, we derive optimal a priori error estimates and provide several numerical results illustrating the good performance of the scheme and confirming the theoretical rates of convergence.

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Key words: Oldroyd–Stokes problem, non-isothermal, fixed-point theory, mixed finite element methods, augmented fully-mixed formulation, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 76R05, 76D07

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A posteriori error analyses for augmented mixed formulations of the Boussinesq model*

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Abstract

In previous works of us, new augmented mixed finite element schemes were developed for the stationary Boussinesq problem describing heat driven flow. Our methodologies consisted of a fixed-point strategy for the variational problems that resulted after introducing the same modified pseudostress tensor as an auxiliary unknown in the Navier-Stokes type system involved in the model and, separately, the normal component of the temperature gradient and a vector depending on the temperature, its gradient and the fluid velocity as auxiliary variables in the advection-diffusion equation describing the heat transfer. In both cases, suitable parameterized redundant Galerkin terms were incorporated to the schemes. The well-posedness of both the continuous and discrete settings, the convergence of the associated Galerkin schemes, as well as a priori error estimates of optimal order were stated there. In this talk we present the corresponding a posteriori error analyses of our aforementioned augmented methods in two and three dimensions. Standard arguments relying on duality techniques, and suitable Helmholtz decompositions are used to derive global error indicators and to show their reliability. A globally efficiency property with respect to the natural norm for each estimator is further proved via usual localization techniques of bubble functions. Finally, adaptive algorithms based on reliable, fully local and computable a posteriori error estimators are proposed, and their performance and effectiveness are illustrated through a few numerical examples in two dimensions.

Key words: Boussinesq model, augmented mixed formulation, a posteriori error analysis, reliability, efficiency, adaptive algorithm.

Mathematics subject classifications (1991): 65N30, 65N12, 65N15, 76D07

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SANTIAGO, CHILE, JUNIO 28 - 30, 2017

The time-harmonic Maxwell equations with impedance boundary conditions

Martin Costabel* Monique Dauge†
Serge Nicaise‡ Jérôme Tomezyk§

Abstract

We review some variational formulations of the time-harmonic Maxwell equations with impedance boundary conditions in smooth and non-smooth domains. Some regularity and a priori error estimates will be presented. Numerical results illustrating our theoretical analysis will be given.

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Wave diffraction by random surfaces: Uncertainty quantification via sparse tensor boundary elements*

PAUL ESCAPIL-INCHAUSPÉ† CARLOS JEREZ-HANCKES‡

Abstract

We consider the numerical solution of time-harmonic scattering of acoustic waves or transverse electric polarized fields from obstacles with uncertain geometries, restricting ourselves to a large variety of small stochastic perturbations of a given relatively smooth nominal shape. Using a first-order shape approximation, we derive deterministic boundary equations for the mean field and the two-point correlation function of the random solution for both sound-soft (resp. sound-hard) obstacle and transmission scattering problems. Taking advantage of the tensor structure of the statistical moments, we derive a sparse tensor Galerkin discretization of these equations, by the so-called *combination technique* and we generalize this hierarchical technique to non-nested meshes thanks to the formalism of shape calculus. At the end, we find an accurate approximation of the two-point correlation field with $\mathcal{O}(Nlog(N))$ degrees of freedom instead of $\mathcal{O}(N^2)$.

Key words: wave scattering, shape calculus, boundary element methods, sparse tensor approximation, uncertainty quantification

Mathematics subject classifications (2000): 35J20, 35R60

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On the DPG method for Signorini problems *

Thomas Führer† Norbert Heuer‡ Ernst P. Stephan§

Abstract

We derive and analyze discontinuous Petrov-Galerkin methods with optimal test functions for Signorini-type problems as a prototype of a variational inequality of the first kind. We present different symmetric and non-symmetric formulations where optimal test functions are only used for the PDE part of the problem, not the boundary conditions. For the symmetric case and lowest order approximations, we provide a simple a posteriori error estimate. In a second part, we apply our technique to the singularly perturbed case of reaction dominated diffusion. Numerical results show the performance of our method and, in particular, its robustness in the singularly perturbed case.

