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A TWO-LANE BIDIRECTIONAL NONLOCAL TRAFFIC MODEL

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Abstract. We propose and study a nonlocal system of balance laws, which models the traffic dynamics on a two-lane and two-way road where drivers have a preferred lane (the lane on their right) and the other one is used only for overtaking. In this model, the convective part is intended to describe the intralane dynamics of vehicles: the flux function includes local and nonlocal terms, namely, the velocity function in each lane depends locally on the density of the class of vehicles traveling on their preferred lane and in a nonlocal form on the density of the class of vehicles overtaking in the opposite direction. The source terms are intended to describe the coupling between the two lanes: the overtaking and return criteria depend on weighted means of the downstream traffic density of the class of vehicles traveling in their preferred lane and of the class of vehicles traveling in the opposite direction on the same lane. We construct approximate solutions using a finite volume scheme and we prove existence of weak solutions by means of compactness estimates. We also show some numerical simulations to describe the behaviour of the numerical solutions in different situations and to illustrate some features of model.

1. Introduction

1.1. **Motivation.** Our main objective is to model vehicular traffic flow on a two-lane and two-way road where drivers have a preferred lane, the lane on their right, and the left one is used only for overtaking slower vehicles, see Figure 1. Lanes are labeled as lane 1 and lane 2 and we denote by $\rho_1 := \rho_1(t,x)$ and $\rho_2 := \rho_2(t,x)$ the density of cars traveling from the left to right on the lane 1 and lane 2, respectively; by $\tilde{\rho}_1 := \tilde{\rho}_1(t,x)$ and $\tilde{\rho}_2 := \tilde{\rho}_2(t,x)$ the density of cars traveling from the right to left on the lane 2 and lane 1, respectively; ρ_1 is the preferred class of vehicle on the lane 1 and $\tilde{\rho}_1$ is the preferred class on the lane 2. In order to extend the classical LWR (Lighthill - Whitham [17] and Richards [18]) traffic model over a two-lane two-way road where overtaking of cars is allowed, we assume that the velocity in each lane depends not only on the density of the priority class of vehicles, but also on the density of the class travelling in the opposite direction, eventually overtaking, which leads the following model

$$
\begin{cases}
\partial_t \rho_1 + \partial_x \left(\rho_1 v_1 \left(\rho_1 + (\rho_{\max} - \rho_1) H_\varepsilon(\tilde{\rho}_2 * \omega_\eta)\right)\right) &= 0,\\
\partial_t \rho_2 + \partial_x \left(\rho_2 v_2 \left(\rho_2 + (\rho_{\max} - \rho_2) H_\varepsilon(\tilde{\rho}_1 * \omega_\eta)\right)\right) &= 0,\\
\partial_t \tilde{\rho}_1 - \partial_x \left(\tilde{\rho}_1 v_1 \left(\tilde{\rho}_1 + (\rho_{\max} - \tilde{\rho}_1) H_\varepsilon(\rho_2 * \tilde{\omega}_\eta)\right)\right) &= 0,\\
\partial_t \tilde{\rho}_2 - \partial_x \left(\tilde{\rho}_2 v_2 \left(\tilde{\rho}_2 + (\rho_{\max} - \tilde{\rho}_2) H_\varepsilon(\rho_1 * \tilde{\omega}_\eta)\right)\right) &= 0,
\end{cases}
\tag{1.1}
$$

where the minus signs in the third and fourth equations in (1.1) indicate that classes $\tilde{\rho}_1$ and $\tilde{\rho}_2$ travel in opposite direction to ρ_2 and ρ_1, respectively. Above, $v_i(\cdot)$, $i = 1, 2$, is the velocity of vehicles, which we assume depends locally on the density of vehicles of the preferential class ρ_i and in a nonlocal form on the downstream density of vehicles of the class coming in opposite direction on the same lane. This nonlocal dependence is introduced to avoid collisions with downstream vehicles traveling in opposite direction on the same lane and is given via the convolution terms $\tilde{\rho}_i * \omega_\eta$ and

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$\rho_i * \tilde{\omega}_\eta$, for $i = 1, 2$, [1, 8, 10] which will be defined later. Likewise, $H_\varepsilon(\cdot)$ is a regularization of the Heaviside function $H(\cdot)$, which keeps the flux smooth to avoid flux discontinuities in the unknown ρ. Indeed, without regularization, if $v(\rho_1, \tilde{\rho}_2) := v_1(\rho_1 + (\rho_{\max} - \rho_1)H(\tilde{\rho}_2 * \omega_\eta)))$, then we have $v(\rho_1, 0) = v_1(\rho_1) > 0$ and $v(\rho_1, \tilde{\rho}_2) = v_1(\rho_{\max}) = 0$ for $\tilde{\rho}_2 > 0$.
We notice that, if $\tilde{\rho}_i = 0$, we recover the local LWR model in the first and second equations in (1.1); the same happens in the third and fourth equations as $\rho_i = 0$.

In addition, we endow to (1.1) with source terms to model overtaking and returning maneuvers. We impose the following rules allowing vehicles to overtake: at the position x, a vehicle can overtake if and only if

- **(C1)** the local velocity $v_i(\rho_i)$ is greater than the velocity of the average of cars in front of it;
- **(C2)** there are no vehicles traveling in the opposite direction neither on the other lane nor on the same lane (overtaking from the other lane);

which is expressed mathematically as follows (see e.g. [11, 15]),

$$S_{\mathrm{O}}(\boldsymbol{\rho}, \mathcal{R}_1, \boldsymbol{\mathcal{R}_2}) = K_1(\rho_{\max} - \rho_2)\rho_1[v_1(\rho_1) - v_1(\mathcal{R}_1)]^+(1 - H_\varepsilon(\mathcal{R}_{2,1} + \mathcal{R}_{2,2})) \tag{1.2}$$

Above, $\boldsymbol{\rho} = (\rho_1, \rho_2)$, $\boldsymbol{\mathcal{R}_2} = (\mathcal{R}_{2,1}, \mathcal{R}_{2,2})$, $K_1 > 0$ is a constant, $[s]^+ = \max\{s, 0\}$ and the nonlocal terms $\mathcal{R}_1(\rho_1) = \omega^1_\eta * \rho_1$, $\mathcal{R}_{2,i} = \omega^2_\delta * \tilde{\rho}_i$ for $i = 1, 2$, describe a weighted mean of the cars traveling in the same direction, in front of the drivers on the lane 1, and a weighted mean of the cars traveling in opposite direction, respectively; we also assume that $\delta > \eta$. Furthermore, we enforce the following rule so that overtaking vehicles return to the preferential lane:

- **(C3)** Lane 2 is only for overtaking, so that the vehicles in lane 2 have to return to lane 1 proportionally to the capacity of it. It means that we can define the return term to lane 1, S_{R}, as follow

$$S_{\mathrm{R}}(\boldsymbol{\rho}) = K_2(\rho_{\max} - \rho_1)\rho_2, \tag{1.3}$$

where $K_2 > 0$ is a constant.
Likewise, we define the terms $\tilde{S}_{\mathrm{O}}(\tilde{\boldsymbol{\rho}}, \tilde{\mathcal{R}}_1, \tilde{\boldsymbol{\mathcal{R}}}_{\mathbf{2}})$ and $\tilde{S}_{\mathrm{R}}(\tilde{\boldsymbol{\rho}})$ for the class of vehicle traveling from right to left, as follows

$$\tilde{S}_{\mathrm{O}}(\tilde{\boldsymbol{\rho}}, \tilde{\mathcal{R}}_1, \tilde{\boldsymbol{\mathcal{R}}}_{\mathbf{2}}) = K_1(\rho_{\max} - \tilde{\rho}_2)\tilde{\rho}_1[v_1(\tilde{\rho}_1) - v_1(\tilde{\mathcal{R}}_1)]^+\Big(1 - H_\varepsilon(\tilde{\mathcal{R}}_{2,1} + \tilde{\mathcal{R}}_{2,2})\Big), \tag{1.4}$$

$$\tilde{S}_{\mathrm{R}}(\tilde{\boldsymbol{\rho}}) = K_2(\rho_{\max} - \tilde{\rho}_1)\tilde{\rho}_2, \tag{1.5}$$

where the nonlocal terms $\tilde{\mathcal{R}}_1 = \hat{\omega}^1_\eta * \tilde{\rho}_1$ and $\tilde{\mathcal{R}}_{2,i} = \hat{\omega}^2_\delta * \rho_i$, $i = 1, 2$, describe the average of cars, in the same direction, in front of the drivers on the lane 2, and average of cars traveling in opposite direction, respectively; here we put $\hat{\omega}^1_\eta(\cdot) := \omega^1_\eta(-\,\cdot)$ and $\hat{\omega}^2_\delta(\cdot) := \omega^2_\delta(-\,\cdot)$.

1.2. **Related Work.** Macroscopic models of vehicular traffic flow with nonlocal fluxes have been extensively studied recently, see e.g. [1, 13, 4, 5, 7, 8, 12, 16]. In this kind of models, the velocity function depends on an integral evaluation of downstream traffic states. In this way, they allow to describe traffic flow dynamics in which drivers adapt their velocity to downstream traffic. Most of works present in the literature consider a one-directional road with one class of vehicles. To model more realistic situations, nonlocal models have been extended to multi-class and multilane settings, for example, in [8] is studied a system of nonlocal conservation laws modeling multi-class traffic

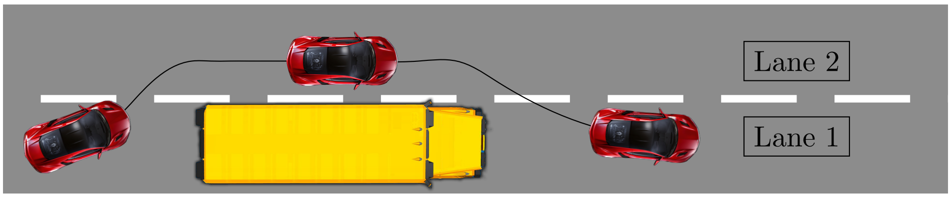

Figure 1. Illustration of the model setting. The red car overtake a slower vehicle, the bus, using the lane 2 and then return to their preferred lane, the lane 1.

flow, providing existence of weak solutions for small times. Concerning multilane flows, Holden and Risebro, [15] proposed a (local) system of balance laws coupled in the source term. In that model, it is assumed that the tendency of drivers to change to a neighboring lane is proportional to the difference in velocity between lanes; the authors proved some bounds for the solutions of the model and show that solutions convergence to a weak solution. This approach was extended in [11] considering nonlocal balance laws where the nonlocal source term was used to describe the lane change rate. More recently, in [9] it has been introduced a system of conservation laws with nonlocal fluxes, coupled in the velocity functions, to describe two classes moving in opposite directions, proving existence of weak solutions for sufficiently small times.

1.3. **Outline of the paper.** This work is organized as follows: In Section 2, we summarize the proposed mathematical model and all the related assumptions. Afterwards, in Section 3, we introduce a Hilliges-Weidlich (HW)-type numerical scheme, as in [5], and we prove fundamental properties such as positivity of approximate solutions, $\mathbf{L}^\infty-$ and $\mathbf{L^1}-$ bounds and $\mathbf{BV}$ estimates, which ensure the convergence of approximate solutions to a weak solution of the proposed model. In Section 4, we present numerical examples illustrating the behavior of the solutions of our model.

2. Mathematical model

The main goal of this work is to study the well-posedness of the nonlocal system of equations

$$\begin{cases} \partial_t\rho_1 + \partial_x(\rho_1 v_1(\rho_1 + (\rho_{\max} - \rho_1)H_\varepsilon(\tilde\rho_2 * \omega_\eta))) = -S_{\mathrm{O}}(\boldsymbol{\rho}, \mathcal{R}_1, \boldsymbol{\mathcal{R}_2}) + S_{\mathrm{R}}(\boldsymbol{\rho}) \\ \partial_t\rho_2 + \partial_x(\rho_2 v_2(\rho_2 + (\rho_{\max} - \rho_2)H_\varepsilon(\tilde\rho_1 * \omega_\eta))) = S_{\mathrm{O}}(\boldsymbol{\rho}, \mathcal{R}_1, \boldsymbol{\mathcal{R}_2}) - S_{\mathrm{R}}(\boldsymbol{\rho}) \\ \partial_t\tilde\rho_1 - \partial_x(\tilde\rho_1 v_1(\tilde\rho_1 + (\rho_{\max} - \tilde\rho_1)H_\varepsilon(\rho_2 * \tilde\omega_\eta))) = -\tilde S_{\mathrm{O}}(\tilde{\boldsymbol{\rho}}, \tilde{\mathcal{R}}_1, \tilde{\boldsymbol{\mathcal{R}}}_{\mathbf{2}}) + \tilde S_{\mathrm{R}}(\tilde{\boldsymbol{\rho}}) \\ \partial_t\tilde\rho_2 - \partial_x(\tilde\rho_2 v_2(\tilde\rho_2 + (\rho_{\max} - \tilde\rho_2)H_\varepsilon(\rho_1 * \tilde\omega_\eta))) = \tilde S_{\mathrm{O}}(\tilde{\boldsymbol{\rho}}, \tilde{\mathcal{R}}_1, \tilde{\boldsymbol{\mathcal{R}}}_{\mathbf{2}}) - \tilde S_{\mathrm{R}}(\tilde{\boldsymbol{\rho}}), \end{cases} \tag{2.1}$$

where $\boldsymbol{\rho}$ and $\tilde{\boldsymbol{\rho}}$, take values in the set

$$\Omega = \left\{\boldsymbol{\rho}, \tilde{\boldsymbol{\rho}} \in \mathbb{R}^2 : 0 \le \rho_i \le \rho_{\max},\ 0 \le \tilde\rho_i \le \rho_{\max},\ i = 1, 2\right\},$$

and the convolution terms in the fluxes are defined as

$$\tilde\rho_i * \omega_\eta := \int_x^{x+\eta} \omega_\eta(y - x)\tilde\rho_i(t, y)\mathrm{d}y, \qquad \rho_i * \tilde\omega_\eta := \int_{x-\eta}^{x} \tilde\omega_\eta(y - x)\rho_i(t, y)\mathrm{d}y,$$

for $i = 1, 2$, where $\tilde\omega_\eta(x) = \omega_\eta(-x)$. The initial conditions satisfy

$$\begin{aligned} \rho_1(x, 0) &= \rho_1^0(x) \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0, \rho_{\max}]), \quad \rho_2(x, 0) = 0, \\ \tilde\rho_1(x, 0) &= \tilde\rho_1^0(x) \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0, \rho_{\max}]), \quad \tilde\rho_2(x, 0) = 0, \end{aligned} \tag{2.2}$$

where the initial conditions $\rho_2(x, 0) = \tilde\rho_2(x, 0) = 0$ means that there is no overtaking initially. In addition, we consider the following assumptions.

