DISCONTINUOUS GALERKIN METHOD FOR HYPERBOLIC EQUATIONS WITH SINGULARITIES

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Abstract. In this talk we will describe our recent work on the study of discontinuous Galerkin (DG) methods for solving hyperbolic equations with singularities in the initial condition, in the source term, or in the solutions. The type of singularities include both discontinuities and δ-functions. Especially for problems involving δ-singularities, many numerical techniques rely on modifications with smooth kernels and hence may severely smear such singularities, leading to large errors in the approximation. On the other hand, the DG methods are based on weak formulations and can be designed directly to solve such problems without modifications, leading to very accurate results. The first part of this talk consists of the joint work with Qiang Zhang, on an error estimate for the explicit Runge-Kutta DG (RKDG) method to solve a linear hyperbolic equation in one dimension with discontinuous but piecewise smooth initial data. The $L^2(\mathbb{R}\backslash\mathcal{R}_t)$-norm error at the final time t is proved to be optimal in both space and time, where $\mathcal{R}_t$ is the pollution region due to the initial discontinuity with the width $\mathcal{O}(\sqrt{t\beta}h^{1/2}\log(1/h))$. Here h is the maximum cell length and β is the flowing speed. The second part of this talk consists of the joint work with Yang Yang, on the development and analysis of DG methods to solve hyperbolic equations involving δ-singularities. Negative-order norm error estimates for the accuracy of DG approximations to δ-singularities are investigated. We first consider linear hyperbolic conservation laws in one space dimension with singular initial data. We prove that, by using piecewise k-th degree polynomials, at time t, the error in the $H^{-(k+1)}(\mathbb{R}\backslash\mathcal{R}_t)$ norm is $(2k+1)$-th order, where $\mathcal{R}_t$ is the pollution region due to the initial singularity with the width of order $\mathcal{O}(h^{1/2}\log(1/h))$ and h is the maximum cell length. As an application of the negative-order norm error estimates, we convolve the numerical solution with a suitable kernel which is a linear combination of B-splines, to obtain L^2 error estimate of $(2k+1)$-th order for the post-processed solution. Moreover, we also obtain high order superconvergence error estimates for linear hyperbolic conservation laws with singular source terms by applying Duhamel's principle. Numerical examples including acoustic equation and the nonlinear rendez-vous algorithms are given to demonstrate the good performance of DG methods for solving hyperbolic equations involving δ-singularities.

Keywords: Runge-Kutta discontinuous Galerkin method; discontinuous solutions; δ-singularities; pollution region; error estimate; hyperbolic problems; negative-order norm; superconvergence; post-processing; rendez-vous.

Mathematics Subject Classifications (2000): 65M60, 65M15

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