

Rigidity results in cellular automata

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Recall: Symbolic Dynamics (in dimension 1)

- Consider A a finite set and $X = A^{\mathbb{Z}}$ the set of two-sided sequences

$$\mathbf{x} = (x_i)_{i \in \mathbb{Z}} = (\dots x_{-i} \dots x_0 \dots x_i \dots)$$

of symbols in A . Analogously one defines $X = A^{\mathbb{N}}$ the set of one-sided sequences in A . Both are called **full-shifts**. For simplicity we restrict to the two-sided case.

- The space X is compact for the product topology and metrizable (two points are close if they coincide near the origin).
- A natural dynamical system is the **shift map** $\sigma : X \rightarrow X$, where $\sigma(\mathbf{x}) = (x_{i+1})_{i \in \mathbb{Z}}$. It is a homeomorphism of X .
- **Subshifts**: if $Y \subset X$ is closed and $\sigma(Y) \subset Y$ it is called a subshift. Consider the orbit closure of points in X as a first example.

Recall: Subshifts of Finite Type (SFT)

Special subshifts are **subshifts of finite type**; they look like Markov chains in probability theory. Y is a subshift of finite type if there is a finite subset \mathcal{W} of words in A of a given length L such that for any $y \in Y$ and $i \in \mathbb{Z}$,

$$y_i \dots y_{i+L-1} \notin \mathcal{W}$$

Example: $A = \{0, 1, 2\}$ and $\mathcal{W} = \{02, 10, 11, 22\}$:

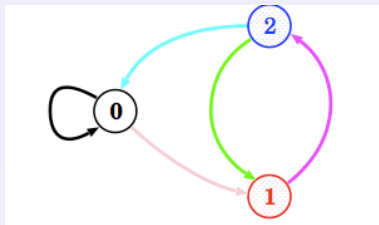


Figure:

Recall: Block maps

- A second kind of important dynamics are given by continuous and shift commuting maps of a subshift Y : $F : Y \rightarrow Y$. That is: F is **continuous** and $F \circ \sigma = \sigma \circ F$.

- They are called **block maps** since there is a **local map**,

$$f : A^{m+a+1} \rightarrow A$$

$a, m \in \mathbb{N}$ (a = anticipation and m = memory respectively), such that $\forall i \in \mathbb{Z}, \forall \mathbf{y} \in Y$

$$F(\mathbf{y})_i = f(y_{i-m}, \dots, y_{i+a})$$

- **Cellular automaton**: Y is a **mixing** shift of finite type (i.e., two words in Y can be glued in a very strong way inside Y), typically the fullshift

Main questions and **evidence** !!!

Randomization evidence (here a CA on $\{0, 1, 2\}^{\mathbb{Z}}$):

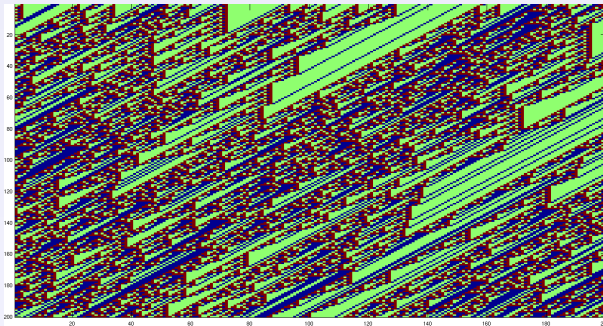


Figure: Iteration of a CA

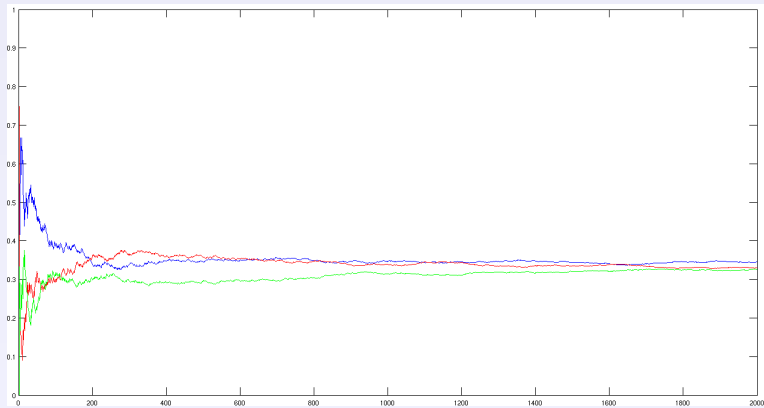


Figure: Frequency of symbols after “Cesàro mean”