Key words: contact problem, Signorini problem, variational inequality, DPG method, optimal test functions, ultra-weak formulation, reaction-dominated diffusion

Mathematics subject classifications (1991): 65N30, 65N12, 35J86, 74M15

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S A N T I A G O   N U M É R I C O   III

Noveno Encuentro de Análisis Numérico de Ecuaciones Diferenciales Parciales

Departamento de Matemática, Pontificia Universidad Católica de Chile

SANTIAGO, CHILE, JUNIO 28 - 30, 2017

A fully discrete scheme for the pressure-stress formulation of a time-domain fluid-structure interaction problem*

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Antonio Márquez§ Salim Meddahi¶

Abstract

We propose an implicit Newmark method for the time integration of the pressure-stress formulation of a fluid-structure interaction problem. The space Galerkin discretization is based on the Arnold-Falk-Winther mixed finite element method with weak symmetry in the solid and the usual Lagrange finite element method in the acoustic medium. We prove that the resulting fully discrete scheme is well-posed and uniformly stable with respect to the discretization parameters and Poisson ratio, and we provide asymptotic error estimates. Finally, we present numerical tests to confirm the asymptotic error estimates predicted by the theory.

Key words: mixed finite elements, fluid-solid interaction, error estimates

Mathematics subject classifications (1991): 65N30, 65M12, 65M15, 74H15

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A mixed virtual element method for the Navier-Stokes equations*

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Abstract

A mixed virtual element method (mixed-VEM) for a pseudostress-velocity formulation of the Navier-Stokes equations with Dirichlet boundary conditions is proposed and analyzed in this work. More precisely, we employ a dual-mixed approach based on the introduction of a nonlinear pseudostress linking the usual linear one for the Stokes equations and the convective term. In this way, the aforementioned new tensor together with the velocity constitute the only unknowns of the problem, whereas the pressure is computed via a postprocessing formula. In addition, the resulting continuous scheme is augmented with Galerkin type terms arising from the constitutive and equilibrium equations, and the Dirichlet boundary condition, all them multiplied by suitable stabilization parameters, so that the Banach fixed point and Lax-Milgram theorems are applied to conclude the well-posedness of the continuous and discrete formulations. Next, we describe the main VEM ingredients that are required for our discrete analysis, which, besides projectors commonly utilized for related models, include, as the main novelty, the simultaneous use of virtual element subspaces for $\mathbf{H}^1$ and $\mathbb{H}(\mathbf{div})$ in order to approximate the velocity and the pseudostress, respectively. Then, the discrete bilinear and trilinear forms involved, their main properties and the associated mixed virtual scheme are defined, and the corresponding solvability analysis is performed using again appropriate fixed point arguments. Moreover, Strang-type estimates are applied to derive the *a priori* error estimates for the two components of the virtual element solution as well as for the fully computable projections of them and the postprocessed pressure. As a consequence, the corresponding rates of convergence are also established. Finally, we follow the same approach employed in previous works by some of the authors and

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introduce an element-by-element postprocessing formula for the fully computable pseudostress, thus yielding an optimally convergent approximation of this unknown with respect to the broken $\mathbb{H}(\mathbf{div})$-norm.

Key words: Navier-Stokes problem, pseudostress-based formulation, augmented formulation, mixed virtual element method

Mathematics subject classifications (1991): 65N30, 65N12, 65N15, 76D07

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A multigrid approach for a class of quasilinear PDEs arising in optimization problems*

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Abstract

In this work, we study a multigrid algorithm for the finite element solution of a class of quasilinear PDEs arising in the numerical solution of several optimization problems. In particular, we focus on optimality systems associated to optimal control problems of equations involving the p-Laplacian operator and PDEs characterizing the solution of a class of quasilinear variational inequalities of the second kind. We analyze the performance of smoothers based on semismooth Newton algorithms and preconditioned descent algorithms, and we discuss the convergence properties of the multigrid algorithm linked to these smoothing algorithms. Finally, several numerical experiments are carried out to show the main features of the proposed method.