Assumptions 2.1. *The system of nonlocal balance laws* (2.1) *is studied under the following assumptions:*

(i) $v_1, v_2 \in \mathbf{C^1}([0,\rho_{\max}];\mathbb{R}^+)$, with $v_1'(\rho) \le 0,\ v_2'(\rho) \le 0,\ \rho \in [0,\rho_{\max}]$;

(ii) $\omega_\eta \in \mathbf{C^1}([0,\eta];\mathbb{R}^+)$ with $\omega_\eta'(x) \le 0,\ \int_0^\eta \omega_\eta(x)\mathrm{d}x = 1,\ \forall \eta > 0.$

(iii) $\omega^1_\eta \in \mathbf{C^1}([0,\eta];\mathbb{R}^+)$ with $(\omega^1_\eta)'(x) \le 0,\ \int_0^\eta \omega^1_\eta(x)\mathrm{d}x = 1,\ \forall \eta > 0$;

(iv) $\omega^2_\delta \in \mathbf{C^1}([0,\delta];\mathbb{R}^+)$ with $(\omega^2_\delta)'(x) \le 0,\ \int_0^\delta \omega^2_\delta(x)\mathrm{d}x = 1,\ \forall \delta > 0$;

(v) $\mathrm{supp}(\omega^1_\eta) \subset \mathrm{supp}(\omega^2_\delta)$, *i.e,* $\delta > \eta$.

Solutions for (2.1)-(2.2) are intended in the following weak sense:

Definition 2.1 (**Weak solution**). *Let* $\rho_i^0, \tilde\rho_i^0 \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)(\mathbb{R};[0,\rho_{\max}])$ *satisfying* (2.2) *for* $i = 1,2$. *We say that* $\rho_i, \tilde\rho_i \in \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R};[0,\rho_{\max}])\right)$, *with* $(\rho_i(t,\cdot),\tilde\rho_i(t,\cdot)) \in (\mathbf{BV}(\mathbb{R};[0,\rho_{\max}]))^2$ *for* $t \in [0,T]$ *and* $i = 1,2$, *are weak solutions to* (2.1)-(2.2) *if, for any* $\varphi \in \mathbf{C^1_c}([0,T[\times\mathbb{R};\mathbb{R})$, *it holds*

$$\int_0^T\!\!\int_{\mathbb{R}} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \varphi_t \,\mathrm{d}x\mathrm{d}t + \int_0^T\!\!\int_{\mathbb{R}} \begin{pmatrix} \rho_1 v_1(\rho_1 + (\rho_{\max}-\rho_1)H_\varepsilon(\tilde\rho_2 * \omega_\eta)) \\ \rho_2 v_2(\rho_2 + (\rho_{\max}-\rho_2)H_\varepsilon(\tilde\rho_1 * \omega_\eta)) \end{pmatrix} \varphi_x \,\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^T\!\!\int_{\mathbb{R}} \begin{pmatrix} -(S_{\mathrm{O}} - S_{\mathrm{R}}) \\ S_{\mathrm{O}} - S_{\mathrm{R}} \end{pmatrix} \varphi \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \varphi(0,x) \begin{pmatrix} \rho_1(0,x) \\ \rho_2(0,x) \end{pmatrix} \mathrm{d}x = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$\int_0^T\!\!\int_{\mathbb{R}} \begin{pmatrix} \tilde\rho_1 \\ \tilde\rho_2 \end{pmatrix} \varphi_t \,\mathrm{d}x\mathrm{d}t - \int_0^T\!\!\int_{\mathbb{R}} \begin{pmatrix} \tilde\rho_1 v_2(\tilde\rho_1 + (\rho_{\max}-\tilde\rho_1)H_\varepsilon(\rho_2 * \tilde\omega_\eta)) \\ \tilde\rho_2 v_2(\tilde\rho_2 + (\rho_{\max}-\tilde\rho_2)H_\varepsilon(\rho_1 * \tilde\omega_\eta)) \end{pmatrix} \varphi_x \,\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^T\!\!\int_{\mathbb{R}} \begin{pmatrix} -\left(\tilde S_{\mathrm{O}} - \tilde S_{\mathrm{R}}\right) \\ \tilde S_{\mathrm{O}} - \tilde S_{\mathrm{R}} \end{pmatrix} \varphi \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \varphi(0,x) \begin{pmatrix} \tilde\rho_1(0,x) \\ \tilde\rho_2(0,x) \end{pmatrix} \mathrm{d}x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Our main result is given by the following theorem, which states the existence of solutions to problem (2.1) - (2.2).

Theorem 2.1. *Let* $\rho_i^0, \tilde\rho_i^0 \in (\mathbf{L^\infty} \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^+)$ *satisfying* (2.2) *for* $i = 1,2$ *and Assumptions* 2.1 *hold. Then, for all* $T > 0$, *the Cauchy problem* (2.1)-(2.2) *admits a weak solution on* $[0,T]\times\mathbb{R}$ *in the sense of the Definition* 2.1.

To prove Theorem 2.1, we propose a finite volume numerical scheme with operator splitting and we derive some important properties of the approximate solutions, as well as compactness estimates that will allow us to conclude by Helly's Compactness Theorem.

3. Numerical scheme

3.1. **Discretization of model** (2.1)-(2.2)**:** We take a uniform space step Δx and a time step Δt subject to a Courant-Friedrichs-Levy (CFL) condition, which will be specified later. For any $j \in \mathbb{Z}$, let $x_{j+1/2} = (j+1/2)\Delta x$ be the cell interfaces and $x_j = j\Delta x$ be the cell centers. We fix $T > 0$, and set $N_T \in \mathbb{N}$ such that $N_T\Delta t \leq T < (N_T+1)\Delta t$ and define the time mesh as $t^n = n\Delta t$, for $n = 0,\dots,N_T$. The initial data are approximated, for $j \in \mathbb{Z}$ and $i = 1,2$ as

$$\rho^0_{i,j} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho^0_i(x)\mathrm{d}x \quad \text{and} \quad \tilde{\rho}^0_{i,j} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{\rho}^0_i(x)\mathrm{d}x.$$

We denote, for $k = 0,\dots,N-1$,

$$\omega^k_\eta := \int_{k\Delta x}^{(k+1)\Delta x} \omega_\eta(y)\mathrm{d}y, \qquad \tilde{\omega}^k_\eta := \int_{(-k-1)\Delta x}^{-k\Delta x} \tilde{\omega}_\eta(y)\mathrm{d}y$$

and set the convolution term, for $i = 1,2$,

$$\tilde{R}_{[i]}(x_{j+1/2},t^n) := (\omega_\eta * \tilde{\rho}_{[i],j})(x_{j+1/2},t^n) \approx \sum_{k=0}^{N-1} \omega^k_\eta \tilde{\rho}^n_{[i],j+k+1},$$

$$R_i(x_{j+1/2},t^n) := (\tilde{\omega}_\eta * \rho_{i,j})(x_{j+1/2},t^n) \approx \sum_{k=0}^{N-1} \tilde{\omega}^k_\eta \rho^n_{i,j-k},$$

where the notation $[\cdot]$, means $[1] = 2$ and $[2] = 1$. Likewise, we define a piecewise constant approximate solution

$$\boldsymbol{\rho}^{\boldsymbol{\Delta}}(t,x) = (\rho^\Delta_1(t,x),\rho^\Delta_2(t,x)) \qquad \text{and} \qquad \tilde{\boldsymbol{\rho}}^{\boldsymbol{\Delta}}(t,x) = (\tilde{\rho}^\Delta_1(t,x),\tilde{\rho}^\Delta_2(t,x)),$$

as follows, for $i = 1,2$

$$\rho^\Delta_i(t,x) = \rho^n_{i,j} \quad \text{and} \quad \tilde{\rho}^\Delta_i(t,x) = \tilde{\rho}^n_{i,j}, \quad (t,x) \in [t^n,t^{n+1}[\times]x_{j-1/2},x_{j+1/2}]. \tag{3.1}$$

The convective terms in (2.1) are obtained via a Hilliges-Weidlich (HW)-type scheme, [3, 2, 14, 6] defined by

$$F_{i,j+1/2} := \rho^n_{i,j} v_1(\rho^n_{i,j+1} + (\rho_{\max} - \rho^n_{i,j+1})H_\varepsilon(\tilde{R}^n_{[i],j+1/2})), \tag{3.2}$$

$$G_{[i],j+1/2} := \tilde{\rho}^n_{[i],j+1} v_2(\tilde{\rho}^n_{[i],j} + (\rho_{\max} - \tilde{\rho}^n_{[i],j})H_\varepsilon(R^n_{i,j+1/2})), \tag{3.3}$$

where the discretizations of convolution terms in the fluxes are given by the expressions

$$\tilde{R}_{[i],j+1/2} := \sum_{k=0}^{N-1} \omega^k_\eta \tilde{\rho}^n_{[i],j+k+1}, \qquad R_{i,j+1/2} := \sum_{k=0}^{N-1} \tilde{\omega}^k_\eta \rho^n_{i,j-k}.$$

Then, we set $\boldsymbol{F}^n_{j+1/2} = [F^n_{1,j+1/2}, F^n_{2,j+1/2}]$ and $\boldsymbol{G}^n_{j+1/2} = [G^n_{1,j+1/2}, G^n_{2,j+1/2}]$.
In order to compute the source terms in (2.1), we first introduce the following notations for the convolutions terms; for kernel functions $\omega^1_\eta \in \mathbf{C}^1_\mathbf{c}([0,\eta])$, $\hat{\omega}^1_\eta \in \mathbf{C}^1_\mathbf{c}([-\eta,0])$ satisfying Assumptions 2.1 for some $N_1 \in \mathbb{N}$ such that $\eta = \Delta x N_1$ and any piecewise constant function u^Δ

$$\mathcal{R}^n_{1,j} := (\omega^1_\eta * u^\Delta)(x_j,t^n) = \int_{x_j}^{x_j+\eta} \omega^1_\eta(y-x_j)u^\Delta(t,y)\mathrm{d}y \approx \sum_{k=0}^{N_1} \gamma_k u^n_{j+k},$$

with coefficients

$$\begin{aligned} &\gamma_0 = \int_0^{\Delta x/2} \omega^1_\eta(y)\mathrm{d}y, \quad \gamma_k = \int_{(k-1/2)\Delta x}^{(k+1/2)\Delta x} \omega^1_\eta(y)\mathrm{d}y, \quad k = 1,\dots,N_1-1; \\ &\gamma_{N_1} = \int_{\eta-\Delta x/2}^{\eta} \omega^1_\eta(y)\mathrm{d}y. \end{aligned} \tag{3.4}$$

We also define

$$\tilde{\mathcal{R}}^n_{1,j} := (\hat{\omega}^1_\eta * u^\Delta)(x_j,t^n) = \int_{x_j-\eta}^{x_j} \hat{\omega}^1_\eta(x_j-y)u^\Delta(t,y)\mathrm{d}y \approx \sum_{k=0}^{N_1} \tilde{\gamma}_k u^n_{j+k},$$

with coefficients

$$\tilde{\gamma}_0 = \int_{-\Delta x/2}^{0} \hat{\omega}_\eta^1(y)\mathrm{d}y, \quad \tilde{\gamma}_i = \int_{-(k+1/2)\Delta x}^{-(k-1/2)\Delta x} \hat{\omega}_\eta^1(y)\mathrm{d}y, \quad k = 1, \dots, N_1 - 1; \\ \tilde{\gamma}_{N_1} = \int_{-\eta}^{-(\eta-\Delta x/2)} \hat{\omega}_\eta^1(y)\mathrm{d}y. \tag{3.5}$$

Similarly, for $\omega_\delta^2 \in \mathbf{C_c^1}([0,\delta])$ and $\hat{\omega}_\delta^2 \in \mathbf{C_c^1}([-\delta,0])$ satisfying Assumptions 2.1 for some $N_2 \in \mathbb{N}$ such that $\delta = \Delta x N_2$, for $i = 1,2$

$$\mathcal{R}^n_{2,i,j} := (\omega_\delta^2 * u^\Delta)(x_j, t^n) = \int_{x_j}^{x_j+\delta} \omega_\delta^2(y - x_j) u^\Delta(t,y)\mathrm{d}y \approx \sum_{k=0}^{N_2} \zeta_k u^n_{j+k},$$

with coefficients ζ_k as in (3.4), and likewise we define

$$\tilde{\mathcal{R}}^n_{2,i,j} := (\hat{\omega}_\delta^2 * u^\Delta)(x_j, t^n) = \int_{x_j-\delta}^{x_j} \hat{\omega}_\delta^2(x_j - y) u^\Delta(t,y)\mathrm{d}y \approx \sum_{k=0}^{N_2} \hat{\zeta}_k u^n_{j+k},$$

with coefficients $\hat{\zeta}_k$ as in (3.5). Finally, for $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, we can compute the source terms (1.2) and (1.3) as

$$S_{\mathrm{O}}(\boldsymbol{\rho}^n_j, \mathcal{R}^n_{1,j}, \boldsymbol{\mathcal{R}^n_{2,j}}) = K_1(\rho_{\max} - \rho^n_{2,j})\rho^n_{1,j}[v_1(\rho^n_{1,j}) - v_1(\mathcal{R}^n_{1,j})]^+ \\ \times \left(1 - H_\varepsilon(\mathcal{R}^n_{2,1,j} + \mathcal{R}^n_{2,2,j})\right),$$

$$S_{\mathrm{R}}(\boldsymbol{\rho}^n_j) = K_2(\rho_{\max} - \rho^n_{1,j})\rho^n_{2,j}$$

In the same way, we can compute the source terms (1.4) and (1.5) as follows

$$\tilde{S}_{\mathrm{O}}(\boldsymbol{\tilde{\rho}^n_j}, \tilde{\mathcal{R}}^n_{1,j}, \boldsymbol{\tilde{\mathcal{R}}^n_{2,j}}) = K_1(\rho_{\max} - \tilde{\rho}^n_{2,j})\tilde{\rho}^n_{1,j}[v_2(\tilde{\rho}^n_{1,j}) - v_2(\tilde{\mathcal{R}}^n_{1,j})]^+ \\ \times \Big(1 - H_\varepsilon(\tilde{\mathcal{R}}^n_{2,1,j} + \tilde{\mathcal{R}}^n_{2,2,j})\Big)$$

$$\tilde{S}_{\mathrm{R}}(\tilde{\boldsymbol{\rho}}^n_j) = K_2(\rho_{\max} - \tilde{\rho}^n_{1,j})\tilde{\rho}^n_{2,j}$$

The values $\boldsymbol{\rho}^n_j = (\rho^n_{1,j}, \rho^n_{2,j})$ and $\tilde{\boldsymbol{\rho}}^n_j = (\tilde{\rho}^n_{1,j}, \tilde{\rho}^n_{2,j})$ are update by using Algorithm 3.1 below, composed of the HW type scheme with operator splitting, to account for the source terms.

Algorithm 3.1.