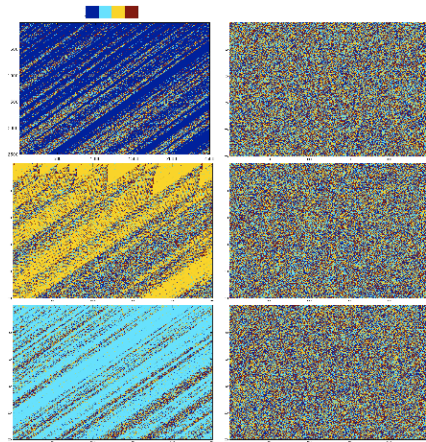


Figure: Other Automata, same phenomena

Recall: Entropy

Classical measure of complexity of the dynamics with respect to an invariant measure μ

$$h_\mu(\sigma) = - \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{a_0, \dots, a_{N-1}} \mu([a_0 \dots a_{N-1}]) \log \mu([a_0 \dots a_{N-1}])$$

where $[a_0 \dots a_{N-1}] = \{\mathbf{y} \in Y : y_0 \dots y_{N-1} = a_0 \dots a_{N-1}\}$.

A measure of **maximal entropy** (for the shift map here) is one for which:

$$h_\mu(\sigma) = \sup_{\nu} h_\nu(\sigma)$$

Let $F : Y \rightarrow Y$ be a surjective or onto block map of a mixing subshift of finite type or cellular automaton.

Question 1: Given a **shift invariant** probability measure μ on Y describe if it exists the limit of the sequence $(F^n \mu : n \in \mathbb{N})$. Every weak limit of a subsequence is invariant for F (and the shift). It is also interested the convergence when $N \rightarrow \infty$ of the Cesàro mean

$$\mathcal{M}_\mu^N(F) = \frac{1}{N} \sum_{n=0}^{N-1} F^n \mu$$

One says F **asymptotically randomizes** μ if the limit of the Cesàro mean converges to the **maximal entropy measure**.

Question 2: Study **invariant measures of F** and for the joint action of **F** and **σ** : i.e. probability measures μ such that for any Borel set $B \in \mathcal{B}(X)$ and $n \in \mathbb{N}, m \in \mathbb{Z}$

$$F^n \mu(B) := \mu(F^{-n} B) = \mu(B)$$

or

$$F^n \circ \sigma^m \mu(B) := \mu(F^{-n} \circ \sigma^{-m} B) = \mu(B)$$

- A natural invariant measure for F is the maximal entropy one for the shift map. In fact F is onto if and only if the maximal entropy measure is F -invariant (Coven-Paul).
- Depending on the subshift Y and dynamical properties of F it is possible to construct other invariant measures; nevertheless in some cases strong rigidities appear (for example when strong forms of expansivity exist) .

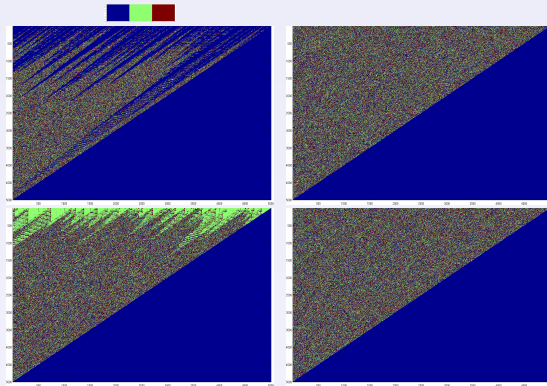
Looking for a good class of examples:

Dichotomy:

- From **Glasner and Weiss** result in topological dynamics one gets essentially that either the map F is **almost equicontinuous** or **sensitive to initial conditions**, and in the last class most interesting known examples (and in fact comes from Nasu's reductions) are **expansive** or **positively expansive maps**.
- In the equicontinuous case or systems with **equicontinuous points**, orbits tend to be periodic and **invariant measures can be more or less described** but are not nice.