Key words: Multigrid methods. Quasilinear PDEs. Variational inequalities. Optimal control problems.

Mathematics subject classifications (1991): 65M55, 49J20, 35J92, 65K15.

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Multiple traces formulation and semi-implicit scheme for modeling packed cells under electrical stimulation*

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Abstract

We model the electrical behavior of several biological cells under external stimuli by extending and computationally improving the semi-implicit multiple traces formulation presented in [3]. Therein, the electric potential and current for a single cell are retrieved through the coupling of boundary integral operators and non-linear ordinary differential systems of equations. Yet, the low-order discretization scheme presented becomes impractical when accounting for interactions among multiple cells. In this note, we consider multi-cellular systems and show existence and uniqueness of the resulting non-linear evolution problem in finite time. Our main tools are analytic semigroup theory along with mapping properties of boundary integral operators. Thanks to the smoothness of cellular shapes, solutions are highly regular at a given time. Hence, spectral spatial discretization can be employed, thereby largely reducing the number of unknowns. Time-space coupling is achieved via a semi-implicit time-stepping scheme shown to be stable and convergent. Numerical results in two dimensions validate our claims and match observed biological behavior for the Hodgin-Huxley dynamical model.

Key words: semi-implicit method, multiple traces formulation, hodgin-huxley model, spectral method

Mathematics subject classifications (2000): 65M38, 65M12, 65R20

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A mixed VEM scheme for a problem with edge and vertex singularities

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Abstract
We introduce and analyze a virtual element method [4] for the mixed formulation of a Poisson problem with right–hand side in L^2 and homogeneous Dirichlet conditions in a non-convex polyhedral domain with edge and vertex singularities, for which, in the presence of the mentioned singularities, it is known that its solution in general is not in H^2 (cfr. [2, 3]). As a consequence, the usual Finite Elements Methods are degraded and we do not obtain an optimal convergence order in the general case. We present a VEM constructing a mesh that combines anisotropic prisms and tetrahedra with pyramids and avoids the use of certain tetrahedra that do not admit anisotropic estimates, recovering the optimal order of convergence. As stated in [1], if we make a subdivision of a general polyhedron Ω only with tetrahedra, then we do not obtain optimal error estimates with Mixed Raviart–Thomas Finite Elements for our problem. That is because there exists a class of anisotropic tetrahedra for which anisotropic estimates needed in the analysis do not hold. For that reason we propose a method which among other things avoids the use of that kind of tetrahedra. In order to deal with general polyhedral domains we need to use mixed meshes, so we present a VEM scheme in a polyhedral mesh $\mathcal{T}_h$ made of tetrahedra, triangular prisms and pyramids. This scheme can be seen as an extension of the method with classical lowest order Raviart–Thomas elements to the case in which the mesh contains pyramids. Besides, it is also an alternative to the generalization of the $H(\text{div})$–conforming elements on pyramids found for instance in [5], whose spaces, in particular, contain rational functions. Incidentally, the number of mesh elements in our method is reduced by a constant factor. We show a discretization method and introduce the corresponding discrete bilinear forms and show that the discrete problem is well posed by proving the discrete $\inf-\sup$ condition. Next we prove that there exists a family of graded meshes $\{\mathcal{T}_h\}_{h\downarrow 0}$ for which we have the optimal estimation $\|\mathbf{u}-\mathbf{u}_h\| \leqslant ch\|f\|, \quad \|p-p_h\| \leqslant ch\|f\|$, with $h \lesssim (1/N)^{1/3}$, where N is the number of elements of the mesh $\mathcal{T}_h$. We show an example of a family of meshes for the Fichera domain that verifies our hipothesis.