Input: approximate solution vectors $\boldsymbol{\rho}^n_j = (\rho^n_{1,j}, \rho^n_{2,j})$ *and* $\tilde{\boldsymbol{\rho}}^n_j = (\tilde{\rho}^n_{1,j}, \tilde{\rho}^n_{2,j})$ *for* $j \in \mathbb{Z}$ *and* $t = t^n$

do $j \in \mathbb{Z}$,

$$\boldsymbol{\rho}^{n+1/2}_j \leftarrow \boldsymbol{\rho}^n_j - \lambda\big(\boldsymbol{F}^n_{j+1/2} - \boldsymbol{F}^n_{j-1/2}\big), \quad \textit{using } (3.2) \\ \tilde{\boldsymbol{\rho}}^{n+1/2}_j \leftarrow \tilde{\boldsymbol{\rho}}^n_j + \lambda\big(\boldsymbol{G}^n_{j+1/2} - \boldsymbol{G}^n_{j-1/2}\big), \quad \textit{using } (3.3). \tag{3.6}$$

enddo

do $j \in \mathbb{Z}$,

$$S^{n+1/2}_j \leftarrow S_{\mathrm{O}}\left(\boldsymbol{\rho^{n+1/2}_j}, \mathcal{R}^{n+1/2}_{1,j}, \boldsymbol{\mathcal{R}^{n+1/2}_{2,j}}\right)_j - S_{\mathrm{R}}\left(\boldsymbol{\rho}^{n+1/2}_j\right),$$

$$\tilde{S}^{n+1/2}_j \leftarrow \tilde{S}_{\mathrm{O}}\left(\boldsymbol{\tilde{\rho}^{n+1/2}_j}, \tilde{\mathcal{R}}^{n+1/2}_{1,j}, \boldsymbol{\tilde{\mathcal{R}}^{n+1/2}_{2,j}}\right) - \tilde{S}_{\mathrm{R}}\left(\tilde{\boldsymbol{\rho}}_j{}^{n+1/2}\right),$$

$$\boldsymbol{\rho}^{n+1}_j \leftarrow \boldsymbol{\rho}^{n+1/2}_j + \Delta t[-S^{n+1/2}_j, S^{n+1/2}_j], \tag{3.7}$$

$$\tilde{\boldsymbol{\rho}}_j^{n+1} \leftarrow \tilde{\boldsymbol{\rho}}_j^{n+1/2} + \Delta t[-\tilde{S}_j^{n+1/2}, \tilde{S}_j^{n+1/2}].$$

enddo

Output: approximate solution vectors $\boldsymbol{\rho}_j^{n+1} = (\rho_{1,j}^{n+1}, \rho_{2,j}^{n+1})$ *and* $\tilde{\boldsymbol{\rho}}_j^{n+1} = (\tilde{\rho}_{1,j}^{n+1}, \tilde{\rho}_{2,j}^{n+1})$ *for* $j \in \mathbb{Z}$ *and* $t = t^{n+1} = t^n + \Delta t$.

Notation: In the following, we will set for simplicity:

$$S_{\mathrm{O},j}^n := S_{\mathrm{O}}\left(\boldsymbol{\rho_j^n}, \mathcal{R}_{1,j}^n, \boldsymbol{\mathcal{R}_{2,j}^n}\right),\ S_{\mathrm{R},j}^n := S_{\mathrm{R}}\left(\boldsymbol{\rho_j^n}\right),\ \tilde{S}_{\mathrm{O},j}^n := \tilde{S}_{\mathrm{O}}\left(\tilde{\boldsymbol{\rho}}_{\boldsymbol{j}}^{\boldsymbol{n}}, \tilde{\mathcal{R}}_{1,j}^n, \tilde{\boldsymbol{\mathcal{R}}}_{\boldsymbol{2,j}}^{\boldsymbol{n}}\right),$$
$$\tilde{S}_{\mathrm{R},j}^n := \tilde{S}_{\mathrm{R}}\left(\tilde{\boldsymbol{\rho}}_{\boldsymbol{j}}^{\boldsymbol{n}}\right).$$

In order to prove the existence of solutions of model (2.1), in the next lemmas we will show some properties of the approximate solutions computed by means of Algorithm 3.1. We start by proving positivity of approximate solutions. From now on, if not stated otherwise, $\|\cdot\|$ defines for simplicity the $\mathbf{L}^\infty$ norm over the underlying space, e.g., $\|v\|_{\mathbf{L}^\infty([0,\rho_{\max}])} := \|v\|$.

Lemma 3.1 (Maximum principle). *Let Assumptions* 2.1 *hold. Then under following CFL condition*

$$\Delta t \le \min\left\{\frac{\Delta x}{\mathcal{C}+\mathcal{D}}, \frac{1}{K}\right\}, \tag{3.8}$$

the approximate solutions computed by means Algorithm 3.1 *satisfies* $0 \le \rho_{i,j}^{n+1}, \tilde{\rho}_{i,j}^{n+1} \le \rho_{\max}$*, for all* $j \in \mathbb{Z}$*,* $i = 1,2$*. Here,* $\mathcal{C} = \max\{\|v_1\|, \|v_2\|\}$*,* $\mathcal{D} = \max\{\rho_{\max}\|v_1'\|, \rho_{\max}\|v_2'\|\}$*, and* $K = \rho_{\max}\max\{K_1, K_2\}$*.*

Proof. We assume that $0 \le \rho_{i,j}^n, \tilde{\rho}_{i,j}^n \le \rho_{\max}$ for all $j \in \mathbb{Z}$, for $i = 1,2$, then for the convective part we first have

$$\begin{aligned}\rho_{i,j}^{n+1/2} &= \left(1 - \lambda v_1(\rho_{i,j+1}^n + (\rho_{\max} - \rho_{i,j+1}^n)H_\varepsilon(\tilde{R}_{[i],j+1/2}^n))\right)\rho_{i,j}^n \\ &\quad + \rho_{i,j-1}^n v_1(\rho_{i,j-1}^n + (\rho_{\max} - \rho_{i,j-1}^n)H_\varepsilon(\tilde{R}_{[i],j-1/2}^n)) \\ &\ge 0,\end{aligned}$$

and similarly we can obtain $\tilde{\rho}_{i,j}^{n+1/2} \ge 0$. Now, to prove that $\rho_{i,j}^{n+1}, \tilde{\rho}_{i,j}^{n+1} \ge 0$, we need to show first that $\rho_{i,j}^{n+1/2}, \tilde{\rho}_{i,j}^{n+1/2} \le \rho_{\max}$. To this end, we will simplify the notation, denoting

$$F(u, w, \tilde{R}) = u\, v_i(w + (\rho_{max} - w)H_\varepsilon(\tilde{R})),$$

for $i = 1,2$, and observe that

$$F(u, \rho_{\max}, \tilde{R}) = u\, v_i(\rho_{\max} + (\rho_{\max} - \rho_{\max})H_\varepsilon(\tilde{R})) = 0$$

and also note that the partial derivatives of F with respect to each argument satisfy the monotonicity conditions

$$\begin{aligned}\partial_1 F &= v_i\left(w + (\rho_{\max} - w)H_\varepsilon(\tilde{R})\right) \ge 0, \\ \partial_2 F &= u\, v_i'\left(w + (\rho_{\max} - w)H_\varepsilon(\tilde{R})\right)\left(1 - H_\varepsilon(\tilde{R})\right) \le 0, \\ \partial_3 F &= u\ v_i'\left(w + (\rho_{\max} - w)H_\varepsilon(\tilde{R})\right)(\rho_{\max} - w)H_\varepsilon'(\tilde{R}) \le 0.\end{aligned} \tag{3.9}$$

With this notation, we can write (3.6) as follows

$$\rho_{i,j}^{n+1/2} = \rho_{i,j}^n - \lambda\left[F(\rho_{i,j}^n, \rho_{i,j+1}^n, \tilde{R}_{[i],j+1/2}) - F(\rho_{i,j-1}^n, \rho_{i,j}^n, \tilde{R}_{[i],j-1/2})\right]. \tag{3.10}$$

By the CFL condition (3.8), we get

$$\begin{aligned}
\rho_{i,j}^{n+1/2} &\leq \rho_{i,j}^n + \lambda\left[F(\rho_{\max}, \rho_{i,j}^n, \tilde{R}_{[i],j-1/2}) - F(\rho_{i,j}^n, \rho_{i,j+1}^n, \tilde{R}_{[i],j+1/2})\right] \\
&\leq \rho_{i,j}^n + \lambda F(\rho_{\max}, \rho_{i,j}^n, \tilde{R}_{[i],j-1/2}) \\
&= \rho_{i,j}^n + \lambda\left[F(\rho_{\max}, \rho_{i,j}^n, \tilde{R}_{[i],j-1/2}) - F(\rho_{\max}, \rho_{\max}, \tilde{R}_{[i],j-1/2})\right] \\
&= \rho_{i,j}^n + \lambda\left[-\partial_2 F(\nu_{j+1/2}^n)(\rho_{\max} - \rho_{i,j}^n)\right] \\
&= \left(1 - \lambda(-\partial_2 F(\nu_{j+1/2}^n))\right)\rho_{i,j}^n - \lambda\partial_2 F(\nu_{j+1/2}^n)\rho_{\max} \\
&\leq \rho_{\max},
\end{aligned}$$

where $\nu_{j+1/2}^n \in \mathcal{I}\left((\rho_{\max}, \rho_{i,j}^n, \tilde{R}_{[i],j-1/2}), (\rho_{\max}, \rho_{\max}, \tilde{R}_{[i],j-1/2})\right)$. In the same way we can compute

$$\begin{aligned}
\tilde{\rho}_{i,j}^{n+1/2} &\leq \left(1 - \lambda(-\partial_2 F(\tilde{\nu}_{j+1/2}^n))\right)\tilde{\rho}_{i,j}^n - \lambda\partial_2 F(\tilde{\nu}_{j+1/2}^n)\rho_{\max} \\
&\leq \rho_{\max},
\end{aligned}$$

where $\nu_{j+1/2}^n \in \mathcal{I}\left((\rho_{\max}, \tilde{\rho}_{i,j}^n, R_{i,j+1/2}), (\rho_{\max}, \rho_{\max}, R_{i,j+1/2})\right)$. Now for the reactive term (3.7), we have the following estimates:

$$\begin{aligned}
\rho_{1,j}^{n+1} &= \rho_{1,j}^{n+1/2} - \Delta t\left(S_{\mathrm{O},j}^n - S_{\mathrm{R},j}^n\right) \\
&\leq \rho_{1,j}^{n+1/2} + \Delta t S_{\mathrm{R},j}^n \\
&= \rho_{1,j}^{n+1/2} + \Delta t K_2(\rho_{\max} - \rho_{1,j}^{n+1/2})\rho_{2,j}^{n+1/2} \\
&= \left(1 - \Delta t K_2 \rho_{2,j}^{n+1/2}\right)\rho_{1,j}^{n+1/2} + \Delta t K_2 \rho_{\max}\rho_{2,j}^{n+1/2} \\
&\leq \left(1 - \Delta t K_2 \rho_{2,j}^{n+1/2}\right)\rho_{\max} + \Delta t K_2 \rho_{\max}\rho_{2,j}^{n+1/2} \\
&= \rho_{\max}
\end{aligned}$$

and also

$$\begin{aligned}
\rho_{1,j}^{n+1} &\geq \rho_{1,j}^{n+1/2} - \Delta t S_{\mathrm{O},j}^n \\
&= \rho_{1,j}^{n+1/2} - \Delta t K_1(\rho_{\max} - \rho_{2,j}^{n+1/2})\rho_{1,j}^{n+1/2}[v_1(\rho_j^n) - v_1(\mathcal{R}_{1,j}^{n+1/2})]^+\left(1 - H_\varepsilon(\mathcal{R}_{2,1,j}^{n+1/2} + \mathcal{R}_{2,2,j}^{n+1/2})\right) \\
&= \left(1 - \Delta t K_1(\rho_{\max} - \rho_{2,j}^{n+1/2})[v_1(\rho_j^n) - v_1(\mathcal{R}_{1,j}^{n+1/2})]^+\left(1 - H_\varepsilon(\mathcal{R}_{2,1,j}^{n+1/2} + \mathcal{R}_{2,2,j}^{n+1/2})\right)\right)\rho_{1,j}^{n+1/2} \\
&\geq 0.
\end{aligned}$$

In the same way,

$$\begin{aligned}
\rho_{2,j}^{n+1} &= \rho_{2,j}^{n+1/2} + \Delta t\left(S_{\mathrm{O},j}^n - S_{\mathrm{R},j}^n\right) \\
&\leq \rho_{2,j}^{n+1/2} + \Delta t S_{\mathrm{O},j}^n \\
&= \rho_{2,j}^{n+1/2} + \Delta t K_1(\rho_{\max} - \rho_{2,j}^{n+1/2})\rho_{1,j}^{n+1/2}[v_1(\rho_j^n) - v_1(\mathcal{R}_{1,j}^{n+1/2})]^+\left(1 - H_\varepsilon(\mathcal{R}_{2,1,j}^{n+1/2} + \mathcal{R}_{2,2,j}^{n+1/2})\right) \\
&= \left(1 - \Delta t K_1 \rho_{1,j}^{n+1/2}[v_1(\rho_j^n) - v_1(\mathcal{R}_{1,j}^{n+1/2})]^+\left(1 - H_\varepsilon(\mathcal{R}_{2,1,j}^{n+1/2} + \mathcal{R}_{2,2,j}^{n+1/2})\right)\right)\rho_{2,j}^{n+1/2} \\
&\quad + \Delta t K_1 \rho_{\max}\rho_{1,j}^{n+1/2}[v_1(\rho_j^n) - v_1(\mathcal{R}_{1,j}^{n+1/2})]^+\left(1 - H_\varepsilon(\mathcal{R}_{2,1,j}^{n+1/2} + \mathcal{R}_{2,2,j}^{n+1/2})\right) \\
&= \left(1 - \Delta t K_1 \rho_{1,j}^{n+1/2}[v_1(\rho_j^n) - v_1(\mathcal{R}_{1,j}^{n+1/2})]^+\left(1 - H_\varepsilon(\mathcal{R}_{2,1,j}^{n+1/2} + \mathcal{R}_{2,2,j}^{n+1/2})\right)\right)\rho_{\max} \\
&\quad + \Delta t K_1 \rho_{\max}\rho_{1,j}^{n+1/2}[v_1(\rho_j^n) - v_1(\mathcal{R}_{1,j}^{n+1/2})]^+\left(1 - H_\varepsilon(\mathcal{R}_{2,1,j}^{n+1/2} + \mathcal{R}_{2,2,j}^{n+1/2})\right)
\end{aligned}$$

$$\leq \rho_{\max},$$

and

$$\begin{aligned}
\rho_{2,j}^{n+1} &\geq \rho_{2,j}^{n+1/2} - \Delta t S_{\mathrm{R},j}^{n} \\
&= \rho_{2,j}^{n+1/2} - \Delta t K_2(\rho_{\max} - \rho_{1,j}^{n+1/2})\rho_{2,j}^{n+1/2} \\
&= \left(1 - \Delta t K_2(\rho_{\max} - \rho_{1,j}^{n+1/2})\right)\rho_{2,j}^{n+1/2} \\
&\geq 0.
\end{aligned}$$