- If the maps are positively expansive they are **conjugate with shifts of finite type** (M-Blanchard, Nasu, M-Boyle), so we have two commuting shifts of finite type with the same maximal entropy measure. In this last case there can still exist an **equicontinuous direction** so invariant measures are as in previous cases.
- Good examples: **(positively) expansive maps without equicontinuous directions**; even if not easy to know a priori how they are constructed, there are some advances by Boyle-Lind and Mike Hochman from the point of view of expansive subdynamics. Main classes with this features correspond to **algebraic maps**.

Another nice example: Let $A = \{0, 1, 2\}$ and consider the CA $F : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ such that $F(0, a) = a$ and $F(1, a) = a$ for any $a \in A$, but $F(2, a) = a + 2 \pmod 3$. We have a strong evidence that it randomizes:



Basic Example: addition modulo 2 or *Ledrappier's three dot problem*

Let $X = \{0, 1\}^{\mathbb{Z}}$ (see X as an Abelian group with coordinatewise addition modulo 2) and $F : X \rightarrow X$ given by $F(\mathbf{x}) = id + \sigma$, where σ is the shift map in X . That is, $F(\mathbf{x})_i = x_i + x_{i+1}$. It is a 2-to-1 onto map.

– **In relation to Question 2:** Natural invariant measures are the uniform Bernoulli measure $\lambda = (1/2, 1/2)^{\mathbb{Z}}$ and measures supported on periodic orbits of F . But there exist other invariant measures of algebraic origin that has been described in works by M. Einsiedler, E. Lindenstrauss, B. Kitchens, K. Schmidt.

– In relation to Question 1: In general the limit does not exist: Pascal triangle modulo 2 in Bernoulli case (we only draw one-sided sequences).

x	=	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_{2n}
$F(x)$	=	x_0+x_1	x_1+x_2	x_2+x_3	x_3+x_4	x_4+x_5	x_5+x_6	x_6+x_7
$F^2(x)$	=	x_0+x_2	x_1+x_3	x_2+x_4	x_3+x_5	x_4+x_6	x_5+x_7
$F^3(x)$	=	$x_0+x_1+x_2+x_3$	$x_1+x_2+x_3+x_4$	$x_2+x_3+x_4+x_5$	$x_3+x_4+x_5+x_6$
$F^4(x)$	=	x_0+x_4	x_1+x_5	x_2+x_6
...
...
$F^{2n-1}(x)$	=	$x_0+\dots+x_{2n-1}$
$F^{2n}(x)$	=	x_0+x_{2n}

— Assume $\mu = (\pi_0, \pi_1)^{\mathbb{Z}}$ be a Bernoulli non-uniform measure on X with $\pi_0 = \mu(x_i = 0)$, $\pi_1 = \mu(x_i = 1)$.

— A simple induction yields to:

$$\mu \left(\sum_{i \in I} x_i = a \right) = \frac{1}{2} \left(1 + (-1)^a (\pi_0 - \pi_1)^{\#I} \right)$$

— Thus,

$$F^n \mu[a]_0 = \mu \left(\sum_{k \in I(n)} x_k = a \right) = \frac{1}{2} \left(1 + (-1)^a (\pi_0 - \pi_1)^{\#I(n)} \right)$$

where $I(n) = \{0 \leq k \leq n : C_n^k = 1 \bmod 2\}$.

— If $a = 0$ for $n = 2^m$ the limit is $\pi_0^2 + \pi_1^2$ and for $n = 2^m - 1$ the limit is $\frac{1}{2}$.