Key words: virtual element method, mixed formulation of Poisson Problem, a priori error analysis

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Mathematics subject classifications (2010): 65N30, 65N50, 65N12, 65N15, 35J05

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Variational formulation of time-fractional parabolic equations*

Michael Karkulik†

Abstract

We consider initial/boundary value problems for time-fractional parabolic PDE of order $1/2 < \alpha < 1$, that is, $\partial_t^\alpha u - \Delta u = f$ where ∂_t^α is a fractional time-derivative. Equations of this kind model diffusion phenomena where the mean-square displacement of a diffusing particle scales non-linear in time (as opposed to e.g., the well-known Brownian motion). Recently, researchers have started to analyze finite element methods with respect to their ability to approximate solutions of fractional PDE. In our talk, based on the work [1], we present a variational formulation of time-fractional parabolic equations which resembles classical results for parabolic PDE. This includes the extension of operators defined on real-valued Sobolev spaces to their Banach space-valued counterparts, the so-called *Sobolev-Bochner spaces*, as well as Sobolev Embedding results. This way, we provide a theoretical underpinning for the numerical analysis of such equations.

Key words: Fractional diffusion, Initial/boundary value problem, Well-posedness

Mathematics subject classifications (1991): 26A33, 35K15, 35R11

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Acoustic interaction between dissipative fluids*

Felipe Lepe† Salim Meddahi‡ David Mora§ Rodolfo Rodríguez¶

Abstract

In this talk we present a finite element method for solving a quadratic eigenvalue problem derived from the acoustic vibration problem for a heterogeneous dissipative fluid [1, 3, 7]. The problem is shown to be equivalent to the spectral problem for a noncompact operator and a thorough spectral characterization is given [6, 8]. The numerical discretization of the problem is based on Raviart-Thomas finite elements [2]. The method is proved to be free of spurious modes and to converge with optimal order [4, 5]. Finally, we report numerical tests which allow us to assess the performance of the method.

Key words: Acoustics, dissipative fluids, spectral problems, error estimates, finite elements.

Mathematics subject classifications (2000): 65N25, 65N30, 76M10.

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SANTIAGO, CHILE, JUNIO 28 - 30, 2017

FEM approximation of sparse optimal control problems with finite–dimensional control space*

Pedro Merino† Alexander Nenjer

Abstract

We consider the numerical approximation by the finite element method of a class of sparse optimal control problems governed by linear partial differential equations of elliptic type, which involves the ℓ_1–norm in the cost function. Optimal control problems with finite dimensional controls are motivated by many practical applications of optimal control problems with PDEs. Here, by technological requirements, controls are finite quantities which can be frequently identified with a vector in $\mathbb{R}^N$. Our research focuses of optimal control problems which entails the following main features:

- Controls are in a finite-dimensional control space
- Sparsity inducting term is considered in order to promote "simple solutions (those with many null entries)
- The state of the optimal control problem is a function where fnitely many pointwise state constraints on the state are imposed.

The problem of deriving error estimates has been addressed in [1] in the case of smooth cost functional, where the problem is translated in terms smooth nonlinear programming theory in finite dimensional and applying stability theory of generalized equations. Since in this case the cost function is non-smooth, the extension of this theory is not directly applicable. We bypass this difficulty by reformulating an alternative problem by exploiting the structure of the ℓ_1-norm, which allows to split the solution into its positive and negative parts. We are able to proof that for a parameter of discretization h, an order of convergence of $h|log(h)|$ is obtained. Numerical experiments are shown to underline our theory.