Following a similar procedure we get $0 \leq \tilde{\rho}_{i,j}^{n+1} \leq \rho_{\max}$. □

Lemma 3.2 ($\mathbf{L^1}$**−bounds).** *Let* $\rho_i^0, \tilde{\rho}_i^0 \in \mathbf{L^1}(\mathbb{R}; [0, \rho_{\max}])$ *satisfying* (2.2) *for* $i = 1, 2$ *and let Assumptions* 2.1 *hold. Under the CFL condition* (3.8)*, the approximate solutions* $\boldsymbol{\rho}^{\boldsymbol{\Delta}}, \tilde{\boldsymbol{\rho}}^{\boldsymbol{\Delta}}$ *constructed by means of Algorithm* 3.1 *satisfy*

$$\left\|\boldsymbol{\rho}^{\boldsymbol{\Delta}}(t,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R})} := \left\|\rho_1^{\Delta}(t,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^{\Delta}(t,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\rho_1^0\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^0\right\|_{\mathbf{L^1}(\mathbb{R})}.$$

Proof. The proof is done by induction. Observe that by conservation and since $\rho_{1,j}^n \geq 0$ and $\rho_{2,j}^n \geq 0$ we get

$$\begin{aligned}
\left\|\rho_1^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} &= \left\|\rho_1^n\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\rho_1^0\right\|_{\mathbf{L^1}(\mathbb{R})}, \\
\left\|\rho_2^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} &= \left\|\rho_2^n\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\rho_2^0\right\|_{\mathbf{L^1}(\mathbb{R})},
\end{aligned}$$

and therefore

$$\left\|\rho_1^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\rho_1^0\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^0\right\|_{\mathbf{L^1}(\mathbb{R})}.$$

Now let us consider the reactive terms (3.7). Note the fact that when we compute $\rho_{1,j}^{n+1} + \rho_{2,j}^{n+1}$ the source terms sum up to 0, for which we get again by positivity

$$\begin{aligned}
\left\|\rho_1^{n+1}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^{n+1}\right\|_{\mathbf{L^1}(\mathbb{R})} &= \left\|\rho_1^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})} \\
&= \left\|\rho_1^0\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_2^0\right\|_{\mathbf{L^1}(\mathbb{R})}.
\end{aligned}$$

In the same way we get

$$\left\|\tilde{\rho}_1^{n+1}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\tilde{\rho}_2^{n+1}\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\tilde{\rho}_1^{n}\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\tilde{\rho}_2^{n}\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\tilde{\rho}_1^0\right\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\tilde{\rho}_2^0\right\|_{\mathbf{L^1}(\mathbb{R})}.$$

□

The next lemma is an important property that will allow us to prove the **BV** estimates later.

Lemma 3.3. *The maps* $S_{\mathrm{O}},\ S_{\mathrm{R}},\ \tilde{S}_{\mathrm{O}},\ \tilde{S}_{\mathrm{R}}$ *defined in* (1.2), (1.3) *and* (1.4), (1.5), *respectively, are Lipschitz continuous in each of their arguments, more precisely,*

$$\begin{aligned}
&|S_{\mathrm{O},j+1} - S_{\mathrm{O},j}| + |S_{\mathrm{R},j+1} - S_{\mathrm{R},j}| \\
&\leq \mathcal{K}\left(|\rho_{1,j+1} - \rho_{1,j}| + |\rho_{2,j+1} - \rho_{2,j}| + |\mathcal{R}_{1,j+1} - \mathcal{R}_{1,j}| + |\mathcal{R}_{2,1,j+1} - \mathcal{R}_{2,1,j}|\right. \\
&\qquad \left. + |\mathcal{R}_{2,2,j+1} - \mathcal{R}_{2,2,j}|\right),
\end{aligned} \tag{3.11}$$

$$\left|\tilde{S}_{\mathrm{O},j+1} - \tilde{S}_{\mathrm{O},j}\right| + \left|\tilde{S}_{\mathrm{R},j+1} - \tilde{S}_{\mathrm{R},j}\right|$$

$$\leq \mathcal{K}\Big(|\tilde\rho_{1,j+1}-\tilde\rho_{1,j}|+|\tilde\rho_{2,j+1}-\tilde\rho_{2,j}|+\Big|\tilde{\mathcal{R}}_{1,j+1}-\tilde{\mathcal{R}}_{1,j}\Big|+\Big|\tilde{\mathcal{R}}_{2,1,j+1}-\tilde{\mathcal{R}}_{2,1,j}\Big| + \Big|\tilde{\mathcal{R}}_{2,2,j+1}-\tilde{\mathcal{R}}_{2,2,j}\Big|\Big), \tag{3.12}$$

for some constant $\mathcal{K}>0$, depending on K_1, K_2 and $\rho_{\max}$.

Proof. For proving this lemma we will show that maps, S_{O}, S_{R}, $\tilde S_{\mathrm{O}}$, $\tilde S_{\mathrm{R}}$, are continuous and their partial derivative with respect to each argument are bounded.
Clearly the maps S_{O}, S_{R}, $\tilde S_{\mathrm{O}}$, $\tilde S_{\mathrm{R}}$, are continuous. Now on the one hand, for derivatives of S_{O} we get the following estimate,

$$\begin{aligned}
|\partial_1 S_{\mathrm{O}}| &\leq \Big|K_1(\rho_{\max}-\rho_2)[v_1(\rho_1)-v_1(\mathcal{R}_1)]^+\left(1-H_\varepsilon(\mathcal{R}_{2,1}+\mathcal{R}_{2,2})\right)\Big| \\
&\quad+\Big|K_1(\rho_{\max}-\rho_2)\rho_1\left(1-H_\varepsilon(\mathcal{R}_{2,1}+\mathcal{R}_{2,2})\right)v_1'(\rho_1)\Big| \\
&\leq K_1\rho_{\max}\|v_1\|+K_1\rho_{\max}^2\|v_1'\|,
\end{aligned}$$

$$\begin{aligned}
|\partial_2 S_{\mathrm{O}}| &= \left|-K_1\rho_1[v_1(\rho_1)-v_1(\mathcal{R}_1)]^+\left(1-H_\varepsilon(\mathcal{R}_{2,1}+\mathcal{R}_{2,2})\right)\right| \\
&\leq K_1\rho_{\max}\|v_1\|,
\end{aligned}$$

$$\begin{aligned}
|\partial_3 S_{\mathrm{O}}| &\leq \left|K_1(\rho_{\max}-\rho_2)\rho_1\left(1-H_\varepsilon(\mathcal{R}_{2,1}+\mathcal{R}_{2,2})\right)\|v_1'(\mathcal{R}_1)\|\right| \\
&\leq K_1\rho_{\max}^2\|v_1'\|,
\end{aligned}$$

$$\begin{aligned}
|\partial_4 S_{\mathrm{O}}| &= \left|-K_1(\rho_{\max}-\rho_2)\rho_1[v_1(\rho_1)-v_1(\mathcal{R}_1)]^+H_\varepsilon'(\mathcal{R}_{2,1}+\mathcal{R}_{2,2})\right| \\
&\leq K_1\rho_{\max}^2\|v_1\|\|H_\varepsilon'\|,
\end{aligned}$$

$$\begin{aligned}
|\partial_5 S_{\mathrm{O}}| &= \left|-K_1(\rho_{\max}-\rho_2)\rho_1[v_1(\rho_1)-v_1(\mathcal{R}_1)]^+H_\varepsilon'(\mathcal{R}_{2,1}+\mathcal{R}_{2,2})\right| \\
&\leq K_1\rho_{\max}^2\|v_1\|\|H_\varepsilon'\|.
\end{aligned}$$

On the other hand, for the derivatives of S_{R} we get the following estimates

$$\begin{aligned}
|\partial_1 S_{\mathrm{R}}| &= |-K_2\rho_2|\leq K_2\rho_{\max}, \\
|\partial_2 S_{\mathrm{R}}| &= |K_2(\rho_{\max}-\rho_1)|\leq K_2\rho_{\max}.
\end{aligned}$$

The previous facts ensure the Lipschitz continuity of the source terms S_{O} and S_{R} with respect to each of their arguments and at the same time allows us to obtain the estimate (3.11). The estimate (3.12) is obtained by means of a similar procedure. □

Remark 3.1. *Let us consider $\mathcal{R}_j=\sum_{k=0}^{N}\gamma_k u_{j+k}$ as any of the convolution terms in the source terms, then, following* [11, Lemma 3.1]*, the following estimate holds*

$$\sum_{j\in\mathbb{Z}}|\mathcal{R}_{j+1}-\mathcal{R}_j|\leq\sum_{j\in\mathbb{Z}}|u_{j+1}-u_j|\,. \tag{3.13}$$

3.2. **BV estimates.**

Proposition 3.1 (BV estimates in space). *Let $\boldsymbol{\rho_j^0},\boldsymbol{\tilde\rho_j^0}\in(\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R};\mathbb{R}^+)$ satisfying* (2.2)*, let Assumptions* 2.1 *and the CFL condition* (3.8) *hold. Then, for all $T>0$, there exist positive* *constants $\mathcal{H}$ and $\mathcal{G}$ such that $\boldsymbol{\rho}^{\boldsymbol{\Delta}}$, $\tilde{\boldsymbol{\rho}}^{\boldsymbol{\Delta}}$ constructed through Algorithm* 3.1 *satisfy the following estimate: for all $n = 0, \ldots, N_T$,*

$$\sum_{i=1}^{2} \left(\mathrm{TV}(\rho_i^n) + \mathrm{TV}(\tilde{\rho}_{[i]}^n)\right) \quad \leq \quad e^{T(5K+\mathcal{H})} \left(\sum_{i=1}^{2} \left(\mathrm{TV}(\rho_i^0) + \mathrm{TV}(\tilde{\rho}_{[i]}^0)\right) + 2T\mathcal{G} \right).$$

Proof. Taking into account (3.10), we can write

$$\rho_{i,j+1}^{n+1/2} = \rho_{i,j+1}^n - \lambda \left[F(\rho_{i,j+1}^n, \rho_{i,j+2}^n, \tilde{R}_{[i],j+3/2}) - F(\rho_{i,j}^n, \rho_{i,j+1}^n, \tilde{R}_{[i],j+1/2}) \right].$$

Setting $\Delta_{i,j+1/2}^{n+1/2} = \rho_{i,j+1}^{n+1/2} - \rho_{i,j}^{n+1/2}$, for all $i = 1, 2$ we compute the following estimates

$$\begin{aligned}
\Delta_{i,j+1/2}^{n+1/2} = {} & \left[1-\lambda \left(\partial_1 F(\xi_{j+1/2}^n) - \partial_2 F(\xi_{j-1/2}^n)\right)\right] \Delta_{j+1/2}^n - \lambda \partial_2 F(\xi_{j+1/2}^n) \Delta_{i,j+3/2}^n \\
& + \lambda \partial_1 F(\xi_{j-1/2}^n) \Delta_{j-1/2}^n - \lambda \partial_3 F(\xi_{j+1/2}^n) \left(\tilde{R}_{[i],j+3/2}^n - \tilde{R}_{[i],j+1/2}^n\right) \\
& + \lambda \partial_3 F(\xi_{j-1/2}^n) \left(\tilde{R}_{[i],j+1/2}^n - \tilde{R}_{[i],j-1/2}^n\right) \\
& \pm \lambda \partial_3 F(\xi_{j-1/2}^n) \left(\tilde{R}_{[i],j+3/2}^n - \tilde{R}_{[i],j+1/2}^n\right) \\
= {} & \left[1-\lambda \left(\partial_1 F(\xi_{j+1/2}^n) - \partial_2 F(\xi_{j-1/2}^n)\right)\right] \Delta_{j+1/2}^n \\
& - \lambda \partial_2 F(\xi_{j+1/2}^n) \Delta_{i,j+3/2}^n + \lambda \partial_1 F(\xi_{j-1/2}^n) \Delta_{j-1/2}^n \\
& + \lambda \partial_3 F(\xi_{j-1/2}^n) \left[\left(\tilde{R}_{[i],j+1/2}^n - \tilde{R}_{[i],j-1/2}^n\right) - \left(\tilde{R}_{[i],j+3/2}^n - \tilde{R}_{[i],j+1/2}^n\right)\right] \\
& + \lambda \left[\partial_3 F(\xi_{j-1/2}^n) - \partial_3 F(\xi_{j+1/2}^n)\right] \left(\tilde{R}_{[i],j+3/2}^n - \tilde{R}_{[i],j+1/2}^n\right),
\end{aligned}$$

where

$$\xi_{j+1/2}^n = \left(u_{j+1/2}^n, w_{j+1/2}^n, \tilde{\sigma}_{j+1/2}^n\right) \in \mathcal{I}\left((\rho_{i,j}^n, \rho_{i,j+1}^n, \tilde{R}_{[i],j+1/2}^n), (\rho_{i,j+1}^n, \rho_{i,j+2}^n, \tilde{R}_{[i],j+3/2}^n)\right).$$

Observe that the coefficient of the first term in the above equality is positive because of the monotonicity properties of F in (3.9) and CFL condition (3.8), so taking absolute values in the above equality we get

$$\begin{aligned}
\left|\Delta_{i,j+1/2}^{n+1/2}\right| \leq {} & \left[1 - \lambda \left(\partial_1 F(\xi_{j+1/2}^n) - \partial_2 F(\xi_{j-1/2}^n)\right)\right] \left|\Delta_{j+1/2}^n\right| \\
& - \lambda \partial_2 F(\xi_{j+1/2}^n) \left|\Delta_{i,j+3/2}^n\right| + \lambda \partial_1 F(\xi_{j-1/2}^n) \left|\Delta_{j-1/2}^n\right| \\
& + \lambda \|\partial_3 F\| \left|\tilde{R}_{[i],j+3/2}^n - 2\tilde{R}_{[i],j+1/2}^n + \tilde{R}_{[i],j-1/2}^n\right| \qquad (3.14) \\
& + \lambda \left|\partial_3 F(\xi_{j-1/2}^n) - \partial_3 F(\xi_{j+1/2}^n)\right| \left|\tilde{R}_{[i],j+3/2}^n - \tilde{R}_{[i],j+1/2}^n\right|. \qquad (3.15)
\end{aligned}$$