• But the Cesàro mean converges:

$$\mathcal{M}_\mu^N(F)[a]_0 = \frac{1}{N} \sum_{n=0}^{N-1} F^n \mu[a]_0 = \frac{1}{2} + \frac{(-1)^a}{2} \frac{1}{N} \sum_{n=0}^{N-1} (\pi_0 - \pi_1)^{\#I(n)}$$

since $\lim_{N \rightarrow \infty} \frac{\#\{0 \leq n < N : \#I(n) \geq \alpha \log \log N\}}{N} = 1$ for some $\alpha \in (0, 1/2)$ (Lucas' lemma)

then the limit is $\frac{1}{2}$ and for ℓ coordinates is $\frac{1}{2^\ell}$. This was observed by D. Lind in 84 for $F = \sigma^{-1} + \sigma$.

This result reinforces the idea that the **uniform Bernoulli measure** $\lambda = (1/2, 1/2)^{\mathbb{Z}}$ must be the unique invariant measure of (F, σ) verifying “some conditions to be determined”.

• **Question 3:** Find conditions to ensure the **maximal entropy measure** is the unique solution to Questions 1 and 2.

— In relation with Question 1 there are two points of view. One is to **consider measures μ of increasing complexity in correlations**: Markov, Gibbs, other chain connected measures; represent them as **“independent processes”** and prove that the limit of the Cesàro mean converges to λ . The other is motivated in harmonic analysis and Lind’s work; the idea is to **define a class of mixing measures** such that the Cesàro mean of any of them converges.

— In relation with Question 2 the type of solutions looks like the **$(\times 2, \times 3)$ -Furstenberg problem** in \mathbb{R}/\mathbb{Z} : **F (or σ) ergodic and σ (resp. F) with positive entropy** for the invariant measure. While ergodicity of one transformation can be changed for a weaker condition the positivity of entropy cannot be dropped for the moment. Proofs strongly rely on **entropy formulas**. These conditions already appear in Rudolph’s solution to $(\times 2, \times 3)$ problem and all recent improvements and related results by Host, **Lindenstrauss**, Einsiedler, ...

Iteration of measures: harmonic analysis point of view

— Lind 84, Pivato-Yassawi 02, 04, Host-M-Martínez 03,
M-Martínez-Pivato-Yassawi 06

Theorem (Pivato,Yassawi 02,04; Ferrari,M,Martínez,Ney 00)

Let A be a finite abelian group. Then if the f_k , $k = -m, \dots, a$, are commuting automorphisms of A and at least two are nontrivial, then F is diffusive in density. Therefore for any harmonically mixing measure μ :

$$\mathcal{M}_\mu(F) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} F^n \mu = \lambda$$

Iteration of measures: regeneration of measures point of view

— Ferrari, M, Martínez, Ney 2000, Host, M, Martínez 2003, M, Martínez, Sobottka 2006

— Let μ be any shift invariant probability measure on $A^{\mathbb{Z}}$ and consider $w = (\dots, w_{-2}, w_{-1}) \in A^{-\mathbb{N}}$. Denote by μ_w the **conditional probability measure** on $A^{\mathbb{N}}$.

— One says that μ has **complete connections** if given $a \in A$ and $w \in A^{-\mathbb{N}}$, $\mu_w([a]_0) > 0$. If μ is a probability measure with complete connections, one define the quantities γ_m , for $m \geq 1$, by

$$\gamma_m = \sup \left(\left| \frac{\mu_v([a]_0)}{\mu_w([a]_0)} - 1 \right| : v, w \in A^{-\mathbb{N}}; v_{-i} = w_{-i}, 1 \leq i \leq m \right)$$

— If $\sum_{m \geq 1} \gamma_m < \infty$ one says μ has **summable decay of correlations**.

Theorem (Ferrari-M-Martínez, Ney 00)

Let $(A, +)$ be a finite Abelian group, μ a probability measure on $A^{\mathbb{Z}}$ with *complete connections and summable decay of correlations*. Let $F : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ as before. Then $\forall w \in A^{-\mathbb{N}} \exists$ Cesàro mean distribution $\mathcal{M}_{\mu_w}(F) = \lambda$.