Key words: Finite element approximation, error estimates, sparse optimal control problems

Mathematics subject classifications (2010): 65M15, 49J20, 49J52

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Departamento de Matemática, Pontificia Universidad Católica de Chile

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Solving unsteady coupled fluid mechanics and convective heat transfer problems by a geometric multigrid finite volume method*

NELSON O. MORAGA† ROBERTO C. CABRALES‡ MARCELO A. MARAMBIO†

Abstract

Complexity in 2D fluid mechanics and natural convection problems arises from the coupling between continuity, linear momentum and energy equations caused by the buoyancy forces driven by density gradients being calculated through temperature differences. Increments on Grashof number increases the influence of the convective acceleration terms in the momentum equations, with velocity and temperature gradients being higher towards the external walls of the fluids container. Motion of the fluid is originated by temperature differences in the container walls. One of the difficulties in the solution of the natural convective problems using finite numerical methods with a single grid is that a high number of nodes is required to achieve accurate solution at high Rayleigh numbers. In such cases, the information provided by the boundary conditions is slowly communicated toward the center of the physical domain. Therefore the development of a novel geometric multigrid method, tested in the solution of three problems of increasing complexity via the finite volume method, is the objective of this paper. The cases studied in square cavities includes natural convection of air with $Ra = 10^3$ and 10^4; mixed convection of air inside a cavity with an inner solid either at the center or in the center of the right upper quarter section with Richardson numbers of 0.1 and 10 and solidification with natural convection of a binary alloy (Al-1.7%Si) with $Ra = 10^4$. The results of velocity profiles and streamlines are used to characterize the fluid mechanics while the temperature distribution allows the description of the convective heat transfer. Computing time required to solve each case with the geometric multigrid method, implemented with V, W and F restriction-prolongation cycles, is compared with the time used to achieve the solution with the same accuracy by using a single grid method. Savings of CPU time with the multigrid geometric method ranged from 12% for natural convection with solid-liquid phase up to 93% for natural convection.

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A mixed VEM scheme for a problem with edge and vertex singularities

Alexis Jawtuschenko* Ariel Lombardi†

Abstract

We introduce and analyze a virtual element method [4] for the mixed formulation of a Poisson problem with right–hand side in L^2 and homogeneous Dirichlet conditions in a non-convex polyhedral domain with edge and vertex singularities, for which, in the presence of the mentioned singularities, it is known that its solution in general is not in H^2 (cfr. [2, 3]). As a consequence, the usual Finite Elements Methods are degraded and we do not obtain an optimal convergence order in the general case. We present a VEM constructing a mesh that combines anisotropic prisms and tetrahedra with pyramids and avoids the use of certain tetrahedra that do not admit anisotropic estimates, recovering the optimal order of convergence. As stated in [1], if we make a subdivision of a general polyhedron Ω only with tetrahedra, then we do not obtain optimal error estimates with Mixed Raviart–Thomas Finite Elements for our problem. That is because there exists a class of anisotropic tetrahedra for which anisotropic estimates needed in the analysis do not hold. For that reason we propose a method which among other things avoids the use of that kind of tetrahedra. In order to deal with general polyhedral domains we need to use mixed meshes, so we present a VEM scheme in a polyhedral mesh $\mathcal{T}_h$ made of tetrahedra, triangular prisms and pyramids. This scheme can be seen as an extension of the method with classical lowest order Raviart–Thomas elements to the case in which the mesh contains pyramids. Besides, it is also an alternative to the generalization of the $H(\mathrm{div})$–conforming elements on pyramids found for instance in [5], whose spaces, in particular, contain rational functions. Incidentally, the number of mesh elements in our method is reduced by a constant factor. We show a discretization method and introduce the corresponding discrete bilinear forms and show that the discrete problem is well posed by proving the discrete $\inf-\sup$ condition. Next we prove that there exists a family of graded meshes $\{\mathcal{T}_h\}_{h\downarrow 0}$ for which we have the optimal estimation $\|\mathbf{u}-\mathbf{u}_h\| \leqslant ch\|f\|, \quad \|p-p_h\| \leqslant ch\|f\|$, with $h \lesssim (1/N)^{1/3}$, where N is the number of elements of the mesh $\mathcal{T}_h$. We show an example of a family of meshes for the Fichera domain that verifies our hipothesis.