Next, the term (3.14) can be estimated as follow

$$\begin{aligned}
& \left|\left(\tilde{R}_{[i],j+3/2}^n - \tilde{R}_{[i],j+1/2}^n\right) - \left(\tilde{R}_{[i],j+1/2}^n - \tilde{R}_{[i],j-1/2}^n\right)\right| \\
= {} & \left| \left(\sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{[i],j+k+2}^n - \sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{[i],j+k+1}^n \right) \right. \\
& \left. - \left(\sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{[i],j+k+1}^n - \sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{[i],j+k}^n \right) \right| \\
= {} & \left| \sum_{k=0}^{N-1} \omega_\eta^k \left(\tilde{\rho}_{[i],j+k+2}^n - \tilde{\rho}_{[i],j+k+1}^n\right) - \sum_{k=0}^{N-1} \omega_\eta^k \left(\tilde{\rho}_{[i],j+k+1}^n - \tilde{\rho}_{[i],j+k}^n\right) \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{k=1}^{N} \omega_\eta^{k-1} \left(\tilde{\rho}^n_{[i],j+k+1} - \tilde{\rho}^n_{[i],j+k} \right) - \sum_{k=0}^{N-1} \omega_\eta^{k} \left(\tilde{\rho}^n_{[i],j+k+1} - \tilde{\rho}^n_{[i],j+k} \right) \right| \\
&= \left| \sum_{k=1}^{N} \left(\omega_\eta^{k-1} - \omega_\eta^{k} \right) \left(\tilde{\rho}^n_{[i],j+k+1} - \tilde{\rho}^n_{[i],j+k} \right) - \omega_\eta^{0} \left(\tilde{\rho}^n_{[i],j+1} - \tilde{\rho}^n_{[i],j} \right) \right| \\
&\leq \sum_{k=1}^{N} \left(\omega_\eta^{k-1} - \omega_\eta^{k} \right) \left| \tilde{\rho}^n_{[i],j+k+1} - \tilde{\rho}^n_{[i],j+k} \right| + \omega_\eta^{0} \left| \tilde{\rho}^n_{[i],j+1} - \tilde{\rho}^n_{[i],j} \right|,
\end{aligned}
$$

and summing over all $j \in \mathbb{Z}$ we get

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} \left| \left(\tilde{R}^n_{[i],j+3/2} - \tilde{R}^n_{[i],j+1/2} \right) - \left(\tilde{R}^n_{[i],j+1/2} - \tilde{R}^n_{[i],j-1/2} \right) \right| \\
&\leq \sum_{k=1}^{N} \left(\omega_\eta^{k-1} - \omega_\eta^{k} \right) \mathrm{TV}\left(\tilde{\rho}^n_{[i]} \right) + \omega_\eta^{0} \mathrm{TV}\left(\tilde{\rho}^n_{[i]} \right) \\
&= 2\Delta x\, \omega_\eta(0) \mathrm{TV}\left(\tilde{\rho}^n_{[i]} \right).
\end{aligned}
$$

Now, for (3.15) we have the following estimates, on the one hand,

$$
\begin{aligned}
\left| \tilde{R}^n_{[i],j+3/2} - \tilde{R}^n_{[i],j+1/2} \right| &= \left| \sum_{k=0}^{N-1} \omega_\eta^{k} \tilde{\rho}^n_{[i],j+k+2} - \sum_{k=0}^{N-1} \omega_\eta^{k} \tilde{\rho}^n_{[i],j+k+1} \right| \\
&= \left| \sum_{k=1}^{N} \left(\omega_\eta^{k-1} - \omega_\eta^{k} \right) \tilde{\rho}^n_{[i],j+k+1} - \omega_\eta^{0} \tilde{\rho}^n_{[i],j+1} \right| \\
&\leq \sum_{k=1}^{N} \left(\omega_\eta^{k-1} - \omega_\eta^{k} \right) \tilde{\rho}^n_{[i],j+k+1} + \omega_\eta^{0} \tilde{\rho}^n_{[i],j+1} \\
&\leq 2\omega_\eta^0 \rho_{\max} \\
&\leq 2\Delta x\, \omega_\eta(0) \rho_{\max},
\end{aligned}
$$

and on the other hand,

$$
\left| \partial_3 F(\xi^n_{j-1/2}) - \partial_3 F(\xi^n_{j+1/2}) \right| \leq \| \nabla \partial_3 F \| \left| \xi^n_{j+1/2} - \xi^n_{j-1/2} \right|,
$$

and by the choice of $\xi^n_{j+1/2}$, the term $\left| \xi^n_{j+1/2} - \xi^n_{j-1/2} \right|$ can be decomposed as follows

$$
\begin{aligned}
\left| \xi^n_{j+1/2} - \xi^n_{j-1/2} \right| = & \left| \theta \rho^n_{i,j+1} + (1-\theta)\rho^n_{i,j} - \mu \rho^n_{i,j} + (1-\mu)\rho^n_{i,j-1} \right| && (3.16) \\
& + \left| \theta \rho^n_{i,j+2} + (1-\theta)\rho^n_{i,j+1} - \mu \rho^n_{i,j+1} + (1-\mu)\rho^n_{i,j} \right| && (3.17) \\
& + \left| \theta \tilde{R}^n_{[i],j+3/2} + (1-\theta) \tilde{R}^n_{[i],j+1/2} - \mu \tilde{R}^n_{[i],j+1/2} \right. \\
& \left. - (1-\mu) \tilde{R}^n_{[i],j-1/2} \right|, && (3.18)
\end{aligned}
$$

so, for (3.16) we obtain

$$
\begin{aligned}
&\left| \theta \rho^n_{i,j+1} + (1-\theta)\rho^n_{i,j} - \mu \rho^n_{i,j} + (1-\mu)\rho^n_{i,j-1} \right| \\
&= \left| \theta (\rho^n_{i,j+1} - \rho^n_{i,j}) + (1-\mu)(\rho^n_{i,j} - \rho^n_{i,j-1}) \right| \\
&\leq \left| \rho^n_{i,j+1} - \rho^n_{i,j} \right| + \left| \rho^n_{i,j} - \rho^n_{i,j-1} \right|.
\end{aligned}
$$

Similarly, for (3.17) we have

$$
\left| \theta \rho^n_{i,j+2} + (1-\theta)\rho^n_{i,j+1} - \mu \rho^n_{i,j+1} + (1-\mu)\rho^n_{i,j} \right| \leq \left| \rho^n_{i,j+2} - \rho^n_{i,j+1} \right|
$$

$$+\left|\rho_{i,j+1}^n-\rho_{i,j}^n\right|,$$

and finally, for (3.18) we get

$$\begin{aligned}
&\left|\theta\tilde{R}^n_{[i],j+3/2}+(1-\theta)\tilde{R}^n_{[i],j+1/2}-\mu\tilde{R}^n_{[i],j+1/2}-(1-\mu)\tilde{R}^n_{[i],j-1/2}\right|\\
=&\left|\sum_{k=1}^{N}\left\{\theta\left(\omega_\eta^{k-1}-\omega_\eta^k\right)+(1-\mu)\left(\omega_\eta^k-\omega_\eta^{k+1}\right)\right\}\tilde\rho^n_{[i],j+k+1}\right.\\
&\left.-\left(\theta\omega_\eta^0+(1-\mu)\omega_\eta^1\right)\tilde\rho^n_{[i],j+1}+(1-\mu)\omega_\eta^0\left(\tilde\rho^n_{[i],j+1}-\tilde\rho^n_{[i],j}\right)\right|\\
\leq&\sum_{k=1}^{N}\left\{\theta\left(\omega_\eta^{k-1}-\omega_\eta^k\right)+(1-\mu)\left(\omega_\eta^k-\omega_\eta^{k+1}\right)\right\}\tilde\rho^n_{[i],j+k+1}\\
&+\left(\theta\omega_\eta^0+(1-\mu)\omega_\eta^1\right)\tilde\rho^n_{[i],j+1}+(1-\mu)\omega_\eta^0\left|\tilde\rho^n_{[i],j+1}-\tilde\rho^n_{[i],j}\right|,
\end{aligned}$$

thus,

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\left|\theta\tilde{R}^n_{[i],j+3/2}+(1-\theta)\tilde{R}^n_{[i],j+1/2}-\mu\tilde{R}^n_{[i],j+1/2}-(1-\mu)\tilde{R}^n_{[i],j-1/2}\right|\\
&\leq\sum_{j\in\mathbb{Z}}\tilde\rho^n_{[i],j}\left(\sum_{k=1}^{N}\left(\omega_\eta^{k-1}-\omega_\eta^{k+1}\right)\right)+2\omega_\eta^0\sum_{j\in\mathbb{Z}}\tilde\rho^n_{[i],j}+\omega_\eta^0\sum_{j\in\mathbb{Z}}\left|\tilde\rho^n_{[i],j+1}-\tilde\rho^n_{[i],j}\right|\\
&\leq 4\,\omega_\eta(0)\|\tilde\rho^n_{[i]}\|_{\mathbf{L^1}(\mathbb{R})}+\Delta x\,\omega_\eta(0)\mathrm{TV}\left(\tilde\rho^n_{[i]}\right).
\end{aligned}$$

Therefore, taking into account all the above estimates, we get

$$\sum_{j\in\mathbb{Z}}\left|\Delta^{n+1/2}_{i,j+1/2}\right|\quad\leq\quad(1+\Delta t\mathcal{H}_1)\,\mathrm{TV}\left(\rho_i^n\right)+\Delta t\mathcal{H}_2\mathrm{TV}(\tilde\rho^n_{[i]})+\Delta t\mathcal{H}_3,$$

where

$$\begin{aligned}
\mathcal{H}_1&=8\omega_\eta^0\|\nabla\partial_3F\|\rho_{\max},\\
\mathcal{H}_2&=2\left(\|\partial_3F\|\omega_\eta(0)+\Delta x(\omega_\eta(0))^2\|\nabla\partial_3F\|\rho_{\max}\right),\\
\mathcal{H}_3&=8(\omega_\eta(0))^2\|\tilde\rho^n_{[i]}\|_{\mathbf{L^1}(\mathbb{R})}\|\nabla\partial_3F\|\rho_{\max}.
\end{aligned}$$

Likewise, we can estimate

$$\sum_{j\in\mathbb{Z}}\left|\tilde\Delta^{n+1/2}_{[i],j+1/2}\right|\quad\leq\quad\left(1+\Delta t\tilde{\mathcal{H}}_1\right)\mathrm{TV}\left(\tilde{\rho_{[i]}}^n\right)+\Delta t\tilde{\mathcal{H}}_2\mathrm{TV}(\rho_i^n)+\Delta t\tilde{\mathcal{H}}_3,$$

with

$$\begin{aligned}
\tilde{\mathcal{H}}_1&=8\hat\omega_\eta(0)\|\nabla\partial_3F\|\rho_{\max},\\
\tilde{\mathcal{H}}_2&=2\left(\|\partial_3F\|\hat\omega_\eta(0)+\Delta x(\hat\omega_\eta(0))^2\|\nabla\partial_3F\|\rho_{\max}\right),\\
\tilde{\mathcal{H}}_3&=8(\hat\omega_\eta(0))^2\|\rho_i^n\|_{\mathbf{L^1}(\mathbb{R})}\|\nabla\partial_3F\|\rho_{\max}.
\end{aligned}$$

Thus we get, for $i=1,2$

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\left(\left|\Delta^{n+1/2}_{i,j+1/2}\right|+\left|\tilde\Delta^{n+1/2}_{[i],j+1/2}\right|\right)=&(1+\Delta t\mathcal{H}_1)\,\mathrm{TV}\left(\rho_i^n\right)+\Delta t\mathcal{H}_2\mathrm{TV}(\tilde\rho^n_{[i]})\\
&+\Delta t\mathcal{H}_3+\left(1+\Delta t\tilde{\mathcal{H}}_1\right)\mathrm{TV}\left(\tilde{\rho_{[i]}}^n\right)\\
&+\Delta t\tilde{\mathcal{H}}_2\mathrm{TV}(\rho_i^n)+\Delta t\tilde{\mathcal{H}}_3
\end{aligned}$$

$$
\begin{aligned}
&= \left(1+\Delta t\left(\mathcal{H}_1+\tilde{\mathcal{H}}_2\right)\right)\mathrm{TV}(\rho_i^n)\\
&\quad+\left(1+\Delta t\left(\mathcal{H}_2+\tilde{\mathcal{H}}_1\right)\right)\mathrm{TV}\left(\tilde{\rho_{[i]}}^n\right)\\
&\quad+\Delta t\left(\mathcal{H}_3+\tilde{\mathcal{H}}_3\right)\\
&\leq (1+\Delta t\mathcal{H})\left(\mathrm{TV}(\rho_i^n)+\mathrm{TV}\left(\tilde{\rho}_{[i]}^n\right)\right)+\Delta t\mathcal{G},
\end{aligned}
$$

where $\mathcal{H}=\max\{\mathcal{H}_1+\tilde{\mathcal{H}}_2,\mathcal{H}_2+\tilde{\mathcal{H}}_1\}$ and $\mathcal{G}=\mathcal{H}_3+\tilde{\mathcal{H}}_3$. If we sum the two lanes, we get

$$
\begin{aligned}
\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n+1/2})+\mathrm{TV}(\tilde{\rho}_{[i]}^{n+1/2})\right) &\leq (1+\Delta t\mathcal{H})\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^n)+\mathrm{TV}\left(\tilde{\rho}_{[i]}^n\right)\right)\\
&\quad+2\Delta t\mathcal{G}. \qquad (3.19)
\end{aligned}
$$

Let us now compute the contribution of the reactive step (3.7)

$$
\begin{aligned}
\Delta_{1,j+1/2}^{n+1} &= \Delta_{1,j+1/2}^{n+1/2}-\Delta t\left((S_{\mathrm{O,j+1}}-S_{\mathrm{O,j}})-(S_{\mathrm{R,j+1}}-S_{\mathrm{R,j}})\right),\\
\Delta_{2,j+1/2}^{n+1} &= \Delta_{2,j+1/2}^{n+1/2}+\Delta t\left((S_{\mathrm{O,j+1}}-S_{\mathrm{O,j}})-(S_{\mathrm{R,j+1}}-S_{\mathrm{R,j}})\right).
\end{aligned}
$$

Now applying absolute value, using the estimates given in Lemma 3.3 on the source terms, and summing over all $j\in\mathbb{Z}$, we get for $i=1,2$