Several generalizations by: M, Martínez, Pivato, Yassawi 06,
M, Martínez, Sobottka 05.

Results on (F, σ) -invariant measures: the basic theorem in the theory concerns our basic example.

Theorem (Basic Theorem: Host-M-Martínez)

Let $F : \{0, 1\}^{\mathbb{Z}} \rightarrow \{0, 1\}^{\mathbb{Z}}$, $F = id + \sigma$. If μ is (F, σ) invariant with $h_{\mu}(F) > 0$ and σ -ergodic, then $\mu = \lambda$.

Some generalizations:

Theorem (Host-M-Martínez)

Let $F : \mathbb{Z}_p^{\mathbb{Z}} \rightarrow \mathbb{Z}_p^{\mathbb{Z}}$ be linear. Let μ be (F, σ) -invariant. If $h_\mu(F) > 0$ and μ is ergodic for σ then μ is the uniform Bernoulli measure.

Theorem (Host-M-Martínez)

Let $F : \mathbb{Z}_p^{\mathbb{Z}} \rightarrow \mathbb{Z}_p^{\mathbb{Z}}$ be linear. Let μ be (F, σ) -invariant. If $h_\mu(F) > 0$, μ is ergodic for (σ, F) and $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{p(p-1)})$, then μ is the uniform Bernoulli measure.

– A block map $F : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ is algebraic if $A^{\mathbb{Z}}$ is a compact abelian topological group and F and the shifts are endomorphisms of such group.

Theorem (Pivato)

Let $F : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ be an algebraic bipermutative block map. Then, if μ is **totally ergodic** for σ , $h_{\mu}(F) > 0$ and $\text{Ker}(F)$ has no shift invariant subgroups, then μ is the Haar measure.

Theorem (Einsiedler)

Let α be an algebraic \mathbb{Z}^d -action of a compact 0-dimensional abelian group, and some additional algebraic conditions. Positive entropy in one direction and **totally ergodicity** of the action imply Haar measure.

Theorem (Sablik)

Let $F : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ be an algebraic bipermutative block map and Σ be a (F, σ) -invariant closed subgroup of $A^{\mathbb{Z}}$. Fix $k \in \mathbb{N}$ such that any prime divisor of $|A|$ divides k . If μ is (F, σ) -invariant with $\text{supp}(\mu) \subseteq \Sigma$ such that:

- μ is ergodic for (σ, F) ,
- $h_{\mu}(F) > 0$,
- $\mathcal{I}_{\mu}(\sigma) = \mathcal{I}_{\mu}(\sigma^{kp_1})$, where p_1 is the smallest common period of the elements in $\text{Ker}(F)$,
- any finite shift invariant subgroup of $\bigcup_{n \in \mathbb{N}} \text{Ker}(F^n) \cap \Sigma$ is dense in Σ ,

then μ is the Haar measure of Σ .

— **Remark.** From last theorems it is possible to deduce the same kind of results for some classes of **positively expansive** and **expansive** block maps of a fullshift, *a priori* not algebraic.

Final Comments:

- Change “complete connections and summable decay of correlations” by some mixing property for the shift map.
- The asymptotic randomization does not require full support of initial measure and positive entropy w.r.t. the shift map: there exist shift invariant measures μ on $\{0, 1\}^{\mathbb{Z}}$ with $h_\mu(\sigma) = 0$ that are asymptotically randomized by $F = id + \sigma$ (Pivato-Yassawi examples 2006).
- **Question:** Cesàro means exist for expansive and positively expansive block maps of a mixing shift of finite type ?; how the limit is related with the unique maximal entropy measure ?
Partial results for classes of right permutative cellular automata: with associative local rules, or N -scaling local rules (Host, M, Martínez); they can be seen as the product of an algebraic CA with a shift: here measures are not asymptotically randomized but the limit are the product of a maximal measure with a periodic measure.

Thanks !!!