Key words: virtual element method, mixed formulation of Poisson Problem, a priori error analysis

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Mathematics subject classifications (2010): 65N30, 65N50, 65N12, 65N15, 35J05

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Simple triangular shell finite elements based on shell theory

Antti H. Niemi*

Abstract

In this work we introduce and analyze simple triangular finite elements for a variational formulation of a refined shallow shell model based on the linear theory of elasticity. The primal unknowns of the formulation are the three displacements of the shell mid-surface and two rotations of its normal. These are defined in terms of local curvilinear coordinate systems which are constructed using the nodal normal vectors assumed as input data. We develop linear elements and employ assumed membrane and transverse shear strain fields to alleviate the problems of membrane and shear locking which are encountered in bending-dominated deformations of shells. Unfortunately, the approach is not uniformly convergent for general shell geometries and mesh configurations but nevertheless leads to higher accuracy than conventional formulations based on flat elements and assumed shear strain fields. The efficiency of the proposed method is assessed numerically in problems involving linear static analysis. The numerical examples feature different shell geometries and the results are compared with analytical and numerical reference solutions and with commercial software solutions.

Key words: assumed strain approach, finite element method, membrane locking, shear locking

Mathematics subject classifications (1991): 65N30, 74S05, 74K25

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SANTIAGO, CHILE, JUNIO 28 - 30, 2017

Unfitted finite element methods for PDEs on evolving surfaces*

Maxim Olshanskii† Arnold Reusken‡ Xianmin Xu§

Abstract

Partial differential equations posed on evolving surfaces appear in a number of applications such as two-phase incompressible flows (surfactant transport on the interface) and flow and transport phenomena in biomembranes. Numerical approaches discussed in this report are based on Eulerian description of the surface PDE problem and employ a time-independent background mesh that is not fitted to the surface. The time-dependent surface $\Gamma(t) \subset \mathbb{R}^3$ is assumed smooth and closed for all $t \in [0, T]$. The evolution of the surface may be given implicitly, for example, by the level set method. As an example of the surface PDE we consider the transport–diffusion equation modelling the conservation of a scalar quantity u with a diffusive flux on $\Gamma(t)$, which is passively advected by a given smooth velocity field $\mathbf{w} : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$,

$$\dot{u} + (\mathrm{div}_\Gamma \mathbf{w})u - \nu\Delta_\Gamma u = f \quad \text{on} \quad \Gamma(t), \quad t \in (0, T], \tag{1}$$

with initial condition $u(\mathbf{x}, 0) = u_0(\mathbf{x})$ for $\mathbf{x} \in \Gamma_0 := \Gamma(0)$. Here $\dot{u}$ denotes the advective material derivative, div_Γ is the surface divergence, Δ_Γ is the Laplace–Beltrami operator, and $\nu > 0$ is the constant diffusion coefficient. In the report, we discuss two unfitted finite element methods based on restrictions of outer (bulk, volumetric) finite element functions to the surface. This methodology is known as the trace finite element method (TraceFEM), see the recent review article [4]. In the first approach from [1, 2], one considers a weak formulation of (1) as a surface PDE on space–time manifold

$$\mathcal{S} = \bigcup_{t \in (0,T)} \Gamma(t) \times \{t\}, \quad \mathcal{S} \subset \mathbb{R}^4,$$

and a weak formulation of (1) as a surface PDE on $\mathcal{S}$. Further, one considers space–time prismatic elements in $\mathbb{R}^4$ and defines finite element counterparts of test and trial functions as traces of outer space–time polynomial finite element spaces on a tetrahedral reconstruction of $\mathcal{S}$. It turns out that the resulting method can be implemented in an efficient time-marching way. It has been also proved to be of the optimal first order (in

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space and time) in an energy norm and of the second order convergence in a weaker norm.Another unfitted finite element method we discuss is the one recently proposed in [3]. The main motivation for this method is to avoid space–time elements or any reconstruction of the space–time manifold. The method is based on trace of time-independent finite element spaces on a sequence od steady 2D surfaces and a hyperbolic solve (such as the Fast-Marching method) to find an extension of a function from a 2D surface to its 3D neighborhood.