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\left|\Delta_{i,j+1/2}^{n+1}\right| &\leq (1+\Delta t\mathcal{K})\sum_{j\in\mathbb{Z}}\left|\Delta_{i,j+1/2}^{n+1/2}\right|\\
&\quad+\Delta t\mathcal{K}\sum_{j\in\mathbb{Z}}\Bigg(\left|\Delta_{[i],j+1/2}^{n+1/2}\right|+\left|\mathcal{R}_{1,j+1}^{n+1/2}-\mathcal{R}_{1,j}^{n+1/2}\right|\\
&\quad+\left|\mathcal{R}_{2,1,j+1}^{n+1/2}-\mathcal{R}_{2,1,j}^{n+1/2}\right|+\left|\mathcal{R}_{2,2,j+1}^{n+1/2}-\mathcal{R}_{2,2,j}^{n+1/2}\right|\Bigg),
\end{aligned}
$$

and by means of (3.13) in Remark 3.1 we get

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\left|\Delta_{i,j+1/2}^{n+1}\right| &\leq (1+2\Delta t\mathcal{K})\sum_{j\in\mathbb{Z}}\left|\Delta_{i,j+1/2}^{n+1/2}\right|\\
&\quad+\Delta t\mathcal{K}\sum_{j\in\mathbb{Z}}\left(\left|\Delta_{[i],j+1/2}^{n+1/2}\right|+\left|\tilde{\Delta}_{i,j+1/2}^{n+1/2}\right|+\left|\tilde{\Delta}_{[i],j+1/2}^{n+1/2}\right|\right), \qquad (3.20)
\end{aligned}
$$

In a similar way for the other class of vehicles we get,

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\left|\tilde{\Delta}_{[i],j+1/2}^{n+1}\right| &\leq (1+2\Delta t\mathcal{K})\sum_{j\in\mathbb{Z}}\left|\tilde{\Delta}_{[i],j+1/2}^{n+1/2}\right|\\
&\quad+\Delta t\mathcal{K}\sum_{j\in\mathbb{Z}}\left(\left|\tilde{\Delta}_{i,j+1/2}^{n+1/2}\right|+\left|\Delta_{i,j+1/2}^{n+1/2}\right|+\left|\Delta_{[i],j+1/2}^{n+1/2}\right|\right), \qquad (3.21)
\end{aligned}
$$

then summing term by term (3.20) and (3.21) we obtain the following estimate

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\left(\left|\Delta_{i,j+1/2}^{n+1}\right|+\left|\tilde{\Delta}_{[i],j+1/2}^{n+1}\right|\right) &\leq (1+3\Delta t\mathcal{K})\sum_{j\in\mathbb{Z}}\left(\left|\Delta_{i,j+1/2}^{n+1/2}\right|+\left|\tilde{\Delta}_{[i],j+1/2}^{n+1/2}\right|\right)\\
&\quad+2\Delta t\mathcal{K}\sum_{j\in\mathbb{Z}}\left(\left|\Delta_{[i],j+1/2}^{n+1/2}\right|+\left|\tilde{\Delta}_{i,j+1/2}^{n+1/2}\right|\right),
\end{aligned}
$$

summing the two lanes and by (3.19) we get

$$
\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n+1})+\mathrm{TV}(\tilde{\rho}_{[i]}^{n+1})\right)\leq(1+5\Delta t\mathcal{K})\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^{n+1/2})+\mathrm{TV}(\tilde{\rho}_{[i]}^{n+1/2})\right)
$$

$$
\begin{aligned}
&\leq (1+5\Delta t\mathcal{K})\left((1+\Delta t\mathcal{H})\sum_{i=1}^{2}\Big(\mathrm{TV}(\rho_i^n)+\mathrm{TV}(\tilde{\rho}_{[i]}^n)\Big)+2\Delta t\mathcal{G}\right)\\
&= (1+5\Delta t\mathcal{K})(1+\Delta t\mathcal{H})\sum_{i=1}^{2}\Big(\mathrm{TV}(\rho_i^n)+\mathrm{TV}(\tilde{\rho}_{[i]}^n)\Big)\\
&\quad+(1+5\Delta t\mathcal{K})2\Delta t\mathcal{G}\\
&\leq e^{\Delta t(5\mathcal{K}+\mathcal{H})}\left(\sum_{i=1}^{2}\Big(\mathrm{TV}(\rho_i^n)+\mathrm{TV}(\tilde{\rho}_{[i]}^n)\Big)+2\Delta t\mathcal{G}\right),
\end{aligned}
$$

then applying an iterative process we get

$$
\sum_{i=1}^{2}\Big(\mathrm{TV}(\rho_i^n)+\mathrm{TV}(\tilde{\rho}_{[i]}^n)\Big)\leq e^{T(5\mathcal{K}+\mathcal{H})}\left(\sum_{i=1}^{2}\Big(\mathrm{TV}(\rho_i^0)+\mathrm{TV}(\tilde{\rho}_{[i]}^0)\Big)+2T\mathcal{G}\right).
$$

□

Corollary 3.1 (BV estimates in space and time). *Let* $\boldsymbol{\rho_j^0}, \boldsymbol{\tilde{\rho}_j^0} \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^+)$ *satisfying* (2.2), *let Assumptions* 2.1 *and the CFL condition* (3.8) *hold. Then, for all* $T>0$, $\boldsymbol{\rho^\Delta}$, $\boldsymbol{\tilde{\rho}^\Delta}$ *constructed through Algorithm* 3.1 *satisfy the following estimate: for all* $n=1,\ldots,N_T$,

$$
\begin{aligned}
&\sum_{n=0}^{N_T-1}\sum_{i=1}^{2}\sum_{j\in\mathbb{Z}}\Delta t\left|\rho_{i,j+1}^n-\rho_{i,j}^n\right|+(T-N_T\Delta t)\sum_{i=1}^{2}\sum_{j\in\mathbb{Z}}\left|\rho_{i,j+1}^{N_T}-\rho_{i,j}^{N_T}\right|\\
&+\sum_{n=0}^{N_T-1}\sum_{i=1}^{2}\sum_{j\in\mathbb{Z}}\Delta x\left|\rho_{i,j}^{n+1}-\rho_{i,j}^n\right|\\
&\leq Te^{T(5\mathcal{K}+\mathcal{H})}(1+2\mathcal{L})\left(\sum_{i=1}^{2}\Big(\mathrm{TV}(\rho_i^0)+\mathrm{TV}(\tilde{\rho}_{[i]}^0)\Big)+2T\mathcal{G}\right)\\
&\quad+2T\mathcal{S}\left(\left\|\rho_i^0\right\|_{\mathbf{L^1}(\mathbb{R})}+\left\|\rho_{[i]}^0\right\|_{\mathbf{L^1}(\mathbb{R})}\right),
\end{aligned}
$$

where $\mathcal{S}=\max\left\{\|\partial_1 S_{\mathrm{O}}\|,\|\partial_2 S_{\mathrm{R}}\|\right\}$

Proof. By means of the **BV** estimate in space in Proposition 3.1, we have

$$
\begin{aligned}
&\sum_{n=0}^{N_T-1}\sum_{i=1}^{2}\sum_{j\in\mathbb{Z}}\Delta t\left|\rho_{i,j+1}^n-\rho_{i,j}^n\right|+(T-N_T\Delta t)\sum_{i=1}^{2}\sum_{j\in\mathbb{Z}}\left|\rho_{i,j+1}^{N_T}-\rho_{i,j}^{N_T}\right|\\
&\leq Te^{T(5\mathcal{K}+\mathcal{H})}\left(\sum_{i=1}^{2}\Big(\mathrm{TV}(\rho_i^0)+\mathrm{TV}(\tilde{\rho}_{[i]}^0)\Big)+2T\mathcal{G}\right).
\end{aligned}
\tag{3.22}
$$

On the other hand, observe that

$$
\left|\rho_{i,j}^{n+1}-\rho_{i,j}^n\right|\leq\left|\rho_{i,j}^{n+1}-\rho_{i,j}^{n+1/2}\right|+\left|\rho_{i,j}^{n+1/2}-\rho_{i,j}^n\right|.
\tag{3.23}
$$

Now we estimate each term on right hand side of the inequality (3.23), beginning with the first term. Observe that $S_{\mathrm{O}}(0,\cdot,\cdot,\cdot,\cdot)=0$ and $S_{\mathrm{R}}(\cdot,0)=0$, then by (3.7) in **Algorithm 3.1** we have

$$
\begin{aligned}
\left|\rho_{i,j}^{n+1}-\rho_{i,j}^{n+1/2}\right|&\leq\Delta t\left|S_{\mathrm{O}}\rho_{1,j}^{n+1/2},\rho_{2,j}^{n+1/2},\mathcal{R}_{1,j}^{n+1/2},\mathcal{R}_{2,1,j}^{n+1/2},\mathcal{R}_{2,2,j}^{n+1/2})-S_{\mathrm{R}}(\rho_{1,j}^{n+1/2},\rho_{2,j}^{n+1/2})\right|\\
&\leq\Delta t\Big[\Big|S_{\mathrm{O}}(\rho_{1,j}^{n+1/2},\rho_{2,j}^{n+1/2},\mathcal{R}_{1,j}^{n+1/2},\mathcal{R}_{2,1,j}^{n+1/2},\mathcal{R}_{2,2,j}^{n+1/2})
\end{aligned}
$$

$$
\begin{aligned}
&- S_{\mathrm{O}}(0, \rho_{2,j}^{n+1/2}, \mathcal{R}_{1,j}^{n+1/2}, \mathcal{R}_{2,1,j}^{n+1/2}, \mathcal{R}_{2,2,j}^{n+1/2}) \Big| \\
&+ \left| S_{\mathrm{R}}(\rho_{1,j}^{n+1/2}, 0) - S_{\mathrm{R}}(\rho_{1,j}^{n+1/2}, \rho_{2,j}^{n+1/2}) \right| \Big] \\
\leq{}& \Delta t \Big[\left| \partial_1 S_{\mathrm{O}}(\theta_{1,j}^{n+1/2}, \rho_{2,j}^{n+1/2}, \mathcal{R}_{1,j}^{n+1/2}, \mathcal{R}_{2,1,j}^{n+1/2}, \mathcal{R}_{2,2,j}^{n+1/2}) \rho_1^{n+1/2} \right| \\
&+ \left| \partial_2 S_{\mathrm{R}}(\rho_{1,j}^{n+1/2}, \theta_{2,j}^{n+1/2}) \rho_{2,j}^{n+1/2} \right| \Big] \\
\leq{}& \Delta t \mathcal{S} \left(\rho_{i,j}^{n+1/2} + \rho_{[i],j}^{n+1/2} \right),
\end{aligned}
$$

where $\theta_{1,j}^{n+1/2} \in (0, \rho_{1,j}^{n+1/2})$ and $\theta_{2,j}^{n+1/2} \in (0, \rho_{2,j}^{n+1/2})$. Then, multiplying by Δx and summing over all $j \in \mathbb{Z}$,

$$
\begin{aligned}
&\Delta x \sum_{j \in \mathbb{Z}} \left| \rho_{i,j}^{n+1} - \rho_{i,j}^{n+1/2} \right| \\
&\leq \Delta t \mathcal{S} \left(\left\| \rho_i^{n+1/2} \right\|_{\mathbf{L^1}(\mathbb{R})} + \left\| \rho_{[i]}^{n+1/2} \right\|_{\mathbf{L^1}(\mathbb{R})} \right).
\end{aligned} \tag{3.24}
$$

Now we analyze the second term on the right hand side of (3.23). Since the numerical flux defined in (3.2) is Lipschitz continuous in all its arguments with Lipschitz constant $\mathcal{L} = \max\{\|\partial_1 F\|, \|\partial_2 F\|, \|\partial_3 F\|\}$, we get

$$
\begin{aligned}
\left| \rho_{i,j}^{n+1/2} - \rho_{i,j}^n \right| &= \left| \lambda \left(F(\rho_j^n, \rho_{j+1}^n, \tilde{R}_{[i],j+1/2}^n) - F(\rho_{j-1}^n, \rho_j^n, \tilde{R}_{[i],j-1/2}^n) \right) \right| \\
&\leq \lambda \mathcal{L} \left(\left| \rho_{i,j}^n - \rho_{i,j-1}^n \right| + \left| \rho_{i,j+1}^n - \rho_{i,j}^n \right| + \left| \tilde{R}_{[i],j+1/2}^n - \tilde{R}_{[i],j-1/2}^n \right| \right).
\end{aligned}
$$

Then, multiplying by Δx and summing over all $j \in \mathbb{Z}$, we have

$$
\begin{aligned}
\Delta x \sum_{j \in \mathbb{Z}} \left| \rho_{i,j}^{n+1/2} - \rho_{i,j}^n \right| \leq \Delta t \mathcal{L} \Bigg(& \sum_{j \in \mathbb{Z}} \left| \rho_{i,j}^n - \rho_{i,j-1}^n \right| + \sum_{j \in \mathbb{Z}} \left| \rho_{i,j+1}^n - \rho_{i,j}^n \right| \\
& + \sum_{j \in \mathbb{Z}} \left| \tilde{R}_{[i],j+1/2}^n - \tilde{R}_{[i],j-1/2}^n \right| \Bigg),
\end{aligned}
$$

and computing an upper bound for the last term in the right hand above

$$
\begin{aligned}
\sum_{j \in \mathbb{Z}} \left| \tilde{R}_{[i],j+1/2}^n - \tilde{R}_{[i],j-1/2}^n \right| &= \sum_{j \in \mathbb{Z}} \left| \sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{[i],j+k+1} - \sum_{k=0}^{N-1} \omega_\eta^k \tilde{\rho}_{[i],j+k} \right| \\
&= \sum_{j \in \mathbb{Z}} \left| \sum_{k=0}^{N-1} \omega_\eta^k \left(\tilde{\rho}_{[i],j+k+1} - \tilde{\rho}_{[i],j+k} \right) \right| \\
&\leq \sum_{j \in \mathbb{Z}} \sum_{k=0}^{N-1} \omega_\eta^k \left| \tilde{\rho}_{[i],j+k+1} - \tilde{\rho}_{[i],j+k} \right| \\
&= \sum_{k=0}^{N-1} \omega_\eta^k \sum_{j \in \mathbb{Z}} \left| \tilde{\rho}_{[i],j+k+1} - \tilde{\rho}_{[i],j+k} \right| \\
&= \mathrm{TV}\left(\tilde{\rho}_{[i]}^n \right) \sum_{k=0}^{N-1} \omega_\eta^k = \mathrm{TV}\left(\tilde{\rho}_{[i]}^n \right),
\end{aligned}
$$

we get the following estimate

$$\Delta x \sum_{j\in\mathbb{Z}} \left|\rho_{i,j}^{n+1/2} - \rho_{i,j}^{n}\right| \leq 2\Delta t \mathcal{L}\left(\mathrm{TV}(\rho_i^n) + \mathrm{TV}(\tilde{\rho}_{[i]}^n)\right). \tag{3.25}$$