Key words: surface PDEs, evolving surfaces, TraceFEM, level set method

Mathematics subject classifications (2010): 65M60, 58J32

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Optimization with respect to order in a fractional diffusion model: analysis, approximation and algorithmic aspects*

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Abstract

We consider an identification problem, where the state u is governed by a fractional elliptic equation and the unknown variable corresponds to the order $s \in (0,1)$ of the underlying operator. We study the existence of an optimal pair $(\bar{s}, \bar{u})$ and provide sufficient conditions for its local uniqueness. We develop semi-discrete and fully discrete algorithms to approximate the solutions to our identification problem and provide a convergence analysis. We present numerical illustrations that confirm and extend our theory.

Key words: optimal control problems, identification problems, fractional diffusion, bisection algorithm, finite elements, stability, fully–discrete methods, convergence.

Mathematics subject classifications (1991): 26A33, 35J70, 49J20, 49K21, 49M25, 65M12, 65M15, 65M60.

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A high order mixed-FEM for the Stokes problem on curved domains*

Ricardo Oyarzúa† Manuel Solano‡ Paulo Zúñiga§

Abstract

In this talk we propose and analize a high order mixed finite element method for a pseudostress-velocity formulation of the Stokes problem with Dirichlet boundary condition on curved domains. The method is based on approximating the fluid domain Ω by a polygonal/polyhedral subdomain D_h, where the Galerkin method is applied to approximate the solution, and on a transferring technique, based on integrating the extrapolated discrete gradient of the velocity, to approximate the Dirichlet boundary data on the boundary of D_h. Considering generic finite dimensional subspaces of $\mathrm{H}(\mathrm{div}, \mathrm{D}_h)$ for the pseudostress and of $\mathrm{L}^2(\mathrm{D}_h)$ for the velocity, we prove that the resulting Galerkin scheme becomes well-posed provided suitable hypotheses on the aforementioned subspaces are assumed. A feasible choice of discrete spaces is given by Raviart–Thomas elements of order $k \geq 0$ for the pseudostress and discontinuous polynomials of degree k for the velocity, yielding optimal convergence whenever the distance between both boundaries is of order h. Moreover, the pressure can be approximated optimally through a simple post-processing of the discrete pseudostress. We also derive an error analysis on the complement $\mathrm{D}_h^c := \Omega \backslash \overline{\mathrm{D}}_h$ for the extrapolated solutions. Finally, we provide some numerical experiments illustrating the good performance of the scheme and confirming the theoretical rates of convergence.

Key words: curved domain, high order, Stokes problem, mixed variational formulation

Mathematics subject classifications (2000): 65N30, 65N12, 65N15

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A high-order singularity subtraction method for the Nyström discretization of boundary integral equations

Carlos Pérez Arancibia* Catalin Turc†

Abstract

We present a high-order singularity subtraction method for the Nyström discretization of Laplace and Helmholtz boundary integral operators and layer potentials. The proposed singularity subtraction approach allows integral operators and layer potentials to be expressed in terms of "smooth" integrands that can be easily and inexpensively evaluated by means of standard quadrature rules. The method relies on the use of Green's third identity and pointwise interpolation of the surface density in terms of homogeneous solutions of the associated PDE (harmonic polynomials in the case of the Laplace equation and plane-waves in the case of the Helmholtz equation). Used in conjunction with the Fast Fourier Transform, for evaluation of the surface derivatives of the density, and the Fast Multipole Method, for evaluation of non-local interactions, the proposed methodology enables second-kind integral equations to be solved in $O(N \log N)$ operations, where N denotes the number of discretization (quadrature) points on the boundary. A variety of numerical examples in two and three spatial dimensions—including smooth and piecewise smooth domains—demonstrate the capabilities of the proposed methodology and its advantages over high-order and spectrally accurate Nyström methods based on specialized quadrature rules.

Key words: Laplace equation, Helmholtz equation, Nyström method, singularity subtraction, Fast Multipole Method.

Mathematics subject classifications (2010): 45A05, 45E99, 30E25, 65R20

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