Collecting together (3.24), (3.25) and summing for $i = 1, 2$ we obtain

$$\begin{aligned}\Delta x \sum_{i=1}^{2}\sum_{j\in\mathbb{Z}} \left|\rho_{i,j}^{n+1} - \rho_{i,j}^{n}\right| &\leq 2\Delta t \mathcal{S}\left(\|\rho_i^n\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_{[i]}^n\right\|_{\mathbf{L^1}(\mathbb{R})}\right)\\ &\quad + 2\Delta t \mathcal{L}\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^n) + \mathrm{TV}(\tilde{\rho}_{[i]}^n)\right),\end{aligned}$$

by using Lemma 3.2 and Proposition 3.1 we get

$$\begin{aligned}&\Delta x \sum_{i=1}^{2}\sum_{j\in\mathbb{Z}} \left|\rho_{i,j}^{n+1} - \rho_{i,j}^{n}\right|\\ &\leq 2\Delta t \mathcal{S}\left(\|\rho_i^0\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_{[i]}^0\right\|_{\mathbf{L^1}(\mathbb{R})}\right)\\ &\quad +2\Delta t \mathcal{L} e^{T(5\mathcal{K}+\mathcal{H})}\left(\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^0) + \mathrm{TV}(\tilde{\rho}_{[i]}^0)\right) + 2T\mathcal{G}\right).\end{aligned} \tag{3.26}$$

Finally, collecting together (3.22), (3.26) and summing for n from 0 until $N_T - 1$ we get the following **BV** bound in space and time

$$\begin{aligned}&\sum_{n=0}^{N_T-1}\sum_{i=1}^{2}\sum_{j\in\mathbb{Z}} \Delta t\left|\rho_{i,j+1}^{n} - \rho_{i,j}^{n}\right| + (T - N_T\Delta t)\sum_{i=1}^{2}\sum_{j\in\mathbb{Z}}\left|\rho_{i,j+1}^{N_T} - \rho_{i,j}^{N_T}\right|\\ &+\Delta x \sum_{i=1}^{2}\sum_{j\in\mathbb{Z}} \left|\rho_{i,j}^{n+1} - \rho_{i,j}^{n}\right|\\ &\leq T e^{T(5\mathcal{K}+\mathcal{H})}(1 + 2\mathcal{L})\left(\sum_{i=1}^{2}\left(\mathrm{TV}(\rho_i^0) + \mathrm{TV}(\tilde{\rho}_{[i]}^0)\right) + 2T\mathcal{G}\right)\\ &\quad +2T\mathcal{S}\left(\|\rho_i^0\|_{\mathbf{L^1}(\mathbb{R})} + \left\|\rho_{[i]}^0\right\|_{\mathbf{L^1}(\mathbb{R})}\right).\end{aligned}$$

□

3.3. **Proof of Theorem 2.1.** The convergence of the approximate solutions constructed by **Algorithm 3.1** towards the weak solution can be proven by applying Helly's compactness theorem. The latter can be applied due to Lemma 3.1 and Corollary 3.1 and states that there exists a sub-sequence of approximate solutions $\boldsymbol{\rho}^{\boldsymbol{\Delta}}$ and $\tilde{\boldsymbol{\rho}}^{\boldsymbol{\Delta}}$ that converges in $\mathbf{L^1}$ to functions $\boldsymbol{\rho}, \tilde{\boldsymbol{\rho}} \in \mathbf{L}^\infty([0, T] \times \mathbb{R}; \mathbb{R}^+)$, respectively.

Now we need to prove that this limit function is indeed a weak solution to (2.1), in the sense of Definition 2.1.

Lemma 3.4. *Let $\boldsymbol{\rho_j^0}, \tilde{\boldsymbol{\rho}}_{\boldsymbol{j}}^{\boldsymbol{0}} \in BV(\mathbb{R}; \mathbb{R}^+)$ satisfying* (2.2)*, and Assumptions* 2.1 *and the CFL condition* (3.8) *hold. Then the piecewise constant approximate solutions $\boldsymbol{\rho}^{\boldsymbol{\Delta}}$, $\tilde{\boldsymbol{\rho}}^{\boldsymbol{\Delta}}$ resulting from the* ***Algorithm* 3.1** *converge, as $\Delta x \to 0$, towards an weak solution of* (2.1).

Proof. Let $\varphi \in C_c^1([0,T];\mathbb{R}^+)$ for some $T>0$. Multiplying first (3.6) by $\Delta x\varphi(t^n,x_j)$ and summing over $j\in\mathbb{Z}$ and over $n=0,\dots,N_T$ yields

$$\underbrace{\Delta x\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N_T}\left(\boldsymbol{\rho_j^{n+1/2}}-\boldsymbol{\rho_j^{n}}\right)\varphi(t^n,x_j)}_{I_1}+\underbrace{\Delta t\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N_T}\left(\boldsymbol{F_{j+1/2}^{n}}-\boldsymbol{F_{j-1/2}^{n}}\right)\varphi(t^n,x_j)}_{I_2}=0.$$

We first consider I_1,

$$I_1=-\Delta t\Delta x\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N_T}\boldsymbol{\rho_j^{n+1/2}}\frac{\left(\varphi(t^{n+1/2},x_j)-\varphi(t^n,x_j)\right)}{\Delta t}-\Delta t\sum_{j\in\mathbb{Z}}\boldsymbol{\rho_j^0}\varphi(0,x_j),$$

and by the the Dominate Convergence Theorem, we get for $i=1,2$

$$I_1\to-\int_0^T\int_{\mathbb{R}}\boldsymbol{\rho_i}(t,x)\partial_t\varphi(t,x)\mathrm{d}x\mathrm{d}t-\int_{\mathbb{R}}\boldsymbol{\rho_j^0}\varphi(0,x)\mathrm{d}x.$$

We now study I_2, this term can be rewritten as

$$I_2\quad=\quad-\Delta t\Delta x\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N_T}\boldsymbol{F_{j+1/2}^{n}}\left(\frac{\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)}{\Delta x}\right),$$

and again by Dominate Convergence Theorem we get

$$I_2\to-\int_0^t\int_{\mathbb{R}}\boldsymbol{F}\partial_x\varphi(t,x)\mathrm{d}x\mathrm{d}t,$$

where $\boldsymbol{F}=\rho_i v_i(\rho_i+(\rho_{\max}-\rho_i)H_\varepsilon(\tilde{\rho}_{[i]}*\omega_\eta))$ for $i=1,2$, thus

$$I_1+I_2\to-\int_0^T\int_{\mathbb{R}}\boldsymbol{\rho_i}(t,x)\partial_x\varphi(t,x)\mathrm{d}x\mathrm{d}t-\int_{\mathbb{R}}\boldsymbol{\rho_j^0}\varphi(0,x)\mathrm{d}x-\int_0^t\int_{\mathbb{R}}\boldsymbol{F}\partial_x\varphi(t,x)\mathrm{d}x\mathrm{d}t.$$

Now, the next step in the proof is to multiply (3.7) by $\Delta x\varphi(t^n,x_j)$ and summing over $j\in\mathbb{Z}$ and over $n=0,\dots,N_T$ yields

$$\Delta x\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N_T}\left(\boldsymbol{\rho_j^{n+1}}-\boldsymbol{\rho_j^{n+1/2}}\right)\varphi(t^n,x_j)-\Delta x\Delta t\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N_T}\left(-\boldsymbol{S_j^{n+1/2}},\boldsymbol{S_j^{n+1/2}}\right)\varphi(t^n,x_j)=0, \tag{3.27}$$

then, by replacing $\boldsymbol{\rho_j^{n+1/2}}$ of (3.6) in (3.27) we get

$$I_1+I_2+I_3=0,$$

where

$$I_3=-\Delta x\Delta t\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N_T}\left(-\boldsymbol{S_j^{n+1/2}},\boldsymbol{S_j^{n+1/2}}\right)\varphi(t^n,x_j),$$

from which we can derive

$$I_3\to-\int_0^T\int_{\mathbb{R}}\left(-\left(\boldsymbol{S_O}-\boldsymbol{S_R}\right),\boldsymbol{S_O}-\boldsymbol{S_R}\right)\varphi(t,x)\mathrm{d}x\mathrm{d}t.$$

Therefore,

$$\int_0^T\int_{\mathbb{R}}\boldsymbol{\rho_i}(t,x)\partial_x\varphi(t,x)\mathrm{d}x\mathrm{d}t+\int_0^t\int_{\mathbb{R}}\boldsymbol{F}\partial_x\varphi(t,x)\mathrm{d}x\mathrm{d}t$$
$$+\int_0^T\int_{\mathbb{R}}\left(-\left(\boldsymbol{S_O}-\boldsymbol{S_R}\right),\boldsymbol{S_O}-\boldsymbol{S_R}\right)\varphi(t,x)\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}}\boldsymbol{\rho_j^0}\varphi(0,x)\mathrm{d}x=0.$$

□

4. Numerical Examples

In the following numerical tests, we will solve (2.1) numerically for $x \in [0,5]$, by using Algorithm 3.1, where we set Δt satisfying CFL condition (3.8). From Example 1 to Example 4 we consider the velocity functions as $v_i(\rho) = V_{\max}(\rho_{\max} - \rho)$, $i = 1, 2$ with maximum speed $V_{\max} = 1$ and maximal density $\rho_{\max} = 1$. We consider kernel functions

$$\omega_\eta(x) = \frac{1}{\eta}, \quad \omega^1_\eta(x) = 2\frac{\eta - x}{\eta^2} \quad \omega^2_\delta(x) = \frac{1}{\delta}$$

with length support $\eta = 0.1$, and $\delta = 0.5$. Regarding the terms K_1 and K_2 in the right hand side (1.2), (1.3), (1.4) and (1.5) we consider $K_2 = 2K_1$, and $K_1 = 10$ in order to get a faster return to the preferred lane of the overtaking vehicles class. Additionally, we consider the following regularization for the indicator function

$$H_\varepsilon(z) = \begin{cases} 0, & \text{if} \quad z < 0 \\ \exp(-50(\frac{z-\varepsilon}{\varepsilon})^2), & \text{if} \quad 0 \le z \le \varepsilon \\ 1, & \text{if} \quad z > \varepsilon, \end{cases}$$

with $\varepsilon = 0.1$. Finally we consider periodic boundary conditions at $x = 0$ and $x = 5$ for all examples.

4.1. Example 1. Overtaking Dynamics only for ρ_1.

In this example we consider only vehicles of the class ρ_1 in its preferential road, more specifically we take initial conditions,

$$\rho^0_1(x) = \begin{cases} 0.5, & \text{if} \quad 0.2 < x < 0.6, \\ 0.9, & \text{if} \quad 1 < x < 2, \end{cases}, \quad \rho^0_2(x) = \tilde{\rho}^0_1(x) = \tilde{\rho}^0_2(x) = 0, \quad x \in [0,5].$$

In Figure 2 we display a heat map of $\boldsymbol{\rho}^{\boldsymbol{\Delta}}(\cdot,t) = [\rho_1(\cdot,t), \rho_2(\cdot,t)]$ for $t \in [0, 2.5]$ computed with $\Delta x = 1/160$. Initially there are two platoons of vehicles in lane 1, the upstream one with medium density and the downstream one with high density, where platoon one is shorter than the second. As time progresses, we can observe that in the head of downstream platoon, a rarefaction wave is formed, which causes the local velocity at each point to be lower than the average velocity of the cars in front, so no produce overtaking vehicles. Only the effect of the overtaking vehicles is observed in the back of downstream platoon and in the upstream platoon, which is due the local velocity is lower in these points. After a short time it is observed that The vehicles rejoin the preferential way again.

4.2. Example 2: No collisions.

This example shows an extreme case in which the class of vehicles ρ_1 travels on lane 1 and there is a lane invasion by the class of vehicles traveling in the opposite direction, i.e., $\tilde{\rho}_2$ is non zero close to ρ_1 on lane 1. The main aim of this example is to show that the proposed model (2.1) doesn't allow crashes among vehicles. As initial conditions for $x \in [0,5]$ we consider

$$\rho^0_1(x) = \begin{cases} 0.9, & 0.5 < x < 1.5, \\ 0, & \text{Otherwise}, \end{cases} \qquad \tilde{\rho}^0_2(x) = \begin{cases} 0.9, & 2.5 < x < 3.5 \\ 0, & \text{Otherwise}, \end{cases}$$

$$\tilde{\rho}^0_1 = \rho^0_2 = 0.$$

In Figure 3, we display density profile of approximate solutions at three different simulation times, $t = 0$, $t = 0.3$ and $t = 1$. At $t = 0$ we can observe an initial platoon for ρ_1 in the preferred lane, and in opposite direction an initial platoon of vehicles of class $\tilde{\rho}_2$ are invading the lane 1. At $t = 0.3$ we can observe in lane 1 the rarefaction wave formed for the vehicles of class ρ_1 in the

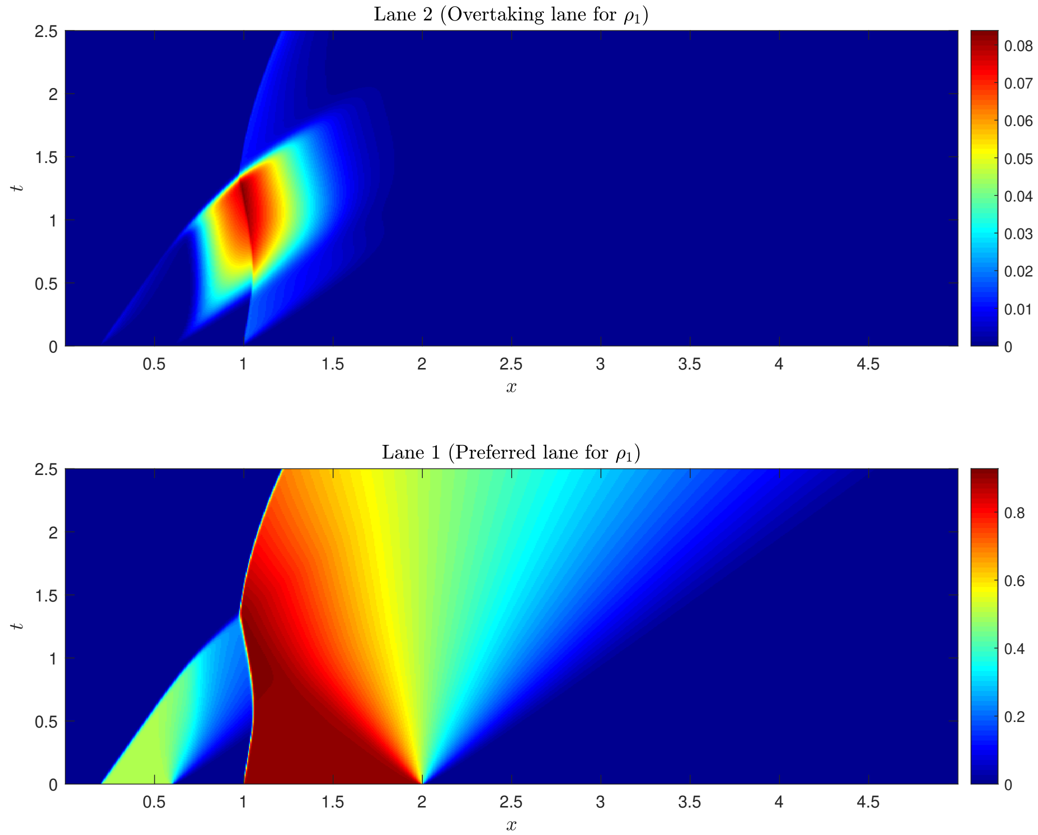

Figure 2. Example 1: heat map of $\boldsymbol{\rho}^{\boldsymbol{\Delta}}(\cdot,t)$ for $t \in [0, 2.5]$. Overtaking Dynamics only for ρ_1. Bottom: dynamics of traffic for ρ_1 in Lane 1. Top: Dynamics of vehicles ρ_2 overtaking in Lane 2.

head of platoon, and some vehicles at the back of the platoon move to lane 2 overtaking others vehicles, and becoming part of the class ρ_2. Meanwhile invading vehicles of class $\tilde{\rho}_2$ return back to preferential lane 2 becoming part of the class $\tilde{\rho}_1$. At $t = 1$ we can observe in lane 1, at the head of the rarefaction wave an increasing in the density of ρ_1, which is due to the fact that on the horizon there are vehicles of class $\tilde{\rho}_2$ invading the lane, these vehicles wait for those coming in the opposite direction to return to their preferred lane. In lane 2, we can observe a rarefaction wave formed by vehicles of class $\tilde{\rho}_1$, moreover we observe that density $\tilde{\rho}_2$ has decreased over time, due to them have moved to preferential lane 2 increasing the density of $\tilde{\rho}_1$ over time.

On the other hand, in Figure 4, we display a heat map of $\boldsymbol{\rho}^{\boldsymbol{\Delta}}(\cdot,t) = [\rho_1(\cdot,t), \rho_2(\cdot,t)]$ and $\tilde{\boldsymbol{\rho}}^{\boldsymbol{\Delta}}(\cdot,t) = [\tilde{\rho}_1(\cdot,t), \tilde{\rho}_2(\cdot,t)]$, $t \in [0, 2.5]$. Initially, we can observe the dynamics vehicles in lane 1 which is affected at $t = 0.5$ close to $x = 2$ increasing the density due to the fact that on the horizon there are vehicles travelling in opposite direction, subsequently continue its trajectory. On the other hand, vehicles invading lane 1, return quickly to its preferential lane, then they continue its trajectory. Finally we can observe the dynamics of vehicles in lane 2 overtaking other vehicles, and how its trajectory after $t = 1.4$ is affected by vehicles traveling in opposite directions, moreover we observe that there are not vehicles try to overtake after this time due to the lane 2 is not empty.

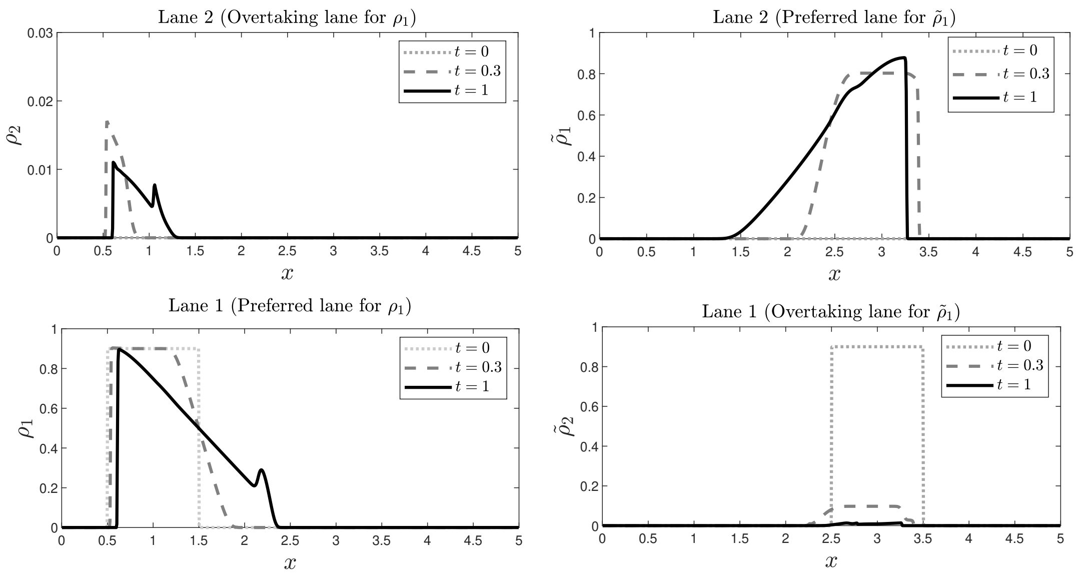

Figure 3. Example 2: numerical solutions of system (2.1) at $t = 0$, $t = 0.3$ and $t = 1$. Profile of numerical solution for (Left-top) ρ_2, (Left-Bottom) ρ_1, (Right-top) $\tilde{\rho}_1$, (Right-Bottom) $\tilde{\rho}_2$

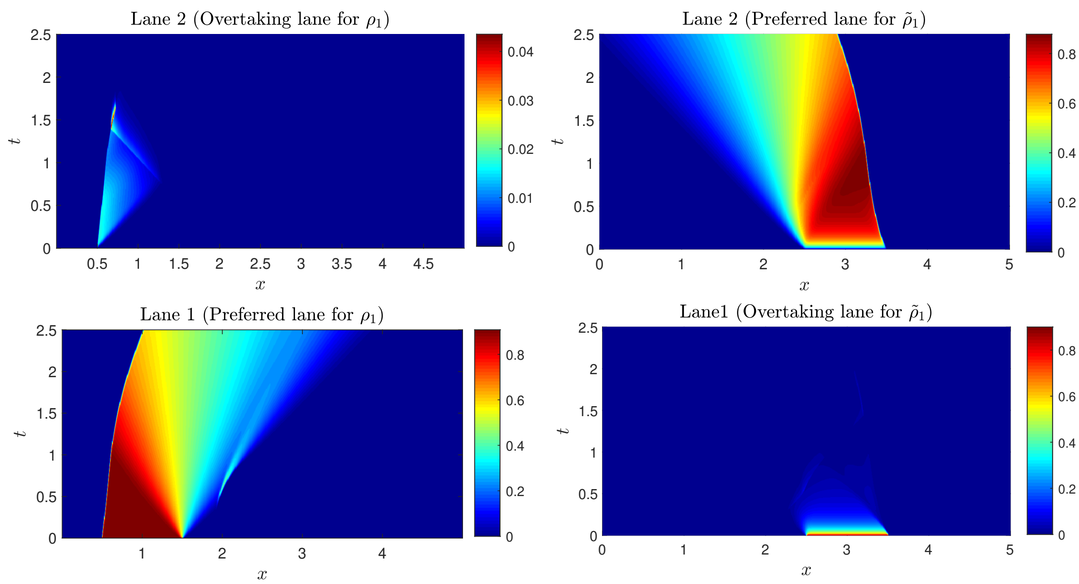

Figure 4. Example 2, heat map of $\boldsymbol{\rho}^{\boldsymbol{\Delta}}(\cdot, t) = [\rho_1(\cdot, t), \rho_2(\cdot, t)]$ and $\tilde{\boldsymbol{\rho}}^{\boldsymbol{\Delta}}(\cdot, t) = [\tilde{\rho}_1(\cdot, t), \tilde{\rho}_2(\cdot, t)]$, $t \in [0, 2.5]$. Dynamics of numerical solution for (Left-top) ρ_2 (Left-Bottom) $\rho_1 2$ (Right-top) $\tilde{\rho}_1$ (Right-Bottom) $\tilde{\rho}_2$

4.3. **Example 3: Convergence Test.** With this example, we illustrate the convergence of the approach obtained with Algorithm 3.1. To this end, we consider the same parameters as in Example 2 at time $t = 2.5$. In Figure 5, we can see several approximate solutions computed by means of Algorithm 3.1 for $\Delta x = 1/20, 1/40, 1/80, 1/160$ and a reference solution corresponding to $\Delta x = 1/640$ computed by means of the same algorithm. As expected, as Δx diminishes, the numerical solutions approach the reference solution, as reported in Table 1, in which the total error, and the

Experimental Order of Convergence (E.O.C.) in $\mathbf{L^1}$ norm are shown. The total error is computed as

$$e_{\Delta x}(u) = e_{\Delta x}(\rho_1) + e_{\Delta x}(\rho_2) + e_{\Delta x}(\tilde{\rho}_1) + e_{\Delta x}(\tilde{\rho}_2),$$

where $e_{\Delta x}(\rho) = \|\rho_{\text{Ref}} - \rho\|_1$ and ρ_{Ref} is the reference solution.

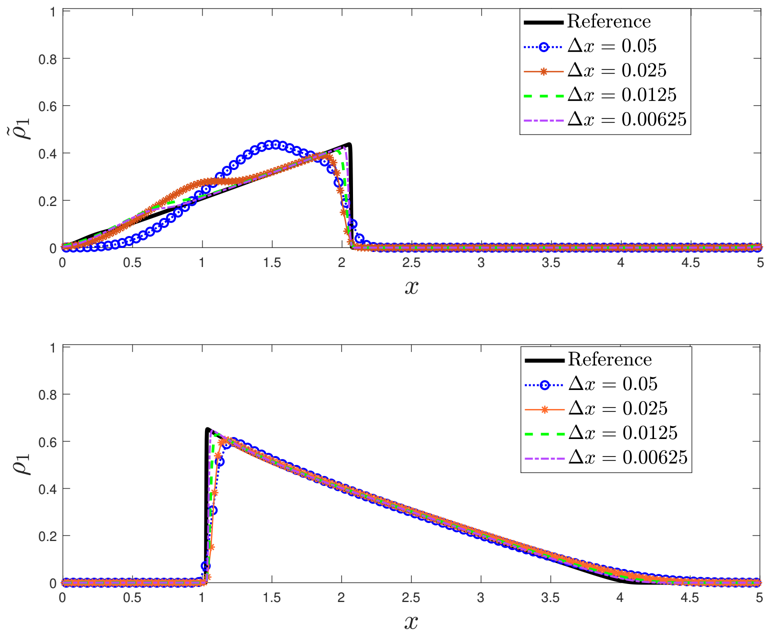

Figure 5. Example 3. Bottom: Convergence of a sequence of approximate solutions ρ_1^Δ to the reference solution. Top: Convergence of a sequence of approximate solutions $\tilde{\rho}_1^\Delta$ to the reference solution.

Table 1. Example 3: Total L^1-error $e_{\Delta x}(u)$.

$T = 2.5$		
$1/\Delta x$	Total $e_{\Delta x}$	E.O.C.
20	0.2173	- - -
40	0.1199	0.8
80	0.0628	0.9
160	0.02978	1.0

4.4. **Example 4: Invariant Region.** The aim of this example is to show numerically that the numerical approach obtained with Algorithm 3.1 preserve a invariant region

$$\tilde{\Omega} := \{(\rho_i, \tilde{\rho}_{[i]}) \in \mathbb{R}^2 : \rho_i + \tilde{\rho}_{[i]} \leq 1, \text{ for } i = 1, 2\}.$$

For this example we consider the initial data given by

$$\rho_1^0(x) = \begin{cases} 0.9 & \text{if } 0.5 < x < 2.5, \\ 0.1 & \text{if } 2.5 < x < 4.5, \end{cases} \qquad \tilde{\rho}_2^0(x) = \begin{cases} 0.1 & \text{if } 0.5 < x < 2.5, \\ 0.75 & \text{if } 2.5 < x < 4.5, \end{cases}$$

and $\rho_2^0 = \tilde{\rho}_1^0 = 0$, i.e., there are vehicles of two classes traveling in opposite directions and occupying the same cells. In Figure 6 we can see that from $x = 2.5$ the total density in lane 1 decreases, while the total density of lane 2 increases. Similarly, for $x \in [0.5, 2.5]$ the total density in lane 1 increases, but in lane 2 decreases, remaining less than $\rho_{\max}$, i.e., sum of densities in each lane remains less than 1 over time.

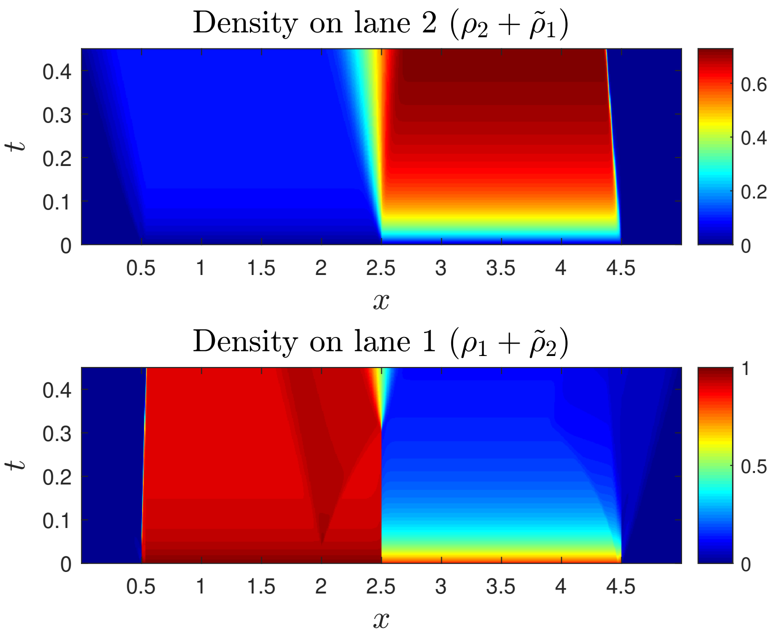

Figure 6. Example 4. Top: Sum of densities on lane 2. Bottom: Sum of densities on lane 1.

5. Conclusions and discussion

In this work, we introduced a system of nonlocal balance laws which describes vehicular traffic flow in a two way and two lane road. Our model allows for vehicles overtaking using the adjacent lane, and returning to the preferred lane, namely, the right lane. We distinguish four classes of vehicles, labeled ρ_1, ρ_2, $\tilde{\rho}_1$, $\tilde{\rho}_2$, according to the direction of travel and the lane used. We provided compactness estimates that allowed us to apply the Helly's Compactness Theorem to prove convergence and existence of weak solutions, in particular, we were able to prove a maximum principle for each class of vehicles considered in the model. Additionally, we show some numerical experiments in which some features of the model are displayed, e.g., no crashes between vehicles traveling in opposite direction in a same lane, the overtaking and returning maneuvers, etc.

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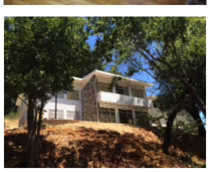

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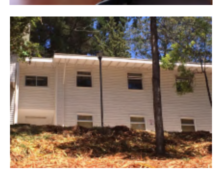

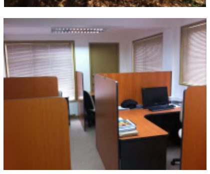

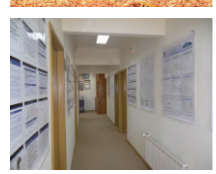

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